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Analysis of the controllability of space-time fractional diffusive and super-diffusive equations

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Abstract

We consider the following class of fractional partial differential equations of evolution in which two parameters are used to sharpen the models:

$$\begin{cases} \mathbb{D}_t^\alpha u(x,t) + (-\Delta)^s u(x,t) = f\chi_{\omega\times(0,T)} & \text{in } \Omega\times(0,T), \\ + \text{ Intial conditions,} \\ + \text{ Boundary or exteriror conditions.} \end{cases} \tag{1}$$

Here, $T > 0$ is a fixed real number (the time), $0 < \alpha \leq 2$, $0 < s \leq 1$, $\Omega \subset \mathbb{R}^N$ is a bounded open set with boundary $\partial\Omega$, $(-\Delta)^s$ is the fractional Laplace operator and $\mathbb{D}_t^\alpha$ denotes a time fractional derivative. In the system (1), $u = u(x,t)$ is the state to be controlled and f is the control function which is localized in a non-empty open set $\omega \subset \Omega$.

In the first part, we clarify which initial and boundary conditions would make the system (1) a well-posed Cauchy problem. In the second part, we study completely the controllability properties of the system. More precisely, we show what is so far known about the null/exact controllability or the approximate controllability of the above system. We conclude by given several open problems. The talk will be delivered for a wide audience avoiding unnecessary technicalities.

Fractional order operators have recently emerged as a modeling alternative in various branches of science. From the long list of phenomena which are more appropriately modeled by fractional differential equations, we mention: viscoelasticity, imaging science, phase field models, Magnetotellurics in geophysics, electrical response in cardiac tissue, diffusion of biological species, data science, anomalous transport and diffusion, hereditary phenomena with long memory, nonlocal electrostatics, the latter being relevant to drug design, and Lévy motions which appear in important models in both applied mathematics and applied probability. A number of stochastic models for explaining anomalous diffusion have been also introduced in the literature; among them

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we quote the fractional Brownian motion; the continuous time random walk; the Lévy flights; the Schneider grey Brownian motion; and more generally, random walk models based on evolution equations of single and distributed fractional order in space. In general, a fractional diffusion operator corresponds to a diverging jump length variance in the random walk. In fact under a very general setting, on can show that there are only two types of heat kernels: diffusion (exponential), or heat kernels for s-stable processes (polynomial). Notice that the fractional Laplace operator is the generator of the so called s-stable Lévy process. These are directly related to stochastic processes.

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