Coloquio IMAFI
Instituto de Matemática y Física
Universidad de Talca
Camino Lircay S/N, Campus Norte, Talca-Chile

On the Pierce–Birkhoff conjecture and related problems.

Mark Spivakovsky*
Institut de Mathématiques de Toulouse, Université Paul Sabatier

Abstract. Let $B = \mathbb{R}[x_1, \ldots, x_n]$ be a polynomial ring over $\mathbb{R}$ in n variables.

Definition 0.1 *A function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be **piecewise polynomial** if $\mathbb{R}^n$ can be covered by a finite collection of closed semi-algebraic sets P_i, $i \in \{1, \ldots, s\}$ such that for each i there exists a polynomial $g_i \in B$ satisfying $g|_{P_i} = g_i|_{P_i}$.*

Piecewise polynomial functions form a ring, containing B, which is denoted by $PW(B)$.

Consider the ring (contained in $PW(B)$) of all the functions obtained from B by iterating the operations of sup and inf. The Pierce–Birkhoff conjecture was stated by M. Henriksen and J. Isbell in the early nineteen sixties ([1] and [3]):

Conjecture 1 (Pierce-Birkhoff) *If $g : \mathbb{R}^n \to \mathbb{R}$ is in $PW(B)$, then there exists a finite family of polynomials $g_{ij} \in B$ such that $f = \sup_i \inf_j (g_{ij})$ (in other words, for all $x \in \mathbb{R}^n$, $f(x) = \sup_i \inf_j (g_{ij}(x))$).*

In this talk, we will discuss our partial results (joint with F. Lucas, J. Madden and D. Schaub) on the Pierce–Birkhoff conjecture and related problesm. We will recall the definition of the real spectrum of a ring Σ, denoted by Sper Σ. We will describe our approach to the problem using valuation-theoretic methods. Time permitting, we will discuss our recent result that, given a real closed field R and a connected semi-algebraic set $D \subset R^n$, if the number of real roots of f, counted with or without multiplicity, is constant for all $x \in D$ then these roots are represented by continuous semi-algebraic functions $\phi_j : D \to R$.

References

[1] G. Birkhoff and R. Pierce, *Lattice-ordered rings.* Annales Acad. Brasil Ciênc. 28, 41–69 (1956).

*e-mail: mark.spivakovsky@math.univ-toulouse.fr

[2] M. Henriksen and J. Isbell, *On the continuity of the real roots of an algebraic equation*, Proc. AMS Vol. 4, pp. 431-434, 1953.

[3] M. Henriksen and J. Isbell, *Lattice-ordered rings and function rings.* Pacific J. Math. 11, 533–566 (1962).

[4] F. Lucas, J.J. Madden, D. Schaub and M. Spivakovsky, *On connectedness of sets in the real spectra of polynomial rings*, Manuscripta Math. 128, 505-547, 2009.

[5] F. Lucas, D. Schaub and M. Spivakovsky, *Approximate roots of a valuation and the Pierce-Birkhoff Conjecture*, Ann. Fac. Sci. Toulouse, Mathématique, Série 6, Vol. XXI, Fasc. 2, 259-342, 2012.

[6] F. Lucas, D. Schaub and M. Spivakovsky, *On the Pierce-Birkhoff Conjecture*, Journal of Algebra 435, (2015), 124-158.

[7] J. J. Madden, *Pierce–Birkhoff rings.* Arch. Math. 53, 565–570 (1989).