Mathematical Developments around Supergravity

Lucas Delage

A thesis submitted in partial fulfillment of the requirements
for the degree of Doctor of Mathematics

Institute of Mathematics
University of Talca

January 2024

Contents

Remerciements

J'aimerais remercier toutes les personnes qui m'ont aidé et soutnenu durant mon parcours. En premier lieu mes deux maîtres de thèse, les professeurs Mokhtar Hassaine et Jorge Zanelli. Mokhtar Hassaine qui dont l'aide fut la plus précieuse lors des mes nombreuses démarches qui ont suivies mon arrivée au Chili - il m'a par exemple avancé mes deux premiers loyers et je l'en remercie - et tout au long de mes quatre années au Chili. Je pense que sans son soutien et sa bienveillance, je ne serai jamais arrivé au bout de ma thèse. Jorge Zanelli que j'ai eu la chance de rencontrer lors d'une conférence à Cuba, rencontre qui fut le début de mon avenutre chilienne. En effet, à la suite de son passionant cours sur les théories de jauge, je lui demanda l'honneur de devenir son étudiant. À ses côtés, j'ai notamment réalisé que la physique ne se limite aux objets mathématiques qui la compose, et j'ai, même si ça ne se voit peut-être pas, profondément changer mon regard personnel sur ce champ d'étude.
Je tient à remercier chaleureusement le professeur et ami Maximiliano Leyton, que je rencontra par un hasard inouï quelques semaines avant mon départ, et qui m'hébergea lors de mes tout premiers jours passé au Chili. On a passé de nombreuses heures ensemble à discuter des mathématiques, de l'univers, de l'université, de la politique, et de bien d'autre sujet, que ce soit sur son canapé ou dans son bureau, ou à d'autres endroits encore. Il m'a fait découvrir de fabuleux endroits de son pays, parfois à bord de Herbie, m'a fait gouté les meilleures viandes et goûté des vins prodigieux. J'ai très volontairement souffert de ses cours d'homologie et de cohomologie. Je remercie aussi sa femme Rachel Rogers, qui m'a également accueillie chez elle, et qui est la cuisinière la plus extraordinaire que je connaisse, en plus d'être incroyablement gentille et attentionnée.
Je remercie les professeurs de l'Université de Talca dont j'ai eu la chance de suivre l'enseignement. Pour les nommer, les professeurs Stephen Griffeth, Steen Ryom-Hansen, Sergei Trofimchuk, Rodrigo Ponce, Manuel O'Ryan, Maria Ronco et Ricardo Baeza. Leurs cours était tout simplement passionnant, et je ne pensais pas apprendre autant en mathématique en à peine un an et demi. Le professeur Stephen Griffeth m'a même fait suffisament confiance pour présenter, au tableau et à la craie, et devant d'autres étudiants, un cours de Victor Kac sur les algèbres de Lie. Je remercie aussi les professeurs Maria Ines Icaza, Alvaro Liendo et Luc Lapointe pour leur soutien durant tout mon séjour à Talca.
Je remercie également les professeurs du Centro de Estudio Cientificos de Valdivia, les professeurs Fabrizio Canfora, Alfredo Perez, Cristián Martinez et Ricardo Troncoso. J'ai réellement appréciés nos discussions qu'elles soient à propos de la physique ou non. Je remercie les professeurs Felipe Barrios et Francisco Sepulveda que je retrouvais avec plaisir au restaurant. Je

remercie Mauricio Valenzuela, qui m'a aidé dans mon apprentissage de la supergravité, et qui m'a aidé à me faire unee vie sociale à Valdivia. Je remercie Sumeet Chougoule, Pulastya Parekh et Perla Medina pour de joyeux moment passé à l'institut et dans les bars des alentours.

Je remercie le différent professeurs qui m'ont appris et passioné durant mon parcours académiques. Les professeurs Renwud Parentani, Hervé Bergeron, Dirk Kreimer et Dominique Manchon. Je remercie mes professeurs de Lycée et de Prépa, en particulier M.Blanc et M. Poublanc

Je remercie ma famille, mes cousins Guillaume, Élise, Amélie, Lucie, Louis, Camille, Charlotte, Matéo et Edgar; mes oncles et tantes Jean, Marilou, Pierre, Pacale, Dom, Fred Patrick et Gaëtane; mes grand-mère Marie-Thérèse et Gaby. C'est une famille absolument formidable, je suis particulièrement chanceux d'avoir grandi à leur côté.

Je remercie Sylvain Roumet, qui m'a gentiment prêté son appartement afin qu'il puisse me servir de bureau pendant la fin de la rédaction de ce manuscrit, ainsi qu'Hervé Gerbault qui s'est occupé de la logistique de cette opération.

Je remercie Gaël, Bobus, Joaqim, Martin, David, Éric et Marion que j'ai eu plaisir à retrouver et à rendre visite en rentrant en France. Ce sont des escaladeurs farouches, des métalleux avant-gardiste, des joueurs convaincus, des amateurs de bières du plus haut échelon, et surtout des amis géniaux.

Ich bedanke Tim, Carla, Eliaz, Vivien und Lisa. Mit ihnen habe ich in Berlin die beste Jahren meiner Lebe gelebt. Wenn Zeit bleib stehen konnte, würde ich beten, um dass genau an dieser Zeitpunkt würde.

Je remercie Lucas Abernot, qui est un homme à la fois improbable et extraordinaire. Il a toujours été la pour me soutenir, même à des milliers de kilomètres de distance, et j'ai vécu avec lui des moments mémorables et des voyages inoubliables. Je remercie aussi sa mère Murielle Launay, qui est d'une gentillesse et d'une volonté inégalée.

Je remercie Raphaël avec qui j'ai toujours plaisir à sortir à l'occasion, rencontrer des gens, refaire le monde, ou simplement discuter.

Je remercie Guillaume Delcluze, qui m'a accueilli chez lui à mon retour en France comme si sept années ne s'était pas écoulées. Je remercie Coline, Valou, Pam-Pam, Léo, Lou, Marie, Archi, Swann, Kayou, avec qui j'ai grandit. Je remercie Gwendola, qui m'a toujours accueilli chez elle avec le sourire.

Je remercie Ganon et Sun, avec qui je discute et je joue très volontiers les soir et les week-end de fainéantise.

Je remercie Paul Michelet, un gaillard pas comme les autres, avec qui j'ai prit plaisir à disputer de nombreuses parties d'échecs. Avec qui j'ai traversé la France, avec qui je suis allé en festival, avec qui j'ai discuté des implications de la physique quantique à des heures improbables et avec qui je me suis retrouvé dans les état les plus singuliers de mon existence.

Je remercie mes parents, Élisabeth Moreau et Jacques Delage. On n'a pas tous la chance d'avoir des parents qui nous aime et nous supporte sur tous nos projets. Je peux cependant certifier que c'est mon cas. Tous mes amis qui les ont rencontré m'assure de la chance folle que j'aie de les avoir comme père et mère, ete je suis plus que d'accord avec eux.

Je remercie Nicolas Delin, l'homme le plus généreux que je connaisse. J'ai avec lui des souvenirs impérissables, des jeux de mots abscons, des parties endiablées, des cafés noirs sans sucre. L'humour a toujours été une composante essentielle de notre amitié, et l'absurdité de certain de nos propos est un sel qui rajoute de la joie dans ma vie. Par ailleurs, on ne s'ennuie jamais avec lui, il trouve toujours le moyen de rendre l'instant présent agréable et relaxant. Il y a aussi un lien très fort qui nous uni aux mathématiques à travers un certain nombre.

Je remercie Florian Nicol. Que serait la vie sans le Dronis ? Sans doute un chemin long et ennuyeux. Heureusement que j'ai pu le rencontrer, et qu'il a depuis égayé mes après-midi et mes soirées. Peu importe le moment, c'est toujours un plaisir de le retrouver et d'aller abuser des sushis à volonté.

Je remercie Valentin Levacher. On est peut-être pas né des mêmes parents, il n'empèche que je le considère comme mon frère. Je n'ai de toute façon pas de souvenir antérieur à notre rencontre. Il m'a accompagné tout au long de ma vie et he l'ai accompagné tout au long de la sienne. Rien ne m'est plus chère que son amitié.

Je remercie pour finir mon frère. J'ai tout partagé avec lui, les mathématiques, le football, les jeux, la musique, beaucoup d'amis et de soirées. Je le remercie pour toute l'aide qu'il m'a apporté lors de la rédaction de ce manuscrit. Sans lui, cette thèse ne serait pas, et sans lui je serais un homme très différent.

Introduction

Theoretical physics is about the study of mathematical models aiming to describe natural phenomena. Mathematics, through the use of numbers, representing physical quantities, answers to a requirement of precision. For example, if we speak about length the most precise and understandable notion the human mind has developed is to compare, using a numerical ratio, everything to a fully apprehended object. No adjectives, like short, big, or huge, would be ever more precise and direct than "1.86m". The clear advantage of the number is that it can accurately describe infinitely many different situations, whereas it appears cumbersome to invent infinitely many words to do the same job. Here the meter is a universal object which serves as reference for all distances; it is called a physical unit. Hence, with an adequate set of physical units - and the same amount of numbers - we can precisely describe any physical object, and the human mind have found no better way to express these accurate descriptions.

Precisely describing a physical object is already very useful but we want more. We want to be also able to foresee his movement, to predict his future behaviour; or simply speaking, we want to describe his evolution. This evolution is expressed by the values the physical quantities, describing the object under consideration, will take in the future. All the relations between these past, present and future physical quantities are given through mathematical laws forming what is called a physical model. Although one could think that each model applies to a particular situation, in fact a same model can describe a variety of apparently completely different phenomena. Hence the study of physical models as purely mathematical objects has become a field of interest in its own, and is the heart of theoretical physics.

As one of the most famous and studied model of physics, general relativity is the modern theory describing the forces of gravity. It is one of the most solid theory, in terms of experimental verification, of today's physics. However, it is the only fundamental theory of classical physics which has no quantum counterpart, and this is interpreted as a problem by many physicists; a problem already in Weinberg's book [Wei72], and still actively discussed nowadays [Gid]. Many ideas has been proposed to tackle this problem. Among them, the Loop Quantum Gravity theory, developed by Carlo Rovelli and its collaborators [Rov04], or String theories. String theories gave birth to a plethora of proposals for fundamental or effective actions and models in theoretical physics, we refer to [Str13] for an introduction. A peculiar aspect of string theory, is that it generally requires, in order to define a well-behaved quantum theory, higher dimensions. A second aspect is that it eventually requires a theory called supersymmetry, forming a sub-classes of string theories called superstring theories. Supersymmetry is a theory that propose that any fundamental particle of nature posses a kind of a dual particle, a so-called supersymmetric partner. Supersymmetry has been introduce in the aim of finding a fundamental theory unifying the theories of electroweak and strong interactions. An excellent book on supersymmetry, detailing all these ideas and more, is [Wei00]. Some of string theories apply supersymmetry to gravity as well, the most famous one doing it might be the M-theory [Wit95]. At low energy and large scale, quantum effects are negligible, hence superstring theories give rise to classical theories combing gravity and supersymmetry: supergravity. The study of supergravity theories on their own have been then an important field of research, and have lead to many new mathematical ideas. our main reference for general supergravity theories is [FV12].

In this manuscript, following a first part devoted to the exposition of some mathematical concepts used in the research area surrounding supergravity, we will present three different works we actively took part in. The first one present a model which describe electroweak theory and gravity together, in the sense that it enjoys the same gauge invariance as these two theories, where all fundamental fields involved are part of a unique super-connection, following the spirit of [AVJ11]. This theory contains more fields than necessary to describe uniquely the electroweak and gravity interactions, hence can be seen as a "mother model", where standard physics should live in a restricted sector. The delimitation of this interesting sector is sought with the help of the notions of symmetry, super-symmetry, spontaneous symmetry breaking, and partially using the field equations of some exotic fields. In fact, a parameterized class of effective theories is obtained and studied. The second work present a family of Chern-Simons supergravity theories, involving a maximal amount of so-called super-charges, for all odd dimensions, in direct continuation of [HR08]. In this theories too, the fundamental fields are all part of a single super-connection. The super-algebras are obtained through an expansion method inspired from [JO03], method which is revisited and geometrically interpreted in the first part of the manuscript. The maximality of these Chern-Simons supergravity theories is carefully shown and their geometric properties allowing for a compact notation enlighted. The third work answer a problematic of [Bag+18], where a tensionless limit of the Polyakov action, describing a string theory, is obtained. In the aforementioned article, no clear path from the standard Polyakov action to its tensionless limit, keeping a Majorana representation of the spinor fields, has been obtained. We show how to obtain these wanted representations, by carefully examining the tensionless limit for all mathematical object involved. A parametrized family of Clifford algebra, Majorana representation, super-field and action are built and shown to all possess a well defined tensionless limit.

CHAPTER 1

Algebra

1. Presentation of the different types of algebras

1.1. Standard definitions.

Let $\mathbb{K}$ be a field. An algebra is a vector space A together with a bilinear map $m : A \times A \to A$. There can also be algebra defined over rings, but we will not use any of them in this manuscript. A morphism between two algebras (A, m_A) and (B, m_B) is a linear map $f : A \to B$ such that

$$\forall\, a, b \in A,\ f(m_A(a,b)) = m_B(f(a), f(b)). \tag{1.1}$$

An isomorphism $A \to A$ is called an automorphism of A. The set of automorphism of A is denoted by Aut(A). It is a group for the composition. A subalgebra B of an algebra A is a subvector space $B \subset A$ such that $m(B,B) \subset B$. An ideal I of an algebra A is a subvector space satisfying $m(A, I) \subset I$. An ideal is in particular always a subalgebra. If I is an ideal of A, then the quotient vector space ${}^{A}\!/_{I}$ inherits of a structure of algebra given by

$$\overline{m}(\overline{a}, \overline{b}) = \overline{m(a,b)} \tag{1.2}$$

where we have used the notation $\overline{a} = a + I$ to denote the equivalent class of a. Furthermore, the projection map $(A, m) \to ({}^{A}\!/_{I}, \overline{m})$ is a morphism of algebra. Conversely, let A, B be algebras and $\psi : A \to B$ be an algebra morphism. Then :

(1) $\mathrm{Ker}(\psi)$ is an ideal of A,

(2) $\mathrm{Im}(\psi)$ is a subalgebra of B,

(3) $\mathrm{Im}(\psi) \simeq {}^{A}\!/_{\mathrm{Ker}(\psi)}$.

An algebra A is unital if it contains a unit, i.e. an element 1 such that

$$\forall\, a \in A,\ m(1, a) = a. \tag{1.3}$$

It is associative if

$$\forall\, a, b, c \in A,\ m(m(a,b), c) = m(a, m(b,c)). \tag{1.4}$$

It is commutative if associative and

$$\forall\, a, b \in A,\ m(a,b) = m(b,a). \tag{1.5}$$

Let A be an algebra and B be a subalgebra of A. The normalizer of B in A, denoted by $N_A(B)$ is the largest subalgebra of A in which B is an ideal. The set of elements commuting with all elements of A is called the center of A and denoted by Z(A).

DEFINITION 1. *Let A be an algebra. A representation of A is a pair (V,ρ) where V is a vector space and ρ a linear map $\rho : A \to End(V)$*

Let A be an algebra. A derivation of A is a linear map $d : A \to A$ such that

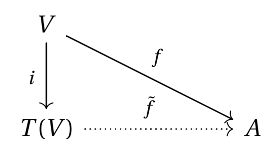
$$d(ab) = d(a)b + ad(b). \tag{1.6}$$

The set of derivation of A is denoted by Der(A).

PROPOSITION 1. *Let D be a nilpotent derivation of an algebra A. Them $e^D \doteq \sum_n \frac{D^n}{n!}$ is an automorphism of A*

1.2. Tensor algebra and tensor product.

There exists a universal associative algebra, i.e. given a vector space V, there exists an algebra, the tensor algebra over V, denoted $T(V)$, satisfying the following universal property: there exists an injective linear map $i : V \to T(V)$ such that, for any linear map f from V into an algebra (A,m), there exists a unique algebra morphism $\tilde{f} : T(V) \to A$ such that the following diagram commutes:



(1.7)

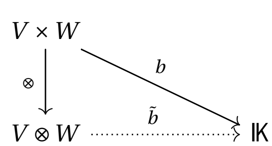
The product of the tensor algebra is called the tensor product and denoted $\otimes$. Its unit is the empty vector . The universal property above assert that any associative algebra is the quotient of $T(V)$ by an ideal identified with Ker($\tilde{f}$), the product being given by $\tilde{f}(a \otimes b)$. Hence, another definition of an algebra is: an algebra is a vector space V together with a linear map $m : A \otimes A \to A$.

Tensor product between different vector spaces can also be formed. Following the preceding definition through the use of a universal property, given two vector spaces V, W over the same field $\mathbb{K}$, the tensor product $V \otimes W$, together with a bilinear map

$$\otimes : V \times W \to V \otimes W, \tag{1.8}$$

$$(v,w) \mapsto v \otimes w, \tag{1.9}$$

is the unique (up to isomorphism) such pair satisfying the following property: for any bilinear map $b : V \times W \to \mathbb{K}$, there is a unique linear map $\tilde{b} : V \otimes W \to \mathbb{K}$ such that $b = \tilde{b} \circ \otimes$. In other words we have the commutative diagram:



(1.10)

There is a natural isomorphism between $V \otimes W$ and $W \otimes V$ given by, for generic $v \in V$ and $w \in W$,

$$\tau : v \otimes w \mapsto w \otimes v. \tag{1.11}$$

1.3. Topological algebras and topological tensor product.

We have seen that the tensor product of two vector spaces A, B is the set of the finite sums

$$\sum a_i \otimes b_i. \tag{1.12}$$

where the a_i's belong to A and the b_i's to B. We can extend these notions to infinite sums by adding a topology and requiring convergence. An algebra in which convergent infinite sum are allowed is called a topological algebra.

An example of topological algbera is a Banach algebra. A Banach algebra is an algebra with a norm, denoted $\|.\|$, the topology is induced by the norm and as topological space it is complete. Given two Banach algebras A, B of infinite dimension, the tensor product $A\otimes B$ is an algebra admitting in general several different norms induced by the norms of A and B. A common one is the so-called π-norm

$$\forall x \in A \otimes B, \pi(x) = \inf\left\{\sum_{i=1}^{n} \|a_i\|\|b_i\| \Big/ x = \sum_{i=1}^{n} a_i \otimes b_i\right\} \tag{1.13}$$

Secondly, the normed algebra so obtained is not complete, i.e. contains Cauchy sequences not converging to an element of $A \otimes B$. Hence it is usual to denote by

$$A \hat{\otimes}_\pi B \tag{1.14}$$

the topological algebra completed with respect to the norm (1.13). Our typical topological algebra is $\mathcal{C}^\infty(\mathbb{R}^n)$, endowed with the sup norm $\|\|_\infty$. In this case, the choice of the π-norm (1.13) for the completion leads to the satisfying isomorphism:

$$\mathcal{C}^\infty(\mathbb{R}^n)\hat{\otimes}_\pi\mathcal{C}^\infty(\mathbb{R}^m) \simeq \mathcal{C}^\infty(\mathbb{R}^{n+m}). \tag{1.15}$$

1.4. Coalgebra.

The dual notion of an algebra is a coalgebra. A coalgebra is a vector space C together with a linear map $\Delta : C \to C \otimes C$, called the coproduct. It is common to use Sweedler's notation for the coproduct Δ:

$$\Delta(c) = \sum c_1 \otimes c_2, \tag{1.16}$$

where the sum is always finite. A counit for a coalgebra C is a map $\epsilon : C \to \mathbb{K}$ such that

$$(\mathrm{Id} \otimes \epsilon) \circ \Delta = (\epsilon \otimes \mathrm{Id}) \circ \Delta = \mathrm{Id}. \tag{1.17}$$

A coalgebra C is coassociative if $\Delta \otimes \mathrm{Id} = \mathrm{Id} \otimes \Delta$, cocommutative if coassociative and

$$\forall c \in C, \Delta(c) = \sum c_1 \otimes c_2 = \sum c_2 \otimes c_1. \tag{1.18}$$

Given an algebra of finite dimension A, there is a natural structure of coalgebra on its linear dual A^*, thanks to the isomorphism $(A \otimes A)^* \simeq A^* \otimes A^*$. The coproduct on A^* is defined by

$$\forall f \in A^*, \forall a, b \in A, \Delta(f)(a \otimes b) = f(m(a, b)). \tag{1.19}$$

When the algebra is of infinite dimension, this definition does not hold in general. It works however on the restricted vector space of linear form vanishing on ideal of finite codimension, called the finite dual, denoted A°. We would like to explain this fact with a simple example. Consider the algebra of polynomial over $\mathbb{R}$, $\mathbb{R}[X]$. As a vector space, it admits the basis

$$\{1, X, X^2, X^3, \dots, X^n, \dots\}. \tag{1.20}$$

Hence, a linear form is uniquely defined by its values on each of the X^n, thus the dual of $\mathbb{R}[X]$ is the vector space of series in one variable $\mathbb{R}[[X]]$, and a basis for a linear form is $\{t_k\}$

$$t_k(X^n) = \begin{cases} 1 & \text{if} k = n \\ 0 & \text{if} k \neq n \end{cases} . \tag{1.21}$$

$\Delta(t_k)$ can be easily computed using the notion of dual coalgebra. Given two polynomials $P = \sum p_i X^i$, $Q = \sum q_j X^j$,

$$t_k(PQ) = \sum_{i+j=k} p_i q_j, \tag{1.22}$$

hence

$$\Delta(t_k) = \sum_{i+j=k} t_i \otimes t_j. \tag{1.23}$$

Now given a linear form f, $\Delta(f)$ would be a finite sum if and only if the expansion $f = \sum f_k t_k$ is finite, which is precisely the same as asking that f vanishes on a ideal of finite codimension. Indeed, if $f = \sum_{k=0}^{K} f_k t_k$, f vanishes at least on the ideal generated by X^{k+1}.

Let C be a coalgebra. A sub-coalgebra D of C is a sub-vector space $D \subset C$ satisfying

$$\Delta(D) \subset D \otimes D \tag{1.24}$$

A coideal J of C is a sub-vector space $J \subset C$ satisfying

$$\Delta(J) \subset J \otimes C + C \otimes J \tag{1.25}$$

In particular, any sub-coalgebra is a coideal. If J is a coideal of C, the quotient vector space C/J equipped with the map $\overline{\Delta}$ defined by

$$\overline{\Delta}(\overline{c}) = \overline{\Delta(c)} \tag{1.26}$$

turn C/J into a coalgebra, and the projection map $(C, \Delta) \to (C/J, \Delta)$ is a coalgebra morphism.

Let A be a finite dimensional algebra. Suppose $A = A_0 \oplus A_1$ where A_0 is a subalgebra. Then for the dual coalgebra, $A^* = A_0^* \oplus A_1^*$ and A_1^* is a coideal of A^*. Reciprocally, if A_1^* is a coideal, A_0 is a subalgebra. In the same fashion, if A_0 is an ideal, A_1^* is a sub-coalgebra and reciprocally. It also means that the dual coalgebra of the subalgebra A_0 is the quotient coalgebra A^*/A_1^* (in the case A_0 is a subalgebra) and the dual coalgebra of the quotient algebra A/A_0 is A_1^* (in the case A_0 is an ideal).

1.5. Tensor product of algebras and coalgebras.

Let (A, m_A) and (B, m_B) be two algebras. Then there is a natural structure of algebra on $(A \otimes B)$ with product defined as

$$m_{A\otimes B} = (m_A \otimes m_B)(\mathrm{Id} \otimes \tau \otimes \mathrm{Id}) \tag{1.27}$$

If (C, Δ_C) and (D, Δ_D) are two coalgebras, there is a natural coproduct

$$\Delta_{C\otimes D} = (\mathrm{Id} \otimes \tau \otimes \mathrm{Id})(\Delta_C \otimes \Delta_D) \tag{1.28}$$

turning $C \otimes D$ into a coalgebra. If A and B are finite dimensional, $(A^* \otimes B^*, {m_{A\otimes B}}^*)$ is the dual coalgebra of $(A \otimes B, m_{A\otimes B})$. If I is an ideal (*resp. a subalgebra*) of A, $I \otimes B$ is an ideal (*resp. a subalgebra*) of $A \otimes B$; if J is a coideal (*resp. a sub-coalgebra*) of C, $J \otimes D$ is a coideal (*resp. a sub-coalgebra*) of $C \otimes D$.

1.6. Lie algebras.

A Lie algebra (over $\mathbb{K}$) $\mathfrak{g}$ is a $\mathbb{K}$-algebra such that product is antisymmetric and for any $a \in \mathfrak{g}$, left multiplication is a derivation of the algebra. As everywhere else in the literature, we write the product of a Lie algebra with a bracket $[,]$. The definition above means that

$$\forall\, x, y \in \mathfrak{g}, \qquad [x, y] = -[y, x] \tag{1.29}$$

$$\forall\, x, y, z \in \mathfrak{g}, \; [x, [y, z]] = [[x, y], z] + [y, [x, z]] \tag{1.30}$$

1.7. Lie algebras and derivations.

Let A be an algebra. Then (Der(A),[,]) is a Lie algebra with

$$[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1 \tag{1.31}$$

A specificity of Lie algebras it that their products define derivations. We let $\mathrm{Inn}(\mathfrak{g}) \subset \mathrm{Der}(\mathfrak{g})$ the set of inner derivations, i.e. the one of the form

$$a \mapsto [b, a] \tag{1.32}$$

for $a, b \in \mathfrak{g}$. $\mathrm{Inn}(\mathfrak{g})$ is an ideal of $\mathrm{Der}(\mathfrak{g})$.

1.8. Bialgebra, Hopf Algebra.

A bialgebra is a vector space endowed with both an unital associative algebra structure and a counital coassociative coalgebra structure, denoted alltogether $(A, m, 1_A, \Delta, \epsilon)$. Furthermore a relation of compatibility between the two structure is required. They are the following:

(1) Δ and ϵ are morphism of coalgebra.
(2) m is a morphism of coalgebra and $\Delta(1_A) = 1_A \otimes 1_A$.
(3) $\epsilon(1_A) = 1$

A Hopf algebra is a bialgebra $(A, m, 1_A, \Delta, \epsilon)$ endowed with an antipode, i.e. a map $S : A \to A$ satisfying:

$$m \circ (S \otimes \mathrm{Id}) \circ \Delta = m \circ (\mathrm{Id} \otimes S) \circ \Delta = 1_A \epsilon \tag{1.33}$$

1.9. Hopf algebras and Groups.

Let G be a group, and $\mathcal{H}(G)$ be the set of function from G to a field $\mathbb{K}$. Then $\mathcal{H}(G)$ is a Hopf $\mathbb{K}$-algebra whose structure is as follows (*in the list, f refers to an element of $\mathcal{H}$ and g refers to an element of G*):

multiplication: $m_{\mathcal{H}}(f_1 \otimes f_2)(g) = f_1(g) f_2(g)$,
unit: $1_{\mathcal{H}} : g \mapsto 1_{\mathbb{K}}$,
comultiplication: $\Delta_{\mathcal{H}} f(g_1 \otimes g_2) = f(g_1 g_2)$,
counit: $\epsilon_{\mathcal{H}} : 1_{\mathbb{K}} \mapsto 1_{\mathcal{H}}$,
antipode: $S_{\mathcal{H}}(f)(g) = f(g^{-1})$.

Conversly, given a Hopf algebra $\mathcal{H}$ over a field $\mathbb{K}$, the set of functions $G(\mathcal{H}) = \{f : \mathcal{H} \to \mathbb{K}\}$ as a natural group structure given by:

$$\forall g_1, g_2, g \in G(\mathcal{H}), \forall h \in \mathcal{H}, \qquad (g_1 \cdot g_2)(h) = m_{\mathbb{K}}(g_1 \otimes g_2)\Delta_{\mathcal{H}}(h)\,, \tag{1.34}$$

$$g^{-1}(h) = g(S_{\mathcal{H}}(h)) \qquad . \tag{1.35}$$

This equivalence is a fundamental result in the theory of algebraic group: [Wat79]

THEOREM 1. *Let $\mathbb{K}$ be a field. There is an equivalence between the category of affine group scheme over $\mathbb{K}$ and the category of Hopf algebras over $\mathbb{K}$.*

1.10. Grading.

Let $(G, +)$ be an abelian group (we have in mind $\mathbb{Z}$, $\mathbb{Z}_2$, eventually $\mathbb{Z}_p$). A G-graded vector space is a vector space V that decompose as

$$V = \bigoplus_{g \in G} V_g \tag{1.36}$$

An element of $v \in V_g$ is called homogeneous of degree g. We denote this degree by $|v|$. A morphism of G-graded vector spaces is a linear map $f : V \to W$, with V, W two G-graded vector spaces, preserving the degree of homogeneous elements:

$$\forall v \in V_g, \quad |v| = |f(v)|. \tag{1.37}$$

It is possible to extend these morphisms by allowing them to have a degree as well. The set of these extended morphisms is what is called an *inner Hom functor* in the categorical vocabulary. Practically, we add the degree of the morphism to the degree of the elements it is applied to. With the preceding notations, if we suppose f has now degree $|f|$ we have

$$|f(v)| = |f| + |v|. \tag{1.38}$$

A G-graded algebra is an algebra (A, m) such that A is a G-graded vector space and such that

$$m(A_{g_1} \otimes A_{g_2}) \subset A_{g_1 + g_2} \tag{1.39}$$

A G-graded coalgebra is an algebra (C, Δ) such that C decompose as

$$C = \bigoplus_{g \in G} C_g \tag{1.40}$$

and such that

$$\Delta(C_g) \subset \bigoplus_{g_1+g_2=g} C_{g_1} \otimes C_{g_2} \tag{1.41}$$

Subalgebras or ideals of graded algebras or coalgebras are also naturally graded.
If A and B are two G-graded algebras (*resp. coalgebras*), then $A \otimes B$ is itself a G-graded algebra (*resp. coalgebra*), with

$$(A \otimes B)_g = \bigoplus_{g_1+g_2=g} A_{g_1} \otimes B_{g_2} \tag{1.42}$$

There is another grading one can consider for the tensor product of two algebras, namely

$$(A \otimes B)_g = A_g \otimes B_g \tag{1.43}$$

Note however, that $\bigoplus_{g\in G}(A \otimes B)_g$ is now only a subalgebra of $A \otimes B$.

1.11. Super-linear algebra.

A super vector space is a $\mathbb{Z}_2$-graded vector space : $V = V_0 \oplus V_1$. We use the notation $|v| \in \mathbb{Z}_2$ to denote the degree of an homogeneous vector v. The super-linear morphism we will consider are the extended morphism, with degree, that we described earlier. There is a subtle point differing from the theory of standard graded vector space. Indeed, usually the natural isomorphism between the tensor product of two vector spaces (1.11) is replaced in the super-linear case by the following: for homogeneous v, w we have

$$\tilde{\tau} : v \otimes w \mapsto (-1)^{|v||w|} w \otimes v. \tag{1.44}$$

This is the sign rule, central in super-linear algebra, which can be resumed by the following sentence : *We add a minus sign whenever we interchange two odd symbols.* This rules applies to vectors as well as morphisms between them. A super-algebra is simply an algebra built over a super-vector space and where the rule (1.44) is used instead of (1.11). For example, an algebra (A, m_A) was commutative when we had the equality

$$m_A = m_A \circ \tau. \tag{1.45}$$

Hence an algebra (A, m_A) is super-commutative if

$$m_A = m_A \circ \tilde{\tau}, \tag{1.46}$$

which can be succinctly written, for $a, b \in A$,

$$ab = -ba. \tag{1.47}$$

As another example, a super-Lie algebra is an algebra $\mathfrak{G}, [,]$ such that, forall $x, y, z \in \mathfrak{G}$,

$$[x, y] = (-1)^{|x||y|}[y, x], \tag{1.48}$$

$$[x, [y, z]] = [[x.y], z] + (-1)^{|x||y|}[y, [x, z]]. \tag{1.49}$$

To give an example with a morphism, a super-derivation δ for a super-algebra A with homogeneous degree $|\delta|$ is a super-linear map satisfying

$$\delta(ab) = \delta(a)b + (-1)^{|\delta||a|} a\delta(b). \tag{1.50}$$

References about super-linear algebras are [CCF10] or [Tuy].

1.12. Real Lie super-algebras.

By a complex Lie super-algebra, we mean a $\mathbb{Z}_2$-graded complex vector space $\mathfrak{G}$ over $\mathbb{C}$ with a linear map

$$[,]:\mathfrak{G}\otimes\mathfrak{G}\to\mathfrak{G}, \tag{1.51}$$

satisfying

$$\forall g_1, g_2, g_3 \in \mathfrak{G}, [g_1,g_2] = (-1)^{1+|g_1||g_2|}[g_2,g_1], \tag{1.52}$$

$$[g_1,[g_2,g_3]] = [[g_1,g_2],g_3] + (-1)^{|g_1||g_2|}[g_2,[g_1,g_3]]. \tag{1.53}$$

By a real structure on a complex Lie super-algebra $\mathfrak{G}$, we mean an anti linear map

$$\mathrm{J}:\mathfrak{G}\to\mathfrak{G}, \tag{1.54}$$

such that

$$\forall z \in \mathbb{C}, \forall g \in \mathfrak{G}, \mathrm{J}(zg) = z^*\mathrm{J}(g), \tag{1.55}$$

$$\forall g_1, g_2 \in \mathfrak{G}, \quad [\mathrm{J}(g_1),\mathrm{J}(g_2)] = \mathrm{J}([g_1,g_2]), \tag{1.56}$$

$$\mathrm{J}^2 = \mathrm{Id}. \tag{1.57}$$

The real form associated to the real structure J is by definition the set of fixed points of J. A real form comes naturally with the structure of a real vector space. By a real Lie super-algebra we mean the real form associated to a real structure defined on a complex Lie super-algebra. Real Lie super-algebras have been studied and classified in [Kac77], [Par80]. There is a generalization of the notion of real structure in which the condition (1.57) is relaxed to

$$\mathrm{J}^2|_{\mathfrak{G}_0} = \mathrm{Id} \tag{1.58}$$

$$\mathrm{J}^2|_{\mathfrak{G}_1} = \pm\mathrm{Id} \tag{1.59}$$

In this case, the notion of real Lie super-algebra is better understood with functors, see [Pel03].

2. Expansion of algebras

2.1. Expanding algebras.

Let $\mathbb{K}[\lambda]$ be the algebra of polynomial in one variable λ over $\mathbb{K}$. Let A be an algebra. We first form $\mathbb{K}[\lambda]\otimes A$. Expansion of the algebra A will be obtained as adequate quotient or subalgebra of $\mathbb{K}[\lambda]\otimes A$. For example, the first and natural quotient to consider is

$$\mathbb{K}[\lambda]\big/\lambda^N\mathbb{K}[\lambda] \otimes A \doteq A(N) \tag{2.1}$$

As a vector space, it is given by N copies of A. Let B be a subalgebra of A. Then $\mathbb{K}[\lambda]\otimes B$ is a subalgebra of $\mathbb{K}[\lambda]\otimes A$. One can also form $B(N)$ which is a subalgebra of $A(N)$. Also remark that $\lambda^M\mathbb{K}[l]\big/\lambda^N\mathbb{K}[\lambda]$, $M<N$, is an ideal (hence a subalgebra) of $\mathbb{K}[l]\big/\lambda^N\mathbb{K}[\lambda]$, thus one can form

$$\lambda^M\mathbb{K}[l]\big/\lambda^N\mathbb{K}[\lambda] \otimes A \doteq A(N,M) \tag{2.2}$$

Combining both subalgebra one forms

$$B(M) + A(N, M) \subset A(N) \tag{2.3}$$

which is a subalgebra of $A(N)$. It structure can be understand as follows: up to order M, one has the structure of B, after order M, one enlarge it to the one of A. One can also consider a tower of subalgebra

$$B_1 \subset B_2 \subset \cdots \subset B_n \subset A \tag{2.4}$$

and a succession of integers

$$0 < M_1 < M_2 < \cdots < M_n < N \tag{2.5}$$

and form the subalgebra

$$B(M_1) + B(M_2, M_1) + \cdots + B(M_n, M_{n-1}) + A(N, M_n) \tag{2.6}$$

If A is $\mathbb{Z}$ graded, one can use the natural $\mathbb{Z}$ grading on $\mathbb{K}[\lambda]$, given by powers of λ to form interesting subalgebras of $A(N)$. For example $(\mathbb{K}[\lambda] \otimes A)_0$. One observes that the grading goes well into the quotient, thus one can consider $A(N)_0$. This construction can be done conjointly with the one we just exposed. One can also consider

$$\bigoplus_{n \in \mathbb{Z}} \lambda^n \mathbb{K} \otimes A_n. \tag{2.7}$$

We can also consider a decomposition

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_n \tag{2.8}$$

with

$$m(A_p \otimes A_q) \subset \bigoplus_{k \leq p+q} A_k \tag{2.9}$$

In that case,

$$\bigoplus_{q \leq n} \lambda^q \bigoplus_{p \leq q} A_p \doteq \mathcal{A} \tag{2.10}$$

is a subalgebra of $\mathbb{K}[\lambda] \otimes A$, and, provided $n < N$, of $A(N)$ (once quotiented). One can thus consider the subalgebra $\mathcal{A} + A(N, n)$, which can be understood as follows: an element of A_p will always appears with power at least p. It is also possible to quotient some higher degree terms in $A(N)$. The easiest case being if one has a decomposition $A = A_0 \oplus A_1$ with A_1 an ideal. Then the highest degree subspace $\lambda^{N-1} A_1$ is an ideal and one can form the quotient algebra $A(N) / \lambda^{N-1} A_1$.

2.2. Equivalent expansion for coalgebras.

Let A be an finite dimensional algebra. We would like to show that the constructions we made in the preceding paragraph can be seen as expansion of the dual coalgebra, i.e. the coalgebra defined over the finite dual. In particular $A(N)$ is dual to

$$A^*[N] \doteq \mathbb{K}[\lambda]_{\leq N} \otimes A^* \tag{2.11}$$

and all subalgebra of $A(N)$ we obtained are dual to quotient algebras of $A^*[N]$, and can be constructed directly as such. In particular, all constructions of [JO03] are just some of the construction of this coalgebra method, presented differently. In this article, the authors wanted to find an expansion in the spirit of the original article of Wigner and Inönü [IW53]. In their constructions, they interpret the indeterminate λ as a deformation parameter. However their presentation is far less direct than our. Furthermore, we will find a geometrical interpretation of this construction at the end of the section devoted to jets.

3. Theory of Lie algebras

We have already defined Lie algebras. We now wish to present some further aspects on their theory. The references we are using for this sections are mainly [Boe63], [Hum72] and lectures notes taught by Victor Kac.

3.1. Nilpotent, solvable, semisimple and simple Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra. Then the series of ideals

$$\mathfrak{g}^{(0)} = \mathfrak{g} \supseteq \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}^{(0)}] \supseteq \mathfrak{g}^{(2)} = [\mathfrak{g}, \mathfrak{g}^{(1)}] \supseteq \ldots \supseteq \mathfrak{g}^{(n)} = [\mathfrak{g}, \mathfrak{g}^{(n-1)}] \supseteq \ldots$$

is called the lower central series and the series of ideals

$$\mathfrak{g}_0 = \mathfrak{g} \supseteq \mathfrak{g}_1 = [\mathfrak{g}_0, \mathfrak{g}_0] \supseteq \mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1] \supseteq \ldots \supseteq \mathfrak{g}_n = [\mathfrak{g}_{n-1}, \mathfrak{g}_{n-1}] \supseteq \ldots$$

is called the derived series. An algebra for which $\mathfrak{g}^{(n)} = 0$ for some n is called nilpotent and an algebra for which $\mathfrak{g}_n = 0$ for some n is called solvable. A Lie algebra is semisimple if it contains no solvable ideals. A Lie algebra is simple if its only ideals are 0 and itself. A well-known proposition states that a semisimple Lie algebra is a direct product of simple Lie algebras

3.2. Killing form.

A standard representation of a Lie algebra $\mathfrak{g}$ is through real- or complex-valued matrices, where the Lie bracket of two of its elements is computed by

$$\forall a, b \in \mathfrak{g}, \qquad [a, b] = ab - ba. \tag{3.1}$$

However another representation always exists for Lie algebras. Indeed the Lie algebra $\mathfrak{g}$ itself is a vector space, and the map

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \tag{3.2}$$

$$a \mapsto [a, -] \doteq \mathrm{ad}_a, \tag{3.3}$$

define a representation of $\mathfrak{g}$ on itself called the adjoint representation. Using this adjoint representation, we can define a bilinear map

$$K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}, \tag{3.4}$$

$$K(a, b) = \mathrm{Tr}(\mathrm{ad}_a \mathrm{ad}_b), \tag{3.5}$$

called the Killing form of the Lie algebra $\mathfrak{g}$. It is symmetric, and invariant in the sense

$$K([a, b], c) = K(a, [b, c]).$$

PROPOSITION 2. *Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra. $\mathfrak{g}$ is semisimple if and only if its Killing form is non degenerate. Furthermore, if $\mathfrak{g}$ is semisimple and $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{h}$ is non degenerate and coincides with the Killing form of $\mathfrak{h}$.*

3.3. Rank of a Lie algebra.

Let $\mathfrak{g}$ be a Lie algebra. For $a \in \mathfrak{g}$ consider the characteristic polynomial of ad_a

$$p_{\mathrm{ad}_a}(\lambda) = \lambda^n + c_{n-1}(a)\lambda^{n-1} + ... + c_0(a). \tag{3.6}$$

The rank of a Lie algebra is the smallest integer r such that

$$\exists a \in \mathfrak{g} \quad c_r(a) \neq 0. \tag{3.7}$$

An element $a \in \mathfrak{g}$ is called regular if $c_r(a) \neq 0$ and singular otherwise. There are other equivalent ways to define the rank of a Lie algebra.

3.4. Weights.

Given a vector space V over $\mathbb{C}$ (or any algebraically closed field) and an endomorphism $f \in$ End(V). Then V decomposes as

$$V = \oplus V_\lambda \qquad V_\lambda = \{v \in V \,/\, \exists n \in \mathbb{N} / (f - \lambda Id_V)^n(v) = 0\} \tag{3.8}$$

In the case of a Lie algebra $\mathfrak{g}$ with a representation (V, ρ), for any given element $a \in \mathfrak{g}$ we write V_λ^a for the decomposition with respect to the endomorphism $\rho(a)$. For example, given a Lie algebra $\mathfrak{g}$ and $a \in \mathfrak{g}$, and ρ is the adjoint representation, we use the notation

$$\mathfrak{g} = \oplus \mathfrak{g}_\lambda^a. \tag{3.9}$$

However a more interesting case happens when λ is not a number but a linear form $\lambda \in \mathfrak{g}^*$, $\mathfrak{g}^*$ the vector space dual of a Lie algebra $\mathfrak{g}$. In this case, the subspace

$$V_\lambda = \{v \in V \,/\, \forall a \in \mathfrak{g}, \rho(a)v = \lambda(a)v\}, \tag{3.10}$$

is called a weight space. If $V_\lambda \neq \{0\}$ then λ is called a weight of the representation (V, ρ). This allows some kind of simultaneous decomposition of V.

The existence of weights is ensured by the following theorem:

THEOREM 2 (Lie's theorem).
Let $\mathfrak{g}$ be a solvable complex Lie algebra and (V, ρ) a finite dimensional representation of it. There exists a non zero weight space.

3.5. Cartan's subalgebra.

Let $\mathfrak{g}$ be a Lie algebra. A Cartan's subalgebra of $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$ which satifies

(1) $\mathfrak{h}$ is nilpotent
(2) $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

A well-known proposition states that a Cartan's subalgebra is a maximal nilpotent subalgebra.

THEOREM 3. ***Cartan's theorem***
Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$ (or any algebraically closed field of characteristic 0). Let $a \in \mathfrak{g}$ be a regular element. Then $\mathfrak{g}_0^a$ is a Cartan subalgebra of $\mathfrak{g}$.

As a remark, we have $\dim(\mathfrak{g}_0^a) = r$ =rank($\mathfrak{g}$). This is simply because the dimension of $\mathfrak{g}_0^a$ is equal to the power of the factor X in the characteristic polynomial $p(X)$ of ad_a, which is equal to the rank of $\mathfrak{g}$ since a is regular. As a second remark, we have that any Lie algebra posses at least one Cartan subalgebra.

PROPOSITION 3. *Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let*

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda \tag{3.11}$$

be the weight space decomposition of the adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$. Then $\mathfrak{g}_0 = \mathfrak{h}$.

Proof : On the one hand, as $\mathfrak{h}$ is nilpotent, $\forall a \in \mathfrak{h}$, $\mathrm{ad}_a|_{\mathfrak{h}}$ is nilpotent showing $\mathfrak{h} \subset \mathfrak{g}_0$. On the other hand, suppose $\mathfrak{h} \neq \mathfrak{g}_0$. Then

$$\mathfrak{g}_0 / \mathfrak{h}$$

is not zero so we can consider the quotient representation $\overline{\mathrm{ad}}$ of $\mathfrak{h}$ on it. As it consist only of nilpotent operators, $\forall a \in \mathfrak{h}$, by first Engel's theorem, there exist

$$\overline{b} \in \mathfrak{g}_0 / \mathfrak{h}$$

such that

$$\forall a \in \mathfrak{h}, \overline{\mathrm{ad}_a}(\overline{b}) = \overline{0}$$

Let b be a preimage of $\overline{b}$ in $\mathfrak{g}$. Then we have $\mathrm{ad}_a(b) \in \mathfrak{h}$ - i.e. $b \in N_{\mathfrak{g}}(\mathfrak{h})$ and $b \notin \mathfrak{h}$, which contradict the fact that $\mathfrak{h}$ is a Cartan subalgebra.

□

The next theorem shows that all Cartan subalgebra are equivalent from a theore tical point of view.

THEOREM 4. ***Chevalley's theorem***
Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$ and let

$$G = \left\{ e^{ad_x}, x \in \mathfrak{g}, ad_x \text{ is nilpotent} \right\}. \tag{3.12}$$

Any two Cartan subalgebra of $\mathfrak{g}$ are conjugated by an element of G.

3.6. Roots space decomposition.

DEFINITION 2. *Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. A weight of the adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$ is called a root. The decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha \tag{3.13}$$

is called the root space decomposition. We call

$$\Delta = \left\{ \alpha \in \mathfrak{h}^* \setminus \{0\} \quad / \quad \mathfrak{g}_\alpha \neq \{0\} \right\}. \tag{3.14}$$

the set of roots, a name that will be explained in the next paragraph. Just before we would like to present some properties of the root space decomposition:

PROPOSITION 4. *Keeping the notation of definition 2,*

a) $\mathfrak{h}$ *is a maximal abelian subalgebra,*
b) $\forall a \in \mathfrak{h}$, ad_a *is diagonalizable.*
c) $dim(\mathfrak{g}_\alpha) = 1$,
d) *If* $\alpha, \beta, \alpha + \beta \in \Delta$, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$,
e) *If* $\alpha \in \Delta$, *then* $n\alpha \in \Delta \Leftrightarrow n = \pm 1$.

In particular, we see that $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ forms an $\mathfrak{sl}_2$ subalgebra. In fact, this proposition shows that the whole structure of the Lie algebra is encoded in its root space decomposition. For example, given two elements $a \in g_\alpha$, $b \in g_\beta$, their bracket $[a, b]$ in non vanishing if and only if $\alpha + \beta$ is either 0 or a root, in which case it belongs to $\mathfrak{g}_{\alpha+\beta}$ (or $\mathfrak{h}$ if $\alpha = -\beta$).

3.7. Root system.

The name "root space" for Δ linked to a mathematical concept called "root system". Let V be a finite dimensional vector space with a non degenerate symmetric bilinear form $\langle , \rangle$. A root system Δ in V is a finite subset of V satisfying the following properties:

(1) $\mathrm{Vect}(\Delta) = V$,
(2) $\forall \alpha \in \Delta$, $\lambda\alpha \in \Delta \Leftrightarrow \lambda = \pm 1$,
(3) $\forall \alpha, \beta \in \Delta$, $\beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha \in \Delta$,
(4) $\forall \alpha, \beta \in \Delta$, $2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.

The link between root space and root system is made by the following celebrated theorem.

THEOREM 5. *Let* $\mathfrak{g}$ *be a semisimple finite dimensional complex Lie algebra,* $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$. *Then* Δ *is a root system of* $\mathfrak{h}^*$.

In order to see Δ as a root system, we need a non degenerate symmetric bilinear form. It is given by the dual K^* of the Killing form K of $\mathfrak{g}$. In details, we define first the isomorphism

$$\nu : \mathfrak{h} \to \mathfrak{h}^*, \tag{3.15}$$

$$a \mapsto K(a, -). \tag{3.16}$$

abd then K^* by

$$K^* : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{K}, \tag{3.17}$$

$$K^*(\alpha, \beta) = K(\nu^{-1}(\alpha), \nu^{-1}(\beta)). \tag{3.18}$$

Conversely, proposition (4) (and a bit of work) shows that any root system give rise to a Lie algebra; in other words the correspondence between root system and Lie algebras is 1-1.

Root systems can be used to apprehend the structure of semisimple Lie algebras. For example, a root system Δ is called decomposable whenever it can be written $\Delta = \Delta_1 \cup \Delta_2$, with $\langle \Delta_1, \Delta_2 \rangle^* = 0$. Otherwise it is called indecomposable. Applied to the root space of a semisimple Lie algebra, this notion gives:

THEOREM 6. *Let $\mathfrak{g}$ be a semisimple finite dimensional complex Lie algebra and Δ its root system. $\mathfrak{g}$ is simple if and only if Δ is indecomposable.*

Hence, in order to classify all simple Lie algebras, it is sufficient to classify all root systems.

3.8. Simple roots.

Let V be a finite dimensional vector space and Δ a root system in V. Let $f \in V^*$ be a linear form such that f does not vanish on any element of Δ. Then f separate Δ in two subsets; Δ_+ containing the elements α for which $f(\alpha) > 0$, called positive roots, and Δ_- those α for which $f(\alpha) < 0$ called negative roots. We can give a partial order on the set of roots. We say that a root α is bigger than another root β, and we write $\alpha \succ \beta$, if $\alpha - \beta$ is a root and if it is positive. A positive root is said decomposable if it is possible to write it as a sum of two other positive roots. A non decomposable positive root is called simple. The set of simple roots form a basis of the surrounding vector space V, and hence of the root system Δ. Given a set $(\alpha_1, \dots, \alpha_n)$ of simple roots of an indecomposable root system, the matrix C whose entries are

$$C_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}, \tag{3.19}$$

is called the Cartan matrix of the root system. Cartan matrices are independent of the linear form f used to define the positive roots. Two roots system with the same Cartan matrix are isomorphic (the definition of isomorphism for root systems can be found in [Hum72]) and give rise to the same simple Lie algebra. Hence the classification of Cartan matrices is equivalent to the classification of simple Lie algebras. There are rules to build Cartan matrices, see [Hum72]. Cartan matrices can be represented by drawings called Dynkin diagrams.

3.9. Weights and irreducible representations of Lie algebra.

We have already given a definition of weights in paragraph 3.4. In view of the next theorem, we give a second definition.

DEFINITION 3. *Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, $\Delta \subset \mathfrak{h}^*$ the root space. A weight is a vector $\lambda \in \mathfrak{h}^*$ such that, $\forall \alpha \in \Delta$,*

$$\frac{2\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \text{ is an integer.} \tag{3.20}$$

A weight is called (strongly) dominant if (3.20) is a (strictly) positive integer. Weights are partially ordered by the following relation

$$\lambda_1 < \lambda_2 \text{ if and only if } \lambda_2 - \lambda_1 \text{ is a sum of positive roots.} \tag{3.21}$$

The fundamental weights are the weights dual to the simple roots; i.e. they are those weights λ_i satisfying

$$\frac{2\langle\alpha_j,\lambda_i\rangle}{\langle\alpha_j,\alpha_j\rangle} = \delta_{ij}, \tag{3.22}$$

where δ_{ij} is the Kronecker symbol and $\{\alpha_j\}$ the set of simple roots. The next two theorems show the correspondence between weights in the sense of (3) and irreducible representations

THEOREM 7. *Let (V,ρ) be an irreducible representation of a simple Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Then there exists a dominant weight λ such that V_λ is a weight subspace of V, has dimension 1 and V decompose as a direct sum of weight spaces V_μ. Furthermore each weight is of the form $\mu = \lambda - \sum k_i\alpha_i$, where k_i are integers and α_i simple roots. In other words, λ is higher than any other weight μ.*

In the above theorem, λ is called the highest weight of the representation.

THEOREM 8. *Let $\mathfrak{g}$ be a simple Lie algebra, and λ a dominant weight. There exists, up two isomorphism, one and only one representation whose highest weight is λ.*

We write $V(\lambda)$ for the irreducible representation with highest λ. Finally, it is possible to show that, given two dominant weights λ, μ

$$V(\lambda+\mu) \subset V(\lambda)\otimes V(\mu). \tag{3.23}$$

Hence, it is possible to construct any irreducible representation from the fundamental representations $V(\lambda_i)$, where the λ_i are the fundamental weights.

3.10. Structure of $\mathfrak{so}(2n)$.

This section present the root space decomposition of the complex Lie algebra $\mathfrak{so}(2n)$, i.e. the set of complex matrices $a \in M_{2n}(\mathbb{C})$ satisfying

$${}^T a + a = 0. \tag{3.24}$$

We define the matrices

$$E = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad X = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \tag{3.25}$$

$$Y = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \tag{3.26}$$

Let

$$\mathfrak{h} = \left\{ \begin{pmatrix} h_1 E & & & \\ & h_2 E & & \\ & & \ddots & \\ & & & h_n E \end{pmatrix}, (h_1, h_2, \dots h_n) \in \mathbb{C}^n \right\}. \tag{3.27}$$

$\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{so}(2n)$. We define $\varepsilon_i \in \mathfrak{h}^*$ by

$$\varepsilon_i \begin{pmatrix} h_1 E & & & \\ & h_2 E & & \\ & & \ddots & \\ & & & h_n E \end{pmatrix} = h_i \tag{3.28}$$

Next we define, for $i < j$,

$$A_{\varepsilon_i - \varepsilon_j} = \begin{pmatrix} & & \\ & & X \\ & & \\ -{}^T X & & \\ & & \end{pmatrix}, \tag{3.29}$$

with X at the (i, j)-th entry and ${}^T X$ at the (j, i)-th one. Similarly, still for $i < j$,

$$A_{-\varepsilon_i + \varepsilon_j} = \begin{pmatrix} & & \\ & & {}^T X \\ & & \\ -X & & \\ & & \end{pmatrix}, \tag{3.30}$$

$$A_{\varepsilon_i + \varepsilon_j} = \begin{pmatrix} & & \\ & & Y \\ & & \\ -{}^T Y & & \\ & & \end{pmatrix}, \tag{3.31}$$

$$A_{-\varepsilon_i - \varepsilon_j} = \begin{pmatrix} & & \\ & & Z \\ & & \\ -{}^T Z & & \\ & & \end{pmatrix}. \tag{3.32}$$

We can check that for any $H \in \mathfrak{H}$,

$$[H, A_\alpha] = \alpha(H) A_\alpha, \qquad \alpha = \pm\varepsilon_i \pm \varepsilon_j \tag{3.33}$$

The set of roots of $\mathfrak{so}(n)$, n even is thus

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j\}. \tag{3.34}$$

To define the set of positive roots, we can take the linear form

$$f : \mathfrak{h}^* \to \mathbb{C}, \tag{3.35}$$

$$\varepsilon_i \mapsto n - i. \tag{3.36}$$

Thus the positive roots are

$$\{\varepsilon_i + \varepsilon_j\}_{i\neq j} \cup \{\varepsilon_i - \varepsilon_j\}_{i<j}. \tag{3.37}$$

The simple roots are

$$\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}. \tag{3.38}$$

Because $\mathrm{Tr}(E) = 2$, we have

$$\nu(h_i E) = \frac{1}{2}\varepsilon_i, \tag{3.39}$$

where ν is the isomorphism defined in (3.16). Thus

$$K^*(\varepsilon_i, \varepsilon_j) = \frac{1}{2}\delta_{ij} \tag{3.40}$$

(δ is the Kronecker symbol). We can compute the Cartan matrix

$$C = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \ddots & 0 & 0 & 0 \\ 0 & -1 & 2 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 & 0 \\ 0 & 0 & 0 & \dots & -1 & 0 & 2 \end{pmatrix} \tag{3.41}$$

The fundamental weights are

$$\begin{gathered}\{\varepsilon_1, \varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \dots, \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{n-2}, \\ \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{n-1} + \varepsilon_n), \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{n-1} - \varepsilon_n)\}.\end{gathered} \tag{3.42}$$

3.11. Spin representations.

The representations associated to the fundamental weights $\{\varepsilon_1, \varepsilon_1 + \varepsilon_2, \dots, \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{n-2}\}$ can all be obtained from the defining representation (3.24). However, the representations associated to $\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{n-1} + \varepsilon_n)$ and $\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{n-1} - \varepsilon_n)$ can not. They are called "spin representations". A similar spin representation exists for the Lie algebra $\mathfrak{so}(2n+1)$. These spin representation play a crucial role in physics. Indeed in quantum mechanics, physical objects are described mathematically through states, which are elements of projective Hilbert spaces (A projective Hilbert space is the set of line of an Hilbert space). As dynamical objects, these physical states must belongs to some representation of the Lorentz group; a projective representation. The projective representations of the Lorentz group are obtained thanks to the spin representations of it ([Wei96] [Bar54]). One of the aim of supergravity is precisely to combine gravity with these physical spin representations, called spinors. The spin representations are themselves obtained as subrepresentation of Clifford algebras, which is the topic of our next chapter.

CHAPTER 2

Clifford algebras and their representations

This Chapter is devoted to the classification of Clifford algebra and their representations, as well as deriving symmetry properties of these representations. Our main reference is [Fig19]

1. Construction

1.1. Quadratic form.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let V be a finite dimensional vector space over $\mathbb{K}$. Let B be a symmetric bilinear form

$$B : V \times V \to \mathbb{K}. \tag{1.1}$$

The associated quadratic form is

$$Q(x) = B(x,x). \tag{1.2}$$

Reciprocally, if Q is a quadratic form, the associated symmetric bilinear form is

$$B(x,y) = \frac{1}{2}(Q(x+y) - Q(x) - Q(y)). \tag{1.3}$$

The pair (V,Q) is called a quadratic vector space.

1.2. Definition.

Let (V,Q) be a quadratic vector space. We already have the tensor algebra $T(V)$ over V. We define the Clifford algebra as a quotient of this tensor algebra by

$$\mathrm{Cl}(V,Q) \doteq {}^{T(V)}\!\big/_{\langle x\otimes x - Q(x)\rangle}. \tag{1.4}$$

where $\langle x\otimes x - Q(x)\rangle$ is the ideal generated by all elements of the form $(x\otimes x - Q(x))$.

1.3. Universal property.

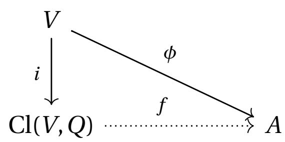
Let (V,Q) be a quadratic vector space and A be a unital associative algebra over $\mathbb{K}$. The $\mathbb{K}$-linear map ϕ : V → A is called Clifford if it satisfies

$$\phi(x)^2 = -Q(x)\mathbb{1}_A. \tag{1.5}$$

The Clifford algebra is universal in the sense that if such a map exists, then there is a morphism of algebras

$$f : \mathrm{Cl}(V,Q) \to A \tag{1.6}$$

such that the following diagram commutes



(1.7)

where $i : V \to \mathrm{Cl}(V,Q)$ denotes the canonical inclusion inherited from $V \hookrightarrow T(V)$.

1.4. Basis.

Let $\{e_i\}$ be a basis of V diagonalizing B and n = dim(v); we write $\gamma_i = i(e_i)$. For a general vector $v \in V$, $v = \sum_{i=1}^n v_i e_i$, we write $\not{v} = i(v) = \sum_{i=0}^n v_i \gamma_i$. As a quotient of the tensor algebra, the Clifford algebra is generated by $\{\gamma_i\}$. A vector space basis of the Clifford algebra is given by

$$\{\mathbb{1}, \gamma_i, \gamma_{i_1 i_2}, \ldots, \gamma_{i_1 \ldots, i_n}\}, \tag{1.8}$$

with

$$\gamma_{i_1 \ldots i_p} = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \gamma_{\sigma(i_1)} \ldots \gamma_{\sigma(i_p)}. \tag{1.9}$$

It has dimension 2^n.

2. Pin group, Spin group and spinorial representations

As the Clifford algebra is unital associative it contains a multiplicative group formed by invertible elements called the Clifford group. We denote this group by $\mathrm{Cl}^\times(V,Q)$. Any element of this group define an inner automorphism by

$$\begin{aligned} \mathrm{Cl}^\times(V,Q) &\to \mathrm{Aut}(\mathrm{Cl}(V,Q)) \\ x &\mapsto [y \mapsto xyx^{-1}] \end{aligned} \tag{2.1}$$

Let v, $w \in V$ with $Q(v) \neq 0$. We check that $\not{v}$ is invertible (with inverse $-\frac{v}{Q(v)}$) and that

$$-\not{v}\not{w}\not{v}^{-1} = \not{w} - 2\frac{B(v,w)}{Q(v)}\not{v}$$

So $\not{v}\not{w}\not{v}^{-1}$ belongs to $i(V)$ and as i is injective it is possible to send this element back to V. Doing so, we define a linear isomorphism of V, that we will denote by $\tilde{\mathrm{Ad}}_v$. One check that $\tilde{\mathrm{Ad}}_v$ is in fact the reflection with respect to the hyperplan orthogonal to v, so that $\tilde{\mathrm{Ad}}_v \in O(V,Q)$ (i.e. $\tilde{\mathrm{Ad}}_v$ preserve the quadratic form Q : $Q(\tilde{\mathrm{Ad}}_v(w)) = Q(w)$).

DEFINITION 4. *We define the following subgroups of $Cl^\times(V,Q)$:*

a) $\tilde{P}(V,Q) = \{x \in Cl^\times(V,Q) / \forall\, v \in V, \tilde{Ad}_x(\not{v}) \in i(V)\}$
b) $P(V,Q)$, *the group generated by* $\{\not{v}\}$ *with* $Q(v) \neq 0$.
c) $Pin(V,Q)$, *the group generated by* $\{\not{v}\}$ *with* $Q(v) = \pm 1$.
d) $Spin(V,Q) = Pin(V,Q) \cap Cl_0(V,Q)$

These definitions are made to clarify the different definitions that one can find in the literature. In fact it is possible show that $\tilde{P}(V,Q)$ take values in $O(V,Q)$ and its kernel is $\mathbb{K}^\times \mathbb{1}_{\mathrm{Cl}(V,Q)}$, with $\mathbb{K}^\times$ are the non zero elements of $\mathbb{K}$ and where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is the field over which V is a vector space. It is also possible to show that $P(V,Q) \to O(V,Q)$ has the kernel $\mathbb{K}^\times \mathbb{1}_{\mathrm{Cl}(V,Q)} \cap P(V,Q)$. The next step is to show that $\mathrm{Ker}\big(\mathrm{Pin}(V,Q) \to O(V,Q)\big) = \pm\mathbb{1}_{\mathrm{Cl}(V,Q)}$ and that $\mathrm{Pin}(V,Q)$ is a double cover of $O(V,Q)$. Finally, using Cartan-Dieudonné theorem, one shows that $\mathrm{Spin}(V,Q)$ is sent to $SO(V,Q)$ by $\tilde{\mathrm{Ad}}$.

3. Classification of Clifford algebras

From now on, the quadratic form Q will be assumed non degenerate.

3.1. The low dimensional real Clifford algebras.

We start with the classification of real Clifford algebras, which means that $\mathbb{K} = \mathbb{R}$. In the diagonalizing basis $\{e_i\}$, B take the form

$$\begin{pmatrix} \mathbb{1}_s & \\ & -\mathbb{1}_t \end{pmatrix}. \tag{3.1}$$

We call Cl(s,t) the associated Clifford algebra.

Cl(0,0) is the algebra consisting of scalar multiples of $\mathbb{1}$. So it is isomorphic to $\mathbb{R}$.
Cl(1,0) is the algebra generated by $\mathbb{1}$ and γ satisfying $\gamma^2 = -\mathbb{1}$. It isomorphic to $\mathbb{C}$ with isomorphism $x\mathbb{1} + y\gamma \mapsto x + iy$.
Cl(0,1) is the algebra generated by $\mathbb{1}$ and γ satisfying $\gamma^2 = \mathbb{1}$. We define $\Gamma_\pm = \frac{\mathbb{1}\pm\gamma}{2}$. $\Gamma_\pm$ are complementary projectors satisfying $\Gamma_\pm^2 = \Gamma_\pm$, $\Gamma_+\Gamma_- = 0$. It isomorphic to $\mathbb{R} \oplus \mathbb{R}$ with isomorphism $x\Gamma_+ + y\Gamma_- \mapsto (x,y)$.
Cl(2,0) is the algebra generated by $\mathbb{1}$, γ_1 and γ_2 satisfying ${\gamma_1}^2 = {\gamma_2}^2 = -\mathbb{1}$, $\gamma_1\gamma_2 = -\gamma_2\gamma_1$. It isomorphic to $\mathbb{H}$ with isomorphism

$$x\mathbb{1} + y\gamma_1 + z\gamma_2 + w\gamma_1\gamma_2 \mapsto x + iy + jz + kw.$$

Cl(1,1) is the algebra generated by $\mathbb{1}$, γ_1 and γ_2 satisfying ${\gamma_1}^2 = -\mathbb{1}$, ${\gamma_2}^2 = \mathbb{1}$, $\gamma_1\gamma_2 = -\gamma_2\gamma_1$. It isomorphic to $M_2(\mathbb{R})$ with isomorphism

$$x\mathbb{1} + y\gamma_1 + z\gamma_2 + w\gamma_1\gamma_2 \mapsto \begin{pmatrix} x+z & y+w \\ -y+w & x-z \end{pmatrix}.$$

Cl(0,2) is the algebra generated by $\mathbb{1}$, γ_1 and γ_2 satisfying ${\gamma_1}^2 = {\gamma_2}^2 = \mathbb{1}$, $\gamma_1\gamma_2 = -\gamma_2\gamma_1$. It isomorphic to $M_2(\mathbb{R})$ with isomorphism

$$x\mathbb{1} + y\gamma_1 + z\gamma_2 + w\gamma_1\gamma_2 \mapsto \begin{pmatrix} x+y & z+w \\ z-w & x-y \end{pmatrix}.$$

3.2. Two propositions regarding matrices.

PROPOSITION 5. *For* $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{H}$, $\forall\ m,n \in \mathbb{N}^*$,

$$M_n(\mathbb{K}) \otimes_{\mathbb{R}} M_m(\mathbb{R}) \simeq M_{mn}(\mathbb{K}).$$

In particular

$$M_n(\mathbb{K}) \simeq \mathbb{K} \otimes_{\mathbb{R}} M_n(\mathbb{R}).$$

Proof : For the second statement, note that the matrices E^i_j, with 0 everywhere except at the (i,j)-entry which is 1, form a basis of $M_n(\mathbb{K})$ which makes the isomorphism trivial ($k \otimes E^i_j \mapsto kE^i_j$). So we just need to show $M_n(\mathbb{R}) \otimes M_m(\mathbb{R}) = M_{nm}(\mathbb{R})$. This follows from the fact that the tensor product of matrices is the Kronecker product and that

$$E^i_j \otimes E^k_l = \begin{pmatrix} & & i & \\ & & \vdots & \\ j & \dots & E^k_l & \\ & & & \end{pmatrix}.$$

PROPOSITION 6.

$$a)\ \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \simeq M_2(\mathbb{C}), \tag{3.2}$$

$$b)\ \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \simeq M_4(\mathbb{R}), \tag{3.3}$$

$$c)\ \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C}.$$

Proof :

a Consider $\mathbb{H}$ as a $\mathbb{C}$-vector space with basis $\{1, j\}$. Let $\Phi : \mathbb{C} \times \mathbb{H} \to \mathcal{L}_{\mathbb{C}}(\mathbb{H})$ defined by $\Phi(z, q)(x) = zx\bar{q}$. Φ is $\mathbb{R}$ bilinear so it define a linear map $\tilde{\Phi} : \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \to \mathcal{L}_{\mathbb{C}}(\mathbb{H})$. Writing any quaternion as $q = z_1 + z_2 j$ we have a natural isomorphism $\mathcal{L}_{\mathbb{C}}(\mathbb{H}) \simeq M_2(\mathbb{H})$. Check that $\tilde{\Phi}$ is a morphism of $\mathbb{R}$-algebra which is injective, and that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ and $M_2(\mathbb{C})$ have the same real dimension.

b Define $\Phi \in \mathbb{H} \times \mathbb{H} \to \mathcal{L}_{\mathbb{R}}(\mathbb{H})$ by $\Phi(q_1, q_2)(x) = q_1 x \bar{q}_2$ and use the same argument.

c Define $p_\pm = (1 \otimes 1 \pm i \otimes i)$. Use ${p_\pm}^2 = p_\pm$, $p_+ p_- = 0$, $p_+ + p_- = \mathrm{Id}$. This gives an isomorphism $\mathbb{C} \oplus \mathbb{C} \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, explicitely $(z_1, z_2) \mapsto z_1 p_+ + z_2 p_-$.

3.3. Classification of all real Clifford algebras.

PROPOSITION 7. *For all n,s,t we have the following isomorphisms:*

a) $Cl(n, 0) \otimes Cl(0, 2) \simeq Cl(0, n+2)$,

b) $Cl(0, n) \otimes Cl(2, 0) \simeq Cl(n+2, 0)$,

c) $Cl(s, t) \otimes Cl(1, 1) \simeq Cl(s+1, t+1)$.

Proof : For a) let $\{e_1, e_2\}$ the standard basis of $\mathbb{R}^{(0,2)}$ and γ_1, γ_2 their image in Cl(0,2). Accordingly discompose $\mathbb{R}^{(0,n+2)} = \mathbb{R}^{(0,n)} \oplus \mathbb{R}e_1 \oplus \mathbb{R}e_2$ and define a map

$$\phi : \mathbb{R}^{(0,n+2)} \to \mathrm{Cl}(n,0) \otimes \mathrm{Cl}(0,2), \tag{3.4}$$

$$x \mapsto \not{x} \otimes \gamma_1\gamma_2, \tag{3.5}$$

$$e_i \mapsto \mathbb{1} \otimes \gamma_i. \tag{3.6}$$

Then we have

$$(\phi(x+\lambda e_1+\mu e_2))^2 = (\not{x} \otimes \gamma_1\gamma_2 + \lambda\mathbb{1} \otimes \gamma_1 + \mu\mathbb{1} \otimes \gamma_2)^2, \tag{3.7}$$

$$= -\not{x}^2 \otimes \mathbb{1} + \lambda^2\mathbb{1} \otimes \mathbb{1} + \mu^2\mathbb{1} \otimes \mathbb{1} + \lambda\not{x} \otimes (\gamma_1\gamma_2\gamma_1 + \gamma_1\gamma_1\gamma_2), \tag{3.8}$$

$$+\mu\not{x} \otimes (\gamma_1\gamma_2\gamma_2 + \gamma_2\gamma_1\gamma_2) + \lambda\mu\mathbb{1} \otimes (\gamma_1\gamma 2 + \gamma_2\gamma_1), \tag{3.9}$$

$$= (Q_{\mathbb{R}^{(n,0)}}(x) - Q_{\mathbb{R}^{(0,2)}}(\lambda e_1 + \mu e_2))\mathbb{1} \otimes \mathbb{1}, \tag{3.10}$$

$$= -Q_{\mathbb{R}^{(0,n+2)}}(x + \lambda e_1 + \mu e_2)\mathbb{1} \otimes \mathbb{1}. \tag{3.11}$$

So ϕ is a Clifford map. Hence there is a unique morphism

$$f : \mathrm{Cl}(0, n+2) \to \mathrm{Cl}(n,0) \otimes \mathrm{Cl}(0,2). \tag{3.12}$$

All the generators $\{\gamma_i \otimes \mathbb{1}\}_{i=1}^{n} \cup \mathbb{1} \otimes \gamma_1 \cup \mathbb{1} \otimes \gamma_2$ are in the image of ϕ (note that $\gamma_i \otimes \mathbb{1} = \phi(\gamma_i\gamma_{n+1}\gamma_{n+2})$. Furthermore the two algebras have the same dimension, finishing the proof. The proofs of b) and c) follow the same line (use the same morphism).

Now with these 2 propositions and the classification of low dimensional Clifford algebras we made before, one get the following table

t \ s	0	1	2	3
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$
1	$\mathbb{R} \oplus \mathbb{R}$	$M_2(\mathbb{R})$	$M_2(\mathbb{C})$	$M_2(\mathbb{H})$
2	$M_2(\mathbb{R})$	$M_2(\mathbb{R}) \oplus M_2(\mathbb{R})$	$M_4(\mathbb{R})$	$M_4(\mathbb{C})$
3	$M_2(\mathbb{C})$	$M_4(\mathbb{R})$	$M_4(\mathbb{R}) \oplus M_4(\mathbb{R})$	$M_8(\mathbb{R})$
4	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$	$M_8(\mathbb{R})$	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$
5	$M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$	$M_4(\mathbb{H})$	$M_8(\mathbb{C})$	$M_{16}(\mathbb{R})$
6	$M_4(\mathbb{H})$	$M_4(\mathbb{H}) \oplus M_4(\mathbb{H})$	$M_8(\mathbb{H})$	$M_{16}(\mathbb{C})$
7	$M_8(\mathbb{C})$	$M_8(\mathbb{H})$	$M_8(\mathbb{H}) \oplus M_8(\mathbb{H})$	$M_{16}(\mathbb{H})$

t \ s	4	5	6	7
0	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$	$M_8(\mathbb{R})$	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$
1	$M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$	$M_4(\mathbb{H})$	$M_8(\mathbb{C}$	$M_{16}(\mathbb{R})$
2	$M_4(\mathbb{H})$	$M_4(\mathbb{H}) \oplus M_4(\mathbb{H})$	$M_8(\mathbb{H})$	$M_{16}(\mathbb{C})$
3	$M_8(\mathbb{C})$	$M_8(\mathbb{H})$	$M_8(\mathbb{H}) \oplus M_8(\mathbb{H})$	$M_{16}(\mathbb{H})$
4	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{C})$	$M_{16}(\mathbb{H})$	$M_{16}(\mathbb{H}) \oplus M_{16}(\mathbb{H})$
5	$M_{16}(\mathbb{R}) \oplus M_{16}(\mathbb{R})$	$M_{32}(\mathbb{R})$	$M_{32}(\mathbb{C})$	$M_{32}(\mathbb{H})$
6	$M_{32}(\mathbb{R})$	$M_{32}(\mathbb{R}) \oplus M_{32}(\mathbb{R})$	$M_{64}(\mathbb{R})$	$M_{64}(\mathbb{C})$
7	$M_{32}(\mathbb{C})$	$M_{64}(\mathbb{R})$	$M_{64}(\mathbb{R}) \oplus M_{64}(\mathbb{R})$	$M_{128}(\mathbb{R})$

In order to complete the table, one uses the following isomorphism

PROPOSITION 8. a) *Cl(n+8,0)* $\simeq$ *Cl(n,0)* $\otimes M_{16}(\mathbb{R})$,
b) *Cl(0,n+8)* $\simeq$ *Cl(0,n)* $\otimes M_{16}(\mathbb{R})$,
c) *Cl(s+4,t+4)* $\simeq$ *Cl(s,t)* $\otimes M_{16}(\mathbb{R})$.

Proof : For a) we use

$$\mathrm{Cl}(n+8,0) \simeq \mathrm{Cl}(2,0)\otimes\mathrm{Cl}(0,2)\otimes\mathrm{Cl}(2,0)\otimes\mathrm{Cl}(0,2)\otimes\mathrm{Cl}(n,0), \tag{3.13}$$

$$\simeq \mathbb{H}\otimes M_2(\mathbb{R})\otimes\mathbb{H}\otimes M_2(\mathbb{R})\otimes\mathrm{Cl}(n,0), \tag{3.14}$$

$$\simeq M_{16}(\mathbb{R})\otimes\mathrm{Cl}(n,0).$$

and similarly for b. For c the proof uses

$$\mathrm{Cl}(s+4,t+4)\simeq\mathrm{Cl}(1,1)^{\otimes 4}\otimes\mathrm{Cl}(s,t)\simeq M_{16}(\mathbb{R})\otimes\mathrm{Cl}(s,t).$$

Corollary :

$$\mathrm{Cl}(s+8,t)\simeq\mathrm{Cl}(s+8-t,0)\otimes\mathrm{Cl}(1,1)^{\otimes t}\simeq\mathrm{Cl}(s-t,0)\otimes\mathrm{Cl}(1,1)^{\otimes t}\otimes M_{16}(\mathbb{R}), \tag{3.15}$$

$$\simeq\mathrm{Cl}(s,t)\otimes M_{16}(\mathbb{R}), \tag{3.16}$$

$$\mathrm{Cl}(s,t+8)\simeq\mathrm{Cl}(s,t)\otimes M_{16}(\mathbb{R}). \tag{3.17}$$

Using this corollary and recording that Cl(1,1) = $M_2(\mathbb{R})$ so that tensoring with it does not change the type of the algebra, we get the following table(n = s + t).

s - t [8]	Cl(s,t)
0	$M_{2^{\frac{n}{2}}}(\mathbb{R})$
1	$M_{2^{\frac{n-1}{2}}}(\mathbb{C})$
2	$M_{2^{\frac{n-2}{2}}}(\mathbb{H})$
3	$M_{2^{\frac{n-3}{2}}}(\mathbb{H}) \oplus M_{2^{\frac{n-3}{2}}}(\mathbb{H})$
4	$M_{2^{\frac{n-2}{2}}}(\mathbb{H})$
5	$M_{2^{\frac{n-1}{2}}}(\mathbb{C})$
6	$M_{2^{\frac{n}{2}}}(\mathbb{R})$
7	$M_{2^{\frac{n-1}{2}}}(\mathbb{R}) \oplus M_{2^{\frac{n-1}{2}}}(\mathbb{R})$

3.4. The subalgebra $\mathrm{Cl}_0(V,Q)$.

The tensorial algebra $T(V)$ is $\mathbb{Z}$-graded and thus $\mathbb{Z}_2$-graded. The ideal

$$I_Q \doteq \langle x\otimes x - Q(x)\langle, \tag{3.18}$$

is homogeneous for the $\mathbb{Z}_2$-grading so the Clifford algebra inherits a $\mathbb{Z}_2$-gradation. Accordingly we decompose

$$\mathrm{Cl}(V,Q) = \mathrm{Cl}_0(V,Q)\oplus \mathrm{Cl}_1(V,Q). \tag{3.19}$$

Note that $\mathrm{Cl}_0(V,Q)$ is a subalgebra of $\mathrm{Cl}(V,Q)$. Its basis is given by the even elements of the basis of $\mathrm{Cl}(V,Q)$, hence its dimension is $\frac{2^n}{2}=2^{n-1}$.

PROPOSITION 9.

$$Cl(s,t)\simeq Cl_0(s+1,t)\simeq Cl_0(t,s+1).$$

In particular $Cl_0(s,t)\simeq Cl_0(t,s)$.

Proof : Let

$$\phi : \begin{array}{ccc} \mathbb{R}^{(s,t)} & \to & \mathrm{Cl}_0(s+1,t) \\ x & \mapsto & \not{x}\otimes\gamma_{s+1} \end{array}. \tag{3.20}$$

ϕ is a Clifford map, hence it extends to a morphism of algebras $\mathrm{Cl}(s,t)\to\mathrm{Cl}_0(s+1,t)$. The image of this extension contains all the generators γ_{ij} so that it is surjective and that both algebras have the same dimension. The proof of the second isomorphism follows the same lines.

3.5. Classification of $\mathrm{Cl}_0(V,Q)$.

With the help of the preceding isomorphism, it is easy to classify the algebras $\mathrm{Cl}_0(s,t)$. (the following classification fails only for (s,t) = (0,0), in which case $\mathrm{Cl}_0(s,t)\simeq\mathbb{R}$.

s - t [8]	$\mathrm{Cl}_0(s,t)$
0	$M_{2^{\frac{n-2}{2}}}(\mathbb{R})\oplus M_{2^{\frac{n-2}{2}}}(\mathbb{R})$
1,7	$M_{2^{\frac{n-1}{2}}}(\mathbb{R})$
2,6	$M_{2^{\frac{n-2}{2}}}(\mathbb{C})$
3,5	$M_{2^{\frac{n-3}{2}}}(\mathbb{H})$
4	$M_{2^{\frac{n-4}{2}}}(\mathbb{H})\oplus M_{2^{\frac{n-4}{2}}}(\mathbb{H})$

3.6. Complex Clifford algebras.

If (V,Q) is a real quadratic vector space, its complexification $(V_{\mathbb{C}},Q_{\mathbb{C}})$ is given by

$$V_{\mathbb{C}} = V\otimes\mathbb{C} \tag{3.21}$$

$$Q_{\mathbb{C}}(v\otimes z) = z^2 Q(v)$$

PROPOSITION 10. $(V_{\mathbb{C}},Q_{\mathbb{C}})\simeq Cl(V,Q)\otimes\mathbb{C}$

Proof : The map $V\times\mathbb{C}\to\mathrm{Cl}(V,Q)\otimes_{\mathbb{R}}\mathbb{C}$; $(v,z)\mapsto \not{v}\otimes_{\mathbb{R}} z$ is $\mathbb{R}$-bilinear, from what we get a linear map $\phi : Vo\times_{\mathbb{R}}\mathbb{C}\to\mathrm{Cl}(V,Q)\otimes_{\mathbb{R}}\mathbb{C}$;

$$\phi(v\otimes_{\mathbb{R}} z) = \not{v}\otimes_{\mathbb{R}} z = (\not{v}\otimes_{\mathbb{R}} 1)z = \phi(v\otimes_{\mathbb{R}} 1)z$$

and thus ϕ is also $\mathbb{C}$-linear. Finally

$$\phi(v \otimes_{\mathbb{R}} 1)^2 = (\not{v} \otimes_{\mathbb{R}} 1)^2 = -Q(v)(\mathbb{1} \otimes_{\mathbb{R}} 1)$$

so the application is Clifford, whence it extends uniquely to a morphism of complex algebras

$$\Phi : \mathrm{Cl}(V_{\mathbb{C}}, Q_{\mathbb{C}}) \to \mathrm{Cl}(V, Q) \otimes_{\mathbb{R}} \mathbb{C}$$

The generating set $\{\not{v} \otimes 1\}$ is in the image of Φ and both algebras have the same dimension, so Φ is an isomorphism.

3.7. Notations.

If (V,Q) is a real quadratic vector space, we call $\mathrm{Cl}_{\mathbb{C}}(V, Q)$ the associated complex Clifford algebra. If (V, Q) is a real quadratic vector space, we can always find a basis of $V_{\mathbb{C}}$ in which $B_{\mathbb{C}}$ is the identity. So there is no signature for complex quadratic forms. Accordingly, complex Clifford algebras are uniquely determined by their dimensions. We will denote them $\mathrm{Cl}_{\mathbb{C}}(n)$.

3.8. The 2-periodicity of complex Clifford algebras.

PROPOSITION 11. $\forall\, n \in \mathbb{N}$, *there is an isomorphism*

$$Cl_{\mathbb{C}}(n+2) \simeq Cl_{\mathbb{C}}(n) \otimes_{\mathbb{C}} M_2(\mathbb{C}). \tag{3.22}$$

Proof : Let $\mathbb{C}^{n+2} = \mathbb{C}^n \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_2$. Define

$$\phi : \begin{array}{ccc} \mathbb{C}^{n+2} & \to & \mathrm{Cl}_{\mathbb{C}}(n) \otimes_{\mathbb{C}} M_2(\mathbb{C}) \\ x & \mapsto & \not{x} \otimes_{\mathbb{C}} \sigma_1 \\ e_1 & \mapsto & \mathbb{1} \otimes_{\mathbb{C}} i\sigma_2 \\ e_2 & \mapsto & \mathbb{1} \otimes_{\mathbb{C}} i\sigma_3 \end{array}, \tag{3.23}$$

where $x \in \mathbb{C}^n$ and

$$\{\sigma_i\} = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \tag{3.24}$$

are the Pauli matrices. Then we check that

$$\begin{aligned} \phi(x + \lambda e_1 + \mu e_2)^2 &= \not{x}^2 \otimes_{\mathbb{C}} {\sigma_1}^2 + \lambda^2 \mathbb{1} \otimes_{\mathbb{C}} (i\sigma_2^2) + \mu^2 \otimes_{\mathbb{C}} (i\sigma_3)^2 + \lambda \not{x} \otimes_{\mathbb{C}} i(\sigma_1\sigma_2 + \sigma_2\sigma_1), \\ &+ \mu \not{x} \otimes_{\mathbb{C}} i(\sigma_1\sigma_3 + \sigma_3\sigma_1) - \lambda\mu \mathbb{1} \otimes_{\mathbb{C}} (\sigma_2\sigma_3 + \sigma_3\sigma_2) \qquad (3.25) \\ &= -Q(x + \lambda e_1 + \mu e_2), \qquad (3.26) \end{aligned}$$

so the map ϕ is Clifford. Thus it extends to a morphism

$$\Phi : \mathrm{Cl}_{\mathbb{C}}(n+2) \to \mathrm{Cl}_{\mathbb{C}}(n) \otimes_{\mathbb{C}} M_2(\mathbb{C}). \tag{3.27}$$

This map is surjective as its image contains the generators of the target algebra, and both algebras have the same dimension.

3.9. Classification of complex Clifford algebras.

Using complexification of real Clifford algebras we obtain the first two complex Clifford algebras

$$\mathrm{Cl}_{\mathbb{C}}(0) \simeq \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C},$$
$$\mathrm{Cl}_{\mathbb{C}}(1) \simeq (\mathbb{R} \oplus \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C}.$$

Then we use the preceding theorem to get

$$\mathrm{Cl}_{\mathbb{C}}(2n) \simeq M_{2^n}(\mathbb{C}),$$
$$\mathrm{Cl}_{\mathbb{C}}(2n+1) \simeq M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C}).$$

3.10. $\mathrm{Cl}_{\mathbb{C},0}(n)$.

As well as for their real equivalent, complex Clifford algebras admit a $\mathbb{Z}_2$-grading, whose even part $\mathrm{Cl}_{\mathbb{C},0}$ is a subalgebra. Furthermore we have

$$\mathrm{Cl}_{\mathbb{C},0}(n) \simeq \mathrm{Cl}_{\mathbb{C}}(n-1) \tag{3.28}$$

A proof of this result can easily be obtain by complexifying real Clifford algebras. From the preceding classification of complex Clifford algebras one obtains the following classification

$$\mathrm{Cl}_{\mathbb{C},0}(2n) \simeq M_{2^{n-1}}(\mathbb{C}) \oplus M_{2^{n-1}}(\mathbb{C}),$$
$$\mathrm{Cl}_{\mathbb{C},0}(2n+1) \simeq M_2^n(\mathbb{C}).$$

4. Classification of representation of Clifford Algebras

4.1. Matrix rings and their representation.

Let R be a ring and M be a R-module. M is simple if and only if the only submodules of M are M and 0

LEMMA 1. ***Schur's Lemma.***
Let R be a ring and M, N be simple R-modules. If $\phi : M \to N$ is a morphism of R-modules, then $\phi = 0$ or ϕ is an isomorphism. For a simple module M the ring $End_R(M)$ is a division ring.

Proof : Ker(ϕ) and Im(ϕ) are submodules of M and N respectively.
For $A \in M_n(\mathbb{K})$, we call

$L_{ij}(A)$ the matrix with all rows being 0 except the j$^{\text{th}}$ one which is the i$^{\text{th}}$ one of A.
$C_{ij}(A)$ the matrix with all columns being 0 except the j$^{\text{th}}$ one which is the i$^{\text{th}}$ one of A.

PROPOSITION 12. $\forall A \in M_n(\mathbb{K})$,

$$E^i_j A = L_{ij}(A),$$
$$E^i_j A = C_{ji}(A),$$
$$E^i_r A E^s_j = a_{rs} E^i_j.$$

with a_{rs} the (r,s)-component of A.

PROPOSITION 13. *$M_n(\mathbb{K})$ is a simple ring (its only bilateral ideals are 0 and itself).*

Proof : Let J be a non-zero bilateral ideal of $M_n(\mathbb{K})$. Then there exist A ∈ J and (r,s) such that $a_{rs} \neq 0$. Then

$$\forall (i,j)\, E^i_j = E^i_r A E^s_j \in J, \tag{4.1}$$

and J = $M_n(\mathbb{K})$.

PROPOSITION 14. *Let $R = M_n(\mathbb{K})$ and $e_i = E^i_i$.*

a *$C_i = Re_i$ is a left ideal of R. In particular it is a left module of R, and it is simple as such.*

b *Any simple left R-module is isomorphic to $\mathbb{K}^n$.*

c *As a left R-module, R is isomorphic to $C_1 \oplus C_2 \oplus ... \oplus C_n$.*

Proof : c is a consequence of a and the fact that any matrix is the direct sum of its columns.

a Let $A \in Re_i$. Then A is of the form (0,0,...,C,...,0) where C is some column. Now for $B \in R$, BA is also of the form (0,0,...,C,...,0) and the first assertion is proved. Now let J be a non zero submodule of Re_i and $A \in J$, $A \neq 0$. So there is r such that $a_{ri} \neq 0$. Thus $E^i_i = \frac{1}{a_{ri}} E^i_r A \in J$ and thus $Re_i \subset J$, which proves the second assertion.

b Let M be a simple left module and $m \in R$, $m \neq 0$. Then

$$1_R \cdot m = (\sum_i e_i) m = \sum_i e_i \cdot m. \tag{4.2}$$

As $m \neq 0, \exists\, i \,/\, e_i \cdot m \neq 0$. Fix such an i. Let

$$\phi : \begin{array}{ccc} Re_i & \rightarrow & M \\ r \cdot e_i & \mapsto & r \cdot e_i m \end{array}. \tag{4.3}$$

ϕ is an morphism of left R-module and $\phi \neq 0$ as $\phi(1_R \cdot e_i) \neq 0$. Hence ϕ is an isomorphism. Finally there is an obvious isomorphism of left R-module $C_i = R \cdot e_i \simeq \mathbb{K}^n$ given by $(0,...,C,...,0) \mapsto C$.

PROPOSITION 15. *Let $R = M_n(\mathbb{K}) \oplus M_n(\mathbb{K})$. Then the simple left R-modules are (up to isomorphism) $(\mathbb{K}^n, 0)$ and $(0, \mathbb{K}^n)$.*

Proof : Let N be a left R-module. Let $p_1 = (\mathbb{1}_{M_n(\mathbb{K})}, 0)$ and $p_2 = (0, \mathbb{1}_{M_n(\mathbb{K})})$. As p_1 and p_2 commutes with any elements of R, p_1N and p_2N are sub-R-modules of N. As ${p_i}^2 = p_i$, $p_1 p_2 = 0$ and $p_1 + p_2 = \mathbb{1}_R$, $N = p_1 N \oplus p_2 N$. Thus if N is simple, $N = p_1 N$ or $N = p_2 N$. But in that case, N is a left $M_n(\mathbb{K})$-module, equally simple as such. Thus $N \simeq \mathbb{K}^n$ as a left $M_n(\mathbb{K})$-module, and this isomorphism can be extended to an isomorphism of left R-module with appropriate action of p_i.

THEOREM 9. *Let $R = M_n(\mathbb{K})$. Then any R-module is completely reducible (or semisimple)*

Proof : Let N be a non zero R module and n ∈ N. Let e_i be the matrix whose only non-zero entry is 1 at the (i,i) entry. We know that Re_i is a simple left R-module. Let $N_i = Re_i n$, it is a simple left R-module as well. So N contains at least one non zero semisimple N-submodule. Let $\mathcal{P}$ be the set of all semisimple N-submodule, ordered by inclusion. Then for any linearly ordered subset $Q = \{M_i\}$ of P, $\oplus_i M_i$ is a maximal element of Q. Thus we can apply Zorn's lemma to get a maximal semisimple submodule M of N. If $M \neq N$, then ${}^{N}\!/_{M}$ is a non trivial R-module. By the above, we can find a simple submodule S of ${}^{N}\!/_{M}$. If we let $\overline{S}$ be its preimage in N, then $\overline{S} \oplus M$ is a semisimple submodule of N strictly containing M, contradicting its maximality. Thus M = N and N is semisimple.

PROPOSITION 16. *Let $R = M_n(\mathbb{K}) \oplus M_n(\mathbb{K})$. Any left R-module is a direct sum of simple R-modules.*

Proof : The proofs of 15 and 9 together.

PROPOSITION 17. *Let $A = M_n(\mathbb{K})$ and $B = M_m(\mathbb{K}')$. Any A-B-bimodule is a direct sum of simple A-B-modules. Any two A-B simple bimodules are isomorphic.*

Proof : Similar, using $A = \oplus_i Ae_i$ and $B = \oplus_i e'_i B$. The simple module is $M_{n,m}(\mathbb{K}")$ where $\mathbb{K}"$ is given by propositions 1 and 2 above. Be careful of the largest commuting subfield of $\mathbb{K}$ and $\mathbb{K}'$.

4.2. Skolem-Noether Theorem. (Originally published in [Sko27].)

THEOREM 10. *Let $A = M_n(\mathbb{K})$ and $\mathbb{K}' = Z(A)$. Any $\mathbb{K}'$-linear automorphism of A is inner, i.e. if f is an automorphism of A, there is an $x \in A$ such that $\forall\, a \in A$, $f(a) = xax^{-1}$*

Proof : Consider A as an A bimodule in two ways, i.e. by $(a \otimes a') \cdot m = ama'$ and $(a \otimes a') \cdot_f m = amf(a')$. Call A_f the second bimodule. It is a direct sum of the simple A-bimodule A, and as both are isomorphic vector space over $\mathbb{K}'$ they are ismorphic as bimodule. Let $\phi : A \to A_f$ be this isomorphism. In particular, ϕ is a morphism of left A-module. So

$$\phi(a) = \phi(a \cdot 1) = a\phi(1) = a \cdot x$$

Now as ϕ is an isomorphism, x need to be invertible. As a morphisme of right A-module, we have

$$\phi(a) = \phi(1 \cdot a) = \phi(1) \cdot_f a = xf(a)$$

.

4.3. Real Clifford algebras.

Let A be a real Clifford algebra. The goal of this section is to classify all finite dimensional representation of A.

DEFINITION 5. *A (real) representation of A is a pair (ρ, V) where V is a $\mathbb{R}$-vector space and ρ is a unital $\mathbb{R}$-linear morphism $A \to End_{\mathbb{R}}(V)$.*

Any representation (ρ,V) of the algebra A gives V a structure of A-module. This leads to several remarks.

DEFINITION 6. *A representation (ρ,V) is called irreducible if the A-module V is simple. Similarly, a representation (ρ,V) is called completely reducible if the A-module V is semisimple.*

From the fact that the Clifford algebras have only semisimple modules, we need only to determined their irreducible representations.

DEFINITION 7. *Let (ρ,V) be a real representation of A. We say that (ρ,V) is of type* $\mathbb{K}$ *if the set of all automorphism of V commuting with ρ is equal to* $\mathbb{K}$.

PROPOSITION 18. *In the preceding definition,* $\mathbb{K}$ *is a division ring (if we add 0).*

Proof : Suppose V is irreducible. An automorphism of V commuting with the representation of A is an automorphism of simple A-module. This result is thus Schur's lemma. Now if V is not irreducible, it can be written as a direct sum of irreducible one. The requirement of being an automorphism exclude morphism which will be 0 on some simple summand and non 0 on another.

We recall the famous Frobenius theorem

THEOREM 11. *The only finite dimensional associative division algebras over the real numbers are the real numbers themselves* $\mathbb{R}$*, the complex* $\mathbb{C}$ *and the quaternions* $\mathbb{H}$.

It follows that the only possibilities for $\mathbb{K}$ are $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. In that case we will say that V is a $\mathbb{K}$-vector space and ρ has target space $End_{\mathbb{K}}(V)$. A special attention need to be done on quaternionic vector spaces. Here scalar act on the right for quaternionic vector spaces and quaternionic matrices act on the left. Moreover, any irreducible representation is given by a simple module. Thus from the classification of simple modules we know the form of irreducible representation. When A = $M_n(\mathbb{K})$, this representation is $\mathbb{K}^n$ and thus the morphism ρ is a morphism $M_n(\mathbb{K}) \to M_n(\mathbb{K})$.

DEFINITION 8. *Two* $\mathbb{K}$*-representations* (ρ_1, V_1)*,* (ρ_2, V_2) *are equivalent if there exist a* $\mathbb{K}$*-linear isomorphism* $\phi : V_1 \to V_2$ *such that*

$$\forall\, a \in A, \phi \circ \rho_1(a) = \rho_2(a) \circ \phi$$

We will start now to classify irreducible representation of Clifford algebras. We begin with the case $A = M_n(\mathbb{K})$. If $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$, we know from Skolem-Noether theorem that any $\mathbb{R}$-morphism $M_n(\mathbb{K}) \to M_n(\mathbb{K})$ can be written as $\rho(a) = xax^{-1}$ i.e. $\rho(a)x = xa$ for some $x \in GL_n(\mathbb{K})$. The morphism defined by x is $\mathbb{K}$-linear with our convention that quaternionic scalars act on the right and quaternionic matrices on the left. Thus any irreducible representation of $M_n(\mathbb{K})$ is equivalent to the natural representation of the algebra of matrices on vectors.

When $\mathbb{K} = \mathbb{C}$, a representation ρ is a $\mathbb{R}$-linear morphism $M_n(\mathbb{C}) \to M_n(\mathbb{C})$. Any $\mathbb{R}$-linear morphism ρ can be written as the sum of a $\mathbb{C}$-linear one and conjugate $\mathbb{C}$-linear one $\rho = \Lambda + C$ with

$$\Lambda(a) = \frac{\rho(a) - i\rho(ia)}{2}, \tag{4.4}$$

$$C(a) = \frac{\rho(a) + i\rho(ia)}{2}. \tag{4.5}$$

ρ is unital so $\rho(\mathbb{1}) = \mathbb{1} = \Lambda(\mathbb{1}) + C(\mathbb{1})$. Also, as algebra isomorphism, ρ needs to preserve the center. As the center is the field $\mathbb{C}$, ρ restricted to the center is a field automorphism of $\mathbb{C}$ preserving $\mathbb{R}$. There are only two of such, the identity or the complex conjugation. So we have

$$\rho(i\mathbb{1}) = \Lambda(i\mathbb{1}) + C(i\mathbb{1}) = 2i\Lambda(\mathbb{1}) - i\mathbb{1}, \tag{4.6}$$

$$\Lambda = \rho \qquad \textit{if } \rho(i\mathbb{1}) = i\mathbb{1}, \tag{4.7}$$

$$\Lambda = 0 \qquad \textit{if } \rho(i\mathbb{1}) = -i\mathbb{1}. \tag{4.8}$$

So either ρ is $\mathbb{C}$-linear or $\mathbb{C}$-antilinear. If ρ is $\mathbb{C}$-linear, we have, by Skolem-Noether theorem $\rho(a) = xax^{-1}$ for some invertible complex matrix x. Similarly, if ρ is $\mathbb{C}$-antilinear $\rho(a) = y\overline{a}y^{-1}$ for some invertible y. In the first case, the representation is equivalent to the tautological representation $\rho(a) = a$ and in the second case it is equivalent to the conjugate representation $\rho(a) = \overline{a}$. Thus we have two inequivalent representations of the real algebra $M_n(\mathbb{C})$.
Finally, if A = $M_n(\mathbb{K}) \oplus M_n(\mathbb{K})$, we know that its simple modules are either $(\mathbb{K}^n, 0)$ or $(0, \mathbb{K}^n)$. Then the same analysis applies to these simple modules. As here we only have $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$, there are only two irreducible representations. This ends our classification of irreducible representations of real Clifford algebras.

4.4. Representations of Complex Clifford Algebras.

Let A be a complex Clifford algebra. We know from the classification above that $A \simeq M_n(\mathbb{C})$ or $A \simeq M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$. Thus the unique simple A-modules are $\mathbb{C}^n$ or $(\mathbb{C}^n, 0)$ and $(0, \mathbb{C})$. As a representation of a complex Clifford algebra is now a $\mathbb{C}$-linear map, A admits respectively only 1 or 2 irreducible representations by Skolem-Noether theorem.

DEFINITION 9. *Let V be a complex vector space. A real structure on V is an antilinear map J such that $J^2 = Id_V$. A quaternionic structure on V is a antilinear map J such that $J^2 = -Id_V$.*

From a complex representation of a complex Clifford algebra $\mathrm{Cl}_{\mathbb{C}}(n)$, one can recover the representations the real algebra $\mathrm{Cl}(s,t)$, where $s+t = n$, with the help of a real or quaternionic structure, commuting with the representation. Of course the existence of such operator is provided only in the case these representations were real or quaternionic respectively.

5. Representation of complex Clifford algebra

5.1. Generating set.

Let (V,Q) be a quadratic vector space. Let $\{e_i\}$ be the orthonormal basis of V w.r.t Q (the one in which B is a diagonal matrix with entry ± 1). We suppose we have a map $\phi : e_i \mapsto \Gamma_i$ where $\{\Gamma_i\}$ is the free set of some matrix algebra $M_n(\mathbb{K})$, where the Γ_i's satisfy

$$\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = -2B_{ij}. \tag{5.1}$$

Then we can extend this map by linearity to all of V and the map thus obtained is automatically Clifford. By universality of the Clifford algebra, we have an algebra morphism $\mathrm{Cl}(V,Q) \to M_n(\mathbb{K})$, i.e. a representation. If we are in the case where we already had an irreducible representation $(\rho, \mathbb{K}^n)$ of the Clifford algebra Cl(V,Q), with ρ taking value in $M_n(\mathbb{K})$, and there is only, up to equivalence, one irreducible representation, we know that there is some matrix X such that $\Gamma_i = X\rho(\gamma_i)X^{-1}$. This argument will be use here to obtain representations enjoying specific symmetry properties.

5.2. Unitary representations.

We will focus on complex Clifford algebra in even dimensions. In that case, we know the Clifford algebra is isomorphic to $M_n(\mathbb{C})$. We consider an irreducible representation $(\rho, \mathbb{C}^n)$ of this algebra. The γ_i's generate a finite multiplicative group G in the Clifford algebra of order 2^n. Also, $\mathbb{C}^n$ has a canonical hermitian form h. We define

$$H(x,y) = \frac{1}{2^n} \sum_{g \in G} h(\rho(g)x, \rho(g)y) \tag{5.2}$$

Any element of G is unitary for the hermitian form H. Any hermitian form can be diagonalized (and rescaled), so let Γ_i the matrix associated to $\rho(\gamma_i)$ in the basis where H is the identity. the γ_i's define an equivalent representation of ρ. So we have shown

PROPOSITION 19. *It is always possible to choose a representation ρ in which every matrix $\rho(\gamma_i)$ is unitary (with respect to the canonical hermitian form of $\mathbb{C}^n$).*

In this paragraph, we have made a distinction between the abstract Clifford element γ_i and a matrix representing it $\Gamma_i = \rho(\gamma_i)$. We will not make this distinction anymore in the sequel, and simply write γ_i for the Clifford generator and its representation, that we assume to be a unitary matrix. (5.1) implies that ${\gamma_i}^2 = \pm\mathbb{1}$. So $\gamma_i = \pm{\gamma_i}^{-1}$. By unitarity of γ_i,

$$\gamma_i = \pm{\gamma_i}^{\dagger}. \tag{5.3}$$

In such a representation, as any other, the γ_i's satisfy (5.1). But then an easy calculation show that $\{{}^t\gamma_i\}$, $\{\gamma_i^*\}$ and $-{\gamma_i}^{\dagger}$ satisfy also (5.1). By the discussion above, there are invertible matrices A, B and C such that

$$ {}^t\gamma_i = A\gamma_i A^{-1} \tag{5.4}$$

$$ {\gamma_i}^* = B\gamma_i B^{-1} \tag{5.5}$$

$$ -{\gamma_i}^\dagger = C\gamma_i C^{-1} $$

5.3. A concrete representation.

We start with complex Clifford algebras. We recall that the Clifford algebra in even dimension d is $M_{2^{d/2}}(\mathbb{C})$ and $M_{2^{(d-1)/2}}(\mathbb{C}) \oplus M_{2^{(d-1)/2}}(\mathbb{C})$ in odd dimension. It is generated by anticommuting elements γ_a, the so-called gamma matrices. In any irreducible representation of the Clifford algebra they can be chosen to be unitary. Their anticommutation relations imply

$$ \gamma_a = {\gamma_a}^\dagger \tag{5.6}$$

For example, we can use the explicit representation

$$ \gamma_1 = \sigma_1 \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}, \tag{5.7}$$

$$ \gamma_2 = \sigma_2 \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}, \tag{5.8}$$

$$ \gamma_3 = \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}, \tag{5.9}$$

$$ \gamma_4 = \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}, \tag{5.10}$$

$$ \ldots, \tag{5.11}$$

$$ \gamma_d = \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_2. \tag{5.12}$$

for even dimensions. For odd dimension the ultimate γ matrix is

$$ \gamma_d = \sigma_3 \otimes \cdots \otimes \sigma_3. \tag{5.13}$$

The σ_i are the Pauli's matrices.

5.4. Basis of irreducible representation and the matrix γ_*.

We will use the common notation

$$ \gamma_{a_1 \ldots a_n} = \frac{1}{n!} \sum_{\sigma \in S_n} \gamma_{a_{\sigma(1)}} \cdots \gamma_{a_{\sigma(n)}}, \tag{5.14}$$

and

$$ \gamma_* = \gamma_1 \ldots \gamma_d. \tag{5.15}$$

For an irreducible representation, a basis of $M_{2^{d/2}}(\mathbb{C})$ is

$$ \{\mathbb{1}, \gamma_a, \ldots, \gamma_{a_1 \ldots a_n}, \ldots, \gamma_*\}, \tag{5.16}$$

for even d and

$$ \{\mathbb{1}; \gamma_a; \ldots; \gamma_{a_1 \ldots a_{\frac{d-1}{2}}}\}, \tag{5.17}$$

for odd d. γ_* satisfies the following commutation relation with the other gamma matrices:

$$ \gamma_* \gamma_{a_1 \ldots a_n} = (-)^{(d-1)n} \gamma_{a_1 \ldots a_n} \gamma_*. \tag{5.18}$$

In particular in odd dimension it commutes with any other gamma matrices, and ,as they form a basis of the algebra of matrices, must be proportional to the identity. Now

$$\gamma_*{}^2 = (-)^{\frac{d(d-1)}{2}} \mathbb{1} \tag{5.19}$$

so

$$\gamma_* = \pm i^{(\frac{d(d-1)}{2})} \mathbb{1} \tag{5.20}$$

In odd dimension there exists two unequivalent irreducible representations of the Clifford algebra, we will see in a moment that they are characterized by the value of γ_*. One go from one to the other with the change :

$$\gamma_a \to -\gamma_a \tag{5.21}$$

It is obvious that if $\{\gamma_a\}$ generates an irreducible representation, then $\{-\gamma_a\}$ generates one as well. To see that they are not equivalent, one of them as its γ_* equal $i^{(\frac{d(d-1)}{2})}\mathbb{1}$. Under the transformation (5.21), γ_* is changed to minus itself, and the to γ_* cannot be similar, as $\mathbb{1}$ is not similar to $-\mathbb{1}$.

5.5. Symmetry of gamma matrices.

5.5.1. *Existence and unitarity of C.*

We start by looking only of the even dimensional case. As $\{\gamma_a{}^T\}$ satisfies the Clifford relations

$$\gamma_a{}^T\gamma_b{}^T + \gamma_b{}^T\gamma_a{}^T = \delta_{ab} \tag{5.22}$$

it can be used define an irreducible representation of the Clifford algebra. By uniqueness of it, there exists some matrix C such that:

$$C\gamma_a C^{-1} = \gamma_a{}^T \tag{5.23}$$

By the same argument, there exists also a matrix $\tilde{C}$ such that:

$$\tilde{C}\gamma_a\tilde{C}^{-1} = -\gamma_a{}^T \tag{5.24}$$

From these two relations one infers

$$C\gamma_{a_1\ldots a_n}C^{-1} = (-)^{\frac{n(n-1)}{2}}\gamma^T_{a_1\ldots a_n} \tag{5.25}$$

$$\tilde{C}\gamma_{a_1\ldots a_n}\tilde{C}^{-1} = (-)^{\frac{n(n+1)}{2}}\gamma^T_{a_1\ldots a_n} \tag{5.26}$$

Next, on the one hand one has

$$\left(C\gamma_{a_1\ldots a_n}C^{-1}\right)^\dagger = (-)^{\frac{n(n-1)}{2}}(C^{-1})^\dagger\gamma^T_{a_1\ldots a_n}C^\dagger \tag{5.27}$$

On the other hand

$$\left(C\gamma_{a_1\ldots a_n}C^{-1}\right)^\dagger = (-)^{\frac{n(n-1)}{2}}(\gamma^T_{a_1\ldots a_n})^\dagger \tag{5.28}$$

$$= \gamma_{a_1\ldots a_n}{}^T \tag{5.29}$$

$$= (-)^{\frac{n(n-1)}{2}}C\gamma_{a_1\ldots a_n}C^{-1} \tag{5.30}$$

Thus

$$C^\dagger C\gamma_{a_1\ldots a_n} = \gamma_{a_1\ldots a_n}C^\dagger C \tag{5.31}$$

So

$$C^\dagger C = \alpha \mathbb{1} \tag{5.32}$$

$C^\dagger C$ is hermitian so $\alpha \in \mathbb{R}$. It is possible to rescale

$$C \to \frac{1}{\sqrt{\alpha}} C \tag{5.33}$$

So we can always choose C to be unitary. This we will do for the rest of this exposition.

5.5.2. *The ϵ-sign, even dimension.*
The corollary of ${\gamma_a}^\dagger = \gamma_a$ is ${\gamma_a}^T = {\gamma_a}^*$. Thus

$$C\gamma_{a_1 \dots a_n} C^{-1} = \gamma^*_{a_1 \dots a_n} \tag{5.34}$$

Using the same argument as above, we find

$$C^* C = \epsilon \mathbb{1} \tag{5.35}$$

Using the unitarity of C,

$$C^* = \frac{1}{\epsilon} C^{-1} = \frac{1}{\epsilon} C^\dagger \tag{5.36}$$

$$C = \frac{1}{\epsilon} C^T = \frac{1}{\epsilon^2} C \tag{5.37}$$

leading to

$$\epsilon = \pm 1 \tag{5.38}$$

Using this fact we get

$$C\gamma_{a_1 \dots a_n} = \epsilon (-)^{\frac{n(n-1)}{2}} (C\gamma_{a_1 \dots a_n})^T \tag{5.39}$$

The same reasoning applies to $\tilde{C}$ i.e. :

$$\tilde{C}^\dagger \tilde{C} = \mathbb{1} \tag{5.40}$$

$$\tilde{C}^T = \tilde{\epsilon} \tilde{C} \qquad \tilde{\epsilon} = \pm 1 \tag{5.41}$$

$$\tilde{C}\gamma_{a_1 \dots a_n} = \tilde{\epsilon} (-)^{\frac{n(n+1)}{2}} \gamma_{a_1 \dots a_n} \tag{5.42}$$

To find the sign of ϵ, we use the following trick: As for $\{\gamma_{a_1 \dots a_n}\}$, $\{C\gamma_{a_1 \dots a_n}\}$ form a basis of the algebra of matrices $M_{2^{d/2}}(\mathbb{C})$. In particular they are linearly independent. Furthermore they are either symmetric or antisymmetric. Now the number of antisymmetric matrices is known to be $\frac{2^{d/2}(2^{d/2}-1)}{2}$. One the other hand, there are

$$\sum_{n=0}^{d} \frac{1}{2} \left[1 - \epsilon (-)^{\frac{n(n-1)}{2}} \right] \binom{d}{n} \tag{5.43}$$

antisymmetric matrices among the $C\gamma_{a_1 \dots a_n}$. So we have to fix ϵ so that this two numbers are equal. To compute (5.43) we go as follow. First we decompose

$$\sum_{n=0}^{d} \binom{d}{n} (-)^{\frac{n(n-1)}{2}} = \sum_{j=0}^{d/2} \binom{d}{2j} (-)^j + \sum_{j=0}^{d/2} \binom{d}{2j+1} (-)^j \tag{5.44}$$

To compute each sum we compute

$$(1+i)^d = 2^{d/2}\left[\cos(\frac{d\pi}{4}) + i\sin(\frac{d\pi}{4})\right] \tag{5.45}$$

$$= \sum_{j=0}^{d/2}\binom{d}{2j}(-)^j + i\sum_{j=0}^{(d/2)-1}\binom{d}{2j+1}(-)^j \tag{5.46}$$

Taking respectively real and imaginary part of the preceding expressions we obtain

$$\sum_{j=0}^{d/2}\binom{d}{2j}(-)^j = 2^{d/2}\cos(\frac{d\pi}{4}) \tag{5.47}$$

$$\sum_{j=0}^{(d/2)-1}\binom{d}{2j+1}(-)^j = 2^{d/2}\sin(\frac{d\pi}{4}) \tag{5.48}$$

From what (5.43) is found to be

$$\frac{1}{2}\left[2^d - 2^{d/2}\epsilon\left(\cos(\frac{d\pi}{4}) + \sin(\frac{d\pi}{4})\right)\right] \tag{5.49}$$

Hence we have

$$\epsilon = \begin{cases} +1 & d \equiv 0,2 \quad [8] \\ -1 & d \equiv 4,6 \quad [8] \end{cases} \tag{5.50}$$

For $\tilde{\epsilon}$ (5.43) has to be replaced by

$$\sum_{n=0}^{d}\frac{1}{2}\left[1 - \tilde{\epsilon}(-)^{\frac{n(n+1)}{2}}\right]\binom{d}{n} \tag{5.51}$$

which is equal to

$$\frac{1}{2}\left[2^d - 2^{d/2}\tilde{\epsilon}\left(\cos(\frac{d\pi}{4}) - \sin(\frac{d\pi}{4})\right)\right] \tag{5.52}$$

and thus the values for $\tilde{\epsilon}$ are:

$$\tilde{\epsilon} = \begin{cases} +1 & d \equiv 0,6 \quad [8] \\ -1 & d \equiv 2,4 \quad [8] \end{cases} \tag{5.53}$$

5.5.3. *The ϵ-sign, odd dimension.*

In odd dimension there are two inequivalent irreducible representations of the Clifford algebra. As $\{\gamma_a^T\}$ and $\{-\gamma_a^T\}$ are related by the transformation (5.21), they generate the two inequivalent irreducible representations. Thus the representation generated by $\{\gamma_a\}$ is either equivalent to the one generated by $\{\gamma_a^T\}$ or to the one generateedd by $\{-\gamma_a^T\}$ but never both. This in turn implies that always one of the two matrices C, $\tilde{C}$ exists, but never both. To see wich one, recall that

$$\gamma_* = \pm i^{(\frac{d(d-1)}{2})}\mathbb{1} \tag{5.54}$$

This implies that

$${\gamma_*}^T = \gamma_* = C\gamma_* C^{-1} = (-)^{\frac{d(d-1)}{2}}{\gamma_*}^T \tag{5.55}$$

Forcing

$$(-)^{\frac{d(d-1)}{2}} = 1 \qquad \Longrightarrow \qquad d \equiv 1 \quad [4] \tag{5.56}$$

Thus the matrix C can only exist in dimension $d \equiv 1 \quad [4]$, and therefore $\tilde{C}$ can only exist for $d \equiv 3 \quad [4]$.

5.5.4. *The different signs of ϵ and $\tilde{\epsilon}$.*

Now we can compute ϵ and $\tilde{\epsilon}$ for the dimensions where the matrices C and $\tilde{C}$ exists. We use the same method as for the even dimension, but in this case we pay attention to the fact that a basis of $M_{2^{(d-1)/2}}(\mathbb{C})$ is given by:

$$\{\mathbb{1}, \gamma_a, \dots, \gamma_{a_1 \dots a_{\frac{d-1}{2}}}\} \tag{5.57}$$

or by

$$\{\gamma_{a_1 \dots a_{\frac{d-1}{2}}}, \dots, \gamma_*\} \tag{5.58}$$

To take this into account one has to multiply (5.43) ((5.51) *for* $\tilde{\epsilon}$) by an additional $\frac{1}{2}$ factor. This gives

$$\frac{1}{4}\sum_{n=0}^{d}\binom{d}{n}\left[1-\epsilon(-)^{\frac{n(n-1)}{2}}\right] = \frac{1}{4}\left[2^d - \epsilon 2^{(d/2}\left(\cos(\frac{d\pi}{4}) + \sin(\frac{d\pi}{4})\right)\right] \tag{5.59}$$

$$\frac{1}{4}\sum_{n=0}^{d}\binom{d}{n}\left[1-\tilde{\epsilon}(-)^{\frac{n(n+1)}{2}}\right] = \frac{1}{4}\left[2^d - \tilde{\epsilon} 2^{(d/2}\left(\cos(\frac{d\pi}{4}) - \sin(\frac{d\pi}{4})\right)\right] \tag{5.60}$$

Which gives

$$\epsilon = \begin{cases} +1 & d \equiv 1 \quad [8] \\ -1 & d \equiv 5 \quad [8] \end{cases} \qquad \tilde{\epsilon} = \begin{cases} +1 & d \equiv 7 \quad [8] \\ -1 & d \equiv 3 \quad [8] \end{cases} \tag{5.61}$$

5.5.5. *Summary.*

The following table and formula condensate the above development.

$$C\gamma_{a_1,\dots,a_n} = \epsilon(-)^{\frac{n(n-1)}{2}}(C\gamma_{a_1,\dots,a_n})^T \tag{5.62}$$

$$\tilde{C}\gamma_{a_1,\dots,a_n} = \tilde{\epsilon}(-)^{\frac{n(n+1)}{2}}(\tilde{C}\gamma_{a_1,\dots,a_n})^T \tag{5.63}$$

$d\,(mod\,8)$	ϵ	$\tilde{\epsilon}$
0	+1	+1
1	+1	
2	+1	-1
3		-1
4	-1	-1
5	-1	
6	-1	+1
7		+1

5.6. Change of representation.

We made our calculations assuming the gamma matrices to be hermitian. In an arbitrary representation,

$$\gamma'_a = P^{-1}\gamma_a P \tag{5.64}$$

the matrix C transform as

$$C' = P^T C P \tag{5.65}$$

The relation

$$C'^T = \epsilon C' \tag{5.66}$$

is conserved. Exactly the same happens to $\tilde{C}$.

Finally let us mention that in the explicit representation given in (5.7)-(5.13), one can use

$$\prod_i \gamma_{2i} \qquad \prod_i \gamma_{2i+1} \tag{5.67}$$

for C and/or $\tilde{C}$. When the dimension is even, one of the two matrix above is C, the other is $\tilde{C}$, when the dimension is odd, both of them are either C or $\tilde{C}$.

6. Representation of real Clifford algebras

6.1. Generators in a given signature, hermitian conjugates.

In the real case, one has to consider the signature of the bilinear form defining our Clifford algebra. We will say it has t minus signs and $s = d - t$ plus signs and we will denote it $\mathrm{Cl}(s,t)$. The set of generators of $\mathrm{Cl}(s,t)$ is obtained by replacing

$$\gamma_a \to i\gamma_a \qquad a = 1\dots t \tag{6.1}$$

We have then

$$\gamma_a{}^\dagger = \begin{cases} -\gamma_a & a = 1\dots t \\ +\gamma_a & a = t+1\dots d \end{cases} \tag{6.2}$$

We introduce a new matrix:

$$A = \gamma_1 \dots \gamma_t \tag{6.3}$$

With it, one can rewrite (6.2) as

$$\gamma_a{}^\dagger = (-)^t A\gamma_a A^{-1} = (-)^{\frac{t(t-1)}{2}} A\gamma_a A \tag{6.4}$$

This is extended to all gamma matrices:

$$\gamma^\dagger_{a_1\dots a_n} = (-)^{nt+\frac{n(n-1)}{2}+1} A\gamma_{a_1\dots a_n} A^{-1} \tag{6.5}$$

Under a similarity transformation (a choice of another equivalent representation), the relation changes as:

$$\gamma'_a = P^{-1}\gamma_a P \tag{6.6}$$

$$\gamma'_a{}^\dagger = (-)^t A'\gamma'_a A'^{-1} \tag{6.7}$$

$$A' = P^\dagger A P \tag{6.8}$$

The following relation stays true in any representation

$$A'^\dagger = (-)^{t(t+1)/2} A \tag{6.9}$$

One can also define the matrix

$$\tilde{A} = \gamma_{t+1}\dots\gamma_d \tag{6.10}$$

It satisfies

$$\tilde{A}\gamma_a\tilde{A}^{-1} = (-)^{d-t+1}\gamma_a^\dagger = \begin{cases} (-)^{t+1}{\gamma_a}^\dagger & \text{when d is even} \\ (-)^t{\gamma_a}^\dagger & \text{when d is odd} \end{cases} \tag{6.11}$$

6.2. The Dirac conjugation.

Let V (that we will identify with $\mathbb{C}^m$) be the irreducible representation like in (6.17), but whose element are odd (Grassmann). We want to build an hermitian form on V invariant under the orthogonal group $SO(s,t)$ associated to the bilinear form defining the Clifford algebra. Such an hermitian form will be of the form:

$$\langle v_1, v_2\rangle = {v_1}^\dagger D v_2 \tag{6.12}$$

where D satisfy

$$D^\dagger = -D \tag{6.13}$$

$$\gamma_{ab}^\dagger D = -D\gamma_{ab} \tag{6.14}$$

We check that the second condition is always satisfied if we take D proportional to A. In regard to the first condition, we can take $D = i^p A$, where p is the opposite parity of $\frac{t(t+1)}{2}$:

$$p \equiv \frac{t(t+1)}{2} + 1\,[2] \tag{6.15}$$

The Dirac conjugation is defined as

$$\overline{v} = v^\dagger D \tag{6.16}$$

6.3. Real and quaternionic structures.

Let V be the vector space defining the irreducible representation. By this we understand that the representation is given by a $\mathbb{C}$-linear map

$$\rho : \mathrm{Cl}(s,t) \to GL(V) \tag{6.17}$$

A real structure for the representation is an antilinear map

$$\mathrm{J} : V \to V \tag{6.18}$$

commuting with ρ and squaring to the identity. A quaternionic structure is the same thing but it square to minus the identity instead[1]. We make the identification $V = \mathbb{C}^{\lfloor d/2\rfloor}$. The representation ρ is now given by what we called earlier the gamma matrices and both the structure we just discussed can be represented by a matrix B:

$$v \mapsto Bv^* \qquad v \in V \tag{6.19}$$

where from now on * will denote complex conjugation. One sees that the squaring property of J implies

$$BB^* = \mathbb{1} \text{ for a real structure} \tag{6.20}$$

$$BB^* = -\mathbb{1} \text{ for a quaternionic structure} \tag{6.21}$$

[1]Physicists often call real representation "Majorana" and quaternionic one "Symplectic-Majorana"

and the "commuting with the ρ" property implies

$$\gamma_{a_1\ldots a_n}{}^* = B^{-1}\gamma_{a_1\ldots a_n}B \tag{6.22}$$

6.4. Under a change of representation, B transforms as

$$\gamma'_a = P^{-1}\gamma_a P \tag{6.23}$$
$$B' = P^{-1}BP^* \tag{6.24}$$

6.5. Existence of real and quaternionic structures.
Suppose such a matrix B exists. Then

$$\gamma_a^T = (-)^t B^\dagger A\gamma_a(B^\dagger A)^{-1} = (-)^{d-t+1}B^\dagger \tilde{A}\gamma_a(B^\dagger\tilde{A})^{-1} \tag{6.25}$$

When d is even, one has

$$\gamma_a^T = (-)^t B^\dagger A\gamma_a(B^\dagger A)^{-1} = (-)^{t+1}B^\dagger \tilde{A}\gamma_a(B^\dagger\tilde{A})^{-1} \tag{6.26}$$

but when d is odd

$$\gamma_a^T = (-)^t B^\dagger A\gamma_a(B^\dagger A)^{-1} = (-)^{t}B^\dagger \tilde{A}\gamma_a(B^\dagger\tilde{A})^{-1} \tag{6.27}$$

which is to be expected because when d is odd, $\tilde{A} = \pm A$. Thus when d is odd we see that there is a limitation. Only one of the two matrices C, $\tilde{C}$ exists, thus B can exist only for $\{t \equiv 0 \bmod 2$ and $d \equiv 1 \bmod 4\}$ or $\{t \equiv 1 \bmod 2$ and $d \equiv 3 \bmod 4$. Furthermore, B is unique. In even dimension however, two B (we call them B and $\tilde{B}$ we they coexist). In other words

$$\begin{array}{lll} d \equiv 0 \bmod 2,\ t \equiv 0 \bmod 2 & B = (CA^{-1})^\dagger & \tilde{B} = (\tilde{C}\tilde{A}^{-1}) \\ d \equiv 0 \bmod 2,\ t \equiv 1 \bmod 2 & B = (\tilde{C}A^{-1})^\dagger & \tilde{B} = (C\tilde{A}^{-1}) \\ d \equiv 1 \bmod 4,\ t \equiv 0 \bmod 2 & B = (CA^{-1})^\dagger & \\ d \equiv 3 \bmod 4,\ t \equiv 1 \bmod 2 & B = (\tilde{C}A^{-1})^\dagger & \end{array} \tag{6.28}$$

Next we compute BB^*. We use a standard unitary representation.

$$(CA^{-1})^\dagger(CA^{-1})^T = AC^{-1}(A^{-1})^T\epsilon C = \epsilon A(C^{-1}A^TC)^{-1} = \epsilon(-)^{\frac{t()t-1}{2}} \tag{6.29}$$
$$(\tilde{C}A^{-1})^\dagger(\tilde{C}A^{-1})^T = \tilde{\epsilon}(-)^{\frac{t(t+1)}{2}} \tag{6.30}$$
$$(C\tilde{A}^{-1})^\dagger(C\tilde{A}^{-1})^T = \epsilon(-)^{\frac{(d-t)(d-t+1)}{2}} \tag{6.31}$$
$$(\tilde{C}\tilde{A}^{-1})^\dagger(\tilde{C}\tilde{A}^{-1})^T = \tilde{\epsilon}(-)^{\frac{(d-t)(d-t+1)}{2}} \tag{6.32}$$

Leading to

$$
\begin{array}{lll}
d \equiv 0 \bmod 4,\ t \equiv 0 \bmod 4 & BB^* = \epsilon & \tilde{B}\tilde{B}^* = \tilde{\epsilon} \\
d \equiv 0 \bmod 4,\ t \equiv 1 \bmod 4 & BB^* = -\tilde{\epsilon} & \tilde{B}\tilde{B}^* = -\epsilon \\
d \equiv 0 \bmod 4,\ t \equiv 2 \bmod 4 & BB^* = -\epsilon & \tilde{B}\tilde{B}^* = -\tilde{\epsilon} \\
d \equiv 0 \bmod 4,\ t \equiv 3 \bmod 4 & BB^* = \tilde{\epsilon} & \tilde{B}\tilde{B}^* = \epsilon \\
d \equiv 1 \bmod 4,\ t \equiv 0 \bmod 4 & BB^* = \epsilon & \\
d \equiv 1 \bmod 4,\ t \equiv 2 \bmod 4 & BB^* = -\epsilon & \\
d \equiv 2 \bmod 4,\ t \equiv 0 \bmod 4 & BB^* = \epsilon & \tilde{B}\tilde{B}^* = -\tilde{\epsilon} \\
d \equiv 2 \bmod 4,\ t \equiv 1 \bmod 4 & BB^* = -\tilde{\epsilon} & \tilde{B}\tilde{B}^* = \epsilon \\
d \equiv 2 \bmod 4,\ t \equiv 2 \bmod 4 & BB^* = -\epsilon & \tilde{B}\tilde{B}^* = \tilde{\epsilon} \\
d \equiv 2 \bmod 4,\ t \equiv 3 \bmod 4 & BB^* = \tilde{\epsilon} & \tilde{B}\tilde{B}^* = -\epsilon \\
d \equiv 3 \bmod 4,\ t \equiv 1 \bmod 4 & BB^* = -\tilde{\epsilon} & \\
d \equiv 3 \bmod 4,\ t \equiv 3 \bmod 4 & BB^* = \tilde{\epsilon} &
\end{array}
\tag{6.33}
$$

Even if it has been computed in a specific representation, this result is independent of it. Also observe that for $d \equiv 0 \bmod 4$, $\epsilon = \tilde{\epsilon}$ while for $d \equiv 0 \bmod 4$, $\epsilon = -\tilde{\epsilon}$. Thus we can resume the preceding table by:

t	BB^*
4k	ϵ
4k +1	$-\tilde{\epsilon}$
4k+2	$-\epsilon$
4k+3	$\tilde{\epsilon}$

For example if we consider the case $t = 1$, i.e. the Lorentzian signature in the mostly plus case, we find that real representations exist in dimension $d \equiv 2, 3, 4\,[8]$ while if we consider the mostly minus signature $t = d - 1$ real representations exist for $d \equiv 0, 1, 2\,[8]$.

6.6. Majorana conjugation and usual physicist's conventions.

In standard convention, the Majorana conjugate ψ^C of a spinor ψ is defined to be

$$
\psi^C = \begin{cases} \psi^T C & \textit{(in even dimensions)} \\ \psi^T \tilde{C} & \textit{(in odd dimensions)} \end{cases}
\tag{6.34}
$$

According to the preceding paragraph, for us a Majorana spinor is a spinor which satisfies

$$
\psi = \mathrm{J}(\psi) = B\psi
\tag{6.35}
$$

while it standard definition is

$$
\overline{\psi} = \psi^C
\tag{6.36}
$$

In order for both to agree we set

$$
B = \begin{cases} (DC^{-1})^T & \textit{(for even t)} \\ (D\tilde{C}^{-1})^T & \textit{(for odd t)} \end{cases}
\tag{6.37}
$$

From now on, whenever we speak of a matrix B, it will be the one of (6.37) (no more reference to α will be made).

6.7. A last remark.

When a real structure exist, it is always possible to go to a real representation. Indeed, B has an eigenvector, call it u. Then u^*, and $v = \frac{u+u^*}{2}$ are also eigenvectors of B. Furthermore, v satisfies $Bv^* = v$. We also have

$$B(\gamma_{a_1\dots a_n}v)^* = BB^{-1}\gamma_{a_1\dots a_n}Bv^* = \gamma_{a_1\dots a_n}v \tag{6.38}$$

Thus we can find a basis $\{v_i\}$ such that $\forall i$, $Bv_i^* = v_i$. Let P be the change-of-basis-matrix, $\{e_i\}$ the original basis and $B' = P^{-1}BP^*$ the B matrix in the new basis. We have:

$$v_i = Pe_i \implies v_i^* = P^*e_i \implies B^{-1}v_i = P^*e_i \implies v_i = BP^*e_i \tag{6.39}$$

$$\implies P^{-1}v_i = P^{-1}BP^*e_i \implies e_i = B'e_i \implies B' = \mathrm{Id} \tag{6.40}$$

CHAPTER 3

Differential Geometry

Differential geometry is the geometrical theory supporting classical field theory. Hence we shall recall its most basics definitions, like the ones of manifolds and fiber bundles. In particular, the theory of connection is of central importance, as gauge fields in physics corresponds to local connection one-form in differential geometry. Their action on other physical fields, like vector fields, is expressed through covariant derivatives, another mathematical object presented in this chapter. We will also present some more advanced topics, like super-manifolds, used in an attempt of a fully geometric description of super-gravity, see [Ede20], or as framework of the super-field approach [Del+99], [Cas18].

1. Manifolds

1.1. Introduction.

Manifold refers to certain class of topological spaces. Somehow, these are the nicest topological spaces after finite dimensional vector spaces and are gluing of them. They have all properties needed to define differentiability. We will start by the notion of real manifold

1.2. Differentiable map.

Let U be an open subset of $\mathbb{R}^n$. An application $f : U \to \mathbb{R}^m$ is called differentiable at $x \in \mathbb{R}^n$ if there exists a linear map $J_x : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{||f(x+h) - f(x) - J_x(h)||}{||h||} = 0. \tag{1.1}$$

($||\cdot||$ is the usual euclidean norm). If f is differentiable at every point $x \in U$, and J_x is invertible for every $x \in U$ as well, then f is called a diffeomorphism. f is two times differentiable if it is differentiable and the application

$$\mathbb{R}^n \to \mathbb{R}^{mn}, \qquad x \mapsto J_x \tag{1.2}$$

is itself differentiable; the definition goes on for k times differentiable. f is smooth if it is k times differentiable for any $k \in \mathbb{N}$.

1.3. Charts and atlases.

Let M be a Hausdorff, secondly countable topological space and let $\mathcal{T}(M)$ denotes its topology. A local chart on M is a open set $U \in \mathcal{T}(M)$ together with an homeomorphism ϕ from U to some subset of $\mathbb{R}^n$. An atlas $\mathcal{A}$ of M is the data of a covering of M by local charts with the restriction that for any two local charts $(U, \phi : U \to \mathbb{R}^n)$, $(V, \psi : V \to \mathbb{R}^m)$, the dimension of the target vector space is the same i.e. $m = n$; and on any non empty intersection of local charts, i.e. if $U \cap V \neq \emptyset$ the map $\phi \circ \psi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism. In case $\phi \circ \psi^{-1}$ is smooth with smooth inverse we speak about a smooth manifold. Given two atlases $\mathcal{A}_1$, $\mathcal{A}_2$ of M, we say that $\mathcal{A}_2$ is a refinement of $\mathcal{A}_1$ if any local chart of $\mathcal{A}_1$ can covered by local charts of $\mathcal{A}_\in$. This gives a partial ordering $\leq$ on the set of atlases on M, and, still in the case where $\mathcal{A}_2$ id a refinement of $\mathcal{A}_1$, we write $\mathcal{A}_1 \leq \mathcal{A}_2$. Two atlases are compatible if there exists an atlas bigger than both of them, and a atlas is called maximal if it is a maximum with respect to this ordering. A topological space with a maximal atlas is called a manifold. A choice of a local chart allows to speak of local coordinates, that we will often denote by $\{x^1, \dots, x^n\}$, which are the image of the natural coordinates of $\mathbb{R}^n$ by the chosen chart.

1.4. Morphisms.

Morphisms of manifolds are applications $f : M \to N$, with M, N two manifolds, such that for any $x \in M$ and local charts $(\phi_\alpha, U_\alpha \ni x)$, $(\psi_\beta, V_\beta \ni f(x)$, the application $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is differentiable (or smooth).

1.5. Partition of the unity.

Let M be a topological space. A partition of the unity on M is a set of continuous maps

$$\{f_\alpha : M \to \mathbb{R}\}$$

such that

$$\forall x \in M, \{\alpha / f_\alpha(x) \neq 0\}$$

is a finite set and

$$\sum_\alpha f_\alpha(x) = 1$$

We recall that the support of a function $f : M \to \mathbb{R}$, denoted supp(f) is the closure of the set :

$$\{x / f(x) \neq 0\}$$

Let M be a manifold. Let (U_α, ϕ_α) be an atlas on M and $\{f_\beta\}$ a partition of the unity on M. We say that $\{f_\beta\}$ is subordinate to (U_α, ϕ_α) if

$$\forall \beta \quad \exists \alpha \quad / \text{supp}(f_\beta) \subset U_\alpha$$

The Hausdorff secondly countable assumptions we imposed in our definition of manifolds ensure the following theorem.

THEOREM 12. *Let M be a manifold and* $\{(U_\alpha, \phi_\alpha)\}$ *an atlas of M. There exists a partition of the unity subordinate to* $\{(U_\alpha, \phi_\alpha)\}$.

2. Sheafs

2.1. Definition.

A presheaf $\mathcal{O}$ of sets over a topological sapce M is the assignment of a set $\mathcal{O}(U)$ to each open set U of M such that, for every open set included into another $U \subset V$ there is a restriction map $r_{V,U} : \mathcal{O}(V) \to \mathcal{O}(U)$ subject to the transitivity condition:

$$U \subset V \subset W \implies r_{W,U} = r_{V,U} \circ r_{W,V} \tag{2.1}$$

and $r_{U,U} = \mathrm{Id}_{\mathcal{O}(U)}$. An element $s_U \in \mathcal{O}(U)$ is called a section of $\mathcal{O}$ over U. A sheaf is a presheaf satisfying the additional condition that given an open set U, covered by a family of open sets (V_i), and a family of section $s_i \in \mathcal{O}(V_i)$, compatible in the sense that they satisfy $r_{V_i,V_i\cap V_j}(s_i) = r_{V_j,V_i\cap V_j}(s_j)$, there exist a unique section $s \in \mathcal{O}(U)$ such that $s_i = r_{U,V_i}(s)$.

2.2. Example.

Here a simple example of a presheaf. Consider a topological space M. Define the constant presheaf $\mathcal{C}$ by $\mathcal{C}(U) = \mathbb{R}$ for any open set U. This presheaf is not a sheaf. Indeed consider two disjoint open sets U and V and the sections $1 \in \mathcal{C}(U)$ and $2 \in \mathcal{C}(V)$. There is no section of $\mathcal{C}(U \cup V)$ whose restriction on U is 1 and whose restriction on V is 2.

2.3. Stalk.

Given a presheaf over a topological space, the stalk $\mathcal{O}_x$ are defined as the direct limit $\varinjlim_{x\in U} \mathcal{O}(U)$, with order given by reverse inclusion. In other words, an element $s_x \in \mathcal{O}_x$ corresponds to an equivalence class of elements $s_U \in \mathcal{O}_U$, with equivalence relation

$$s_U \sim s_V \equiv \exists W \,/\, W \subset U \cap V \,/\, r_{U,W}(s_U) = r_{V,W}(s_W). \tag{2.2}$$

The elements $s_x \in \mathcal{O}_x$ are called germs at x of sections of $\mathcal{O}$.

When the sets $\mathcal{O}(U)$ are rings, one speaks about a (pre)sheaf of rings. A topological space M together with a sheaf of rings is called a ringed space. If furthermore the stalks are local rings (i.e. they have a unique maximal ideal), one speaks about a locally ringed space.

2.4. Manifolds as locally ringed spaces.

An alternative definition for a manifold is the following. A manifold is a locally ringed space $(M,\mathcal{O})$, Hausdorff and second countable, such that for any point $x \in M$, there exist a neighborhood U, $x \in U \subset M$, and an open space $V \subset \mathbb{R}^n$, such that $\mathcal{O}(U)$ is isomorphic (as a ring) to $\mathcal{C}^\infty(V)$. Under this isomorphisms, the maximal ideal of the stalk $\mathcal{O}_x$ is identified with the set of germs of functions vanishing at x. We denote this ideal by $\mathcal{I}_{x,0}$. In fact, we have the isomorphism

$$\mathcal{O}_x \simeq \mathbb{R}[[X_1,\ldots,X_n]], \tag{2.3}$$

where $\mathbb{R}[[X_1,\ldots X_n]]$ denotes the ring of formal series in $n = \dim(M)$ variables. This isomorphism identifies any germ at x with the Taylor expansion at x of the function it represents. Thus, we have the decomposition

$$\mathcal{O}_x \simeq \mathbb{R} \oplus \mathcal{I}_{x,0} \tag{2.4}$$

and the isomorphism

$$\mathcal{I}_{x,0} \simeq (X_1 + X_2 + \cdots + X_n)\mathbb{R}[[X_1,\ldots X_n]] \tag{2.5}$$

It is also interesting to pay attention at the finite dual $\mathcal{O}^\circ(M)$ of the ring of global sections $\mathcal{O}(M)$. We recall the definition of the finite dual

$$\mathcal{O}^\circ(U) = \{\phi : \mathcal{O}(U) \to \mathbb{R} \big/ \phi \text{ vanishes on an ideal of finite codimension}\}, \tag{2.6}$$

and that it naturally comes with a structure of coalgebra. We could consider a sheaf of finite dual $\mathcal{O}^\circ$, but the restriction 2.6 is so restrictive that such a sheaf would not have great interest.

Indeed the first element of interests of this dual are the group-like elements, i.e. elements ev_x satisfying

$$\Delta(\mathrm{ev}_x) = \mathrm{ev}_x \otimes \mathrm{ev}_x. \tag{2.7}$$

Are they name suggest, they are exactly the evaluation at a point x of a function, and are thus in one-to-one correspondence with the points of the manifold. Indeed, the property 2.7 is equivalent to say that such an ev_x preserves the multiplication:

$$\mathrm{ev}_x(fg) = \mathrm{ev}_x(f)\mathrm{ev}_x(g), \tag{2.8}$$

for some functions f, g. And it is a well-known proposition, called "Milnor's exercise" that only evaluation at points satisfies this property.

2.5. Super-manifold.

This alternative definition of manifold is perfectly suited for generalisation of manifolds. Indeed, we see that we a natural generalisation of the concept of manifold is obtained after extending the ring isomorphic to $\mathcal{O}(U)$. A supermanifold is thus a locally ringed space $(M,\mathcal{O})$, Hausdorff and second countable, such that for any point $x \in M$, there exist a neighborhood U, $x \in U \subset M$, and an open space $V \subset \mathbb{R}^n$, such that

$$\mathcal{O}(U) \simeq \mathcal{C}^\infty(V) \otimes \Lambda(\theta^1, \ldots \theta^p), \tag{2.9}$$

where $\Lambda(\theta^1, \ldots \theta^p)$ is the exterior algebra generated by $\{\theta^1, \ldots \theta^p\}$. In other words, $\Lambda(\theta^1, \ldots \theta^p)$ is the exterior algebra of a vector space whose one possible basis is $\{\theta^1, \ldots \theta^p\}$.

3. Tangent space

3.1. Tangent space at a point.

Let M be a manifold of dimension n, and $\gamma : \mathbb{R} \to M$ a curve, whose parameter is called t. We choose a system local chart (ϕ, U) and write $\phi \circ \gamma(t) = \{x^\mu(t)\}$. The tangent vector at $t = 0$ of this curve in $\mathbb{R}^n$ is $\frac{d}{dt}\phi \circ \gamma|_{t=0}$. We want to define the abstract tangent vector $\dot\gamma(0) = \frac{d}{dt}\gamma|_{t=0}$ independently of the chosen chart. For this we introduce the equivalence relation "$\sim_x$" among the curves satisfying $\gamma(0) = x$, with $\gamma_1 \sim_x \gamma_2$ if and only if there is a chart (U,ϕ) with $x \in U$ such that $\frac{d}{dt}\phi\circ\gamma_1|_{t=0} = \frac{d}{dt}\phi\circ\gamma_2|_{t=0}$, and we set $\dot\gamma(0) = [\gamma]_{\sim_x}$, the equivalence class for $\sim_x$. The tangent space at x, denoted T_xM is the set of all tangent vector at x. The (total) tangent space is $TM = \bigcup_{x\in M} T_xM$. It is a manifold as well.

3.2. Vector field.

An application which send (smoothly) to each $x \in M$ a vector of T_xM is called a (smooth) vector field. (We will always assume our vector fields to be smooth.) If this application is only defined on an open subset $U \subset M$, we speak of a local vector field, by opposition to global vector fields. The choice of local coordinates $\{x^i\}$ gives a local basis (sometimes called natural basis, although this denomination has nothing to do with naturality in the categorical sense) for vector fields $\frac{\partial}{\partial x^i} \doteq \partial_i$. In other words any vector field (global or local) can locally be written as

$$X(x) = X^i(x)\partial_i, \quad x \text{ in some open set U} \tag{3.1}$$

Given a function f and a vector field X, we form a new function $X(f)$, over which we can apply a new vector field Y to form $Y(X(f))$. One then shows that although $Y \circ X$, defined by $(Y \circ X)(f) = Y(X(f))$, is not a vector field (it does not define a derivation), the Lie bracket $[X, Y] = X \circ Y - Y \circ X$ does. Hence the set of vector field over a given manifold define a Lie algebra.

3.3. Functoriality.

Any morphism of manifold $f : M \to N$ extends to a morphism of vector bundle $Tf : TM \to TN$, such that $T(f \circ g) = Tf \circ Tg$. In local coordinates $\{x^i\} \subset M$, $\{y^j\} \subset N$ Tf is expressed as

$$Tf(\{x^i\},\{X^i\}) = (\{f^j(x^1,\ldots,x^n),\partial_i f^j X^i\} \tag{3.2}$$

3.4. Sheaf definition.

Let $(M, \mathcal{O})$ be a manifold with its local ring, as presented above. We can alternatively define $T_x M$ as the set of derivation of the stalks $\mathcal{O}_x$, i.e. linear maps $X : \mathcal{O}_x \to \mathbb{R}$ satisfying

$$\forall u, v \in \mathcal{O}_x, X(uv) = X(u)v + uX(v), \tag{3.3}$$

i.e. $T_x M \simeq \mathrm{Der}(\mathcal{O}_x)$. More algebraically, we can say that tangent vectors are the elements X of the finite dual $\mathcal{O}^\circ(M)$ satisfying

$$\Delta X = X \otimes \mathrm{ev}_x + \mathrm{ev}_x \otimes X, \tag{3.4}$$

i.e. a tangent vector at x is a primitive element with respect to one ev_x. Now we can define the tangent space TM as the union of the tangent spaces $T_x M$ properly topologized, and vector fields as smooth maps as above, but the sheaf structure allows for a more direct definition. We define the sheaf $\mathbf{Der}(\mathcal{O})$ of derivation of $\mathcal{O}$ with values in $\mathcal{O}$. More precisely, for any open set $U \subset M$, we let $\mathbf{Der}(\mathcal{O})(U)$ be the set of $\mathbb{R}$-linear maps $\mathcal{O}(U) \to \mathcal{O}(U)$ satisfying the derivation property (3.3, but adapted for u, v in $\mathcal{O}(U)$ instead). A (local) vector field is a section of that sheaf. One then show that the stalk of $\mathbf{Der}(\mathcal{O})$ at some $x \in M$ coincide with the space $T_x M$ defined just above.

3.5. Tangent space of super-manifolds.

This definition admits a straightforward generalization in the case of supermanifolds: the super-tangent space of a supermanifold $(M, \mathcal{O})$ is the space of super-derivations of the stalk of it defining sheaf at a given point; (local) super-vector fields are section of the sheaf of super-derivations of the structure sheaf $\mathcal{O}$ of the supermanifold with values in $\mathcal{O}$. In local coordinates $\mathcal{O}(U) \sim \mathcal{C}^\infty(V) \otimes \Lambda\mathbb{R}^p$, $U \subset M$, $V \subset \Lambda(\theta^1, \ldots \theta^p)$ both open, U sufficiently small, it can be shown that a super-vector field $X \in \mathbf{Der}(\mathcal{O})(U)$ admit an expansion similar to (3.1):

$$X(x) = X^i \frac{\partial}{\partial x^i} + X^\alpha \frac{\partial}{\partial \theta^\alpha}. \tag{3.5}$$

3.6. Submersions and immersions.

A smooth map $\pi : M \to N$ is called a submersion if the tangent map $\pi_* : TM \to TN$ is surjective. A smooth map $\iota : M \to N$ is called an immersion if the tangent map $\iota_* : TM \to TN$ is injective.

4. Lie groups and Super-Lie groups

4.1. Lie groups.

A Lie group is a set G which is at the same time a group and a manifold, with the additional assumption that the multiplication $m_G : G \times G \to G$ and the inverse $i : G \to G$ are morphism of (smooth) manifolds. The unit of the group is commonly written e.

4.2. Fundamental vector fields.

The multiplication on the left by an element g being a smooth map $L_g : G \to G$, it gives rise to a tangent map $L_{g*} : TG \to TG$. Given any vector tangent at the identity $X \in T_eG$, we can use this tangent map to define a vector field $\tilde{X}$ by $\tilde{X}(g) = L_{g*}X$. This vector field is called a left fundamental vector field, because it is defined using a left action. Similarly, there exists right fundamental vector fields. The tangent space at the identity, thus isomorphic to a particular sub-Lie algebra of vector fields, (either left fundamental or right fundamental) is called the Lie algebra of the Lie group. The left fundamental vector fields $\tilde{X}$'s are left invariant, in the sense that $L_{g*}\tilde{X} = \tilde{X}$, and similarly for right fundamental vector fields.

4.3. Sheaf definition and super-Lie groups.

A Lie group is a manifold $(G, \mathcal{O})$ whose structure sheaf is not only a sheaf of rings but a sheaf of Hopf algebra. As explained in the section "Hopf algebras and groups", the product and inverse of the Lie group are given respectively through the coproduct, denoted Δ_m in the sequel, and the antipode, denoted S in the sequel, of the Hopf algebra structure. A super-Lie group is a super-manifold $(G, \mathcal{O})$ whose structure sheaf is a sheaf of super-commutative graded Hopf algebras.

4.4. The super-Lie algebra of a super-Lie group.

The standard left-invariance property of vector fields is given in the sheaf point of view by

$$(\mathbb{1} \otimes X) = \Delta_m X. \tag{4.1}$$

The left invariant vector fields of a (super-)Lie group form a (super-)Lie algebra. This (super-)Lie algebra is also related to the tangent space at the identity e by

$$T_eG \ni X_e \mapsto (\mathbb{1} \otimes X_e)\Delta_m \in \mathrm{Lie}(G) \tag{4.2}$$

where $\mathrm{Lie}(G)$ is the (super-)Lie algebra of left invariant vector fields. A similar construction holds for right-invariant vector fields. For more information on this topic, including a detailed proof of this last fact, see [CCF10], chapter 7.

5. Fiber bundle

5.1. Definition.

A fiber bundle is a collection (P, M, F, π) of three manifolds (P, M, F) and a surjective submersion $\pi : P \to M$ such that $\forall x \in M$, $\pi^{-1}(x) \simeq F$. Furthermore, P is required to fulfill the local trivialization property: for any $x \in M$, there is a neighborhood U_α of x such that there is a (smooth) diffeomorphism $\psi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times F$. ψ_α is called a local trivialisation and we say that P is locally trivialised over U_α. Whenever $x \in U_\alpha \cap U_\beta$, the (smooth) diffeomorphism $g_{\alpha\beta}(x) = \psi_\alpha|_x \circ \psi_\beta|_x^{-1} : F \to F$ is called a transition function. A fibre bundle is called trivial if admit a trivialisation of the form $P \simeq M \times F$.

5.2. Principal and Vector bundles.

We are interested mainly in two types of fibre bundles: vector bundles and principal bundles. A vector bundle is a fibre bundle (P, M, F, π) for which the fibre F is a vector space and such that the transition function $g_{\alpha\beta}(x) : F \to F$ belongs to $GL(F)$ for any $x \in U_\alpha \cap U_\beta$. A principal bundle is a fibre bundle such that the fibre is a Lie group G, called the structure group, and such that there is a right action of G on P preserving the fibre. The two notions are related. First, for any vector bundle (P, M, V, π) we can construct a principal bundle $(P, M, GL(V), \bar{\pi})$ in the following manner. Let $\{U_\alpha\}$ be a cover of M over which P is locally trivialised. Consider the disjoint union $\bigsqcup_\alpha U_\alpha \times G$ form the quotient space under the equivalence relation $(x, g)_\alpha \sim (x, g_{\alpha\beta} g)_\beta$. This quotient space is a principal bundle with structure group $GL(v)$. On the other hand, given a principal bundle (P, M, G, π) and a representation (ρ, V) of G on a vector space V, one can construct the associated vector bundle ${P \times V}/{\sim_\rho}$, where the equivalence relation is $(z, v) \sim_\rho (zg, \rho(g^{-1})v)$.

5.3. Sections.

Let (P, M, F, π) be a fiber bundle and U an open set in M. A local section is a continuous map $s : U \to P$ such that $\pi \circ s = \mathrm{Id}_U$. From now on, we will always assume our sections to be smooths. A global section is a local section with $U = M$. A principal bundle is trivial if it admits a global section. A vector bundle is trivial if it admits n linearly independent global section, with n the dimension of the fiber. We denote by $\Gamma(P)$ the space of smooth sections $M \to P$.

5.4. Morphisms.

A morphism of fiber bundle $f : (P, M, F, \pi) \to (Q, N, H, \tau)$ is a morphism of manifold $f : P \to Q$ sending fibers onto fibers and such that the induced map $\bar{f} : M \to N$, $\bar{f}(x) = \tau \circ f \circ \pi^{-1}(x)$ is a morphism of manifold as well. Here π^{-1} denotes any section $M \to P$. In the section "Characteristic classes", we will furthermore ask that the morphism induces an isomorphism between the fibers.

5.5. Building new fiber bundles from old ones.

Let (P, N, F, π) be a fiber bundle and $f : M \to N$ a morphism. It is possible to construct a new bundle over M, called pullback bundle of P by f and denoted f^*P. It consists of pairs (x, z), $x \in M$, $z \in P$ such that $f(x) = \pi(z)$. If (E_1, M, V_1, π_1) and (E_2, M, V_2, π_2) are two vector bundles over the same base manifold, one can constructs their Whitney sum $E_1 \oplus_M E_2$ with fiber $V_1 \oplus V_2$ and tensor product $E_1 \otimes_M E_2$ with fiber $V_1 \otimes V_2$. As sets, they correspond to the disjoint unions $\bigsqcup_{x \in M} E_{1x} \oplus E_{2x}$, $\bigsqcup_{x \in M} E_{1x} \oplus E_{2x}$ respectively. They are then topologized so that they become fiber bundle over M, by constructing the base for their topologies using the bases of E_1 and E_2 so that the local trivialization conditions are satisfied. Using tensor product, it is possible to build any "associative algebra bundle" as quotient of a finite tensor algebra bundle $E^{\otimes n}$. In particular, we will be able to construct the bundle of differential form as an exterior algebra bundle, the Clifford algebra bundle, or a symmetric algebra bundle as the space for pseudo-Riemannian metrics.

5.6. Sheaf definition of a vector bundle.

The sheaf counterpart of vector bundle is a rank-r locally free sheaf. Let $(M, \mathcal{O})$ be a manifold with its structure sheaf. A sheaf $\mathcal{F}$ on M is called locally free of rank r if, for any $x \in M$, there is a neighborhood $U \ni x$ such that the restriction of $\mathcal{F}$ to U, $\mathcal{F}|_U$, is a sheaf of free module of rank r of $\mathcal{O}|_U$, i.e., for all open subsets $V \subset U$, $\mathcal{F}(V)$, is isomorphic (as $\mathcal{O}(V)$-module) to $\mathcal{O}(V)^r$. If (E, M, π) is a standard vector bundle, then for any open set $U \subset M$ we can define a set of smooth sections $\mathcal{C}^\infty(U, E)$, which has naturally the structure of a $\mathcal{C}^\infty(U)$-module. As U varies through the open set of M, we can give the family $\mathcal{C}^\infty(U, E)$ the structure of a locally free sheaf. We denote by Σ_E this sheaf. Thus we have shown that any vector bundle structure (E, M, π) give rise to a locally free sheaf over M of constant rank. Conversely, given a manifold $(M, \mathcal{O})$ and a locally free sheaf of constant rank r Σ, we can use the locally free condition on a sufficiently fine open cover $\{U_\alpha\}$ of M to obtain transition functions $g_{\alpha\beta} : \Sigma(U_\alpha \cap U_\beta) \to \Sigma(U_\alpha \to U_\beta)$. We can then use this transition function to patch together the different $U_\alpha \otimes \mathbb{R}^r$.

The above definition does not mimic the traditional case given by two manifolds and a projective submersion $P \xrightarrow{\pi} M$, or only indirectly. Hence we can also define a vector bundle as a triple $\{(P, \mathcal{O}_P), (M, \mathcal{O}_M), \pi\}$, where $\{(P, \mathcal{O}_P)\}$ and $\{M, \mathcal{O}_M\}$ are differentiable manifolds, $\pi : P \to M$ is a projective submersion and we have the local triviality condition: fro each z in P, there exists an open $U \subset M$ with $z \in V \doteq \pi^{-1}(V)$ and an isomorphism of manifold $\phi : (V, \mathcal{O}_P|_V) \simeq (U \times \mathbb{R}^n, \mathcal{O}_M|_U \hat{\otimes} \mathcal{C}^\infty)$, where $\mathcal{C}^\infty$ is the natural sheaf of C^∞ functions on $\mathbb{R}^n$.

5.7. Sheaf definition of principal bundles.

We can similarly define a principal bundle. First we need the notion of action of a group. Let $(\mathcal{G}, \mathcal{O}_\mathcal{G})$ be a Lie group. A right action of $(\mathcal{G}, \mathcal{O}_\mathcal{G})$ on a manifold $(M, \mathcal{O}_M)$ is a map $r : (M, \mathcal{O}_M) \times (\mathcal{G}, \mathcal{O}_\mathcal{G}) \to (M, \mathcal{M})$ such that the morphism of commutative algebra $r^* : \mathcal{O}_M(M) \to \mathcal{O}_M(M) \hat{\otimes} \mathcal{O}_\mathcal{G}(\mathcal{G})$ endows $\mathcal{O}_M(M)$ with a structure of right $\mathcal{O}_\mathcal{G}(\mathcal{G})$-comodule. In term of coalgebra structures, this can be written by the equalities:

$$(\mathrm{Id} \otimes \Delta_\mathcal{G}) \circ r^* = (r^* \otimes \mathrm{Id}) r^*, \qquad (\mathrm{Id} \otimes \epsilon_\mathcal{G}) \circ r^* = \mathrm{Id}. \tag{5.1}$$

A right action $r : (M, \mathcal{O}_M) \times (\mathcal{G}, \mathcal{O}_\mathcal{G}) \to (M, \mathcal{M})$ is said free if for each $x \in M$ the morphism $r_{x*} : (\mathcal{G}, \mathcal{O}(\mathcal{G}))^\circ \to (M, \mathcal{O}_M)^\circ$ is injective. We recall that $\mathcal{O}_M(M)^\circ$ is the finite dual of $\mathcal{O}_M(M)$. We are ready to state the definition of a principal bundles in terms of sheafs. Let $(\mathcal{G}, \mathcal{O}_\mathcal{G})$ be a Lie group. A $\mathcal{G}$-principal bundle is a quadruple $\{(P, \mathcal{O}_P), (M, \mathcal{O}_M), \pi, r\}$ where $(P, \mathcal{O}_P)$ and $(M, \mathcal{O}_M)$ are manifolds, $\pi : (P, \mathcal{O}_P) \to (M, \mathcal{O}_M)$ is a projective submersion, $r : (P, \mathcal{O}_P) \times (\mathcal{G}, \mathcal{O}_\mathcal{G}) \to (P, \mathcal{O}_P)$ is a free right action, and we have the local triviality condition: for each $z \in P$, there exists an open $U \subset M$ such that $z \in V \doteq \pi^{-1}(U)$ and an isomorphism $\phi : (V, \mathcal{O}_P|_V) \simeq (U \times$

$\mathcal{G}, \mathcal{O}_M|_U \hat{\otimes} \mathcal{O}_{\mathcal{G}})$, such that furthermore the isomorphism of algebra ϕ^* is an isomorphism of $\mathcal{O}_{\mathcal{G}}(\mathcal{G})$-comodules. The comodule structure of $\mathcal{O}_M(U)\hat{\otimes}\mathcal{O}_{\mathcal{G}}(\mathcal{G})$ is given by $id \otimes \Delta_{\mathcal{G}}$.

5.8. Tangent and cotangent bundle.

A natural vector bundle that we can consider over any manifold M is the tangent bundle TM previously described. It is a natural bundle in the sense that it defines a functor T from the category of manifolds to the category of vector bundles. The image of a smooth map f by this functor is called the (associated) tangent map or pushforward, denoted Tf or more frequently here f_*. The projection $\pi : TM \to M$ assign x to the pair (x, v), where $v \in T_x M$. Vector fields are simply section of this tangent bundle.

The tangent space $T_x M$ is a vector space, hence we can consider its dual space, called cotangent space, denoted $T_x^* M$. It naturally admits the structure of vector bundle over M as well, and T^* can be seen as a contravariant functor from the category of manifolds to the category of vector bundles, i.e. it reverses the direction of the morphisms. The image of a smooth map f by this functor is called (associated) cotangent map or pullback; and can be written $T^* f$ or f^*.

5.9. The frame bundle.

Another natural bundle over any manifold M is the frame bundle LM. The fiber $L_x M$ over $x \in M$ consists in all possible basis of the tangent space $T_x M$ and is isomorphic to $GL_m(\mathbb{R})$, $m = \dim(M)$. It is a principal bundle and TM is a vector bundle associated to it.

5.10. Differential form.

From the cotangent bundle one form the exterior algebra $\Lambda T^* M$. It is the vector bundle over M whose fiber over x is $\Lambda T_x^* M$, and it is the Whitney's sum of the $\Lambda^k T^* M$. A smooth section of $\Lambda^k T^* M$ is called a differential k-form or simply k-form. The space of differential k-form is denoted by $\Omega^k(M)$ and the total space of differential form simply $\Omega(M)$.

5.11. Exterior derivative.

The pairing between a k-form ω and k vector fields $X_1, \dots X_k$ defines a new function $\omega(X_1, \dots X_k) : M \to \mathbb{R}$, over which we can apply another vector field X_0, forming yet another function $X_0(\omega(X_1 \dots X_k))$. This action is used to define the exterior derivative d. We recall its formula: for vector fields $v_1, \dots v_{k+1}$ and a k-form ω we have

$$\begin{aligned} \mathrm{d}\omega(v_1, \dots, v_{k+1}) = &\sum_{i=1}^{k+1} v_i \left(\omega(v_1, \dots, \hat{v}_i, \dots v_{k+1})\right) \\ &+ \sum_{1 \le i < j \le k+1} (-)^{i+j} \omega([v_i, v_j], v_1, \dots, \hat{v}_i, \dots \hat{v}_j, \dots v_{k+1}). \end{aligned} \tag{5.2}$$

An important property of the exterior derivative is that it commutes with pullback

5.12. Sheaf definition and super-differential form.

Let $\Lambda\mathbf{Der}(\mathcal{O})$ the exterior algebra built out of the algebra $\mathbf{Der}(\mathcal{O})$. A differential form is an element of $\mathrm{Hom}(\Lambda\mathbf{Der}(\mathcal{O}), \mathcal{O})$, where morphism here are $\mathcal{O}$-module morphisms. The exterior derivative d can be define exactly as in (5.2).

5.13. Form with values in a vector space.

Let V, W be vector spaces. A linear map $V \to W$ is the same tings as an element of $V^* \otimes W$, where V^* is the (linear) dual of V. Hence by tensoring $T_x^* M$ with a vector space V, for each $x \in M$, we obtain a collection of linear maps $T_x M \to V$. We can form the bundle $T^* M \otimes V$, an element of which is called a *one-form with values in V*. We can go further and consider the bundle $\Lambda T^* M \otimes V$, whose sections are called differential forms with values in V. We denote by $\Omega(M, V)$ the space of differential form with values in V.

5.14. Maurer-Cartan form.

As an example of a famous one form with values in a vector, we describe the Maurer-Cartan one-form θ of a Lie group G. It is a one-form taking values in the Lie algebra $\mathfrak{g}$ of the group. We said earlier that any Lie group admits left fundamental vector fields $\tilde{X}_{(g)} = L_{g*} X$ for some $X \in \mathfrak{g}$. By definition

$$\theta_{(g)}(X_{(g)}) = X. \tag{5.3}$$

The Maurer-Cartan form is then extended by linearity. In other words, for a vector $V \in T_g G$, the Maurer-Cartan form is given by

$$\theta_{(g)}(V) = L_{g^{-1}*} V. \tag{5.4}$$

5.15. Tensors.

The tensor bundle construction explained above applied to the tangent and cotangent bundle form what we will simply call tensor bundles. They are of the form

$$T^{*\otimes n} M \otimes T^{\otimes p} M, \tag{5.5}$$

for some positive integers $n, p \in \mathbb{N}$. Its sections are called tensors. These tensors can be seen as maps, at each $x \in M$, from $T_x^{\otimes n} M$ to $T_x^{\otimes p} M$.

5.16. Metrics.

As an example of tensor is the metric $g_{\mu\nu}$. It is a symmetric rank two tensor, a section of $T^{*\otimes 2} M$. It is non degenerate: the determinant of the $\dim(M) \times \dim(M)$ matrix $g_{ab}(x)$ never vanishes, whatever $x \in M$. Here g_{ab} denotes the expression of the matrix in some local basis $\{e_a\}$ which cam be thought as a local section of LM. It is always possible to choose this local section such that

$$g_{ab} = \begin{pmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \ddots & \\ & & & \pm 1 \end{pmatrix}. \tag{5.6}$$

If all the ± 1 are $+1$, we call the metric Riemannian. Often, in mathematical works, metrics are required to have only $+1$ in the diagonal, while the ones with one or several -1 are called pseudo metrics; but we won't use this denomination, and instead speak of Riemannian metrics for the "true" ones and just simply metrics for the "pseudo" ones. A manifold with a Riemannian metric is called a Riemannian manifold. The number of $+$ and $-$ signs in (5.6) is called the signature of the metric, hence a metric of signature (p,q) is a metric which can be put in the form

$$\begin{pmatrix} \mathbb{1}_p & \\ & -\mathbb{1}_q \end{pmatrix}. \tag{5.7}$$

Riemannian metrics are thus those metrics of signature $(m,0)$. Metrics of signature $(m-1,1)$ or $(1,m-1)$ are called Lorentzian metrics.

5.17. Volume form and Hodge operator. Let e_a be a local basis in which the metric g_{ab} takes the form 5.6, and e^a the dual basis. The n–form (n =dim(m))

$$\omega = e^1 \wedge e^2 \wedge \cdots \wedge e^n, \tag{5.8}$$

is called the volume form. (This appellation is abusive as it is a local object, we should call it "a local volume form".) Because the space of top form $\Omega^n(M)$ is "one dimensional", the volume form allows us to define a pairing

$$\begin{aligned} &\Omega^k(M) \times \Omega^{n-k}(M) \to \mathcal{C}^\infty(M), \\ &(\alpha,\beta) \mapsto \lambda, \quad \text{with } \lambda \text{ defined through } \alpha\wedge\beta = \lambda\omega. \end{aligned} \tag{5.9}$$

This pairing gives an isomorphism

$$\Omega^k(M) \simeq \Omega^{n-k}(M)^\star \tag{5.10}$$

Now the (inverse) metric also induces an isomorphism

$$\begin{aligned} &\Omega^k(M) \to \Omega^k(M)^\star, \\ &\alpha_{\mu_1\ldots\mu_k} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k} \mapsto g^{\nu_1\mu_1}\ldots g^{\nu_k\mu_k}\alpha_{\mu_1\ldots\mu_k}\partial_{\nu_1}\wedge\cdots\wedge\partial_{\nu_k}. \end{aligned} \tag{5.11}$$

with $\{\partial_\nu\}$ a dual local basis of $\{dx^\mu\}$. The successive application of the first isomorphism and the inverse of the second one leads to an isomorphism

$$* : \Omega^k(M) \to \Omega^{n-k}(M) \tag{5.12}$$

called Hodge isomorphism.

6. Homology and cohomology of manifolds - Orientation

6.1. Homology.

A complex $\mathcal{C}_\bullet$ is a sequence of abelian groups indexed by integers

$$\ldots \xrightarrow{\partial_{n+1}} \mathcal{C}_{n+1} \xrightarrow{\partial_n} \mathcal{C}_n \xrightarrow{\partial_{n-1}} \mathcal{C}_{n-1} \xrightarrow{\partial_{n-2}} \mathcal{C}_{n-2} \xrightarrow{\partial_{n-3}} \ldots \tag{6.1}$$

with boundary maps $\partial_n : \mathcal{C}_{n+1} \to \mathcal{C}_n$ such that $\partial_n \circ \partial_{n-1} = 0$. ($n$ denotes an integer.) In fact, we will see these abelian groups $\mathcal{C}_n$ as R-modules for some ring R. The homology groups associated to such a complex are the groups

$$H_n = {}^{\mathrm{Im}(\partial_{n+1})}\!\big/\!{}_{\mathrm{Ker}(\partial_n)} \tag{6.2}$$

The homology groups of a topological space M are defined to be those associated to the complex of singular simplices. The standard n-simplex is

$$\Delta^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n / 0 \le x_i \le 1 \forall i\} \tag{6.3}$$

A singular n-simplex is (an abstract sum of) continuous maps form the standard n simplex to the topological space M under consideration. Given a ring R, the complex of singular simplices is formed by the free R-modules C_n with one generator for each singular n-simplex. The ring R is called a coefficient ring. The boundary map of a singular n-simplex σ is obtained by restricting σ to the boundary of Δ^n.

Using standard method it is possible to show for example that homology groups of the sphere S^n are

$$H_n(S^n, \mathbb{Z}) = H_0(S^n, \mathbb{Z}) = \mathbb{Z}, \quad H_i(S^n, \mathbb{Z}) = 0 \text{ if } i \neq n, 0. \tag{6.4}$$

6.2. Cohomology.

Given a complex $\mathcal{C}_\bullet$ of R-modules with boundary maps $\partial_\bullet$, we can look at the complex of cochains $\mathcal{C}^\bullet$

$$\ldots \xrightarrow{\mathrm{d}_{n-1}} \mathcal{C}^{n-1} \xrightarrow{\mathrm{d}_n} \mathcal{C}^n \xrightarrow{\mathrm{d}_{n+1}} \mathcal{C}^{n+1} \xrightarrow{\mathrm{d}_{n+2}} \ldots, \tag{6.5}$$

formed by their dual

$$\mathcal{C}^n \doteq \mathrm{Hom}_R(\mathcal{C}_n, R), \tag{6.6}$$

and the coboundary maps $\mathrm{d}_n : \mathcal{C}^{n-1} \to \mathcal{C}^n$ defined by

$$\forall u \in \mathcal{C}^{n-1}, \forall a \in \mathcal{C}_n, \quad \mathrm{d}_n u(a) = u(\partial_n a) \tag{6.7}$$

It is straightforward to show that

$$\mathrm{d}_{n+1} \circ \mathrm{d}_n = 0. \tag{6.8}$$

The cohomolgy groups of such complex are

$$H^n(\mathcal{C}^n, R) = {}^{\mathrm{Ker}(\mathrm{d}_{n+1})}\!/_{\mathrm{Im}(\mathrm{d}_n)} \tag{6.9}$$

In fact, any complex of cochain of the form (6.5) with coboundary map satisfying 6.8 give rise to cohomology groups through (6.9).

6.3. Functoriality.

An important fact about homology is that the construction is natural in the sense it defines a functor from the category of topological spaces and continuous maps to the category of abelian groups and group morphisms. In particular map, any continuous map $f : M \to N$ between two topological spaces M and N give rise to a group morphism $f_ast : H_n(M) \to H_n(N)$. Sometimes we also write $H_n(f)$ or $H_\bullet(f)$ instead of f_ast. The same remark holds for cohomology. But in the case cohomology, arrows go in the opposite direction : f gives rise to $f^ast : H^n(N) \to H^(M)$. f^* can also be written $H^n(f)$ or $H^\bullet(f)$.

6.4. Homotopical invariance of homology.

Let M and N be two topological spaces and $f, g : M \to N$ two continuous maps. An homotopy between f and g is a continuous map $h : [0,1] \times M \to N$ such that $h(0,-) = f$ and $h(1,-) = g$. If there exists an homotopy between f and g we say that f and g are homotopic. Homology (as well as cohomology) is invariant under homotopy in the sense that if f and g are two homotopic maps then $H_\bullet(f) = H_\bullet(g)$ (and $H^\bullet(f) = H^\bullet(g)$). For example a contractile space is a topological space M such that there is an homotopy between the identity map Id_M and a constant map $x_0 : M \to M$; $x \mapsto x_0$. Homotopy invariance of homology implies in this case that the homology of M is equal to the homology of the space $\{x_0\}$ which is, with R the coefficient ring, $H_0(x_0, R) = R$ and $H_n(x_0, R) = 0$ $\forall n \neq 0$. This is the case for example for $\mathbb{R}^n$ where an homotopy between $\mathrm{Id}_{\mathbb{R}^n}$ and the map $0_{\mathbb{R}^n}$ sending any point to 0. Let (E, M, π) be a vector bundle with fiber $\mathbb{R}^n$. Let σ_0 be the zero section of E. Then the homotopy between $\mathrm{Id}_{\mathbb{R}^n}$ and $0_{\mathbb{R}^n}$ extend to an homotopy between the identity of E and $\sigma_0 \circ \pi$, showing the isomorphism of cohomology $H^n(E, R) \simeq H^n(M, R)$ for any n.

6.5. Relative Homology.

Let M be a topological space, $A \subset M$ a subspace and consider the quotient

$$C_n(M) \big/ C_n(A). \tag{6.10}$$

Because the boundary of a singular n-simplex in A is a singular $(n-1)$-simplex in A the boundary map

$$\partial_n : C_n(M) \to C_{n-1}(M) \tag{6.11}$$

passes well to the quotient and defines a boundary map

$$\partial_n : C_n(M) \big/ C_n(A) \to C_{n-1}(M) \big/ C_{n-1}(A) \tag{6.12}$$

The homology groups of the complex so obtained are called relative homology groups and denoted $H_n(X, A)$ or $H_n(X, A; R)$ if we specify the coefficient ring. Exactly the same definition holds for cohomology.

6.6. Cup product.

The front n-face of the standard $(n+m)$-simplex is $\{(x_1, x_2, \ldots, x_{n+m}) \in \Delta^{n+m} / x_{n+1} = x_{n+2} = \cdots = x_{n+m} = 0\}$. Similarly its back m-face is $\{(x_1, x_2, \ldots, x_{n+m}) \in \Delta^{n+m} / x_1 = x_2 = \cdots = x_n = 0\}$. Let α_n be the projection to the front n-face and β_m the projection to the back m-face. Let $c_1 \in C^n(M)$, $c_2 \in C^m(M)$ be two cocycles representing the cohomology classes $[c_1]$, $[c_2]$ and $\sigma \in C_n(M)$ a cycle representing an homology class $[\sigma]$. We define the cup product $\cup$ by

$$([c_1] \cup [c_2])(\sigma) = c_1(\sigma \circ \alpha_n) \cdot c_2(\sigma \circ \beta_m). \tag{6.13}$$

The cohomology complex $H^\bullet(M) = \bigoplus_n H^n(M)$ already had the structure of an abelian group, with the sum of two cohomology classes being the class of the sum of any two of their representative if they belongs to the same $H^n(M)$ or just their abstract sum if they belong to different $H^n(M)$, $H^m(M)$. The addition of the cup product turn $H^\bullet(M)$ into a ring.

6.7. Orientation for a Manifold.

Whereas homology is a device to extract global information about a given topological space M, relative homology is a way to extract local information. For example, the relative homology group $H_n(M, M\setminus x)$ only depends of the local topology around the point $x \in M$. If M is a manifold of dimension m, we know that this local topology is equal to the local topology of $\mathbb{R}^m$ around 0, hence the isomorphism $H_n(M, M\setminus x) \simeq H_n(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\})$. Using the long exact sequence of relative homology, and the fact that $\mathbb{R}^m \setminus \{0\}$ is homotopic to S^m, it is straightforward to show that $H_n(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}; \mathbb{Z})$ is 0 when $n \neq m$ and isomorphic to $\mathbb{Z}$ when $n = m$. A generator $\mu_x \in H_n(M, M \setminus x; \mathbb{Z})$ is called a local orientation around x. An orientation of M is an assignment $x \mapsto \mu_x$, μ_x a generator of $H_n(M, M\setminus \{x\}; \mathbb{Z})$ such that $\forall x \in M$, there exists a compact neighborhood K of x and a class $\mu_K \in H_n(M, M\setminus K)$ such that the restriction $r_{K,x}(\mu_K) = \mu_x$. Here, the restriction map $r_{K,x}$, is derived from the canonical inclusion

$$C_n(X)/C_n(U) \hookrightarrow C_n(X)/C_n(\{x\}). \tag{6.14}$$

This definition of orientation should be understood in the following way. An orientation of $\mathbb{R}^n$ is the choice of an ordered basis that we can see as the edges of Δ^n. Because locally M is like $\mathbb{R}^n$, we can look at a singular simplex with image around $x \in M$ as a map $\mathbb{R}^n \to \mathbb{R}^n$. The one with non vanishing determinant fall into two classes, choosing the generator μ_x is exactly like choosing which one has positive determinant.

6.8. Orientation for vector bundle.

Let (E, M, π) be a vector bundle with fiber $\mathbb{R}^n$. An orientation for E is a choice of an orientation for each of the fiber in a continuous way. The orientation of the fiber is given by a homology class $\mu_F \in H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}, \mathbb{Z})$. The locality condition is that for any trivializing chart $\pi^{-1}(U) \simeq U \times \mathbb{R}^n$ there should exist an homology class $\mu_U \in H_n(\pi^{-1}(U), \pi^{-1}(U)_0; \mathbb{Z})$ such that under restriction to any fiber, μ_U is mapped to μ_F. Here $\pi^{-1}(U)_0$ means $\pi^{-1}(U)$ minus the 0 section. When a possible choice for an orientation of E exists, we say that E is orientable. A proposition states that a manifold is orientable (as a manifold) if and only if its tangent bundle is orientable (as a vector bundle).

6.9. De Rahm Cohomology.

Let M be a manifold. Then the differential forms on M form a complex of cochain

$$0 \xrightarrow{\mathrm{d}} \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Omega^{\dim(M)}(M) \xrightarrow{\mathrm{d}} 0 \tag{6.15}$$

whose coboundary map is the exterior derivative. The cohomology groups $H^n_{dR}(M)$ of this complex are called de Rahm cohomology group, named after Georges de Rahm, who proved, among other things, that for compact manifolds these de Rahm cohomology groups are isomorphic to the singular cohomology groups [Rah31]. The isomorphism is still valid for paracompact manifold, as has been shown by Weil [Wei52]. However the coefficient ring of the de Rahm cohomology groups is necessarily $\mathbb{R}$ because the space of differential form $\Omega(M)$ is a real vector space. Furthermore, under this isomorphism, the cup product is mapped to the wedge product, turning it into an isomorphism of graded ring

$$H^\bullet(M;\mathbb{R}) \simeq H^\bullet_{dR}(M) \tag{6.16}$$

6.10. Orientation with Differential form.

We have seen that the orientation of a manifold can be defined by an ordering of the basis $\partial_1, \partial_2, \dots \partial_n$ associated to a local frame $(x^1, x^2, \dots x^n)$. Now, the space $\Lambda^m T^* M$, $m = \dim(M)$ is one-dimensional. If $\{\tilde{x}^i\}$ denotes another local basis, we have

$$\mathrm{d}\tilde{x}^1 \wedge \mathrm{d}\tilde{x}^2 \wedge \dots \wedge \mathrm{d}\tilde{x}^n = \det(f) \quad \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \dots \wedge \mathrm{d}x^n, \tag{6.17}$$

where f is the endomorphism sending $\{x^i\}$ to $\{\tilde{x}^i\}$. $\{\tilde{x}^i\}$ defines the same orientation as $\{\tilde{x}^i\}$ if and only if $\det(f)$ is positive. From this remark, we understand that a nowhere vanishing top form $\omega \in \Omega^m(M)$ defines an orientation at any point. If ω' is another nowhere vanishing top form, then there is a nowhere vanishing function $F \in \mathcal{C}^\infty(M)$ such that $\omega' = F \cdot \omega$. If F is positive, ω' defines the same orientation as ω, if F is negative, it defines the reverse orientation.

6.11. Orientation with de Rahm Cohomology.

A top form ω defining an orientation does not necessarily define a non trivial cohomology class in $H^m_{\mathrm{dR}}(M)$. Indeed, the prototype of orientable manifold of dimension m is $\mathbb{R}^m$ and

$$H^m_{\mathrm{dR}}(\mathbb{R}^m) = 0. \tag{6.18}$$

As explained earlier, what has to be considered is compactly supported cohomology

$$H^m_{\mathrm{dR},c}(M) = \varinjlim_{K \subset M} H^m_{\mathrm{dR}}(M, M \setminus K) \tag{6.19}$$

represented by form vanishing outside some compact $K \subset M$. We will simply say that for a smooth paracompact manifold of dimension m,

$$H^n_{\mathrm{dR},c}(M) = \begin{cases} \mathbb{R} & \text{if } M \text{ is orientable,} \\ 0 & \text{if } M \text{ is not orientable.} \end{cases} \tag{6.20}$$

For any compact K of M with inclusion map $\iota_K : K \hookrightarrow M$, the top form ω defining the orientation is mapped to a generator of $H^m_{\mathrm{dR}}(M, M \setminus K)$ by $\iota_K^* \omega$, linking orientation defined through top form with orientation defined through homology.

7. Reduction of principal bundle

7.1. Reductive subgroup.

Let G be a Lie group and $H \subset G$ a sub-Lie group. H is called reductive if there exist a decomposition of the Lie algebra $\mathfrak{g}$ of G:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{7.1}$$

as the direct sum of the Lie algebra $\mathfrak{h}$ of H and an Ad_H-invariant subspace $\mathfrak{m} \subset \mathfrak{g}$.

7.2. Example.

The pseudo-orthogonal group $SO(p,q)$, $p+q=n$ is a reductive subgroup of $GL_n(\mathbb{R})$, the group of invertible $m\times m$ real matrices. Indeed, let η be the standard bilinear form on $\mathbb{R}^n$ of signature (p,q) and let ^{t} denote the matrix transposition with respect to it, i.e., for any $M\in M_n(\mathbb{R})$, $u,v\in\mathbb{R}^n$,

$$\eta(u,Mv)=\eta({}^tMu,v) \tag{7.2}$$

the Lie algebra $\mathfrak{gl}_n(\mathbb{R})=M_n(\mathbb{R})$ of $GL_n(\mathbb{R})$ can be decomposed as

$$\mathfrak{gl}_n(\mathbb{R})=\mathfrak{so}(p,q)\oplus\mathfrak{s}' \tag{7.3}$$

with $\mathfrak{s}'$ the set of matrices symmetric with respect to η. $\mathfrak{s}'$ is indeed $\mathrm{Ad}_{SO(p,q)}$-invariant, as, for any $M\in\mathfrak{s}'$, $O\in SO(p,q)$,

$${}^t(OMO^{-1})={}^t(OM{}^tO)=O^tM^tO=OMO^{-1} \tag{7.4}$$

7.3. Reduction.

Let (P,M,π,G) be a principal bundle and H a subgroup of G. In general, an H-reduction, or simply reduction, is a subspace $Q\subset P$ such that $(Q,M,\pi|_Q,H)$ is a principal H-bundle. We denote by $i_{Q\to P}$ the canonical inclusion map. In this thesis, we will impose the supplementary condition that H is a reductive subgroup of G.

7.4. Reduction as a section of an associated bundle.

PROPOSITION 20. *A reduction $Q\subset P$ as described above is equivalent to the data of a global section $\sigma: M\to P\times_G {}^G\!/_H$.*

Proof : Suppose $Q\subset P$ is an H-reduction. Define a map $\bar\sigma: P\to {}^G\!/_H$ by $\bar\sigma(q)=eH$ for all $q\in Q$, and $\bar\sigma(zg)=g^{-1}\bar\sigma(z)$ for all $z\in P$ and $g\in G$. We get a global section $\sigma: M\to P\times_G {}^G\!/_H$ by defining

$$\sigma(x)=[z,\bar\sigma(z)],\quad \pi(z)=x \tag{7.5}$$

where $[z,\bar\sigma(z)]$ denote the equivalence class $\{zg,g^{-1}\sigma(z)\}_{g\in G}$.
Conversely suppose given a global section $\sigma: M\to P\times_G {}^G\!/_H$. Taking equations (7.5) as a reverse definition, we obtain a G-equivariant map $\bar\sigma: P\to {}^G\!/_H$, and a reduction $Q\subset P$ by setting $Q\bar\sigma^{-1}(eH)$. It remains to show that Q is an principal H-bundle. Let $q_1,q_2\in Q$ such that $\pi(q_1)=\pi(q_2)$. Because q_1,q_2 are in the same fiber of P, there exists $g\in G$ such that $q_1=q_2g$. Hence we just need to show that $g\in H$. This follows from $\bar\sigma(q_1)=g^{-1}\bar\sigma(q_2)=eH$. The number of + and minus − signs is called the signature of the metric.

7.5. Reduction for Riemannian Manifolds.

The frame bundle LM of a Riemannian manifold admits a natural reduction to $SO(M)$, a principal bundle with structure group $SO(m)$, $m=\dim(M)$. We can decompose LM into local trivializing charts $\{U_\alpha\times GL(m)\}$. On each U_α, we can choose, using the Riemannian metric, local orthonormal frame $\{e^a_\alpha\}$. We can order these frames so that they respect the manifold's orientation. On overlaps $U_\alpha \cap U_\beta$, the transition maps $g_{\alpha\beta}$ sending $\{e^a_\alpha\}$ to $\{e^a_\beta\}$ belongs to $SO(m)$ and satisfy the cocycle property

$$g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = \mathrm{Id}_{U_\alpha \cap U_\beta \cap U_\gamma}, \tag{7.6}$$

so we can use them to define the reduced principal bundle $SO(M)$. The local orthonormal frames $\{e^a_\alpha\}$ are called "vielbein" or "tetrad". We can use them to define local orthonormal basis of the tangent or cotangent manifolds through

$$e^a(x) = e^a_\alpha(x) dx^\alpha \in T^*_x M, \qquad E_a(x) e^a_\alpha(x) = \partial_\alpha \in T_x M \tag{7.7}$$

7.6. The tangent space of a reduction.

Let $Q \subset P$ be a reduction and $i_{Q\to P}$ the canonical inclusion. The pullback bundle $i^*_{Q\to P}(TP) = TP \times_P Q$, which can be understood as the restriction of TP to Q, admits a canonical decomposition

$$i^*_{Q\to P}(TP) = TQ \oplus \mathcal{V}. \tag{7.8}$$

$\mathcal{V}$ can be seen as generated by the fundamental vector fields associated to $\mathfrak{m}$ in the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. In particular, any vector field $P \to TP$ decomposes, once restricted to Q, to a vector field tangent to Q and a transverse vector field.

8. Lie derivative

8.1. Flow.

Let M be a manifold and X a vector field over M. At each point $x \in M$, there is an interval $I_x \subset \mathbb{R}$ and a curve $\gamma^{(x)} : I \to M$ satisfying

$$\forall\, t \in I,\ \frac{d}{ds}\gamma^{(x)}(s)|_{t=s} = X_{(\gamma^{(x)}(t))}. \tag{8.1}$$

This is the theorem about existence of solution of differential equation in manifolds, whose proof can be found in many books, including [Lan02]. Let I be the infimum, with respect to the order defined by inclusion, $\inf_{x\in M}(I_x)$. We can define an automorphism Φ^X_t, for each $t \in I$, by

$$\Phi^X_t(x) = \gamma^{(x)}(t). \tag{8.2}$$

The theory of ordinary differential equations insures that, for $t_1, t_2 \in I$ such that $t_1 + t_2 \in I$

$$\Phi^X_{t_1} \circ \Phi^X_{t_2} = \Phi^X_{t_1+t_2}, \tag{8.3}$$

a result which follows directly from the unicity of the solution of ordinary differential equation. This turn the set $\{\Phi^X_t\}_{t\in I}$ almost into a 1-parameter group, the group law being the composition, the identity being Φ^X_0 and the inverse of Φ^X_t being Φ^X_{-t}. $\{\Phi^X_t\}_{t\in I}$ is group only when I is itself an additive group, i.e. when $I = \mathbb{R}$ or $I = \{0\}$. In this case, $\{\Phi^X_t\}_{t\in I}$ is called a 1-parameter group. If M is compact, then it is possible to take $I = \mathbb{R}$.

The map $t \mapsto \Phi^X_t$, $t \in I$, is called the flow of the vector field X. The map $t \mapsto \Phi^X_t(x)$, $t \in I_x$ is called the local flow. The local flow always is always defined for sufficiently small $t > 0$, which is all what we need for the next paragraph.

8.2. Lie derivatives.

Keeping the same notation, the Lie derivative by the vector field X $\mathcal{L}_X$ of a function $f : M \to \mathbb{R}$ is

$$\mathcal{L}_X(f) \doteq \frac{d}{dt}|_{t=0} f \circ \Phi_t^X. \tag{8.4}$$

A small computation shows this definition leads to

$$\mathcal{L}_X(f) = X(f). \tag{8.5}$$

The Lie derivative $\mathcal{L}_X$ of another vector field Y is

$$\mathcal{L}_X(Y) \frac{d}{dt}|_{t=0} \Phi_{-t*}^X Y \circ \Phi_t^X. \tag{8.6}$$

This definition leads to

$$\mathcal{L}_X(Y) = [X, Y]. \tag{8.7}$$

The Lie derivative $\mathcal{L}_X$ of a one-form ω is

$$\mathcal{L}_X(\omega) \doteq \frac{d}{dt}_{t=0} \Phi_{-t}^{X*} \omega \circ \Phi_t^X \tag{8.8}$$

Lie derivatives are, as their name suggests, derivations. This means that, for $f, g \in \mathcal{C}^\infty(M)$, X, Y vector fields and ω a one-form,

$$\mathcal{L}_X(fg) = \mathcal{L}_X(f)g + f\mathcal{L}_X(g), \tag{8.9}$$

$$\mathcal{L}_X(fY) = \mathcal{L}_X(f)Y + f\mathcal{L}_X(Y), \tag{8.10}$$

$$\mathcal{L}_X(f\omega) = \mathcal{L}_X(f)\omega + f\mathcal{L}_X(\omega). \tag{8.11}$$

8.3. Lie derivatives of tensors.

Lie derivatives extend well to tensors. For example, the Lie derivative of $Y_1 \otimes \cdots \otimes Y_n \otimes \omega_1 \otimes \cdots \otimes \omega_p \in TM^{\otimes n} \otimes T^*M^{\otimes p}$ is

$$\mathcal{L}_X(Y_1 \otimes \cdots \otimes Y_n \otimes \omega_1 \otimes \cdots \otimes \omega_p) = \frac{d}{dt}(\Phi_{-t*}^X Y_1 \otimes \cdots \otimes \Phi_{-t*}^X Y_n \otimes \Phi_{-t}^{X*} \omega_1 \otimes \cdots \otimes \Phi_{-t}^{X*} \omega_p) \circ \Phi_t^X, \tag{8.12}$$

$$= \mathcal{L}_X Y_1 \otimes \cdots \otimes \mathcal{L}_X Y_n \otimes \mathcal{L}_X \omega_1 \otimes \cdots \otimes \mathcal{L}_X \omega_p. \tag{8.13}$$

This formula shows that Lie derivatives are derivations for the tensor product.

Lie derivatives for sections of sub-bundles of tensor bundles are also well defined, by the same above formula. For example, we can compute Lie derivatives of sections of the frame bundle, which is a sub-bundle of $TM^{\otimes 2}$. However we have to keep in mind that these Lie derivatives may not stay inside this sub-bundle. Hence, the Lie derivative of a smooth frame can be computed, but may not be a frame.

8.4. Lie derivatives for algebra bundles. Lie derivative of sections of algebra bundles defined as quotient of tensor bundles, i.e. bundles of the type

$$TM^{\otimes n} \otimes T^*M^{\otimes p} \Big/ I, \tag{8.14}$$

I an ideal, can be defined if and only if Lie derivatives of sections of I, seen as a sub-bundle of $TM^{\otimes n} \otimes T^*M^{\otimes p}$, stay in I. For example, differential form are sections of the exterior algebra of the cotangent bundle; it is a quotient $T^*M^{\otimes m}$, $m = \dim(M)$, by the ideal I_Λ generated by $\{\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1\}$. Because

$$\mathcal{L}_X(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1) = (\mathcal{L}_X\omega_1 \otimes \omega_2 + \omega_2 \otimes \mathcal{L}_X\omega_1) + \omega_1 \otimes \mathcal{L}_X\omega_2 + \mathcal{L}_X\omega_2 \otimes \omega_1 \tag{8.15}$$

the Lie derivative of a section of I_Λ is a section of I_Λ, consequently Lie derivatives pass well to quotient, and thus we can define Lie derivative of differential forms. However, in the case of the Clifford bundle, the ideal I_C defining it is generated by $Y_1 \otimes Y_2 + Y_2 \otimes Y_1 - 2g(Y_1, Y_2)$, with g the metric. In that case the Lie derivative of a section of I_C is not a section of I_C, as can be seen from

$$\begin{aligned}\mathcal{L}_X(Y_1 \otimes Y_2 + Y_2 \otimes Y_1 - 2g(Y_1, Y_2)) &= \mathcal{L}_X Y_1 \otimes Y_2 + Y_2 \otimes \mathcal{L}_X Y_1 - 2g(\mathcal{L}_X Y_1, Y_2)\\ &+ Y_1 \otimes \mathcal{L}_X Y_2 + \mathcal{L}_X Y_2 \otimes Y_1 - 2g(Y_1, \mathcal{L}_X Y_2) - 2(\mathcal{L}_X g)(Y_1, Y_2).\end{aligned} \tag{8.16}$$

The appearance of the Lie derivative of the metric prevent any satisfying definition for Lie derivative of sections of the Clifford bundle.
Because Lie derivatives are derivations for the tensor product, they are also derivation for all other product obtained from it by quotient. In particular, they are derivations for the exterior product of differential forms. Concretely, for ω_1, ω_2 two differential forms,

$$\mathcal{L}_X(\omega_1 \wedge \omega_2) = \mathcal{L}_X(\omega_1) \wedge \omega_2 + \omega_1 \wedge \mathcal{L}_X(\omega_2). \tag{8.17}$$

8.5. The Cartan's formula.

Let X be a vector field. We define a map $\iota_X \omega^k(M) \to \omega^{k-1}(M))$, called insertion by X, by

$$\iota_X \omega(Y_1, \dots Y_{k-1}) \doteq \omega(X, Y_1, \dots Y_{k-1}). \tag{8.18}$$

Using the insertion map, we have the very famous Cartan's formula.

PROPOSITION 21. *Let ω be a differential form and X be vector field, then*

$$\mathcal{L}_X \omega = (d\iota_X + \iota_X d).\omega \tag{8.19}$$

8.6. Lie derivatives for super-manifolds.

The formulas (8.5), (8.7) and (8.19) are so simple that they usually serve as definition in super-differential geometry, together with their property of super-derivations. More precisely if $M, \mathcal{O}$ is a super-manifold, $f, g \in \mathcal{O}(M)$, $X, Y \in \mathrm{Der}\mathcal{O}(M)$, $\omega \in \Omega(M)$, all supposed homogeneous, the following equalities serve as definitions

$$\mathcal{L}_X(f) = X(f),\ \mathcal{L}_X(fg) = \mathcal{L}_X(f)g + (-1)^{|X||f|} f \mathcal{L}_X(g), \tag{8.20}$$

$$\mathcal{L}_X(Y) = [X, Y]\ \mathcal{L}_X(fY) = \mathcal{L}_X(f)Y + (-1)^{|X||f|} f \mathcal{L}_X(Y), \tag{8.21}$$

$$\mathcal{L}_X(\omega) = (\mathrm{d}\iota_X + \iota_X \mathrm{d})\omega\ \ \mathcal{L}_X(f\omega) = \mathcal{L}_X(f)\omega + (-1)^{|X||f|} f \mathcal{L}_X(\omega), \tag{8.22}$$

$$\mathcal{L}_X(Y \otimes \omega) \quad = \mathcal{L}_X(Y) \otimes \omega + (-1)^{|X||f|} Y \otimes \mathcal{L}_X(\omega). \tag{8.23}$$

9. Connections

9.1. Distributions.

Given a smooth manifold M, a distribution is a the data, for each $x \in M$, of a subspace $E_x \subset T_x M$. We will call a distribution regular if all the subspaces V_x have the same dimension. A distribution $\{E = \sqcup E_x\}$ is called smooth if for every $x \in M$ there exists some smooth vector fields $X_1, \dots, X_n$ defined in a neighborhood of x such that for each y in this neighborhood $X_1(y), \dots, X_n(y)$ span E_y. An integral manifold for a distribution V in a manifold M is a pair (N, ι) where N is a manifold and $\iota : N \to M$ is an immersion such that for each $x \in N$, $\iota_*(T_x N) = E_{\iota(x)}$. An integral manifold is called maximal if it is maximal with respect to the order defined by inclusions. A distribution is called integrable if for any point $x \in M$ there is an integrable manifold containing it. A distribution is called involutive if the Lie brackets of any two vector fields of the distribution is again a vector field of the distribution. The Froebenius theorem states that a smooth regular distribution is integrable if and only if it is involutive. For supermanifolds, we say that a distribution is regular if their even and odd dimensions are separately constant.

9.2. Vertical, horizontal and G-invariant distributions.

Let (P, M, π) be a fiber bundle. The set $V = \mathrm{Ker}(\pi_*) \subset TP$ form a smooth regular integrable distribution called the vertical distribution. Its integral manifolds are simply the fiber of the fibration. The choice of complementary space H_z of V_z form what is called an horizontal distribution, also named connection. It is automatically regular, and is usually required to be smooth. If (P, M, π, G) is a principal fiber bundle with right action r, we say that a distribution E is G-equivariant (sometimes also called G-invariant) if $E_{xg} = r_{g*} E_x$. A principal connection is a smooth G-equivariant horizontal distribution. Still in the case of a principal bundle, the right action $r : P \times G \to P$ give rise, when restricted to a fixed $z \in P$, to an isomorphism r_z between G and the fiber containing z. The tangent map at the identity of this isomorphism defines another isomorphism $r_{z*}|_e$, between the Lie algebra $\mathfrak{g}$ of G and the vertical subspace V_z. Those vertical and horizontal subspaces can be respectively regrouped into two new bundle, denoted VP and HP. The construction of VP can be extended to any bundle, not necessarily principal. For HP we can extend it generically if we forget about G-equivariance, or we can define it on associated bundle using the associated structure.

9.3. Connection one-forms.

The choice of an horizontal distribution H_z at z is equivalent to the choice of a projection

$$\Phi_z : T_z P \to V_z, \tag{9.1}$$

with $H_z = \mathrm{Ker}(\Phi_z)$. Φ_z can be seen as an element of $T_z^* P \otimes V_z$. When the horizontal distribution is smooth, the assignment $z \mapsto \Phi_z$ define a one-form Φ with values in V: $\Phi \in \Omega(P, VP)$, and reciprocally. In the case of a principal bundle, we can use the isomorphism $r_{z*}|_e$ defined above to have a connection one-form with values in $\mathfrak{g}$ instead of VP

$$\tilde{\Phi} : z \mapsto (r_{z*}|_e)^{-1} \circ \Phi_z. \tag{9.2}$$

Using pullback of sections, we define a connection one-form over the base manifold

$$A = s^* \tilde{\Phi}, \qquad s : M \to P \text{ a section.} \tag{9.3}$$

This connection one-form may be defined only locally, if s is a local section. It is this definition that we will mainly use in this thesis, and that we will refer to simply as "connection" and we will also always assume that the connection is G-equivariant.

9.4. Gauge transformations of the connection.

Because this definition of the connection A depends on the choice of a section, it is important to understand how A changes when we change this choice. Given two sections s_1, s_2, defining two connections $A_1 = {s_1}^* \tilde{\Phi}$, $A_2 = {s_2}^* \tilde{\Phi}$, the principal bundle structure ensures the existence of a smooth map $g : M \to G$ such that

$$s_2 = s_1 \cdot g. \tag{9.4}$$

Hence we have to relate $s_1^* \tilde{\Phi}$ with $(s_1 \cdot g)^* \tilde{\Phi}$. We start a digression. Suppose we have a smooth map $f : M \times N \to P$ between three smooth manifolds M, N, P. Let li_x and ri_y be the left- and right-insertion map

$$li_x : N \to M \times N, \quad y \mapsto (x, y), \tag{9.5}$$

$$ri_y : M \to M \times N, \quad x \mapsto (x, y). \tag{9.6}$$

Their tangent maps are given by, at some $y_0 \in N$, $x_0 \in M$,

$$li_{x*}|_{y_0} : T_{y_0} N \to T_x M \times T_{y_0} N, \quad Y \mapsto (0, Y), \tag{9.7}$$

$$ri_{y*}|_{x_0} : T_{x_0} M \to T_{x_0} M \times T_y N, \quad X \mapsto (X, 0). \tag{9.8}$$

Thus the tangent map of f at (x, y) is

$$f_*|_{(x,y)}(X, Y) = (f \circ li_x)_*|_y(Y) + (f \circ ri_y)_*|_x(X) \tag{9.9}$$

Applying this observation to the right action r of G on P we obtain that

$$(r \circ (s_1, g))_* = r_{g*} \circ s_{1*} + r_{s_1*} \circ g_*, \tag{9.10}$$

where we have used the short-hand notation $r_g = r \circ ri_g$, $r_{s_1} = r \circ li_{s_1}$. Dualizing, we get

$$(s_1 \cdot g)^* = s_1^* {r_g}^* + g^* {r_{s_1}}^* \tag{9.11}$$

This equality simply reflect the derivation rule:

$$\partial_i (s_1(x) \cdot g(x)) = (\partial_i s_1(x)) \cdot g(x) + s_1(x) \cdot (\partial_i g(x)). \tag{9.12}$$

Note that the above formula requires a well defined action (we have denoted it $\cdot$ as well) of G on the tangent space $T_{s_1(x)} P$ and of the tangent space $T_{g(x)} G$ on P. This would be the case, for example, if G is a matrix group, see below.

The gauge invariance of the connection is by definition the equality

$$r_{g*} H_z = H_{zg}. \tag{9.13}$$

This means that given a tangent vector X, its decomposition $X = X^{vert} + X^{hor}$ is preserved by the tangent map r_{g*}

$$r_{g*}(X^{vert}) = (r_{g*}X)^{vert}, \quad r_{g*}(X^{hor}) = (r_{g*}X)^{hor}. \tag{9.14}$$

For the one-form Φ, this last equality implies, for any tangent vector X, $r_{g*}\Phi(X) = \Phi(r_{g*}X)$, or simply

$$r_{g*}\Phi = r_g{}^*\Phi. \tag{9.15}$$

Now let $\bar{X}$ be a vector in the Lie algebra $\mathfrak{g} = T_eG$. From

$$r_{g*}|_z r_{z*}|_e(\bar{X}) = \frac{d}{dt}(r_g(ze^{t\bar{X}})) = \frac{d}{dt}(zgg^{-1}e^{t\bar{X}}g) \tag{9.16}$$

we obtain the equality

$$r_{g*}|_z r_{z*}|_e(\bar{X}) = r_{zg*}|_e(\mathrm{Ad}_{g^{-1}}\bar{X}). \tag{9.17}$$

To follow carefully the end of the computation, we denote be $\Phi_{(z)}$ the value of Φ at z, i.e. $\Phi_{(z)} \in T^*_zP$. By definition $\Phi_{(z)} = r_{z*}|_e\tilde{\Phi}_{(z)}$, hence

$$r_{g*}|_z\Phi_{(z)} = r_{g*}|_z r_{z*}|_e\tilde{\Phi}_{(z)}, \tag{9.18}$$

$$= r_{zg*}|_e\mathrm{Ad}_{g^{-1}}\tilde{\Phi}_{(z)}. \tag{9.19}$$

Using equation (9.15), we get

$$r_g{}^*|_{zg}\Phi_{(zg)} = r_{zg*}|_e\mathrm{Ad}_{g^{-1}}\tilde{\Phi}_{(z)}. \tag{9.20}$$

Finally, applying $r_{zg*}|_e{}^{-1}$ on both sides of the preceding equation, we obtain $r_g{}^*|_{zg}\tilde{\Phi}_{(zg)} = \mathrm{Ad}_{g^{-1}}\tilde{\Phi}_{(z)}$ or simply, as this equality hold for all $z \in P$,

$$r_g{}^*\tilde{\Phi} = \mathrm{Ad}_{g^{-1}}\tilde{\Phi}. \tag{9.21}$$

Hence we have worked out the first part of the computation

$$A_2 = \mathrm{Ad}_{g^{-1}}A_1 + g^* r_{s_1}{}^*\tilde{\Phi}. \tag{9.22}$$

For any vector $X_{(g)} \in T_gG$,

$$r_z{}^*|_{zg}\tilde{\Phi}_{(zg)}(X_{(g)}) = r_z{}^*|_{zg} r_{z*}|_e{}^{-1}\Phi_{(zg)}(X_{(g)}), \tag{9.23}$$

$$= r_{z*}|_e{}^{-1}\Phi_{(zg)}(r_{g*}|_g X_{(g)}) \tag{9.24}$$

$$= r_{z*}|_e{}^{-1} r_{g*}|_g X_{(g)}, \tag{9.25}$$

where the last equality holds because $r_{g*}|_g X_{(g)}$ is a vertical vector. Now of course

$$r_{z*}|_e{}^{-1} r_{g*}|_g X_{(g)} r_{z*}|_e{}^{-1} r_{g*}|_g \tilde{l}_{g*}|_e \tilde{l}_{g^{-1}*}|_g X_{(g)}, \tag{9.26}$$

where $\tilde{l}_g$ denotes the left multiplication by g. Because r is a group action $r_z(g_1g_2) = r_{zg_1}(g_2)$, or, in other words, $r_{zg} = r_z \circ \tilde{l}_g$. Hence

$$r_z{}^*|_{zg}\tilde{\Phi}_{(zg)}(X_{(g)}) = \tilde{l}_{g^{-1}}(X_{(g)}) \tag{9.27}$$

showing that ${r_z}^*\tilde{\Phi}$ is simply the Maurer-Cartan form of G. Hence the second term in 9.22 is the pullback of the Maurer-Cartan form by the map $g: M \to G$. Hence

$$A_2 = \mathrm{Ad}_{g^{-1}} A_1 + g^*\theta \tag{9.28}$$

Now suppose G is a matrix group, in which we can multiply tangent vectors with group elements. Then, for a vector $X \in T_x M$,

$$g^*\theta(X) = \theta_{(g(x))}(g_* X) = L_{g^{-1}(x)*} g_* X = g^{-1}(x)\partial_i g(x) X^i \tag{9.29}$$

Hence we arrive to the physicist's notation of gauge transformation

$$A_2 = \mathrm{Ad}_{g^{-1}} A_1 + g^{-1}\mathrm{d}g \tag{9.30}$$

9.5. Local description.

Let (P, M, π) be any bundle with fiber F, Φ a connection one-form and $\phi_\alpha : P \to U_\alpha \times F$ a local trivializing chart. In this chart, a tangent vector X_z at $z \in P$ can be decomposed as a tangent vector of M and a tangent vector of f:

$$\phi_{\alpha *}(X_z) = (X_x, X_f), \qquad \phi_\alpha(z) = (x, f), \qquad X_x \in T_x M, \quad X_f \in T_f F. \tag{9.31}$$

The vertical vectors are those tangent to the fiber, i.e. the "X_f's". Hence applying the connection one form, we have

$$\phi_{\alpha *}\Phi\phi_{\alpha *}^{-1}(X_x, X_f) = (0_x, \hat{\Gamma}_{(x,f)}(X_x + X_f)). \tag{9.32}$$

For a linear homomorphism $\hat{\Gamma}_{(x,f)} : T_x M \oplus T_f F \to T_f F$. This homomorphism is a projection onto the $T_f F$, hence

$$\hat{\Gamma}_{(x,f)}(X_x + X_f) = \Gamma_{(x,f)}(X_x) + X_f. \tag{9.33}$$

Thus a connection can also be defined charts by charts, by saying how much a vector tangent to M in a chart is vertical, which is precisely what $\Gamma_{(x,f)}(X_x)$ tells. In the case (P, M, π, G) is a principal bundle, the equivariance of the connection is the expressed through Γ by

$$r_{g*}\Gamma_{(x,\tilde{g})} = \Gamma_{(x,\tilde{g}g)}. \tag{9.34}$$

In the case, the link between Γ and A_α is given by

$$A_\alpha = l_{g_\alpha^{-1}*}(\Gamma_{x,g_\alpha} + g_{\alpha *}). \tag{9.35}$$

In this last equation, the section s_α defining A_α is given in local coordinates by

$$\phi_\alpha \circ s_\alpha(x) = (x, g_\alpha(x)), \tag{9.36}$$

and $l_{g_\alpha *}$ denotes the pushforward of the left multiplication by g_α in G.

9.6. Connections and reductions.

If (P, M, π, G) is a principal bundle, A_α a connection form on P (rigorously, we should write $\{A_\alpha\}$, associated with some cover $\{U_\alpha\}$ of M), and $(Q, M, \tilde{\pi}, H)$ a reduced principal bundle, with $Q \subset P$ and H a subgroup of G, then A_α defines a connection on Q if and only its image is in the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ of H.

9.7. Connections on associated bundles.

If (P, M, π, G) is a principal bundle, Φ a G-equivariant connection on P and $E = P \times_G F$ a associated bundle, it is possible to construct a connection for E. We will do it locally on trivializing charts using the form Γ of the previous paragraph. For simplicity, we will write the right action of G on P and its left action on F using dots. We will also not write explicitly the charts map ϕ_α as well as their pushforward, and indentify $P \simeq U_\alpha \times G$, $U_\alpha \subset M$. In this set-up the quotient map defining E takes the form

$$U_\alpha \times G \times F \simeq P \times F \to E \simeq U_\alpha \times F, \tag{9.37}$$

$$(x, g, v) \mapsto (x, g \cdot v). \tag{9.38}$$

At the level of tangent spaces,

$$TP \times_{TG} TF \to TE, \tag{9.39}$$

$$(x, g, v; X_x, X_g, X_v) \mapsto (x, g \cdot v; X_x, X_g \cdot v + g \cdot X_v). \tag{9.40}$$

hence a tangent vector of E can be written as an equivalence class

$$(x, v; X_x, X_v) = [(x, g, g^{-1} \cdot v; X_x, X_g, g^{-1} \cdot X_v - g^{-1} \cdot X_g \cdot g^{-1} \cdot V)]. \tag{9.41}$$

What we can do is apply the connection Φ to any vector of this equivalence class to obtain its vertical part, express it in terms of the G-equivariant Γ and apply the quotient map (9.40). We obtain a local definition for a connection form Φ^E on E

$$\Phi^E(x, v; X_x, X_v) = (x, v; 0_x, \Gamma^E_{(x,v)}(X_x) + X_v), \tag{9.42}$$

$$\Gamma^E_{(x,v)}(X_x) = \Gamma_{x,g}(X_x) \cdot g^{-1} \cdot v. \tag{9.43}$$

9.8. Covariant derivatives.

The covariant derivative correspond the "horizontal projection after taking exterior derivative". So far we have defined $\Phi : TP \to VP$. Equivalently, we have the horizontal projection $h = \Phi - \mathrm{Id} : TP \to HP$. The dual endomorphism acts on differential forms $h^* : \Omega(P) \to \Omega(P)$, whose image are those differential forms which vanish on vertical vectors. The covariant derivative ∇ is the composition of h^* with d

$$\nabla \doteq h^* \circ \mathrm{d}. \tag{9.44}$$

It is a derivation as d is a derivation and h^* is an endomorphism for the exterior algebra of forms.

9.9. Basic forms.

Suppose we have a representation $\rho : G \to F$ of G onto a vector space F and consider the differential form with values in F, $\Omega(P, F) \ni \omega$. Among these forms, horizontal forms are the one satisfying

$$h^* \omega = \omega. \tag{9.45}$$

Invariant forms are the one satisfying

$$r_g^* \omega = \rho(g^{-1}) \cdot \omega. \tag{9.46}$$

A form which is at the same time horizontal and invariant will be called basic, their space will be denoted $\Omega_{\text{Bas}}(P,F)$. Basic forms are related to form with values in the space of sections of the associated bundle $P\times_G F$ through the isomorphism

$$\Omega_{\text{Bas}}(P,F) \simeq \Omega(M, P\times_G F) \tag{9.47}$$

where $\Omega(M,P\times_G F) = \Omega(M,\Gamma(P\times_G F)) = \Omega(M)\otimes\Gamma(P\times_G F)$. We will refer to elements of $\Omega(M,P\times_G F)$ as forms-sections.

Proof : Let $\overline{\omega} = \overline{\omega}^a \otimes f_a \in \Omega_{\text{Bas}}(P,F)$. We cover M by local patches $\{U_\alpha\}$ such that we have a local section s_α on each patch. We define $\omega \in \Omega(M,P\times_G F)$ by

$$\omega(x) = \left(s_\alpha^*\overline{\omega}^a\right)(x)\otimes[s_\alpha(x),f_a] \text{ if } x\in U_\alpha, \tag{9.48}$$

where $[s_\alpha(x),f_a]$ denotes the equivalence class defining an element of $P\times_G F$ over x. We show that this definition does not depends on the chosen section s_α, which will at the same time show that ω is single-valued on overlap $U\alpha\cap U_\beta$. Let $\tilde{s}_\alpha$ be another local section defined on U_α. Then there exists a map $g_\alpha : M\to G$ such that $\tilde{s}_\alpha = s_\alpha\cdot g_\alpha$. Using (9.11),

$$\begin{aligned}&\left((s_\alpha\cdot g_\alpha(x))^*\overline{\omega}^a\right)(x)\otimes[s_\alpha(x)\cdot g_\alpha(x),f_a] = \\ &\qquad\left((s_\alpha^* r_{g_\alpha}^* + g_\alpha^* r_{s_\alpha}^*)\overline{\omega}^a\right)(x)\otimes[s_\alpha(x)g_\alpha(x),f_a].\end{aligned} \tag{9.49}$$

Because $\overline{\omega}$ is horizontal, $r_{s_\alpha}^*\overline{\omega}=0$. Using invariance of $\overline{\omega}$, we have

$$\left(\tilde{s}_\alpha^*\overline{\omega}^a\right)(x)\otimes[\tilde{s}_\alpha(x),f_a] = \left(s_\alpha^*\overline{\omega}^a\right)(x)\otimes[s_\alpha(x)g_\alpha,\rho(g_\alpha^{-1})^b{}_a f_b], \tag{9.50}$$

which is what was needed to be shown.
Conversely suppose given $\omega\in\Omega(M,P\times_G F)$. Then we define $\overline{\omega}(z)\in\Omega_{\text{Bas}}(P,F)$ by the formula

$$\overline{\omega}^a(z)\otimes f_a = \pi_z^*\omega^a(\pi(z))\otimes[z,f_a], \tag{9.51}$$

where π_z^* is the pullback $T_\pi^*(z)M\to T_z^*P$. This form is horizontal because of π^*. Its invariance follows form the equalities $\pi_{zg}^* = r_g^*\pi_z^*$ and $[z,f_a] = [zg,\rho(g^{-1})^b{}_a f_b]$.

9.10. Covariant derivatives of basic forms.

Let $\omega\in\Omega_{\text{Bas}}(P,F)$ be a basic form, ρ the action of G on F, Φ a connection and ∇ the covariant derivative. Then

$$\nabla\omega = \mathrm{d}\omega + \rho_*(\tilde{\Phi})\wedge\omega. \tag{9.52}$$

We show this equality following [KolMicSlo]. We calculate what gives the two sides of the equality when applied to k vectors $(X_1,\ldots,X_k)$ supposing ω is a differential form of degree $k-1$. We separate in two cases: when all vectors are horizontal and when at least one vector is vertical. In the first case,

$$\nabla\omega(X_1,\ldots X_k) = h^*\mathrm{d}\omega(X_1,\ldots,X_k), \tag{9.53}$$

$$= \mathrm{d}\omega(X_1\ldots,X_k), \tag{9.54}$$

$$= \mathrm{d}\omega(X_1\ldots,X_k) + \rho_*(\tilde{\Phi})\wedge\omega(X_1\ldots,X_k), \tag{9.55}$$

since $\tilde{\Phi}$ applied on any horizontal vector gives 0. The second case is equivalent to applying ι_X to both side of the equation, where X is a vertical vector and ι the insertion operator. Because X is vertical, and both ω and $\nabla\omega$ are horizontal,

$$\iota_X \omega = \iota_X \nabla \omega = \iota_X h^* \mathrm{d}\omega = 0. \tag{9.56}$$

Using Cartan's formula, we have

$$\iota_X \mathrm{d}\omega = (\iota_x \mathrm{d} + \mathrm{d}\iota_X)\omega = \mathcal{L}_X \omega \tag{9.57}$$

We can write $X = r_e^* a$ for some $a \in \mathfrak{g}$. So

$$\mathcal{L}_X \omega_z = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \Phi_{-t}^{X*} \omega_{\Phi_t^X(z)}, \tag{9.58}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} r^*_{e^{-ta}} \omega_{ze^{ta}}, \tag{9.59}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \rho(e^{ta}) \omega_{ze^{ta}}, \tag{9.60}$$

$$= \rho_*(a)\omega_z, \tag{9.61}$$

$$= \rho_*(\tilde{\Phi}(X))\omega_z. \tag{9.62}$$

Hence

$$h^* \mathrm{d}\omega(X, \ldots) = \iota_X h^* \mathrm{d}\omega(\ldots) = \tag{9.63}$$

9.11. Covariant exterior derivatives in associated bundles.

The isomorphism between basic forms and and forms-sections, together with the formula for covariant derivative of basic forms, lead to a natural definition for covariant derivative of forms-sections. Let $\omega = \omega^a s_a$ be a form-section. Using the notation of the preceding paragraph, e.g. $s_a = [s_\alpha, f_a]$, we perform the following transformations

$$\Omega(M, P \times_G F) \ni \omega^a \otimes [s_\alpha, f_a] \mapsto \pi^*_{s_\alpha} \omega^a \otimes f_a \in \Omega_{\mathrm{Bas}}(P, F), \tag{9.64}$$

$$\pi^*_{s_\alpha} \omega^a \otimes f_a \mapsto \nabla(\pi^*_{s_\alpha} \omega^a \otimes f_a) = \mathrm{d}\pi^*_{s_\alpha} \omega^a \otimes f_a + (-1)^k \pi^*_{s_\alpha} \omega^a \wedge \rho_* \tilde{\Phi} f_a, \tag{9.65}$$

where k is the degree of ω^a (assuming it is of homogeneous degree) and the $(-1)^k$ comes from the inversion of $\tilde{\Phi}$ and ω^a. We continue

$$\begin{aligned} &\mathrm{d}\pi^*_{s_\alpha} \omega^a \otimes f_a + (-1)^k \pi^*_{s_\alpha} \omega^a \wedge \rho_* \tilde{\Phi} f_a, \\ &\mapsto s_\alpha^* \mathrm{d}\pi^*_{s_\alpha} \omega^a \otimes [s_\alpha, f_a] + (-1)^k s^*_\alpha \pi^*_{s_\alpha} \omega^a \wedge [s_\alpha, \rho_* s^*_\alpha \tilde{\Phi} f_a]. \end{aligned} \tag{9.66}$$

The operator d commutes with pullbacks, $s^*_\alpha \pi^*_{s_\alpha}$ =Id and s^*_α is A_α by definition. Hence, after simplification, we obtain our definition for covariant derivative of forms-sections:

$$\nabla\omega \doteq \mathrm{d}\omega + \rho_* A_\alpha \wedge \omega, \qquad \omega \in \Omega(U_\alpha, P \times_G F) \tag{9.67}$$

This last formula is widely used in physics, because what we call forms-sections are usually used to describe matter fields. It also explains why the connection expressed in its local form A_α is preferred by physicists (rather than the global Φ).

9.12. The covariant derivative is a derivation.

It is direct to show from (9.67) that the covariant derivative fulfill a derivative property. If $\omega \in \Omega(M)$ is a k-form and $\sigma \in \Omega(M, P \times_G F)$ is a p-form-section, then $\omega \wedge \sigma$ is a $k+p$-form-section and

$$\nabla(\omega \wedge \sigma) = \mathrm{d}\omega \wedge \sigma + (-1)^k \omega \wedge \nabla\sigma. \tag{9.68}$$

9.13. Matrix form of the connection.

In this part we assume that the representation ρ of G on F is transitive. Let $\{e_a\}$ be a local basis of E, i.e. $\{e_a)(x)\}$ is a basis of the fiber E_x for all x in some open subset of M. Then we can write e_a in the form

$$e_a(x) = [s_\alpha(x), f_a], \tag{9.69}$$

for some local section s_α of P. The preceding development shows that

$$\nabla e_a = (A_\alpha)^b{}_a e_b, \tag{9.70}$$

where $(A_\alpha)^b{}_a$ is the matrix form of $\rho_*(A_\alpha)$. If $\sigma \in \Omega(M, E)$ is any local form-section, then we can decompose $\sigma = \sigma^a e_a$ with $\sigma^a \in \Omega(M)$ and by the derivation property of the covariant derivative

$$\nabla\sigma = (\mathrm{d}\sigma^a)e_a + \sigma^a(\nabla e_a), \tag{9.71}$$

$$= \left(\mathrm{d}\sigma^a + (A_\alpha)^a{}_b\sigma^b\right) e_a. \tag{9.72}$$

We can write down the form indices as well, in which case we have, for 0- and 1-forms-sections,

$$\nabla_\mu \sigma^a = \partial_\mu \sigma^a + A^a{}_{b\mu}\sigma^b, \tag{9.73}$$

$$(\nabla\sigma^a)_{\mu\nu} = \partial_\mu \sigma^a{}_\nu - \partial_\nu \sigma^a_\mu + A^a_{b\mu}\sigma^b{}_\nu - A^a{}_{b\nu}\sigma^b{}_\mu, \tag{9.74}$$

We have introduced a convention that we will keep throughout this manuscript: matrix indices are written before form indices. We also recall the convention that for a 2-form ω,

$$\omega = \frac{1}{2}\omega_{\mu\nu} dx^\mu \wedge dx^\nu. \tag{9.75}$$

10. Curvature

10.1. Definition.

For any fiber bundle (P, M, π), the vertical distribution is an integrable distribution, because of the formula, for $X, Y \in \Gamma(VP)$,

$$\pi^*([X,Y]) = [\pi^*(x), \pi^*(Y)]. \tag{10.1}$$

However the horizontal distribution is not necessarily integrable; the vertical part of the Lie bracket of two horizontal vector fields is correspond to the curvature of the distribution. More precisely, we define the curvature two form R at $x \in W_\alpha \subset P$, for two local vector fields $x, Y \in \Gamma(TW_\alpha)$ defined around x, by

$$R_x(X,Y) = \Phi([h(X), h(Y)])_x. \tag{10.2}$$

Intuitively, this definition of the curvature can be understood in the following way. Pick two horizontal direction, say west and north, and let's imagine the vertical direction is the altitude. If there is non a vanishing curvature, making first an infinitesimal step to the west then an infinitesimal step to the north versus making first an infinitesimal step to the north and then an infinitesimal step to the west (this would represent a Lie bracket $[X_{\text{West}}, Y_{\text{North}}]$) will lead you to the same location but not at the same altitude. This difference of altitude is the curvature and it is precisely the quantity expressed by (10.2). A short calculation shows that the definition (10.2) is equivalent to

$$R = h^* \mathrm{d}\Phi. \tag{10.3}$$

10.2. Standard Formula for curvature.

We assume (P, M, π, G) is a principal G-bundle and the connection one-form Φ is a principal connection. We show that our definition is equivalent to the standard curvature's formula

$$R = \mathrm{d}\Phi + \frac{1}{2}[\Phi \wedge \Phi]. \tag{10.4}$$

The notation $[\wedge]$ denotes the product obtained from the tensor product of the two algebras $\Omega(M)$ and the Lie algebra of vector fields, recall ref(EARLIER). The proof of (10.4) is given in [SpinJMF] and consists in deriving the equality

$$h^* \mathrm{d}\Phi(X, Y) = \mathrm{d}\Phi(X, Y) + \frac{1}{2}\Phi(X) \wedge \Phi(Y), \tag{10.5}$$

in three cases:

i) X and Y both horizontal,
ii) X vertical and Y horizontal,
iii) X and Y both vertical.

In case i), both sides of (10.5) are equal to

$$\mathrm{d}\Phi(X, Y). \tag{10.6}$$

In case ii), the left-hand-side of (10.5) vanishes whereas its right-hand-side equals to

$$\mathrm{d}\Phi(X, Y) = X(\Phi(Y)) - Y(\Phi(X)) + \Phi([X, Y]). \tag{10.7}$$

$X(\Phi(Y))$ and $Y(\Phi(Y))$ both vanishes because Φ is constant on both the horizontal and vertical subspaces: it is either the Identity or the 0 map there. $[X, Y]$ is horizontal because of the G-equivariance of the connection

$$[X, Y] = \mathcal{L}_X(Y) = \frac{d}{dt}|_{t=0}\Phi^X_{t*}(Y) = \frac{d}{dt}|_{t=0} r_{e^{ta}*} Y, \tag{10.8}$$

where we have assumed that X is a fundamental vector field generated by $a \in \mathfrak{g}$. The push-forward of the right action of G on P sends horizontal vectors to horizontal vectors, hence the right hand side of (10.5) vanishes as well in case ii). In case iii), the left-hand-side of (10.5) vanishes, and the right-hand-side is equal to

$$-\Phi([X, Y]) + \Phi \wedge \Phi(X, Y), \tag{10.9}$$

where $X\Phi(Y) = Y\Phi(X) = 0$ for the same reason as in case ii). The Lie bracket of two vertical vector fields is vertical hence

$$\Phi([X,Y]) = [X,Y] \tag{10.10}$$

The definition of the wedge product implies

$$\Phi \wedge \Phi([X,Y]) = [\Phi(X), \Phi(Y)], \tag{10.11}$$

which is equal to $[X,Y]$ as well as X and Y are vertical, finishing the proof.

10.3. Local curvature.

We gives the formula for the curvature in terms of the local form for the connection A_α. So let (P, M, π, G) be a principal fiber bundle witch connection Φ, a local section $s_\alpha : U_\alpha \to M$ defined on a an open $U_\alpha \in M$ and A_α the "physicist's" connection. Applying $s_\alpha^* r_{s_\alpha *}|_e^{-1}$ to both sides of equation (10.4), we obtain a formula expressing the curvature in terms of A_α

$$F_\alpha = \mathrm{d}A_\alpha + \frac{1}{2}[A_\alpha \wedge A_\alpha]. \tag{10.12}$$

where we have defined

$$F_\alpha \doteq s_\alpha^* r_{s_\alpha *}|_e^{-1} R. \tag{10.13}$$

It this version of the curvature that we will mostly use. In the case G and its Lie algebra $\mathfrak{g}$ are a matrix group and a matrix algebra respectively, and writing explicitly the matrix and form indices, and forgetting the local section index α, the last equation becomes

$$F^a{}_{b\mu\nu} = \partial_\mu A^a{}_{b\nu} - \partial_\nu A^a{}_{b\mu} + \frac{1}{2}\left(A^a{}_{c\mu} A^c{}_{b\nu} - A^a{}_{c\nu} A^c{}_{b\mu}\right) \tag{10.14}$$

10.4. Equivariance of the curvature.

The curvature satisfies the same equivariance property as the connection

$$r_g^* R = r_{g*} R, \qquad \forall g \in G. \tag{10.15}$$

Indeed, equivariance of Φ implies equivariance of h, hence h^* commutes with r_g^*. d commutes with pullbacks hence commutes with r_g^*. Using these two properties and equivariance of Φ in $R = h^* \mathrm{d}\Phi$ shows the result. Express in term of the $\mathfrak{g}$-valued connection

$$\tilde{R}_z = r_{z*}|_e^{-1} R = h^* \mathrm{d}\tilde{\Phi}, \tag{10.16}$$

this property is written

$$r_g^* \tilde{R} = \mathrm{Ad}_{g^{-1}} \tilde{R}, \tag{10.17}$$

showing that the curvature $\tilde{R}$ is a basic form $\tilde{R} \in \Omega_{\mathrm{Bas}}(P, \mathfrak{g})$. Hence F_α is the expression on local patch of section 2-form of the associated bundle $P \times_{\mathrm{Ad}_G} \mathfrak{g}$.

10.5. Bianchi Identity.

The Bianchi identity expresses the fact that the covariant derivative of the curvature vanishes

$$h^* \mathrm{d}R = 0. \tag{10.18}$$

This is shown by direct calculation using (10.3). Because $\tilde{R}$ is a basic form, formula (9.52) directly shows that

$$\mathrm{d}\tilde{R} + [\tilde{\Phi} \wedge \tilde{R}] = 0. \tag{10.19}$$

Similarly, for F_α,

$$\mathrm{d}\tilde{F}_\alpha + [A_\alpha \wedge F_\alpha] = 0. \tag{10.20}$$

10.6. Covariant derivative and curvature.

Let (P, M, π, G) be a principal bundle and $E = P \times_G F$ an associated bundle with $\rho : G \to GL_n(F)$ the defining representation. Let A_α be a connection on P and ∇ the associated covariant derivative on E. Let $\omega \in \Omega(U_\alpha, E)$ be a (locally defined) form-section. Using (9.67) two times we have

$$\nabla\nabla\omega = \mathrm{d}^2\omega + \rho_*(A_\alpha) \wedge \mathrm{d}\omega + \mathrm{d}\rho_*(A_\alpha) \wedge \omega \tag{10.21}$$

$$-\rho_*(A_\alpha) \wedge \mathrm{d}\omega + \rho_*(A_\alpha) \wedge \rho_*(A_\alpha) \wedge \omega \tag{10.22}$$

$$= \rho_*(\mathrm{d}A_\alpha + \frac{1}{2}[A_\alpha \wedge A_\alpha]) \wedge \omega, \tag{10.23}$$

hence,

$$\nabla\nabla\omega = \rho_*(F_\alpha) \wedge \omega. \tag{10.24}$$

Another link between the curvature and the covariant derivative can be expressed through the operator

$$\nabla_X \doteq \iota_X \nabla, \tag{10.25}$$

where X is a vector field on M. Using the derivation property of ι_X and d, two vector fields X, Y on M and a form-section ω, a computation of $\nabla_X \nabla_Y \omega$ shows that

$$\iota_Y \iota_Y F_\alpha = [\nabla_X, \nabla_Y] - \nabla_[X, Y]. \tag{10.26}$$

10.7. Connection and curvature for super-manifolds. We end this section by briefly presenting how the theory of connection and curvature extends to the case of super-manifold. We follow [Sta98], and we refer to it for further details. However here we call "super" what the author of the aforementioned article calls "graded". Let $(M, \mathcal{O})$ be a super-manifold. A super-distribution is a subsheaf $\mathcal{D}$ of **Der**$(\mathcal{O})$ of constant graded dimension i.e. the dimension of the even and odd part of $\mathcal{D}$ are separately constant. The distribution is called regular if the dimension of the even and odd part of the stalk $\mathcal{D}_x$, $x \in M$ are also separately constant. In [Sta98] is shown

THEOREM 13. *Let $(P, \mathcal{O}_P)$ be a super-principal bundle over the super manifold $(M, \mathcal{O}_M)$ with structure group $G, \mathcal{O}_G$. The action of $(G, \mathcal{O}_G)$ on $(P, \mathcal{O}_P)$ induces a regular super-distribution of vertical derivations that we denote* ***Ver****.*

A graded connection is then defined as a regular distribution $\mathbf{Hor} \subset \mathbf{Der}(\mathcal{O}_P)$ of (even and odd) dimensions equal to the ones of $M, \mathcal{O}_M$ such that

1) $\mathbf{Hor} \oplus \mathbf{Ver} = \mathbf{Der}(\mathcal{O}_P)$,
2) **Hor** is $(G, \mathcal{O}_G)$ invariant,

where we recall that the action of $(G, \mathcal{O}_G)$ on **Hor** is given by the comodule structure of $\mathcal{O}_P$. Later it is shown that such a distribution can be equivalently described by a one-form ω with values in the super-Lie algebra $\mathfrak{g}$ of G of total $\mathbb{Z}_2$ degree 0,

$$\omega = \omega^a_{(0)} X_{a(0)} + \omega^\alpha_1 X_{\alpha(1)}, \tag{10.27}$$

where $\omega^a_{(0)}$ (*resp.* $\omega^\alpha_{(1)}$) is a purely even (*resp. odd*) 1-form and $X_{a(0)}$ (*resp.* $X_{\alpha(1)}$) is a generator of the even (*resp. odd*) part of $\mathfrak{g}$. From ω, it is possible to define a super-covariant derivative and a super-curvature exactly as in formulas (9.67) and (10.12).

11. Riemannian Geometry

11.1. Riemannian Manifold.

Stricto sensu, a Riemannian manifold is an orientable manifold M together with a Riemannian metric. However, in this work we will usually call "Riemannian manifolds" orientable manifolds with pseudo metrics. We recall some notation regarding Riemannian manifolds and introduce some new ones. First the metric is denoted

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu. \tag{11.1}$$

However, when expressed in a local orthonormal basis $\{e^a\}$ we denote the coefficients by η_{ab}

$$g = \eta_{ab} e^a \otimes e^b \tag{11.2}$$

We shall always use greek indices to refer to coefficients expressed in a natural frame while latin indices refer to coefficients expressed in an orthonormal frame. The orthonormal dual basis e^a is called vielbein, as well a the matrix e^a_μ relating it to the natural basis dx^μ. E^a_μ denotes the inverse matrix of e^a_μ. If we summarize

$$e^a = e^a_\mu dx^\mu, \qquad E_a = E^\mu_a \partial_\mu, \tag{11.3}$$

$$E^\mu_a e^a_\nu = \delta^\mu_\nu, \qquad E^\mu_a e^b_\mu = \delta^a_b, \tag{11.4}$$

$$\eta_{ab} e^a_\mu e^b_\nu = g_{\mu\nu}, \qquad g_{\mu\nu} E^\mu_a E^\nu_b = \eta_{ab}. \tag{11.5}$$

The orthonormal tangent vectors E_a as well as the coefficients E^μ_a are also called "inverse vielbein". There is also an inverse metric g^{-1}, with upper indices.

$$g^{-1} = g^{\mu\nu} \partial_\mu \otimes \partial_\nu = \eta^{ab} E_a \otimes E_b. \tag{11.6}$$

We don't write the power $^{-1}$ for the coefficient, hence the reader should remember that a $g^{\mu\nu}$ (or η^{ab}) with upper indices refers to the inverse metric.

11.2. Connections for manifolds.

Let M be a manifold. We will call a connection on M a connection on LM, the frame bundle of M. This kind of connection gives rise to a covariant derivative on TM. Let us recall that for any vector field X, and any point $x \in M$, ∇X is an element of $T_xM \otimes T^*_xM$. Thus we can expand it in the natural basis $\{\partial_\nu \otimes dx^\mu\}$

$$\nabla X = (\nabla X)^\nu{}_\mu \quad \partial_\nu \otimes dx^\mu. \tag{11.7}$$

In particular, applying this formula to the vector field ∂_ρ itself

$$(\nabla \partial_\rho) = (\nabla \partial_\rho)^\nu{}_\mu \quad \partial_\nu \otimes dx^\mu \doteq \Gamma^\nu{}_{\rho\mu} \quad \partial_\nu \otimes dx^\mu \tag{11.8}$$

In equation (11.8), we have define the coefficient of the linear connection $\Gamma^\nu_{\rho\mu}$ expressed in a natural basis. It is important for the reader to remember that whenever he sees the connection expressed with greek indices and denoted by the letter Γ, we refer to formula (11.8). It may also be important to remember that the convention we adopt is that the first two indices are the matrix indices while the last one is the form index.

11.3. The spin connection.

In the Riemannian case, we have access to the orthonormal frame E_a, hence we can express the matrix coefficients of the connection in this basis. Doing an analysis similar to the one of the preceding paragraph, but using the orthonormal frame as basis of the tangent space (but keeping the natural frame as basis of the cotangent space), we get

$$\nabla E_a = (\nabla E_a)^b{}_\mu \quad E_b \otimes dx^\mu = \omega^b{}_{a\mu} \quad E_b \otimes dx^\mu \tag{11.9}$$

Expressed in this basis, we always denotes the connection-form $\omega^a{}_{b\mu}$. $\omega^a{}_{b\mu}$ is sometimes called the spin connection. The relation between $\Gamma^\nu{}_{\rho\mu}$ and $\omega^a{}_{b\mu}$ is obtained by expending $E_a = E^\nu_a \partial_\nu$ in (11.9). Using the derivation property of the covariant derivative

$$\nabla(E^\nu_a \partial_\nu) = \mathrm{d}E^\nu_a \partial_n u + E^\nu_a \nabla \partial_\nu, \tag{11.10}$$

we arrive at

$$\omega^a{}_{b\mu} E^\nu_a = \partial_\mu E^\nu_b + E^\rho_b \Gamma^\nu{}_{\rho\mu}. \tag{11.11}$$

This equation is sometimes called "vielbein postulate" in the literature.

11.4. Connection for Riemannian manifolds.

Let M be a Riemannian manifold, with metric g. and connection form A. We recall that the metric induces of the principal frame bundle LM to the (pseudo-)orthonormal frame bundle $SO(M)$. The connection A is called "compatible with the metric" if it define a connection for the reduced bundle $SO(M)$, in other words, if A takes values in the Lie subalgebra $\mathfrak{so}(p,q) \subset \mathfrak{gl}(p+q)$. In term of $\omega^a{}_b$, the fact that the image of the connection is inside $\mathfrak{so}(p,q)$ can be written

$$\eta_{ac}\omega^c{}_b + \eta_{bc}\omega^c{}_a = 0 \tag{11.12}$$

Because η is constant, this last relation immediately implies that

$$\nabla g = \mathrm{d}\eta_{ab} + \omega^c{}_a \eta_{ca} + \omega^c{}_b \eta_{ac} = 0 \tag{11.13}$$

But this relation must hold in any basis, hence we have

$$\nabla_\mu g_{\nu\lambda} = \partial_\mu g_{\nu\lambda} - \Gamma^\rho{}_{\sigma\mu} g_{\rho\lambda} - \Gamma^\rho{}_{\lambda\mu} g_{\sigma\rho} = 0. \tag{11.14}$$

11.5. Torsion.

A particularity of the tangent space in the whole family of tangent bundle, is that the frontier between form-sections and tensors is not so clear. In fact, the same object can be seen through different angle, but the way we look at it is important when we compute its covariant derivative. Among those objects, the identity of the tangent space

$$\mathrm{Id} : TM \to TM, \tag{11.15}$$

can be seen as a 1-form with values in TM, i.e. an object of $\Omega(M, TM)$. When so considered, it is called the soldering form. We can write it

$$\underline{e} = e^a \otimes E_a. \tag{11.16}$$

The covariant derivative of $\underline{e}$ need not to vanish; in fact this covariant derivative is called the torsion of the connection and is denoted T

$$T = \nabla \underline{e} = T^a{}_{\mu\nu} dx^\mu \wedge dx^\nu \otimes E_a, \tag{11.17}$$

$$T^a{}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^a{}_{b\mu} e^b_\nu - \omega^a{}_{b\nu} e^b_\mu\right). \tag{11.18}$$

11.6. Levi-Civita Connection.

THEOREM 14. *Let M be a Riemannian manifold. There exist on M a unique metric compatible connection whose torsion vanishes at every point, called the Levi-Civita connection.*

In term of the metric, the Levi-Civita connection is given by

$$\Gamma^\mu{}_{\nu\lambda} = \frac{1}{2} g^{\mu\rho}\left(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\rho\nu} - \partial_\rho g_{\nu\lambda}\right) \tag{11.19}$$

11.7. Riemann, Ricci and scalar curvature.

The Riemann curvature is by definition the curvature of the Levi-Civita connection. We denote it

$$R = \frac{1}{2} R^\mu_{\nu\lambda\rho} \partial_\mu \otimes dx^\nu \otimes dx^\lambda \wedge dx^\rho. \tag{11.20}$$

The matrix coefficient of the Riemann curvature can also be expressed in the orthonormal frame, in which case we write them $R^a{}_{b\lambda\rho}$ or even

$$R^{ab}{}_{\lambda\rho} \doteq \eta^{bc} R^a{}_{c\lambda\rho}. \tag{11.21}$$

Because the $^\mu$ and the $_\lambda$ of (11.20) are dual to each other, we can pair them, obtaining a trace leading to the Ricci tensor

$$\mathrm{Ric}_{\nu\rho} = R^\mu{}_{\nu\mu\rho}, \tag{11.22}$$

where we have used Einstein's summation convention. Finally, the scalar curvature is

$$\mathcal{R} = g^{\mu\nu} \mathrm{Ric}_{\mu\nu}. \tag{11.23}$$

In some physical models, we shall sometimes also use an arbitrary SO(p,q) connection. We will still write it Γ or ${\Gamma^{\mu}}_{\nu\lambda}$, and still call the tensors defined by equations (11.20), (11.22) and (11.23) using this arbitrary Γ in the right-hand side the Riemann, Ricci and scalar curvature.

12. Lie derivatives of spinor fields

12.1. The strategy.

A natural definition for Lie derivative of spinors fields would have been to see them as section of a bundle associated to the Clifford bundle, if we had a natural definition for Lie derivative of sections of this Clifford bundle. However we have seen this is not the case, hence we need another idea. The method we will use is to see the spinor bundle as a bundle associated to the spin bundle, i.e. the principal bundle whose structure group is the spin group. The spin bundle however is not a natural bundle, in the sense it is not a sub-bundle of a tensor bundle. But the (pseudo)-orthogonal bundle is; and the 2-to-1 covering $Spin \to SO$ is sufficiently simple, so that we can "lift" the definition of Lie derivatives on $SO(M)$ to $Spin(M)$. Once we have a definition of Lie derivatives on $Spin(M)$, we have one for the spinor-bundle by the associated bundle construction.

12.2. Projectable vector fields.

Let (F,M,π) be a fiber bundle. A vector field $X : F \to TF$ is vector field satisfying $\pi_* X_{z_1} = \pi_* X_{z_2}$ whenever $\pi(z_1) = \pi(z_2)$. In that case, denoting $\xi = \pi_* X$ ($\pi_* X$ is taken at any point of the fiber), we say that X is projectable over ξ.

12.3. The vertical vector bundle.

Let (E,M,π) be a vector bundle. Then the vertical bundle VE is isomorphic to $E\times_M E$,i.e. the Whitney sum of E with itself, the canonical isomorphism v_E being given by

$$v_E : E\times_M E \to VE \tag{12.1}$$

$$(x,u_x,v_x) \mapsto \frac{d}{dt}|_{t=0}(x,u_x+tv_x) \tag{12.2}$$

12.4. Lie derivatives for general vector bundles.

Let (E,M,π) be a vector bundle and $\sigma : M \to E$ a section. Let $X : M \to TM$ be a vector field and $\xi : E \to TE$ a projectable vector field over M. Then the expression $\sigma_* X - \xi\circ\sigma$ define a section of the vertical bundle VE. Indeed, $\pi_*\sigma_* = \mathrm{Id}$ and $\pi_*\xi = X$ at any point of E, showing that $\pi_*(\sigma_* X - \xi\circ\sigma) = 0$. Using the preceding observation, we have, for all $x \in M$,

$$\sigma_* X - \xi\circ\sigma(x) = (\sigma(x), \mathcal{L}_{\xi,X}(\sigma)(x)) \in E\times_M E_x. \tag{12.3}$$

$\mathcal{L}_{\xi,X}(\sigma)(x)$ is the generalized Lie derivative of σ with respect to (ξ,X). Hence the generalization needs the additional data of a projectable vector field over X.

12.5. Recovering the standard definition.

When the vector bundle is $E = TM$ and the section σ is a vector field Y, we have a natural projectable vector field $X_* \in TTM$ over X. We then recover that $\mathcal{L}_X(Y)$ is the second component of $Y_*X - X_*Y$ in the decomposition (12.3). The same remark applies for Lie derivatives of one forms (where the projectable vector field is X^*) or Lie derivatives general tensors.

12.6. General Lie derivatives.

We now give the general definition of the Lie derivative of any map $f : M \to N$ between two manifolds, with respects to two vector fields $X : M \to TM$, $\xi : N \to TN$,

$$\tilde{L}_{\xi,X}(f) = f_*X - \xi \circ f. \tag{12.4}$$

12.7. Kosmann Lift.

Let M be a manifold, and $X : M \to TM$ a vector field. There is a natural lift of X to the frame bundle LM defined as follows. Let Φ_t^X the flow of X and $(x;(e_1,\ldots,e_m))$ a point of LM, i.e. $\{e_a\}_{a=1}^m$ is a local basis of T_xM, where $m = \dim(M)$. We can extend Φ_t^X to $\tilde{\Phi}_t^X$ by

$$\tilde{\Phi}_t^X((x;(e_1,\ldots,e_m))) \doteq (\Phi_t^X(x);(\Phi_{t*}^X e_1,\ldots,\Phi_{t*}^X e_m)), \tag{12.5}$$

which define a one-parameter group of diffeomorphism of LM. Differentiating with respect $\tilde{\Phi}_t^X$ to t at $t = 0$, we obtain a vector field $\tilde{X} : LM \to TLM$. In local coordinates,

$$\tilde{X}(x;\{e_a\}) = [(x;\{e_a\});(X(x);e_a(X^b)\frac{\partial}{\partial {X^a}_b})]. \tag{12.6}$$

This natural lift is invariant under the action of $GL(m)$. From it, it is quite simple to construct a natural orthonormal lift, called Kosmann lift,

$$X_K(x;\{e_a\}) = (X(x), \frac{1}{2}(\eta^{ac}e_c(X^b) - \eta^{bc}e_c(X^a))J_{ab}) \tag{12.7}$$

where

$$J_{ab} = \frac{1}{2}(\eta_{bc}\frac{\partial}{\partial {X^a}_c} - \eta_{ac}\frac{\partial}{\partial {X^b}_c}) \tag{12.8}$$

denote a set of generators of $\mathfrak{so}(p,q)$. An important point is that the Kosmann lift is $SO(p,q)$-invariant. We recall that a vector field $X : P \to TP$ on a principal bundle P with structure group G is G-invariant if

$$\forall (z,g) \in P \times G, \quad r_{g*}X(z) = X(zg). \tag{12.9}$$

12.8. Covering projection.

We recall that, given two topological space A,B, a covering projection is a surjective map $p : B \to A$ such that, for all $a \in A$, there exists an open subset $U_a \subset A$ and a discrete space D (not depending on a) such that

$$p^{-1}(U_a) = \bigsqcup_{d\in D} V_{d,a}, \tag{12.10}$$

where each $V_{d,a}$ is homeomorphic to a. When such p exists, B is called a covering space for A. As a matter of facts, a covering space of a manifold is a manifold, a covering space of a Lie group is a Lie group.

12.9. Covering lift of vector fields.

Let M be a manifold, G a Lie group and $p : \tilde{G} \to G$ a covering projection such that Ker(p) is normal in $\tilde{G}$ (equivalently, ker(p) is contained in the center of $\tilde{G}$). Let (P, M, G, π), $(\tilde{P}, M, \tilde{G}, \tilde{\pi})$ be principal bundle over M such that we have a map $\underline{p} : \tilde{P} \to P$ satisfying

$$\forall \tilde{z} \in \tilde{P}, \forall \tilde{g} \in \tilde{G}, \underline{p}(\tilde{z} \cdot \tilde{g}) = \underline{p}(\tilde{z}) \cdot p(\tilde{g}). \tag{12.11}$$

In this set-up, it holds that any G-invariant vector field of P is projectable over a vector field of M, and can be lifted uniquely to a $\tilde{G}$-invariant vector field of $\tilde{P}$. In particular, if we consider the principal spin bundle $Spin(M)$, covering the orthonormal bundle $SO(M)$ as in (12.11), then the Kosmann lift (12.7) can be lifted again to a $Spin(p,q)$-invariant vector field, that we still call Kosmann lift. From the practical point of view, this Kosmann lift to the spin manifold takes the same form as (12.7), but with the generators J_{ab} being now generators of the $\mathfrak{spin}(p,q)$ Lie algebra, typically represented by $\frac{1}{2}\gamma_{ab}$, the γ_a's being the generators of the Clifford algebra.

12.10. Lie derivative of spinor fields.

If (P, M, π, G) is a principal vector bundle and $E = P \times_G F$ an associated vector bundle, then a G-invariant vector field X_P defines a vector field X_E on E through

$$X_E(e) = [(z, X_P(z)), (e, 0)], \tag{12.12}$$

where [,] denote the equivalence class defining TE as the associated bundle $TE = TP \times_{TG} TF$, and reciprocally. Hence if SM denotes a spinor bundle $\psi : M \to SM$ a spinor field and X a vector field on M, we obtain a vector field $\overline{X_K}(\psi(x))$ projecting on $X(x)$, using the Kosmann lift X_K for X_P in (12.12). Applying (12.3), we obtain

$$\mathcal{L}_X(\psi)(x) \equiv \mathcal{L}_{\overline{X_K},X}(\psi)(x) = X^a e_a(\psi) + \frac{1}{8}(\eta^{ac} e_c(X^b) - \eta^{bc} e_c(X^a))\gamma_{ab}\psi. \tag{12.13}$$

13. Characteristic classes

This whole section is a resumé of [MS74] until the paragraph "Chern-Weil Homomorphism". Then we use [Nak91].

13.1. Thom isomorphism theorem.

It is also possible to define orientation in term of cohomology rather than homology. Exactly like homology, we have that $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}) = \mathbb{Z}$, and an orientation is given by the continuous choice of a generator u_F for this cohomology group at each fiber F in accordance with local trivializing charts like in the homological definition. For oriented vector bundle we have the Thom isomorphism theorem

THEOREM 15. *Let (E, M, π) be a vector bundle with fiber $\mathbb{R}^n$. Then $H^i(E, E_0; \mathbb{Z}) = 0$ for $i < n$ and $H^n(E, E_0; \mathbb{Z})$ contains one and only one class u such that for any fiber $\mathbb{R}^n \xhookrightarrow{i_F} F \subset E$,*

$$i_F^* u = u_F \in H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}), \tag{13.1}$$

where u_F is the preferred generator at F defining the orientation. Furthermore the application

$$H^k(E;\mathbb{Z}) \ni a \mapsto a \cup u \in H^{k+n}(E,E_0,\mathbb{Z}), \tag{13.2}$$

define an isomorphism $H^k(E;\mathbb{Z}) \simeq H^{k+n}(E,E_0,\mathbb{Z})$.

Finally because $H^k(E,\mathbb{Z}) \simeq H^k(M,\mathbb{Z})$, $\cup u$ leads to an isomorphism between $H^k(M,\mathbb{Z})$ and $H^k(E;\mathbb{Z}) \simeq H^{k+n}(E,E_0,\mathbb{Z})$.

13.2. Euler class.

Keeping the same notations, the inclusion $\iota : (E,) \hookrightarrow (E,E_0)$ gives a morphism $H^k(E,E_0;\mathbb{Z}) \overset{\iota^*}{\hookrightarrow} H^k(E;\mathbb{Z}) \underset{\sim}{\overset{\pi^*}{\longrightarrow}} H^k(M;\mathbb{Z})$. The image of the orientation u just defined under this morphism is called the Euler class, we denote it "e". The Euler class is our first example of a characteristic class: a cohomological class of the base manifold but describing a certain type of vector bundle. The Euler class fulfill some important properties among which we find:

i) The Euler class is natural: If $f : M \to M'$ is a smooth map covered by a morphism of vector bundle $\tilde{f}E \to E'$, then $e(E) = f^* e(E')$.
ii) The Euler class of a Whitney sum is $e(E \oplus E') = e(E) \cup e(E')$.

Here a morphism of vector bundle means a morphism of manifold such that its restriction to any fiber defines an isomorphism $\mathbb{R}^n \to \mathbb{R}^n$. The first property implies that the Euler class of a trivial vector bundle vanishes. Indeed, if (E,M,π) is such a bundle, there is a morphism of vector bundle sending E to $\mathbb{R}^n$ covering a map sending M to a point (and the cohomology of a point is trivial). The second property implies that if (E,M,π) admits a nowhere vanishing section, its Euler class is trivial. Indeed, in that case we can decompose $E = E' \oplus L$ where L is the trivial line bundle defined by this nowhere vanishing section. Then $e(E) = e(E') \cup e(L) = e(E') \cup 0$.

13.3. Complex vector bundle.

By a complex vector bundle, we mean here a vector bundle (A,M,π,F) such that each fiber F is isomorphic the the complex vector space $\mathbb{C}^n$ for some n. The realification $A_{\mathbb{R}}$ of a complex vector bundle A is the real vector bundle with the same total space A but whose fiber are now the real vector spaces $\mathbb{C}^n_{\mathbb{R}} \simeq \mathbb{R}^{2n}$. The complexification $E_{\mathbb{C}}$ of a real vector bundle E is the vector bundle obtained form E whose each fiber have been complexified $F \mapsto F \otimes_{\mathbb{R}} \mathbb{C}$. If (E,M,π) is a real vector bundle, a complex structure on E is a continuous assignment $x \mapsto J_x$, $x \in M$ where $J_x : F_x \to F_x$, F_x the fiber over x, is a real linear map squaring to $-\mathrm{Id}$. A real vector bundle can be turned into a complex one, by declaring on each fiber the action of $\mathbb{C}$ to be

$$(a+ib)\cdot v_x = a \cdot v_x + b \cdot J_x v_x, \quad v_x \in F_x,\ a+ib \in \mathbb{C}. \tag{13.3}$$

If (A,M,π) is a complex vector bundle, its realification admits a canonical complex structure induced by the multiplication by i. A word of cautious: for the real vector bundle (TM,M,π), what we have presented as a complex structure is called an almost complex structure. It is only when this almost complex structure is integrable, i.e. when it allows to see M as a complex manifold, that this almost complex structure is called simply a complex structure. But in this case the complex-real duality which is studied concern the base manifold itself, not the vector bundle, hence the discrepancy in the denomination.

If (A, M, π) is a complex vector bundle, its realification $A_{\mathbb{R}}$ is orientable. Indeed, consider any basis $a_1, \ldots, a_n$ of the fiber $\mathbb{C}^n$. Declaring the basis $a_1, ia_1, a_2, ia_2, \ldots, a_n, ia_n$ of $\mathbb{R}^{2n} \simeq \mathbb{C}^n_{\mathbb{R}}$ to be ordered in this sequence does define an order on $\mathbb{R}^{2n}$. This order does not depends on the chosen basis of $\mathbb{C}_n$ because mainly a permutation of any two a_i's will lead into permuting simultaneously two elements of the induced real basis.

13.4. Grassmann Manifold and universal bundle.

The Grassmann manifold $G_n(\mathbb{R}^{n+k})$ is the set of n-planes in $\mathbb{R}^{n+k}$. It is a direct generalization of the projective space $P\mathbb{R}^n = G_1(\mathbb{R}^{n+1})$, the set of lines in $\mathbb{R}^n$. The Grassmann manifold $G_n(\mathbb{R}^{n+k})$ is a compact manifold of dimension nk. The infinite Grassmann is the direct limit

$$G_n^\infty = \bigcup_{k\in\mathbb{N}^*} G_n(\mathbb{R}^{n+k}), \tag{13.4}$$

($\mathbb{N}^*$ is the set of strictly positive integers), where each $G_n(\mathbb{R}^{n+k})$ is seen as a subset of the bigger $G_n(\mathbb{R}^{n+k'})$, $k \le k'$, thanks to the canonical inclusion $\mathbb{R}^{n+k} \subset \mathbb{R}^{n+k'}$. In other words we have

$$G_n(\mathbb{R}^n) \subset G_n(\mathbb{R}^{n+1}) \subset G_n(\mathbb{R}^{n+2}) \subset \cdots \subset G_n^\infty \tag{13.5}$$

and a set in G_n^∞ is open if and only if its intersection with any $G_n(\mathbb{R}^{n+k})$ is open. G_n^∞ is an infinite dimensional paracompact "manifold" (we will not define infinite dimensional manifold). A paracompact space is an Hausdorff topological space such any cover admits a locally finite refinement. A cover is locally finite if each point meets only finitely many open set of the cover. Over each $G_n(\mathbb{R}^{n+k})$, and over G_n^∞ we define a vector bundle γ_n^{n+k} (or γ_n^∞), with fiber $\mathbb{R}^n$ through

$$\text{point in } \gamma_n^{n+k} = (n-\text{plane in } \mathbb{R}^{n+k},\ \text{point in that } n-\text{plane}). \tag{13.6}$$

The fact that γ_n^{n+k} and γ_n^∞ respect the local triviality condition can be found in []. In a completely analogous way we can $G_n(\mathbb{C}^{n+k})$, $G_{n,\mathbb{C}}^\infty$, $\gamma_{n,\mathbb{C}}^{n+k}$, $\gamma_{n,\mathbb{C}}^\infty$ where $\mathbb{R}$ is everywhere replaced by $\mathbb{C}$. (E.g. $\gamma_{n,\mathbb{C}}^{n+k}$ is the set of ($n-$plane in $\mathbb{C}^{n+k}$, point in that $n-$plane)), etc.)

13.5. Universality of γ_n^∞.

The bundle γ_n^∞ is universal in the following sense :

THEOREM 16. *Let E be a vector bundle over a paracompact base with fiber isomorphic to $\mathbb{R}^n$. Then there is, up to homotopy, a unique morphism of vector bundle from E to γ_n^∞.*

The same theorem holds if we replace $\mathbb{R}$ by $\mathbb{C}$. The existence of such a morphism of vector bundle means that we can see E as the pullback bundle

$$E = \bar{f}^*(\gamma_n^\infty) \tag{13.7}$$

where $\bar{f}$ in the underlying map $\bar{f} : M \to G_n^\infty$. In other words, a vector bundle over M with fiber $\mathbb{R}^n$ is nothing but an homotopy class of a smooth map $[\bar{f} : M \to G_n]$. Also, this theorem implies that for any real vector bundle E with fiber $\mathbb{R}^n$ there is a morphism

$$f_E^* : H^\bullet(\gamma_n^\infty) \to H^\bullet(E) \tag{13.8}$$

13.6. Chern Classes.

This paragraph presents one possible definition of the Chern classes $c_1, c_2, \ldots$. Let $(A, M, \pi, \mathbb{C}^n)$ be a complex vector bundle. The top Chern class is defined as the Euler class of the underlying real vector bundle

$$c_n(A) \doteq e(A_{\mathbb{R}}) \tag{13.9}$$

For the next one, we define an $n-1$-vector bundle over A whose fiber over $(x, v) \in A$, $x \in M$, $v \in F \simeq \mathbb{C}^n$ is ${C^n}/{\langle v \rangle}$. The Chern class c_{n-1} is defined as the Euler class of the realification of this bundle. The $n-2$, $n-3$, etc Chern classes are defined with iteration of this construction. The total Chern class is the sum

$$c = 1 + c_1 + c_2 + \cdots + c_n \tag{13.10}$$

With this construction we can see directly that the Chern classes tells us how many independent nowhere vanishing sections of A we can find. If there exists one, c_n vanishes, if there exist two, c_{n-1} vanishes, etc.

13.7. Universality of Chern classes.

Chern classes are universal because of the following properties:

(1) Chern classes are natural: Given two vector bundles E, E' and a morphism of vector bundle $f : E \to E'$, we have $c_i(E) = f^* c_i(E')$.
(2) Chern classes generates the ring $H^\bullet(\gamma_{n,\mathbb{C}}^\infty)$.

In other words, any cohomology class of $H^\bullet(\gamma_{n,\mathbb{C}}^\infty)$ can be written as a polynomial in the $c_i(\gamma_{n,\mathbb{C}}^\infty)$. If any cohomology class would be a universal class among the complex vector bundles, by universality of $\gamma_{n,\mathbb{C}}^\infty$, it would need be a cohomology class of $H^\bullet(\gamma_{n,\mathbb{C}}^\infty)$, because of the naturality property and the universality of $\gamma_{n,\mathbb{C}}^\infty$, and hence be expressed as a polynomial of the Chern classes. Finally, we can add that there are no polynomial relations for Chern classes.

13.8. Pontrjagin classes.

We have seen that Chern classes are all the characteristic classes for complex vector bundles. For real vector bundles; the story slightly more complicated because not all real vector bundles are orientable. For example the projective space $\mathbb{R}P^n$ is not orientable when n is even, hence the direct copy paste of the construction of Chern classes will not leads to universal cohomology class with integral coefficient. But for orientable real vector bundle, we can define characteristic classes as Chern classes of their complexification i.e. something like $\tilde{c}_i(E) = c_i(E_{\mathbb{C}})$. We remark that this definition gives $\tilde{c}_i(E) \in H^{2i}(E; \mathbb{Z})$. One more fact: Chern classes are independent elements and we would like to keep this property. Let A be a complex vector bundle; we denote its complex conjugate by $\overline{A}$, which is the set of all $(x, \bar{v})$ with $(x, v) \in E$. It is possible to show that Chern classes satisfies

$$c_i(\overline{A}) = (-1)^i c_i(A) \tag{13.11}$$

But if E is a real vector bundle, then $\overline{E_{\mathbb{C}}} = \overline{E}$, showing that $2c_i(E_{\mathbb{C}}) = 0$ in that case, leading to the following definition. Let (E, M, π) be a real vector bundle. The Pontrjagin class $p_i(E) \in H^{4i}(M; \mathbb{Z})$ are

$$p_i(E) \doteq (-1)^i c_{2i}(E_{\mathbb{C}}) \tag{13.12}$$

The total Pontrjagin class is

$$p = 1 + p_1 + p_2 + \cdots + p_n \tag{13.13}$$

for a bundle of fiber dimension n. The Pontrjagin classes are natural because Chern classes are, i.e. $f^*(p_i(E')) = p_i(E)$ for any morphism of vector bundle $f : E \to E'$. Like the Chern classes, the Pontrjagin classes generates a universal cohomology ring, but not $H^\bullet(\gamma_n^\infty, \mathbb{Z})$. Indeed the torsion two elements $c_{2i+1}(E_{\mathbb{C}})$ cannot be generated by the Pontrjagin classes, but we can get rid of it using a coefficient ring in which 2 is invertible, like $\mathbb{Z}[\frac{1}{2}]$ or $\mathbb{R}$. Furthermore, γ_n^∞ has to be replaced $\tilde{\gamma}_n^\infty$, the universal oriented vector bundle, in order for the property to hold. Finally, in even dimension, the top Pontrjagin has to be replaced by the Euler class in the set of generators. because in that case

$$p_n = e^2. \tag{13.14}$$

If we resume, let R be a ring containing $\frac{1}{2}$

$\{p_1(\tilde{\gamma}_n^\infty), p_2(\tilde{\gamma}_n^\infty), \ldots, p_n(\tilde{\gamma}_n^\infty)\}$ generates $H^\bullet(\tilde{G}_n^\infty)$ if n is odd,
$\{p_1(\tilde{\gamma}_n^\infty), p_2(\tilde{\gamma}_n^\infty), \ldots, p_{n-1}(\tilde{\gamma}_n^\infty), e(\tilde{\gamma}_n^\infty)\}$ generates $H^\bullet(\tilde{G}_n^\infty)$ if n is even.

13.9. Stiefel-Whitney classes.

Another type of characteristic classes in the real case can be obtained by changing the coefficient ring by $\mathbb{Z}_2$. Indeed, in that case $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}_2)$ has a unique generator (equivalently: all manifolds are $\mathbb{Z}_2$-orientable). This leads to the Thom isomorphism:

THEOREM 17. *Let (E, M, π) be a real vector bundle and E_0 the obtained from E after removing the image of the* 0*-section. The cohomology groups $H^i(E, E_0; \mathbb{Z}_2)$ are all* 0 *for $i < n$, and $H^n(E, E_0; \mathbb{Z}_2)$ is generated by a unique class u, whose restriction to any fiber gives the unique generator of $H^n(\mathbb{R}, \mathbb{R}^n \setminus \{0\})$. The map*

$$H^k(B; \mathbb{Z}_2) \to H^k(E; \mathbb{Z}_2) \to H^{n+k}(E, E_0, \mathbb{Z}_2), \tag{13.15}$$

$$a \mapsto \pi^*(a) \mapsto \pi^*(a) \cup u. \tag{13.16}$$

is an isomorphism, called Thom isomorphism.

It is then possible to construct the Stiefel-Whitney classes from the class u of the theorem. We won't detail the construction. Let us say that we obtain one class $w_i(\mathbb{Z}_2)$. These classes contains global information about the vector bundle. For example $(E, M\pi)$ is orientable if and only if $w_1(E) = 0$ and, if it is the case, admits a spin structure if and only if $w_2(E) = 0$.

13.10. Gauss-Bonnet theorem.

The perhaps most famous link between cohomology and differential form is the Gauss-Bonnet theorem which relates the Euler characteristic $\chi(M)$ of a Riemann surface to the integral of its Gaussian curvature K.

$$\chi(M) = \int \mathrm{d}^2 x K. \tag{13.17}$$

The Euler characteristic is simply the Euler class evaluated on the fundamental homology class μ_M defining the orientation

$$\chi(M) = \langle e, \mu_m \rangle. \tag{13.18}$$

The direct generalization of the Gauss-Bonnet theorem, the Chern-Gauss-Bonnet theorem, states that the for a $2n$ dimensional Riemannian manifold M, with Riemann curvature R, the Euler class is

$$e = \frac{1}{(2\pi)^n} \mathrm{Pf}(R). \tag{13.19}$$

The Pfaffian "Pf" is like the square root of the determinant. For a $2n \times 2n$ skew symmetric matrix X it is defined by

$$Pf(X) = \frac{(-1)^n}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} (-1)^{|\sigma|} X_{\sigma(1)\sigma(2)} X_{\sigma(3)\sigma(4)} \dots X_{\sigma(2n-1)\sigma(2n)}. \tag{13.20}$$

13.11. Chern Weil homomorphism.

The equivalent of Chern classes in de Rahm cohomology can be computed as integral of certain polynomials in the curvature of any connection. This fact is a consequence of the Chern-Weil homomorphism which states the following: let (E, M, π) be a vector bundle, associated to a principal bundle whose structure group is G; then any Ad_G-invariant polynomial define a de Rahm cohomology class by evaluating it on the curvature of any covariant derivative of E. Here we will consider polynomials $P(X)$ with the indeterminate X evaluated in a (super-)Lie algebra $\mathfrak{g}$. In order for this to make sense, we see $\mathfrak{g}$ as a subalgebra of an algebra of (super-)matrices. In this setting, and with G a group whose Lie algebra is $\mathfrak{g}$, P is called Ad_G invariant if

$$P(\mathrm{Ad}_G X) = P(X). \tag{13.21}$$

If (E, M, π) is a vector bundle, $\nabla = \mathrm{d} + A$ a covariant derivative on E, F the associated curvature and G the group such that A takes values in $\mathfrak{g} = \mathrm{Lie}(G)$, we can evaluate a polynomial in F using the product rule

$$F^2 = F^\alpha \wedge F^\beta \otimes X_\alpha X_\beta. \tag{13.22}$$

X_α, X_β are Lie algebra elements and their product is the matrix product as just said. If P is an Ad_G-invariant polynomial, then

(1) $P(F)$ is closed i.e. $\mathrm{d}P(F) = 0$,

(2) If A' is another connection with curvature F', then $P(F') - P(F)$ is exact, i.e. is the exterior derivative of a differential form.

In other words, $P(F)$ define a de Rahm cohomology class, which does not depend on the choice of the connection one-form A.

13.12. Proof of the Chern-Weil homomorphism.

Let us prove this statement as we will need a result from the proof of the second statement. It is sufficient to prove the proposition for homogeneous polynomial, so suppose P is homogeneous of degree r and write

$$P(F) = P_r(\underbrace{F, F, \dots, F}_{r-\text{times}}). \tag{13.23}$$

P_r, called the polarization of P, is the symmetric polynomial which, when all its argument are identical, gives P. Infinitesimal Ad_G invariance implies, for $X \in \mathfrak{g}$

$$P_r([X,F],F,\dots,F) + P_r(F,[X,F],\dots,F) + \cdots + P_r(F,F,\dots,[X,F]) = 0. \tag{13.24}$$

Because d is a derivation, we have

$$\mathrm{d}P_r(F,F\dots,F) = P_r(\mathrm{d}F,F,\dots,F) + P_r(F,\mathrm{d}F,\dots,F) + \cdots + P_r(F,F\dots,\mathrm{d}F), \tag{13.25}$$

$$= P_r(\mathrm{d}F,F,\dots,F) + P_r(F,\mathrm{d}F,\dots,F) + \cdots + P_r(F,F\dots,\mathrm{d}F) \tag{13.26}$$

$$+P_r([A,F],F,\dots,F) + P_r(F,[A,F],\dots,F) + \cdots + \tag{13.27}$$

$$+P_r(F,F,\dots,[A,F]), \tag{13.28}$$

$$= P_r(\nabla F,F,\dots,F) + P_r(F,\nabla F,\dots,F) + \cdots + \tag{13.29}$$

$$+P_r(F,F\dots,\nabla F) + P_r([A,F],F,\dots,F) = 0, \tag{13.30}$$

using the Bianchi identity $\nabla F = 0$. For the second statement, define interpolating connection and curvature

$$A_t = A + tA_\Delta, \qquad A_\Delta = A' - A, \tag{13.31}$$

$$F_t = F + t\nabla A_\Delta + t^2 {A_\Delta}^2. \tag{13.32}$$

In this calculation, $\nabla = \mathrm{d} + A\wedge$ is the covariant derivative with respect to A. Then

$$P(F') - P(F) = \int_0^1 \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t} P(F) = r\int_0^1 \mathrm{d}t P_r(\frac{\mathrm{d}}{\mathrm{d}t}F_t, F_t, \dots F_t) \tag{13.33}$$

$$= r\int_0^1 \mathrm{d}t \Big(P_r(\nabla A_\Delta, F_t, \dots F_t) + 2tP_r({A_\Delta}^2, F_t \dots F_t)\Big). \tag{13.34}$$

Using Bianchi identity and invariance of P a short calculation gives

$$\mathrm{d}P_r(A_\Delta, F_t, \dots F_t) = \nabla P_r(\nabla A_\Delta, F_t, \dots F_t) + (r-1)tP_r(A_\Delta, [A_\delta, F_t], F_t, \dots, F_t). \tag{13.35}$$

Invariance of P again implies

$$2P_r({A_\Delta}^2, F_t \dots F_t) + (r-1)P_r(A_\Delta, [A_\delta, F_t], F_t, \dots, F_t) = 0. \tag{13.36}$$

Hence

$$P_r(F') - P_r(F) = \mathrm{d}r\,T_P(A, A'), \tag{13.37}$$

$$T_P(A, A') = \int_0^1 \mathrm{d}t P_r(A' - A, F_t, \dots F_t) \tag{13.38}$$

The form $T_P(A, A')$ is called a transgression form.

13.13. Analytical Chern and Pontrjagin classes.

Let (E, M, π) be a complex vector bundle of rank n, with a covariant derivative ∇ and the associated curvature F. Then the total Chern class $c(E)$ is equal to

$$c(E) = \det(\mathrm{Id} + \frac{F}{2\pi i}) \tag{13.39}$$

Expanding the determinant, we obtain the Chern classes c_i. For example

$$c_1(E) = \frac{i}{2\pi} F, \tag{13.40}$$

$$c_2(E) = \frac{1}{4\pi^2}(\mathrm{Tr}(F \wedge F) - \mathrm{Tr}(F) \wedge \mathrm{Tr}(F)), \tag{13.41}$$

$$c_n(E) = \left(\frac{i}{2\pi}\right)^n \det(F). \tag{13.42}$$

Now if (E, M, π) is a real vector bundle, with covariant derivative ∇ and curvature F, the total Pontrjagin class is given by

$$p(E) = \det(A + \frac{F}{2\pi}). \tag{13.43}$$

The expansion of this formula gives

$$p_1(E) = -\frac{1}{8\pi^2}\mathrm{Tr}(F^2), \tag{13.44}$$

$$p_2(E) = \frac{1}{128\pi^4}\left(\left(\mathrm{Tr}(F^2)\right)^2 - 2\mathrm{Tr}(F)^4\right), \tag{13.45}$$

$$\vdots \tag{13.46}$$

$$p_{\lfloor n/2 \rfloor} = \left(\frac{1}{2\pi}\right)^n \det(F). \tag{13.47}$$

13.14. Proof for Chern classes.

Let us outline the proof of (13.39). The first step is to prove it for a compact complex line bundle L. This is done using the Gauss-Bonnet theorem, which links the Euler class to the Riemannian curvature. This ends the first step as in that case $c = 1 + e$. The second step is to prove it for arbitrary Whitney sum of compact complex line bundles $E = L_1 \oplus L_2 \oplus \cdots \oplus L_n$. In this case we can write a connection as a sum of connections on each Line bundle, leading to a diagonal curvature

$$F = \begin{pmatrix} F_1 & & & \\ & F_2 & & \\ & & \ddots & \\ & & & F_n \end{pmatrix} \tag{13.48}$$

Then the property of Chern classes with respect to Whitney sum

$$c(E_1 \oplus E_2 \oplus \cdots \oplus E_k) = c(E_1) \cup c(E_2) \cup \cdots \cup c(E_n) \tag{13.49}$$

is used to show that (13.39) holds in this case. The last step is to show that the pullback of the morphism of vector bundle

$$f : \underbrace{\gamma^\infty_{1,\mathbb{C}} \oplus \gamma^\infty_{1,\mathbb{C}} \oplus \cdots \oplus \gamma^\infty_{1,\mathbb{C}}}_{n-\text{times}} \to \gamma^\infty_{n,\mathbb{C}}, \tag{13.50}$$

maps $H^\bullet(\gamma^\infty_{n,\mathbb{C}}, \mathbb{C})$ monomorphically into $H^\bullet(\gamma^\infty_{1,\mathbb{C}} \oplus \cdots \oplus \gamma^\infty_{1,\mathbb{C}})$. As a consequence, the result holds for $\gamma^\infty_{n,\mathbb{C}}$. Because the pullback of the Chern classes are the Chern classes of the pullback, and that the pullback of a curvature 2-form is a curvature 2-form, if the result holds for $\gamma^\infty_{n,\mathbb{C}}$, it holds for any bundle.

13.15. Chern-Simons forms.

Let P be a $2k$-form associated with a characteristic class. Because P is closed, there exists, at least locally, a $(2k-1)$-form Q whose exterior derivative is P. Such a form is called a Chern-Simons form. Chern-Simons forms can be constructed explicitly using the transgression form T_P. Indeed, putting $A' = 0$ in (13.37), we obtain

$$Q = k \int_0^1 P(A, F_t, \ldots F_t), \tag{13.51}$$

with now

$$F_t = tF + (t^2 - t) A \wedge A. \tag{13.52}$$

As an example, the Chern-Simons form associated to the second Chern-Class is

$$Q_3 = \frac{1}{8\pi^2} \mathrm{Tr}(A\delta A + \frac{2}{3} A \wedge A \wedge A) \tag{13.53}$$

where Tr denotes an invariant bilinear form of the Lie algebra in which the connection-form A takes values, like the trace for a faithful irreducible representation.

14. Jets

14.1. Multi-index notation.

A mutli-index is a finite sequence of positive integer which we will denote using an underbar

$$\underline{\mu} = \{\mu_1, \ldots \mu_n\}. \tag{14.1}$$

It will serve for example to denote succinctly derivatives with respect to multiple variables

$$\partial_{\underline{\mu}} \doteq \partial_{\mu_1} \ldots \partial_{\mu_n}. \tag{14.2}$$

Given a multi-index $\underline{\mu}$, we denote by $|\underline{\mu}|$ the sum of all index it contains

$$|\underline{\mu}| = \mu_1 + \mu_2 + \cdots + \mu_n. \tag{14.3}$$

14.2. Jets.

Let M and N be two manifolds. We define equivalence classes indexed by positive integers $\sim_k$ on the set of functions from $x \in M$ to $y \in N$ by the following. Two functions $f, g : M \to N$, $f(x) = g(x) = y$ are equivalent, and we write it $f \sim_k g$ if and only if for any function $\psi : \mathbb{R} \to M$ with $\psi(0) = x$ and any real-valued function ϕ defined on a neighborhood of y, the derivatives of $\phi \circ f \circ \psi$ and $\phi \circ g \circ \psi$ at 0 coincides up to order k. An equivalence class for this relation is called a k-jet at x, denoted by $j^k_x f$ (or sometimes only $j^1 f$ if the origin x is clear), and the set of all these jets, which is easily seen as a vector space, is denoted $J^k_x(M,N)_y$. By an abuse of language, we can say that two function define the same k-jet at x if their respective Taylor expansions at x agree up to order k. The following simple proposition should make the reader's mind clear:

PROPOSITION 22. *Two functions defines the same germ at x if and only if they define, $\forall\, k \in \mathbb{N}$, the same k-jet at x.*

14.3. Tangent vectors and jets.

Given a manifold M and a point $x \in M$, we can identify the tangent space at x $T_x M$ and the Jet space from $\mathbb{R}$ to M $J^1_0(\mathbb{R}, M)_x$. Hence we can easily extend the notion of tangent vectors to second-order, third-order, k-order tangent vectors as elements of $J^1_0(\mathbb{R}, M)_x$, $J^2_0(\mathbb{R}, M)_x$ and $J^k_0(\mathbb{R}, M)_x$ respectively. We can also speak of tangent bi-vectors, or second-order tangent tri-vectors as elements of $J^1_0(\mathbb{R}^2, M)_x$ and $J^2_0(\mathbb{R}^3, M)_x$ respectively. This suggests the notation $T^k_{rx} M$ for the space $J^k_0(\mathbb{R}^r, M)_x$. The union of all the spaces $T^k_{rx} M$ form a bundle over M, denoted $T^k_r M$. Its typical fiber is the space $L^k_{rm} \doteq J^k_0(\mathbb{R}^r, \mathbb{R}^m)_0$.

There is a natural pairing between $J^1_0(\mathbb{R}, M)_x$ and $J^1_x(M, \mathbb{R})_0$ given by

$$\langle j^1 f, j^1 g\rangle = \frac{d}{dt}(f \circ g)(t)|_{t=0}, \tag{14.4}$$

showing that the space $J^1_x(M, \mathbb{R})_0$ can be identified with the cotangent space $T^*_x M$. Analogously of what is done for the tangent space, we will denote by T^{*k}_{rx} the space $J^k_x(M, \mathbb{R}^r)_0$ and $T^{*k}_r M$ the bundle over M their union form. Its typical fiber is the space L^k_{mr}.

14.4. The group G^k_m.

We define the group G^k_m of k-jets of diffeomorphism $\phi : \mathbb{R}^m \to \mathbb{R}^m$ sending 0 to 0, i.e. G^k_m is a subset of $J^k_0(\mathbb{R}^m, \mathbb{R}^m)_0$. The group multiplication is given by composition:

$$\forall\, j^k f, j^k g \in G^k_m, j^k f \cdot j^k g = j^k (f \circ g) \tag{14.5}$$

The group G^k_m can be identified with the group of polynomial functions from $\mathbb{R}^m$ to $\mathbb{R}^m$ with at most degree k or equivalently to

$$GL(m)[X_1, \ldots, X_m] \big/ \mathcal{I}, \tag{14.6}$$

where

$$\mathcal{I} \doteq \Big(\sum_{|\underline{\mu}|=k+1} \underline{X}^{\underline{\mu}}\Big) GL(m)[X_1, \ldots, X_m] \tag{14.7}$$

is the ideal generated by elements of total polynomial degree strictly bigger than k.

14.5. Jet prolongation of manifolds.

Jet prolongation of manifolds are defined as jet prolongation of the local charts. Hence we define F^kM, the bundle of k-jets of smooth diffeomorphism from $\mathbb{R}^m$ to M. It is a principal bundle whose structure group is G^k_m. The right action is given by composition of representatives.

$$r : F^kM \times G^k_m \to F^kM \tag{14.8}$$

$$(j^k\phi, j^k\psi) \mapsto j^k(\phi \circ \psi). \tag{14.9}$$

As notation suggest, this bundle is the natural extension of the frame bundle FM. The bundles T^k_rM, T^{*k}_rM defined above can be seen as associated bundles of the principal bundles F^kM, exactly like the tangent bundle is a bundle associated to the frame bundle. The left actions of the structure group G^k_m on the typical fibers L^krm, L^kmr are given respectively by

$$\rho_1 : G^k_m \times L^k_{rm} \to L^k_{rm} \tag{14.10}$$

$$(j^k\phi, j^k\psi) \mapsto j^k(\phi \circ \psi), \tag{14.11}$$

$$\rho_2 : G^k_m \times L^k_{mr} \to L^k_{mr} \tag{14.12}$$

$$(j^k\phi, j^k\psi) \mapsto j^k(\psi \circ \phi^{-1}). \tag{14.13}$$

14.6. Geometrical interpretation of Lie algebra expansions.

If (G, m_G, i, e) is a Lie group, then the tangent group $(TG, Tm_G, Ti, (e, 0))$ is a Lie group as well. This fact is still valid for the extended tangent spaces T^kG. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^{[k]}$ the one of T^kG. Then

$$\mathfrak{g}^{[k]} = \mathfrak{g}[X]\big/_{\langle X^{k+1}\rangle}. \tag{14.14}$$

Let us show it. The first point is to compute the prolongation j^km of the multiplication law of the Lie group. In order to avoid complicated notation, we will write everything as if we were in the case of a matrix Lie group. The notation can then be adapted by the reader to the general case. Hence let

$$(g_1, X_1, A_1, \dots U_1), (g_2, X_2, A_2, \dots, U_k) \in T^k_m(G). \tag{14.15}$$

Then there are two local curves $\varphi_1, \varphi_2 : [-1, 1] \to G$ such that

$$g_1 = \varphi_1(0) \qquad g_2 = \varphi_2(0), \tag{14.16}$$

$$X_1 = \frac{d}{dt}|_{t=0}\varphi_1 \qquad X_2 = \frac{d}{dt}|_{t=0}\varphi_2, \tag{14.17}$$

$$A_1 = \frac{d^2}{dt^2}|_{t=0}\varphi_1 \qquad A_1 = \frac{d^2}{dt^2}|_{t=0}\varphi_2, \tag{14.18}$$

$$\dots \tag{14.19}$$

$$U_1 = \frac{d^k}{dt^k}|_{t=0}\varphi_1 \qquad U_2 = \frac{d^k}{dt^k}|_{t=0}\varphi_2. \tag{14.20}$$

The result of the sought product is obtained by computing the k-first derivatives of $m(\varphi_1,\varphi_2)$. Hence

$$j^k m\big((g_1, X_1, A_1, \dots U_1);(g_2, X_2, A_2, \dots, U_k)\big), = (g_1\cdot g_2, g_1\cdot X_2 + X_1\cdot g_2,$$
$$g_1\cdot A_2 + A_1\cdot g_2 + 2X_1\cdot X_2, \dots, g_1\cdot U_2 + U_1\cdot g_2 + \dots). \quad (14.21)$$

We have not write the full product at the k^{th} element for the sake of clarity. Next we compute the tangent of the left multiplication $l^{[k]}$ associated with this product. We need an element a of the Lie algebra $\mathfrak{g}^{[k]}$ expressed as a derivative

$$a = \frac{d}{dt}|_{t=0}(e^t W, tY, tB, \dots, tV). \quad (14.22)$$

We have

$$l^{[k]}{}_{(g,X,A,\dots U),*}(a) = \frac{d}{dt}|_{t=0}\left[l^{[k]}{}_{(g,X,A,\dots U)}(e^t W, tY, tB, \dots, tV)\right], \quad (14.23)$$
$$= (g\cdot W, g\cdot Y + X\cdot W, g\cdot B + A\cdot W + 2X\cdot Y, \dots, g\cdot V + U\cdot W + \dots) \quad (14.24)$$

We can similarly compute the tangent of the right multiplication $r^{[k]}{}_*$. The last step is to compute the Lie bracket of two elements $a, b \in \mathfrak{g}^{[k]}$ using

$$[a,b]_{\mathfrak{g}^{[k]}} = \mathrm{ad}_a(b) = \frac{d}{dt}|_{t=0}\mathrm{Ad}_{g(t)}(b), \quad (14.25)$$
$$a = \frac{d}{dt}|_{t=0}g(t), \quad (14.26)$$
$$\mathrm{Ad}_g = l^{[k]}{}_{g*}\circ r^{[k]}{}_{g*}. \quad (14.27)$$

The result is

$$\big[(W_1, X_1, A_1, \dots, U_1), (W_2, X_2, A_2, \dots, U_2)\big]_{\mathfrak{g}^{[k]}} = \Big([W_1, W_2]_{\mathfrak{g}}, [W_1, X_2]_{\mathfrak{g}} + [X_1, W_2]_{\mathfrak{g}}$$
$$[W_1, A_2]_{\mathfrak{g}} + [W_2, A_1]_{\mathfrak{g}} + 2[X_1, X_2]_{\mathfrak{g}}, \dots [W_1, U_2]_{\mathfrak{g}} + [U_1, W_2]_{\mathfrak{g}} + \dots\Big), \quad (14.28)$$

where we have denoted by $[]_{\mathfrak{g}}$ the Lie bracket of $\mathfrak{g}$. Hence the Lie bracket $[]_{\mathfrak{g}^{[k]}}$ is the exactly the one of $\mathfrak{g}/\langle X^{k+1}\rangle$.

14.7. Jet prolongation of fiber bundles.

There are two kinds of jet prolongation of fiber bundle. The first category consists in prolonging the sections. Hence, given a fiber bundle (F, M, π) we define the set $J^k F$ whose elements are k-jets $j^k_x\sigma$ of sections $\sigma : M \to F$ defined around a point $x \in M$. There is a natural structure of nested fiber bundles

$$M \xleftarrow{\pi} F \xleftarrow{p_{1,0}} J^1 F \xleftarrow{p_{2,1}} \dots \xleftarrow{p_{r,r-1}} J^r F, \quad (14.29)$$

where the projection $p_{r,r-1}$ is just given by the truncation of the Taylor expansion. These jets bundle are the one to consider in field theory. Indeed, when we consider a Lagrangian depending on fields and its derivatives

$$\mathcal{L}(\phi, \partial_\mu\phi, \dots) \quad (14.30)$$

it is an interesting point of view to see in fact the fields and its derivatives as jets, as they are like simple coordinates, and the whole lagrangian as a function

$$\mathcal{L} : J^r F \to \lambda^m M \tag{14.31}$$

instead of a functional

$$\tilde{\mathcal{L}} : S(M,F) \to \lambda^m M \tag{14.32}$$

where $S(M,F)$ denotes here the space of smooth sections. The former point of view simplify the formalism, in particular when we consider the Euler-Lagrange derivatives

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \dots \tag{14.33}$$

which are now simple derivatives instead of variation of functional with respect to functions. In particular no δ function appears in (14.33).

CHAPTER 4

Physical Models

1. Gauge symmetries

1.1. The harmonic oscillator.

One of the most basic model in theoretical physics is the harmonic oscillator. It describes a large class of system including for example pendulum's oscillations in the small angle limit. The equation of an harmonic oscillator is

$$\ddot{q} + \omega_0^2 q = 0, \tag{1.1}$$

whose solution is

$$q(t) = A\cos(\omega_0 t + \phi). \tag{1.2}$$

Equation (1.1) is referred as the equation of motion of the system. q is the dynamical variable under consideration, it also called coordinate, and belongs to the configuration space. A and ϕ are two integration constant determined by the problem one is considering, ω_0 is the frequency. The model of the harmonic oscillator can be used to describe the movement of a pendulum with small oscillations. In this case, q is the angle between the pendulum and the vertical axis and $\omega_0 = \sqrt{\frac{g}{L}}$, g being the gravitational acceleration and L the length of the pendulum. In this model, like almost all other model of classical physics, the dynamical parameter is the time t. Here, q is also called a physical degree of freedom. It correspond to a parameter which can in principle evolves in any way, but whose behaviour is completely fixed by the equation of motion. The equation (1.1) does not depends explicitly on time, thus is invariant under time translation $t \mapsto t + t_0$. This kind of symmetry, which correspond to a global transformation of a physical quantities, is called a global symmetry.

1.2. Example of gauge symmetry.

In this thesis we will be interested in another kind of symmetries called gauge symmetries. There are the symmetries of the equations of motions involving an arbitrary function of time. For example consider the system

$$\begin{cases} \dot{x} - \frac{x\dot{x}+y\dot{y}}{x^2+y^2}x + y = 0 \\ \dot{y} - \frac{x\dot{x}+y\dot{y}}{x^2+y^2}y - x = 0 \end{cases}. \tag{1.3}$$

It is invariant under

$$x(t) \mapsto e^{\lambda(t)}x(t), \qquad y(t) \mapsto e^{\lambda(t)}y(t), \tag{1.4}$$

with λ an arbitrary function. Indeed, the general solution is given by

$$x(t) = e^{\lambda(t)}\cos(t), \qquad y(t) = e^{\lambda(t)}\sin(t). \tag{1.5}$$

Here, the transformation 1.4 is called a gauge transformation and λ is called a *pure gauge quantity*. It is a variable which is still completely free after solving the equation of motion, and which does not depend on the initial conditions; and thus cannot be given any physical interpretation. Indeed the physical system we are considering does not provide any way of measuring it. As it cannot be provided any physical sense, it is convenient to fix its value to a given value in order to simplify the problem. For example we could choose $\lambda(t) = 0$. This procedure is called *gauge fixing*. It is worthwhile to note that going to another system of coordinates makes the arbitrariness of unphysical coordinate more explicit. Indeed, choosing polar coordinate (r, α), the system 1.3 can be written

$$\dot{\alpha} = 1, \tag{1.6}$$

making it clear that $r = e^{\lambda}$ does not play any role. Once in this form, we can get rid of r definitively and the system does not contains any pure gauge quantities anymore. However, in many physical problems this kind of simplification may not occur, and it is often more convenient to work keeping the pure gauge quantities, solving them only in the last steps of the resolution of a physical proble. In this example, we saw that only the angular coordinate was sensible to the initial data, necessary to solve the differential equation. Such a quantity is called an "on-shell degree of freedom". On the other hand, x and y are called "off-shell degrees of freedom". They correspond to the number of expected functions to be found solving equation of motion after a quick and naive look at them. One important step in any physical problem is to count the number of degrees of freedom, This has to be understood as counting the minimal amount of initial data that are necessary to solve the equations of motion.

1.3. Gauge symmetry in shape dynamics.

The following example is taken from [Wil22]. We consider a body that we approximate in the following manner: we cut it into N pieces, that we then reduce to a single point, concentrating all the mass of that piece. Hence the body is seen as a collection of N point $\{x_i\}_{i=1}^{N}$, each having a mass m_i. We consider then a motion of the body, which consist of a rotation each point x_i around the center of mass, hence:

$$x_i^{\mu}(t) = R^{\mu\nu}(\alpha_i) x_i^{\nu}(0), \tag{1.7}$$

for a matrix of rotation $R^{\mu\nu}$ with some angle of rotation α_i. In the notation we adopt the convention that two repeated indices are summed. This convention is called Einstein's summation convention and will always be assumed in this manuscript. In order to study deformation of the body, we introduce the reference coordinate s_i, which is the position the point x_i would have taken if the body were not deformed during its motion. Once again, we consider that the motion consists only of rotations, hence:

$$x_i^\mu(t) = U^{\mu\nu}(\theta_i) s_i^\nu(t), \tag{1.8}$$

$$s_i^\mu(t) = T^{\mu\nu}(\beta_i) x_i^n u(0), \tag{1.9}$$

$$R^{\mu\nu}(\alpha_i) = U^{\mu\nu}(\theta_i) T^{\mu\nu}(\beta_i), \tag{1.10}$$

$$\tag{1.11}$$

with two new rotation matrices $U^{\mu\nu}$ and $T^{\mu\nu}$. We consider that $s_i(0) = x_i(0)$, i.e. the body is not deformed at the beginning of the motion. For the sake of clarity, we will no more write the angles (α_i), (β_i), (θ_i) in argument of the rotation matrices. We are interested in the analysis of the torque induced by the motion. We recall that the torque correspond to the derivative with respect to time of the angular momentum, which is itself given by:

$$L^{\mu\nu} = \sum_{i=1}^{N} m_i (x_i^\mu \dot{x}_i^\nu - x_i^\nu \dot{x}_i^\mu), \tag{1.12}$$

with the time derivative denoted by a dot. We introduce the angular velocity:

$$\omega_{x_i}^{\mu\nu} = \dot{R}^{\mu\rho} (R^{-1})^{\rho\nu}, \tag{1.13}$$

with which the velocity can be expressed as:

$$\dot{x}_i^\mu = \omega_{x_i}^{\mu\nu} x_i^\nu, \tag{1.14}$$

and, with the help of the moment of inertia:

$$I^{\mu\nu;\rho\sigma}(x_i) = m_i \left(\delta^{\mu\sigma} x_i^\nu x_i^\rho + \delta^{\nu\rho} x_i^\mu x_i^\rho - \delta^{\mu\rho} x_i^\nu x_i^\sigma - \delta^{\nu\sigma} x_i^\mu x_i^\rho \right), \tag{1.15}$$

(where $\delta^{\mu\nu}$ is the Kronecker's tensor) express the angular momentum as:

$$L^{\mu\nu} = \sum_{i=1}^{N} I^{\mu\nu;\rho\sigma}(x_i) \omega_{x_i}^{\rho\sigma}. \tag{1.16}$$

We now wish to separate the contribution from what we will call the true movement of the body, which is given by the s_i's, and the deformation of the body, given by the motion of the x_i's relatively to the s_i's. For this, we find the relations between the respective moments of inertia and angular velocities:

$$I^{\mu\nu;\rho\sigma}(x_i) = U^{\mu\alpha} U^{\nu\beta} U^{\rho\gamma} U^{\sigma\delta} I^{\alpha\beta;\gamma\delta}(s_i), \tag{1.17}$$

$$\omega_{x_i}^{\mu\nu} = U^{\mu\rho} U^{\nu\sigma} \omega_{s_i}^{\rho\sigma} + \dot{U}^{\mu\rho} (U^{-1})^{\rho\nu}. \tag{1.18}$$

The angular momentum can thus be written as:

$$L^{\mu\nu} = U^{\mu\alpha} U^{\nu\beta} I^{\alpha\beta;\rho\sigma}(s_i) \left(\tilde{\omega}_i^{\rho\sigma} + \omega_{s_i}^{\rho\sigma} \right), \tag{1.19}$$

where $\tilde{\omega}_i^{\rho\sigma} = (U^{-1})^{\rho\gamma} \dot{U}^{\gamma\sigma}$ will be called in this section the "connection". The gauge freedom of this problem is the choice of the reference coordinate s_i. Indeed, nothing here tells us what is the true motion of the undeformed body, and it is our free choice the set it through the coordinates s_i, or equivalently, through the choice of the global rotation T. A gauge transformation is a choice of another s_i:

$$s'_i = (V^{-1})s_i. \tag{1.20}$$

It implies a change in the connection:

$$\tilde{\omega}'_i = V^{-1}\tilde{\omega}V + V^{-1}\dot{V} \tag{1.21}$$

This is precisely the kind of transformations followed by connections in fibre bundles, as we will be shown later. Finally, the torque of the body is expressed as:

$$\frac{dL^{\mu\nu}}{dt} = U^{\mu\alpha}U^{\nu\beta}\mathrm{D}_t I^{\alpha\beta;\rho\sigma}(\tilde{\omega}_i^{\rho\sigma} + \omega_{s_i}^{\rho\sigma}). \tag{1.22}$$

We have introduced the covariant derivative

$$\mathrm{D}_t I^{\alpha\beta;\rho\sigma}(\tilde{\omega}_i^{\rho\sigma} + \omega_{s_i}^{\rho\sigma}) = (\delta^{\alpha\mu}\delta^{\beta\nu}\frac{d}{dt} + \tilde{\omega}^{\alpha\mu}\delta^{\beta\nu} + \delta^{\alpha\mu}\tilde{\omega}^{\beta\nu})I^{\mu\nu;\rho\sigma}(\tilde{\omega}_i^{\rho\sigma} + \omega_{s_i}^{\rho\sigma}). \tag{1.23}$$

1.4. Lagrangian formalism.

Models in physics are often cast into a formalism called Lagrangian formalism. It correspond to an integrated form of the equations of motions, which are then obtained by varying the dynamical coordinates or the dynamical fields, following the least action principle. Here, a theory is specified by an action, denoted $\mathcal{S}$, which is the integral over time of the Lagrangian, denoted $\mathcal{L}$,

$$\mathcal{S} = \int_{t_A}^{t_B} \mathcal{L}(q_n(t), \dot{q}_n(t), \ddot{q}_n(t), ...)dt, \tag{1.24}$$

where $q_n(t)$, $n \in \{1, \dots, N\}$ are the dynamical coordinates. The action principle states that among all possible path going from $\{q_n(t_A)\}$ to $\{q_n(t_B)\}$, the physical one is the one for which the action take an extremal value. Thus the equation of motions are

$$\delta\mathcal{S} = 0. \tag{1.25}$$

The variation $\delta\mathcal{S}$ is given by the integral of the variation of the Lagrangian, which in turn is given by

$$\delta\mathcal{L} = \left[\frac{\partial\mathcal{L}}{\partial q_n} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_n} + \frac{d^2}{dt^2}\frac{\partial\mathcal{L}}{\partial\ddot{q}_n} + \dots\right]\delta q_n \tag{1.26}$$

$$+\frac{d}{dt}\left[\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_n} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\ddot{q}_n} + \dots\right)\delta q_n + \left(\frac{\partial L}{\partial\ddot{q}_n} + \dots\right)\delta\dot{q}_n + \dots\right]. \tag{1.27}$$

In computing the equation of motions, the variation is always assumed to leave the extremal points fixed

$$\delta q_n(t_A) = \delta q_n(t_B) = 0. \tag{1.28}$$

Hence the integral of the total derivative (1.27) gives 0. Furthermore, in most of the situation encountered in physics, the Lagrangian depends explicitly only of q and $\dot{q}$. Thus we arrive at the celebrated Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial q_n} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_n} = 0. \tag{1.29}$$

while the derivative term reduce to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\delta q_n\right). \tag{1.30}$$

1.5. Noether symmetries.

Lagrangian formalism permits to make the link between global symmetries and conserved quantities. Global symmetries are those which are parametrized by a global constant parameter. We said earlier that the equations of motion of the harmonic oscillator were unchanged after translation in time. We propose to show here that, using the Lagrangian formalism, a system enjoying such a symmetry possesses an associated conserved quantities called the energy. Translation in time form a one-parameter group of transformation. When dealing with such a kind of transformation, it is best advised to look at infinitesimal transformation. Let $\mathcal{L}(q_n(t),\dot{q}_n(t))$ be a Lagrangian not depending explicitly on time. Under infinitesimal time translation $t\mapsto t+\varepsilon$, $\mathcal{L}$ is transformed as

$$\mathcal{L}+\varepsilon\frac{d}{dt}\mathcal{L}. \tag{1.31}$$

At the same time, we showed that the have

$$\delta_\varepsilon\mathcal{L}=\left(\frac{\partial\mathcal{L}}{\partial q_n}-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_n}\right)\delta_\varepsilon q_n+\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_n}\delta_\varepsilon q_n\right) \tag{1.32}$$

Here, $\delta_\varepsilon q_n = q_n(t+\varepsilon)-q_n(t)\simeq\varepsilon\dot{q}_n$. We are interested at properties of a physical object, for which the equations of motion hold, hence we assume the first term of the right-hand-side of (1.32) to vanish. However, in this computation we don't require the boundaries to be fix : here they are shifted as well $(q_n(t_A)\mapsto q_n(t_A+\varepsilon))$. Equating the variations obtained in (1.31) and in (1.32), we get

$$\varepsilon\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_n}\dot{q}_n-\mathcal{L}\right)=0 \tag{1.33}$$

Showing that $\mathcal{E}=\frac{\partial L}{\partial\dot{q}_n}\dot{q}_n-\mathcal{L}$ is conserved.

More generally, a Noether symmetry is any transformation which transform the Lagrangian by a total derivative, i.e. for which $\delta_\varepsilon\mathcal{L}=\frac{d}{dt}\mathcal{K}$, for some functional $\mathcal{K}$ and for which the infinitesimal parameter ε is constant. Then, a computation totally similar as the one we just made shows that the quantity

$$\mathcal{Q}=\frac{\partial L}{\partial\dot{q}_n}\delta_\varepsilon q_n-\mathcal{K} \tag{1.34}$$

is conserved. This conserved quantity Q is called the *charge* of the symmetry.

1.6. Gauge symmetries in Lagrangian formalism.

Let us point out that in the case of time-independent Lagrangian, the Euler-Lagrange equations (1.29) can be rewritten as

$$\ddot{q}_m\frac{\partial^2\mathcal{L}}{\partial\dot{q}_m\partial\dot{q}_n}=\frac{\partial\mathcal{L}}{\partial q_n}-\dot{q}_m\frac{\partial^2\mathcal{L}}{\partial q_m\partial\dot{q}_n}. \tag{1.35}$$

Hence, the acceleration $\ddot{q}_n$ can explicitly compute from the $\{q_n\}$'s and $\{\dot{q}_n\}$'s if and only if the matrix $\frac{\partial^2\mathcal{L}}{\partial\dot{q}_m\partial\dot{q}_n}$ is invertible. In the case it is, the system can be exactly solved, in the opposite case, the solutions will contains arbitrary functions of time, meaning that we have the appearance of a gauge symmetry.

1.7. Hamiltonian formalism.

The Hamiltonian formalism starts by defining canonical momenta associated to the dynamical coordinates (or dynamical fields).

$$p_n = \frac{\partial \mathcal{L}}{\partial \dot{q}_n}. \tag{1.36}$$

In case of gauge symmetries, these momenta are not independent: they satisfy some conditions, denoted ϕ_k, $k \in \{1, \dots K\}$ called "constraints", that we assume independent. Next we introduce the Hamiltonian as the Legendre transform of the Lagrangian:

$$\mathcal{H} = \dot{q}_n p^n - \mathcal{L}. \tag{1.37}$$

The Hamiltonian, as well as The constraints ϕ_k, are functions of p_n and q_n only and does not depend on the velocities $\dot{q}_n$. When non trivial constraints exist, the change of variables

$$\{q_n, \dot{q}_n\} \to \{q_n, p_n\}, \tag{1.38}$$

is not invertible, and map the whole $(q_n, \dot{q}_n)$ space to an hypersurface, called the constraint surface", in the (q_n, p_n) space, which is itself called the "phase space". The following regularity condition on the constraints are usually required: the rank of the matrix $\frac{\partial \phi_k}{\partial (q_n, p_n)}$ is maximal. This regularity condition implies that locally, the constraints ϕ_k can be seen (rigorously only locally, but here we assume globally) as coordinates complementary to the $\{q_n, p_n\}$. In other words, there exists some functions x_l, $l \in \{K+1, \dots, 2N\}$ such that the Jacobian matrix $\frac{\partial(\phi_k, x_l)}{\partial(q_n, p_n)}$ is invertible.

The fact that the phase space variable are required to satisfy the equations $\phi_k(q_n, p_n) = 0$ means that the Hamiltonian is not a well defined function of these same variables; the following proposition teach us that it could be changed by

$$\mathcal{H} \mapsto \mathcal{H} + c^k \phi_k \tag{1.39}$$

where the c_k's are abitrary functions of the phase space variables.

PROPOSITION 23. *If a smooth phase space function $G(q_n, p_n)$ vanishes on the constraint surface, then $G = g^k \phi_k$ for some functions g^k*

(Taken from [HT94])

A tangent vector to the constraint surface can be written as

$$v^l \frac{\partial}{\partial x_l} = v^l \frac{\partial q_n}{\partial x_l} \frac{\partial}{\partial q_n} + v^l \frac{\partial p_n}{\partial x_l} \frac{\partial}{\partial p_n}. \tag{1.40}$$

To simplify the notation, in the following paragraph we write

$$\delta q_n = v^l \frac{\partial q_n}{\partial x_l}, \qquad \delta p_n = v^l \frac{\partial p_n}{\partial x_l}. \tag{1.41}$$

In this direction, the Hamiltonian changes as

$$\delta\mathcal{H} = \dot{q}_n p_n - \frac{\partial \mathcal{L}}{\partial q_n}\delta q_n \tag{1.42}$$

implying

$$\left(\frac{\partial \mathcal{H}}{\partial q_n} - \frac{\partial \mathcal{H}}{\partial q_n}\right)\delta q_n + \left(\frac{\partial \mathcal{H}}{\partial p_n} - \dot{q}_n\right)\delta p_n = 0 \tag{1.43}$$

This last equation is true for any vector tangent to the constraint surface; we obtain at each point $2N-K$ equations with $2N$ unknowns, one possible basis of solutions of which is $\{\frac{\partial \phi_k}{\partial q_n}, \frac{\partial \phi_k}{\partial p_n}\}$ because of the regularity conditions. Thus we obtain

$$\dot{q}_n = \frac{\partial \mathcal{H}}{\partial p_n} + c^k \frac{\partial \phi_k}{\partial p_n}, \tag{1.44}$$

$$\frac{\partial \mathcal{L}}{\partial q_n} = -\frac{\partial \mathcal{H}}{\partial q_n} - c^k \frac{\partial \phi_k}{\partial q_n}, \tag{1.45}$$

with c^k some arbitrary smooth functions. Now we can use the Euler-Lagrange equations to change (1.45) into

$$\dot{p}_n = -\frac{\partial \mathcal{H}}{\partial q_n} - c^k \frac{\partial \phi_k}{\partial q_n}. \tag{1.46}$$

1.8. Poisson Bracket.

Equations (1.44,1.46), together with the constraint equations $\phi_k(q_n, p_n) = 0$ are called "Hamilton's equations of motion". For any two function on the phase space, we introduce the notion of Poisson bracket, denoted $\{\cdot,\cdot\}$:

$$\{F(q_n, p_n), G(q_n, p_n)\} = \frac{\partial F}{\partial q_n}\frac{\partial G}{\partial p_n} - \frac{\partial F}{\partial p_n}\frac{\partial G}{\partial q_n}. \tag{1.47}$$

With it, (1.44,1.46) can be written as:

$$\dot{q_n} = \{\mathcal{H}, q_n\} + c^k\{\phi_k, p_n\}, \qquad \dot{p_n} = -\{\mathcal{H}, p_n\} - \{\phi_k, q_n\}, \tag{1.48}$$

and more generally

$$\dot{F}(p_n, q_n) = \{\mathcal{H}, F\} + c^k\{\phi_k, F\} \tag{1.49}$$

1.9. Constraint algorithm.

Constraints are time-independent. Hence, on the constraint surface, $\dot{\phi}_k = 0$. In term of Poisson brackets:

$$\dot{\phi}_k = \{\mathcal{H}, \phi_k\} + c^j\{\phi_j, \phi_k\}. \tag{1.50}$$

If this expression in not a linear combination of the already existing constraints, we obtain a new constraint. Then we have to check that $\ddot{\phi}_k = 0$, and so on. We keep the notation ϕ_k for the constraints, including the new ones we just added to the list, k belonging now to a bigger set of indices. During the process, equations (1.50) is solved for the c^j's, which lose arbitrariness. We can always redefine $c^j = u^j + v^j_a c'^a$, where u^j and v^j_a are some functions which has been explicitly found soling (1.50), whereas c'^a are unknown functions, corresponding to the remaining arbitrariness. Thus we rewrote our system of constraints:

$$\mathcal{H} + c^k\phi_k \to \mathcal{H}' + c'^a\phi'_a \tag{1.51}$$

$$\mathcal{H}' = \mathcal{H} + u^j\phi_j, \qquad \phi'_a = v^j_a\phi_j. \tag{1.52}$$

1.10. First class constraints.

A quantity is called *first class* if its Poisson bracket with any constraint vanishes on the constraint surface, or, according to proposition 23, is proportional to the constraints. It is straightforward to show that the Poisson bracket of two first class quantities is again first class. A quantity which is not first class is called *second class*. The Hamiltonian $\mathcal{H}'$ defined in (1.52) is first class. We use now the notation ϕ_a for a first class constraint and χ_a for a second class constraint. The matrix of Poisson bracket for the different constraints (on the constraint surface) is:

$$\begin{array}{cc} & \begin{array}{cc}\phi_a & \chi_a\end{array} \\ \begin{array}{c}\phi_b\\ \chi_b\end{array} & \begin{bmatrix} 0 & 0 \\ 0 & C_{ab}\end{bmatrix}\end{array}. \tag{1.53}$$

If C_{ab} is not invertible, there exist a linear combination $\lambda^a C_{ab} = 0$. Then $\lambda^a\chi_a$ is seen to be a first class constraint. Thus we can assume that the separation between first class and second class constraints has been properly made (no combination of the second class constraints yields a first class one), and that the matrix C_{ab} is invertible. C_{ab} is antisymmetric, thus there is an even number of second class constraints.

1.11. Gauge symmetries.

We have seen that constraints enter in the Hamiltonian multiplied by arbitrary functions. Let us now explore how the dynamic is changed under different choices for these arbitrary functions. Let c^a and $\tilde{c}^a$ be two such different functions. Let $F(q_n, p_n)$ be any dynamical quantity. Its evolution is governed by the equation

$$\dot{F} = \{F, \mathcal{H}\} + c^a\{\phi_a, F\}. \tag{1.54}$$

After an infinitesimal displacement in time, the difference between the two dynamics is

$$\delta F = (c^a - \tilde{c}^a)\{\phi_a, F\}dt. \tag{1.55}$$

Thus, we can interpret the constraints as generators of transformations $F \mapsto F + \epsilon^a\{\phi_a, F\}$. First class constraints generate transformations that do not take physical quantities out of the constraint surface, as they preserve the equality $\phi_a = 0$ on the this surface. Thus they can be interpreted as true gauge transformation, as defined earlier. Second class constraints, on the other hand, do not. However, it is possible to remove them from the formalism, by modifying the Poisson bracket. The new bracket, called Dirac bracket, is:

$$\{F, G\}_{\text{Dir}} = \{F, G\} - \{F, \chi_a\}(C^{-1})^{ab}\{\chi_b, G\}. \tag{1.56}$$

The dynamics is given by the exact same equations as before, but with the Poisson bracket everywhere replaced by the Dirac bracket.

1.12. Gauge fixing.

A gauge fixing procedure is exactly like a constraint but added by hand. It solve a first class constraint by forming with it a pair of second class constraint, which as we have seen, does not allow for arbitrary function in the physical solutions of the motion. The converse is also true: removing one of the second class constraint will turn one other (or a linear combination of the other) into a first class constraint.

1.13. Degrees of freedom.

In this formalism, the counting of degrees of freedom is immediate. Starting with N physical coordinates q_n, then going to $2N$ canonical coordinates (q_n, p_n), we remove one degree of freedom for each constraint. Furthermore, we have seen that each first class constraint generates a gauge transformation, so we have to remove an additional degree of freedom for each first class constraint. In the end

Number of degrees of freedom = (Number of canonical variables - 2· Number of first class constraints - Number of second class constraints) /2.

1.14. The free relativistic particle.

Let us illustrate our preceding discussion by a very basic example of physics: he free massive relativistic particle. Its action is:

$$\mathcal{S} = -m \int ds \sqrt{g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu}, \tag{1.57}$$

with $\mu \in \{0, 1, 2, 3\}$, and we call the dynamical parameter s and not t to emphasize the fact that it is not time (time is usually q^0). The associated momenta are

$$p_\mu = -m \frac{\dot{q}_\mu}{|\dot{q}|}. \tag{1.58}$$

There is one constraint $p^2 + m^2 = 0$. The Hamiltonian vanishes. Thus there is no further constraint, and the constraint is first class. It generates the transformation:

$$q^\mu \mapsto q^\mu - 2m\epsilon \frac{\dot{q}^\mu}{|\dot{q}|}. \tag{1.59}$$

Accordingly the Lagrangian changes as

$$\mathcal{L} \mapsto \mathcal{L} - 2\dot{\epsilon}\mathcal{L}. \tag{1.60}$$

Using the equality $\frac{d\epsilon}{ds} ds = d\epsilon$ we see that the gauge symmetry correspond to the freedom in the choice of the parameter λ used to describe the particle. The number of physical degrees of freedom is 3.

1.15. Field theories.

Field theories describe the evolution of dynamical fields instead of dynamical coordinates. Concretely, this means that there are now several dynamical parameter instead of just one. Mathematically, the problem is changed: the equations of motions are partial differential equations, not ordinary differential equations. For example, if we consider a theory describing a scalar field $\varphi(\underline{x})$, depending on four space-time variables $\underline{x} \equiv \{x^\mu\}_{\mu=0,1,2,3}$, through a Lagrangian $\mathcal{L}(\varphi(\underline{x}), \partial_\mu \varphi(\underline{x}))$, the Euler-Lagrange equations are

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \varphi\right)} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0. \tag{1.61}$$

Usually, initial data for these partial differential equations are given on a surface at fixed time. Hence space-time variables are decomposed into space variables $\overrightarrow{x}$ and the time variable t. Concretely, solving the Euler-Lagrange equations (1.61) amount to find the time evolution $\varphi(\overrightarrow{x}, t)$, knowing $\varphi(\overrightarrow{x}, 0)$ and its time derivative $\partial_t \varphi(\overrightarrow{x}, 0)$.

1.16. Gauge symmetries in field theories.

In field theories, the momentum field associated to the physical field φ is

$$\pi(x,t) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}, \tag{1.62}$$

The Hamiltonian $\mathcal{H}$ is defined a the space integral of an Hamiltonian's density H

$$\mathcal{H}(t) = \int d^3x\, H(x,t), \qquad H = \pi\dot{\phi} - \mathcal{L}. \tag{1.63}$$

The constraints functions turns into smeared constraints functionals $\mathcal{F}$, which are integrals of local functionals

$$\mathcal{F} = \int d^3x\, F\left(\varphi(x,t), \pi(x,t)\right). \tag{1.64}$$

The Poisson bracket for two functional depending of the fields ans their momenta is

$$\left\{F\left(\varphi(x,t),\pi(x,t)\right), G\left(\varphi(x',t),\pi(x't)\right)\right\} = \int \left(\frac{\delta F(x,t)}{\delta \varphi x", t}\frac{\delta G(x',t)}{\delta \pi x", t} - \frac{\delta F(x,t)}{\delta \pi x", t}\frac{\delta G(x',t)}{\delta \varphi x", t}\right) d^3x", \tag{1.65}$$

where the variation show the appearance of delta-functions

$$\frac{\varphi(x,t)}{\varphi(x',t)} = \delta(x-x'). \tag{1.66}$$

The formalism - distinction between first and second class constraint, generation of gauge transformation by the first class constraints, etc. - stay globally the same, but the variational calculus implies a lot of technical difficulties in particular in the quantum theory. Hence the attempt of using jets to mathematically simplify the calculus, [GM].

1.17. Yang-Mills theories. The undoubtedly most famous gauge field theories are the Yang-Mills theories. The Yang-Mils action take the form

$$\mathcal{S}_{\text{Yang-Mills}} = \int_M \mathcal{F} \wedge * \mathcal{F} \tag{1.67}$$

where M is a (usually Lorentzian) manifold, $*$ is the Hodge operator defined in 5.12, and $\mathcal{F}$ the curvature 10.12 associated to a connection defined over a G-principal bundle over M. Three of the four fundamental interactions are described by Yang-Mills theories, namely electromagnetism with gauge group $G = U(1)$, weak interaction with gauge group $G = SU(2)$ and strong interaction with $G = SU(3)$, hence their importance in mathematical physics.

2. Gravity

2.1. The Einstein-Hilbert action.

In general relativity, the physical space-time is described as a four dimensional manifold. On it, gravity is the effect produced by metric field g of Lorentzian signature. The physical action is the Einstein-Hilbert action

$$\mathcal{S}_{\text{EH}} = -\frac{1}{16\pi G}\int \sqrt{-g}\mathcal{R} d^4x, \tag{2.1}$$

where the integral is over the whole space-time, $\mathcal{R}$ is the scalar curvature presented in (11.23), and G is the gravitational constant. It is possible to add to this action a constant Λ, called the cosmological constant

$$\mathcal{S}_{\Lambda\text{EH}} = -\frac{1}{16\pi G}\int \sqrt{-g}(\mathcal{R}+\Lambda) d^4x. \tag{2.2}$$

The Euler-Lagrange equations derived from it are the Einstein equations (+ cosmological constant)

$$\text{Ric}_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{2.3}$$

So far, we have presented a vacuous theory, with no matter in it. Adding matter amount to add a matter Lagrangian to the action,

$$\mathcal{S}_{\Lambda\text{EH}} = \int \sqrt{-g}(\mathcal{R}+\Lambda) d^4x + \mathcal{L}_{\text{matter}}, \tag{2.4}$$

and the Einstein equations become

$$\text{Ric}_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}. \tag{2.5}$$

Here

$$T_{\mu\nu} = \frac{\delta \mathcal{L}_{\text{matter}}}{\delta g^{\mu\nu}}, \tag{2.6}$$

is the energy momentum tensor, describing the distribution of matter. It is convenient to work in a system of unit in which $-\frac{1}{16\pi G} = 1$ so that we do not have to write this constant in the equations anymore; we adopt this convention. All of the material presented here can be found in [Wei72].

2.2. Gravity is a peculiar gauge theory.

Like Yang-Mills theories, the Einstein-Hilbert action of gravity presents gauge symmetries, as can be seen explicitly in the ADM formalism [ADM59]. However, in Yang-Mills theories the algebra of infinitesimal gauge transformations can be factorized as

$$\mathfrak{g} \otimes \mathcal{C}^{\infty}(M) \tag{2.7}$$

where $\mathfrak{g}$ is the Lie algebra of the so-called gauge group G - the structure group of the principal fibre bundle - and M the manifold over which the theory is defined. In gravity the group of gauge transformation is the group of diffeomorphism of the manifold. Its Lie algebra is the Lie algebra of smooth vector fields and does not factorize as $\mathbb{C}^{\infty}(M)$ times a finite dimensional Lie algebra. This fact leads to many complications, in particular for the quantum theory of gravity.

2.3. First order formalism.

It is common to consider that in general relativity, the only one fundamental field is the metric g. This approach is called the second-order formalism of general relativity, because the Einstein equations (2.3) involve derivative of order two of the metric. On the other hand, the first order formalism consider two fundamental fields for the theory, the metric g and the linear connection Γ. This linear connection is similar to the Levi-Civita, but do is not related to the metric by 11.19, in particular its torsion does not necessarily vanishes. It is in fact the new equations of motion, obtained by varying the connection field Γ, that impose the vanishing of the torsion. In other words, the connection solution in the first order formalism (with action 2.1) is the Levi-Civita connection. The remaining equations of motion are the Einstein equation 2.3, unchanged.

The use of an arbitrary connection implies the appearance of another gauge symmetry (different of the one of the previous paragraph). This is because the connection form Γ only appears in the connection through its curvature, curvature which is invariant under gauge transformations as defined as in paragraph 9.4. Another way of understanding this new gauge symmetry is the following. In the first order formalism, the fundamental object is an orthonormal frame (the vielbein) whereas in the second order formalism the fundamental object is the metric. There are several choices of an orthonormal frame for a given metric; the freedom in in the choice of the orthonormal frame is encoded by this new gauge symmetry.

2.4. Vielbein as fundamental fields.

In the first order formalism, it is common to use the vielbein e^a_μ defined in 11.3 instead of the metric g as fundamental field. The connection form is accordingly put with orthonormal indices, i.e. we use the $\omega^a{}_b$ of 11.12 instead of Γ. In term of this fields, the Einstein-Hilbert action takes the form

$$\mathcal{S}_{\text{EH}} = \int \varepsilon_{abcd}\; R^{ab} \wedge e^c \wedge e^d, \tag{2.8}$$

where R^{ab} is the Riemann tensor, while the one with cosmological constant becomes

$$\mathcal{S}_{\Lambda\text{EH}} = \int \varepsilon_{abcd} \left(R^{ab} \wedge e^c \wedge e^d + \Lambda e^a \wedge e^b \wedge e^c \wedge e^d \right). \tag{2.9}$$

The integral is written as the integral over a four-form instead of a zero one, hence the appearance of wedge products, as the fact that the metric determinant do not appears explicitly. Finally, we have also introduced the Levi-Civita tensor ε_{abcd}, whose values is the sign of the signature of the permutation sending $\{0,1,2,3\}$ to $\{a,b,c,d\}$. [Pro] contains all the standard convention we adopt concerning it. In particular

$$\varepsilon^{abcd} = \eta^{ae}\eta^{bf}\eta^{gc}\eta^{hd}\varepsilon_{efgh}, \tag{2.10}$$

$$\varepsilon_{\mu\nu\lambda\rho} = e^a_\mu e^b_\nu e^c_\lambda e^d_\rho \varepsilon_{abcd}, \tag{2.11}$$

$$\varepsilon^{\mu\nu\lambda\rho} = E^\mu_a E^\nu_b E^\lambda_c E^\rho_d \varepsilon^{abcd}. \tag{2.12}$$

In this manuscript, we will mainly use this form of the Einstein-Hilbert action. In this formalism, the gauge transformation

2.5. Lorentz gauge transformations. Expressed in term of vielbein and spin connection, the gauge symmetries evoked in (2.3) are, for $\mathcal{S}_{\text{EH}}$

$$\delta\omega^{ab} = \mathrm{d}\lambda^{ab} + \omega^a{}_c\lambda^{cb} + \omega^b{}_c\lambda^{ac}, \tag{2.13}$$

$$\delta\epsilon^a = \lambda^a{}_b e^b. \tag{2.14}$$

These transformations are the geometric gauge transformations, as presented in paragraph (9.4), for a connection-form ω^a_b of a principal bundle whose structure group is the Lorentz group. The field e^a transforms as a section of an associated vector bundle. In fact, introducing the generators J_{ab}, P_a of the Poincaré algebra

$$[J_{ab}, J_{cd}] = \eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}, \tag{2.15}$$

$$[J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, \qquad [P_a, P_b] = 0, \tag{2.16}$$

we can write (2.13) more compactly

$$\delta\omega = \mathrm{d}\lambda + \frac{1}{2}[\omega\wedge\lambda], \qquad \delta e = \frac{1}{2}[\lambda\wedge e], \tag{2.17}$$

$$\lambda = \frac{1}{2}\lambda^{ab}J_{ab}, \qquad \omega = \frac{1}{2}\omega^{ab}J_{ab}, \qquad e = e^a P_a \tag{2.18}$$

2.6. (Anti-)de Sitter connection.

In the case of a non vanishing cosmological constant, the symmetry algebra is changed: the Lorentz Lie algebra $\mathfrak{so}(3,1)$ (we assume the dimension to be 4) is extended to the de Sitter Lie algebra $\mathfrak{so}(4,1)$ (if $\Lambda > 0$) or anti-de Sitter Lie algebra $\mathfrak{so}(3,2)$. The Poincaré generators P_a are replaced with (anti-)de Sitter generators J_a with commutation relations

$$[J_a, J_b] = \pm\frac{1}{\ell^2}J_{ab} \tag{2.19}$$

In that case, it is possible to view the vielbein e^a as part of the connection as well. In other word, it is possible to define the connection

$$A = \frac{1}{2}\omega^{ab} J_{ab} + \frac{1}{\ell} e^a J_a. \tag{2.20}$$

The gauge symmetry of action $\mathcal{S}_{\Lambda \mathrm{EH}}$ can be cast as a gauge transformation of the connection A. Using a gauge parameter $\lambda = \frac{1}{2}\lambda^{ab} J_{ab} + \lambda^a J_a$ it reads

$$\delta_\lambda A = \mathrm{d}A + \frac{1}{2}[A \wedge \lambda]. \tag{2.21}$$

The action $\mathcal{S}_{\Lambda \mathrm{EH}}$ is not directly built from the connection one-form A. However, if we add the Gauss-Bonnet invariant of paragraph (13.10), we can achieve such a construction. In details, we define the curvature

$$F = \frac{1}{2}\left(R^{ab} \pm \frac{1}{\ell^2} e^a \wedge e^b\right) J_{ab} + \frac{1}{\ell} T^a, \tag{2.22}$$

where T^a is the torsion two-form. Using this curvature, we build the action

$$\mathcal{S} = \varepsilon_{abcd}\left(R^{ab} \pm \frac{1}{\ell^2} e^a \wedge e^b\right)\left(R^{cd} \pm \frac{1}{\ell^2} e^c \wedge e^d\right). \tag{2.23}$$

This action contains the Einstein-Hilbert action with cosmological constant, plus the Gauss-Bonnet term $\varepsilon_{abcd} R^{ab} \wedge R^{cd}$. This term does not contributes to the equation of motions, as it is purely topological. Hence we recover from the action (2.23) the standard equation of motions of general relativity.

2.7. Chern-Simons gravity.

The Einstein-Hilbert action can be extended to any number of space-time dimension

$$\mathcal{S}^{(d)}_{\Lambda \mathrm{EH}} = \varepsilon_{a_1 a_2 a_3 \dots a_d}\left(R^{a_1 a_2} + \Lambda e^{a_1} \wedge e^{a_2}\right) \wedge e^{a_3} \wedge \cdots \wedge e^{a_d}. \tag{2.24}$$

The case $d = 3$ presents the peculiarity of being a Chern-Simons action. A Chern-Simons action is an action whose Lagrangian is a Chern-Simons form, as defined in section (13.15). Here, we can write $\mathcal{S}^{(3)}_{\Lambda \mathrm{EH}}$ as the difference of two integral of the form (13.53) if we set (we follow [Bañ])

$$A^a = -\frac{1}{2}\varepsilon^a{}_{bc}\omega^{bc} + \frac{i}{\ell} e^a, \tag{2.25}$$

$$\overline{A}^a = -\frac{1}{2}\varepsilon^a{}_{bc}\omega^{bc} - \frac{i}{\ell} e^a. \tag{2.26}$$

The connections $A = A^a X_a$, $\overline{A} = \overline{A}^a X_a$ are two $\mathrm{Sl}(2,\mathbb{C})$ connections. The exact relation between the Einstein-Hilbert action and the Chern-Simons forms built from A and $\overline{A}$ is

$$\mathcal{S}^{(3)}_{\Lambda \mathrm{EH}} = \mathcal{S}^{(3)}_{CS}(A) - \mathcal{S}^{(3)}_{CS}(\overline{A}), \tag{2.27}$$

$$\mathcal{S}^{(3)}_{CS}(A) = \int d^3x \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\right) \tag{2.28}$$

where the trace is taken over the representation

$$X_0 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad X_1 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad X_2 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{2.29}$$

We refer to [HZ16] or [Wit88] for more details. Similar Chern-Simons action can be built in any dimension and can be seen as useful toy-model of gauge-invariant systems in higher dimension. However, only in 3 dimensions it matches the Einstein-Hilbert action.

In the first order formulation of gravity, there is an additional gauge symmetry which, this time, behaves like the Yang-Mills gauge symmetry.

3. Supergravity

3.1. The Rarita-Schwinger action.

A common feature of all supergravity theories, is the presence of a spin-$\frac{3}{2}$ field $\Psi = \Psi_\mu dx^\mu$, called the gravitino. Physically, it is the supersymmetric partner of the vielbein; geometrically, it is one-form-section of a spinor bundle. One of the most simple action using such Rarita Schwinger action is

$$\mathcal{S}_{\text{RS}} = \int \overline{\Psi}_\mu \gamma^{\mu\nu\rho} \partial_\rho \Psi, \tag{3.1}$$

where $\overline{\Psi}$ is the Majorana conjugate of Ψ. This action is defined over the Minkowski space. In other words, the metric is the Minkowski metric, and the gamma matrices γ^μ are assumed to be constant (although we used greek indices). The Rarita-Schwinger action is invariant under the infinitesimal transformation

$$\Psi_\mu \mapsto \Psi_\mu + \partial_\mu \epsilon. \tag{3.2}$$

Transformation (3.2) is an example of a supersymmetry transformation, where the symmetry parameter is fermionic.

3.2. A supergravity action.

A simple supergravity action in four dimension can be obtained by adding to the Einstein-Hilbert action a slightly deformed version of the Rarita-Schwinger action. This supergravity action is

$$\mathcal{S} = \int \varepsilon_{abcd} R^{ab} \wedge e^c \wedge e^d + \overline{\Psi} \wedge e^a \wedge (\mathrm{d} + \frac{1}{4}\omega^{ab}\gamma_{ab})\Psi. \tag{3.3}$$

As such, the action is not supersymmetric. However, it is possible to make it invariant under a supersymmetry transformation by going back two a second-order formalism, where the spin connection ω^{ab} is no more seen as an invariant field but instead is determined by the fields e^a and Ψ. Its expression is obtained through its equations of motion

$$\omega^a{}_{b\mu} = \omega^a{}_{b\mu}(e) + K^a{}_{b_\mu}(\psi), \tag{3.4}$$

where $\omega^a_{b\mu}(e)$ is the torsionless Levi-Civita connection, depending exclusively on the vielbein field e^a_μ, and $K^a_{b\mu}(\psi)$ is the contorsion tensor whose dependence from the gravitino is (eq. 9.21 in[FV12])

$$K_{\nu\rho\mu}(\psi) = -\frac{1}{4}(\overline{\Psi}_\mu\gamma_\rho\Psi_\nu - \overline{\Psi}_\nu\gamma_\mu\Psi_\rho + \overline{\Psi}_\rho\gamma_\nu\Psi_\mu). \tag{3.5}$$

In the last equation, μ is the form index. To go from the natural to the orthonormal indices, we have to use equation (11.11). When the action (3.3) is expressed with $\omega^a{}_b$ satisfying (3.4), it invariant under the supersymmetry transformations

$$\delta e^a = \frac{1}{2}\bar{\epsilon}\gamma^a\Psi, \qquad \delta\Psi_\mu = (\mathrm{d} + \frac{1}{4}\omega^{ab}\gamma_{ab})\epsilon. \tag{3.6}$$

Working with (3.4) is sometimes called the 1.5 formalism. This denomination comes from the fact that we are working in the second-order formalism but with fields expressed in a notation usually used in the first-order formalism.

3.3. Geometric supergravity actions.

Earlier, we have shown that we can see the fundamental fields of gravity as related with the Poincaré (or (anti-)de Sitter) algebra. This fact can be extended to supergravity. For example, the field Ψ^α appearing in (3.3) can be associated with a super-generator Q_α. The algebra spanned by the generators $\{J_{ab}, P_a, Q_\alpha\}$ is called the super-Poincaré algebra. The new non-vanishing commutation relations are

$$[J_{ab}, Q_\alpha] = -\frac{1}{2}(\gamma_{ab})^\beta{}_\alpha Q_\beta, \qquad \{Q_\alpha, Q_\beta\} = -\frac{1}{2}(\gamma^a)_{\alpha\beta}P_a. \tag{3.7}$$

We have used the notation $\{,\}$ to denote the symmetric anti-commutator of two fermionic (i.e. with degree 1) generators. The fields involved in 3.3 can be put together forming a "super-connection one-form"

$$\mathbb{A} = \frac{1}{2}\omega^{ab}J_{ab} + e^a P_a + \Psi^\alpha Q_\alpha, \tag{3.8}$$

however 3.3 cannot be written as an integral over a functional in $\mathbb{A}$ (unlike a Yang-Mills action); and one of the main motivation of the two following works we are about to present is to write down actions starting from a very geometrical point of view, explicitly using a super-connection $\mathbb{A}$. This idea is not new, as an example, the MacDowell-Mansouri action for supergravity uses this formalism [MM77]. Also, Chern-Simons gravity can be extended to Chern-Simons supergravity, where the connection one form A of 2.28 is replaced by a super-connection $\mathbb{A}$. But before going to the explicit geometric action, we would like to further develop some points about real super-algebras and their connections.

3.4. Super-Lie algebras for AdS supergravities.

3.4.1. *The complex AdS super-algebra.*

$\mathfrak{sl}(m|1,\mathbb{C})$ of $(m+1)\times(m+1)$ is the super-algebra of super-traceless complex super matrices. The supercharges are represented by

$$\overline{\mathbf{Q}}_\alpha = \left[\begin{array}{c|c} 0_{m\times m} & E_{\alpha 1} \\ \hline 0_{1\times m} & 0 \end{array}\right], \qquad \mathbb{Q}^\alpha = \left[\begin{array}{c|c} 0_{m\times m} & 0_{m\times 1} \\ \hline E_{1\alpha} & 0 \end{array}\right]. \tag{3.9}$$

where $E_{1\alpha}$ denotes the elementary matrix with entry 1 in the $(1,\alpha)$ position and 0 everywhere else. The supercharges will be associated to the physical spinors of the theory. The basis of the even part is chosen as:

$$\mathbf{J}^{\mathbb{C}}_{a_1\dots a_n} = \left[\begin{array}{c|c} \gamma^{a_1\dots a_k} & 0_{m\times 1} \\ \hline 0_{1\times m} & 0 \end{array}\right], \quad n\geq 1, \qquad \mathbf{J}^{\mathbb{C}}_0 = \left[\begin{array}{c|c} \mathbb{1} & 0_{m\times 1} \\ \hline 0_{1\times m} & m \end{array}\right]. \tag{3.10}$$

3.4.2. *The real AdS super-algebra.*
We consider the following real structure:

$$\mathrm{J}(X) = -\left[\begin{array}{c|c} A & \\ \hline & 1 \end{array}\right]^{-1} X^{\dagger} \left[\begin{array}{c|c} A & \\ \hline & 1 \end{array}\right], \tag{3.11}$$

where A is the matrix defining the Dirac conjugate of a spinor, usually $A = \gamma_0$. This real structure is chosen for three reasons:

(i) It preserves the decomposition of the even basis as in (3.10) (modulo factors of i),
(ii) It preserves the generators of the Lorentz algebra

$$\mathbf{J}_{a_1 a_2} = \left[\begin{array}{c|c} \frac{1}{2}\gamma^{a_1 a_2} & 0_{m\times 1} \\ \hline 0_{1\times m} & 0 \end{array}\right], \tag{3.12}$$

(iii) The spinors associated to the real supercharges can be reassembled into a Dirac spinor and its Dirac conjugate (see the next paragraph).

The basis of the obtained real super-algebra is given by

$$\mathbf{J}_{a_1\dots a_n} = \left[\begin{array}{c|c} \frac{1}{2}\gamma^{a_1\dots a_n} & 0_{m\times 1} \\ \hline 0_{1\times m} & 0 \end{array}\right], \quad n \equiv 1,2 \mod 4, \tag{3.13}$$

$$\mathbf{J}_{a_1\dots a_n} = i\left[\begin{array}{c|c} \frac{1}{2}\gamma^{a_1\dots a_n} & 0_{m\times 1} \\ \hline 0_{1\times m} & 0 \end{array}\right], \quad n \equiv 0,3 \mod 4, \tag{3.14}$$

$$\mathbf{J}_0 = i\left[\begin{array}{c|c} \mathbb{1} & 0_{m\times 1} \\ \hline 0_{1\times m} & m \end{array}\right], \tag{3.15}$$

for the bosonic part, and

$$\mathbf{T}_{1\alpha} = \left[\begin{array}{c|c} 0_{m\times m} & E_{\alpha(m+1)} \\ \hline E_{(m+1)\alpha}A & 0 \end{array}\right], \qquad \mathbf{T}_{2\alpha} = \left[\begin{array}{c|c} 0_{m\times m} & iE_{\alpha(m+1)} \\ \hline -iE_{(m+1)\alpha}A & 0 \end{array}\right], \tag{3.16}$$

for the fermionic part. We have

$$[\mathbf{J}_{a_1}, \mathbf{J}_{a_2}] = \mathbf{J}_{a_1 a_2}. \tag{3.17}$$

Therefore we call this algebra an AdS super-algebra. Using the fact that there is a representation in which (see for example [FV12])

$$A = \gamma_0 = \sigma_1 \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \sim \mathrm{Id}_{\frac{m}{2},\frac{m}{2}}, \tag{3.18}$$

the AdS super-algebra is classified as $\mathfrak{su}(\frac{m}{2}, \frac{m}{2}|1)$.

3.4.3. *Using the complex generators.*

The AdS super-algebra is a real algebra and all its structure constants are real as it can be verified with this example

$$[\mathbf{J}_{ab}, \mathbf{T}_{1,\alpha}] = \mathbf{T}_{1,\beta}(\frac{1}{2}\mathrm{Re}(\gamma_{ab})^{\beta}{}_{\alpha} + \frac{1}{2}\mathrm{Im}(\gamma_{ab})^{\beta}{}_{\alpha}). \tag{3.19}$$

In order to build a real Lagrangian, one should associate real spinor-fields with the generators $\mathbf{T}_1$ and $\mathbf{T}_2$, let us call them ϕ_1 and ϕ_2. They are real in the most basic sense: $\phi_i^* = \phi_i$, where * stands for the standard complex conjugation. Under a Lorentz transformation, these fields are multiplied by real and imaginary parts of gamma matrices, as dictated by (3.19). Because of these transformations rules, the fields ϕ_i will not be interpreted as physical fields. This explains why it is more convenient to introduce the complex spinor field $\psi = \phi_1 + i\phi_2$. Indeed, it is straightforward to show that ψ so defined will, under a Lorentz transformation (with parameter λ^{ab}), changes as $\psi \mapsto \psi + \frac{1}{4}\lambda^{ab}\gamma_{ab}\psi$. In a connection for the $\mathfrak{su}(\frac{m}{2},\frac{m}{2}|1)$, we obtain

$$\mathbb{A}_{\mathfrak{su}(\frac{m}{2},\frac{m}{2}|1)} = \cdots + \mathbf{T}_1\phi_1 + \mathbf{T}_2\phi_2 = \ldots \overline{\mathbf{Q}}\psi + \overline{\psi}\overline{\mathbf{Q}}, \tag{3.20}$$

with $\mathbb{Q}, \overline{\mathbf{Q}}$ the generators given in (3.9). In other words, the $\mathbb{Q}$ and $\overline{\mathbf{Q}}$ generators can be used in the connection as long as the spinor fields multiplying them are Dirac conjugate of each others. Whereas for a general complex connection, the spinor fields ψ, $\overline{\psi}$ associated to $\overline{\mathbf{Q}}$, $\mathbb{Q}$ are totally independent of each other.

3.4.4. *Representation of Higher $\mathcal{N}$ algebra.*

The fundamental representation of $\mathfrak{sl}(m|M,\mathbb{C})$ is given by

$$\mathbf{J}^{\mathbb{C}}_{a_1\ldots a_n} = \left[\begin{array}{c|c} \gamma^{a_1\ldots a_k} & 0_{m\times M} \\ \hline 0_{M\times m} & 0_{M\times M}\end{array}\right], \quad n\geq 1, \qquad \mathbf{J}^{\mathbb{C}}_0 = \left[\begin{array}{c|c} \mathrm{Id}_{m\times m} & 0_{m\times M} \\ \hline 0_{M\times m} & \frac{m}{N}\mathrm{Id}_{M\times M}\end{array}\right], \tag{3.21}$$

$$\overline{\mathbf{Q}}^{i}_{\alpha} = \left[\begin{array}{c|c} 0_{m\times m} & E_{\alpha i} \\ \hline 0_{M\times m} & 0_{M\times M}\end{array}\right], \qquad \mathbb{Q}^{\alpha}_{i} = \left[\begin{array}{c|c} 0_{m\times m} & 0_{m\times M} \\ \hline E_{i\alpha} & 0_{M\times M}\end{array}\right], \tag{3.22}$$

$$\mathbf{K}_I = \left[\begin{array}{c|c} 0_{m\times m} & 0_{m\times M} \\ \hline 0_{M\times m} & K_I\end{array}\right], \tag{3.23}$$

where K_I denotes the standard representation of $\mathfrak{su}(M)$. One then uses the same real structure as in (3.11), with the 1 in the bottom-right corner changed by an $\mathrm{Id}_{M\times M}$, and get $\mathfrak{su}(\frac{m}{2},\frac{m}{2}|M)$. We will be in particular interested in $\mathfrak{su}(2,2|2)$.

3.4.5. *Reduction to Majorana spinors.*

It was already shown in [TZ99] how to reduce the Dirac spinors to (symplectic) Majorana spinors. Here, we would like to propose a similar construction, but in a way that does care for the reality of the algebra at every step. We will work here with $\mathfrak{sl}(\frac{m}{2},\frac{m}{2}|1)$, the cases of the different Poincaré super-algebra can be inferred from it. We first start with the Majorana case. As we are back in the case $t = 1$, a quick glance at the preceding table forces us to put tilde over our C and ϵ. In order to perform the reduction, one needs to find a new generator

$\mathbf{S}$, satisfying (3.11) such that, for any real spinor (in the sense $\phi^* = \phi$), one has:

$$\phi\mathbf{S} = \overline{\mathbf{Q}}\psi + (-)^{q}\overline{\psi}\mathbb{Q}, \tag{3.24}$$

where ψ is a Majorana spinor. Suppose that such a generator $\mathbf{S}_\alpha$ exists and write it

$$\mathbf{S}_\alpha = \left[\begin{array}{c|c} & S_\alpha \\ \hline S_\alpha^\dagger D & \end{array}\right]. \tag{3.25}$$

The Majorana condition $\psi^c = \psi^\dagger$ implies the same condition on S_α. We obtain it by considering an invertible matrix M satisfying

$$M^\dagger D = M^T \tilde{C}, \tag{3.26}$$

and then by setting:

$$S_\alpha = M E_{\alpha,(m+1)}. \tag{3.27}$$

For example one can take $M = \frac{1+B}{2}$ (if invertible). The next step to go to an $\mathcal{N} = 1$ supergravity theory is to find a subalgebra with the $\mathbf{S}_\alpha$'s generators. The anti commutator is found to be

$$\left\{\mathbf{S}_\alpha, \mathbf{S}_\beta\right\} = \sum \frac{(-)^{n(n-1)/2}}{2^k n!}(1 + \tilde{\epsilon}(-)^{n(n+1)/2})\left(M^T \tilde{C}\gamma_{a_1\ldots a_n} M\right)_{\beta\alpha}(-i)^q \mathbf{J}_{a_1\ldots a_n}. \tag{3.28}$$

with

$$q \equiv nt + \frac{n(n-1)}{2} + 1 \text{ mod } 2. \tag{3.29}$$

We check that the coefficient
$(-i)^q\left(M^T\tilde{C}\gamma_{a_1\ldots a_n}M\right)_{\beta\alpha}$ is real. Then, we need to check that all generators appearing in the right-hand-side of (3.28) together with the generators S_α form a subalgebra. For the $\mathbf{J}_{a_1\ldots a_n}$ generators, we compute

$$(\tilde{C}[\gamma_{a_1\ldots a_n}, \gamma_{b_1\ldots b_m}])^T = -\tilde{\epsilon}\tilde{C}[\gamma_{a_1\ldots a_n}, \gamma_{b_1\ldots b_m}], \tag{3.30}$$

where in the left-hand-side the labels n, m are such that $(\tilde{C}\gamma_{a_1\ldots a_n})^T = \tilde{C}\gamma_{a_1\ldots a_n}$. In order for the right-hand-side to have the same symmetry as the left-hand-side, we need $\epsilon = -1$. We finally check the commutator

$$[\mathbf{J}_{a_1\ldots a_n}, \mathbf{S}_\alpha] = \left(M^{-1}\frac{1}{2}i^q\gamma_{a_1\ldots a_n}M\right)^\beta{}_\alpha \mathbf{S}_\beta. \tag{3.31}$$

In this computation, we have to be sure that the structure constant appearing in the right-hand-side is real, otherwise it would require the introduction of a new generator (*e.g.* iS_α), and in that case, the algebra will not close. Using the defining properties of M (3.26), one verifies that the coefficient is indeed real. The symplectic Majorana case is very similar. The first difference is that one need to introduce an index i associated to higher $\mathcal{N}$ theories. Then S^i_α needs to be a "symplectic Majorana column" making the matrix M to satisfy

$$M^T \epsilon C = M^\dagger D. \tag{3.32}$$

The anti commutator then becomes

$$\{\mathbf{S}^i_\alpha, \mathbf{S}^j_\beta\} = \sum \frac{(-)^{n(n-1)/2}}{2^m n!}(1 - \epsilon(-)^{n(n\pm1)/2})(M^T \epsilon C\gamma_{a_1\ldots a_n}M)^{ji}_{\beta\alpha} J_{a_1\ldots a_n}. \tag{3.33}$$

Relation (3.30) is still valid, however now we want only matrices satisfying the condition $(C\gamma_{a_1\ldots a_n})^T = -C\gamma_{a_1\ldots a_n}$, and this implies $\epsilon = 1$. If we look at the commutator between even and odd generators, we obtain again (3.31). The reality condition of the structure constants can be checked as before.

CHAPTER 5

$\mathcal{N} = 2$ Extended MacDowell-Mansouri Supergravity

1. Conditional symmetries

Yang-Mills and Einstein-Hilbert field theories are the two main theories of fundamental physics. Although both are gauge theories, they present an important difference. In Yang-Mills theories, the fundamental fields of the theories are the so-called *gauge fields* geometrically interpreted as (smooth) sections of a principal G-bundle, G, the *gauge group* (usually $U(1)$ or $SU(N)$) is a finite dimensional Lie group. In contrast, the fundamental field of Einstein-Hilbert theory, in his most common presentation, is the metric field $g_{\mu\nu}$, which does not take values in the gauge group of the theory. Furthermore, in this textbook picture of gravity theory, the gauge group is the *infinite dimensional* Lie[1] group of diffeomorphism of the manifold over which is defined the metric. Using the first order formalism for gravity, it is possible to give a Yang-Mills-like aspect for the Einstein Hilbert theory. In this case, the fundamental fields are the vielbein (replacing the metric) and the spin connection, which both take values in the finite dimensional Poincaré Lie group. However, only the Lorentz (structure) group is a gauge invariance of the Einstein-Hilbert action, while

We have seen that in first order gravity, the fundamental fields - the vielbein and the spin connection - takes values in the Poincaré Lie algebra. We have also seen that the theory possesses a Lorentz gauge invariance which does not require additional conditions and is therefore called an *off-shell* symmetry. On the other hand local translation invariance is a broken symmetry [Reg86]. In order to enforce local translation invariance of the action, it is necessary to appeal to the so-called *torsion constraint*, which is a consequence of the field equations and it is therefore referred to as an *on-shell* symmetry. In *pure supergravity* [FNF76; DZ76]—composed by the Einstein-Hilbert and the Rarita-Schwinger actions—supersymmetry remains on-shell [FN76] up to a torsion constraint, $T^a \cong \bar{\psi}\gamma^a\psi$, like in first order gravity. The introduction of auxiliary fields [SW78; Fv78] makes it possible to realize the off-shell fermionic symmetry (for further details see *e.g.* [FV12; RV20; DRV21]). Leaving aside the introduction of auxiliary fields, on-shell and off-shell symmetries play different roles in (super)gravity. As is well known, off-shell symmetries can be represented by a principal bundle. Broken off-shell symmetries on the other hand, which are preserved when some constraints are imposed, could be understood as sections of an associated vector bundle. Indeed, as both on- and off-shell symmetries form a group, there is a natural representation of the structure group on the generators of infinitesimal on-shell symmetries.

[1] If we do not restrict Lie group to be of finite dimensions

Symmetries that are realized on the surface of the field equations and which belong to certain Lie (super)group that also contains (unbroken) off-shell symmetries are often referred to as "on-shell symmetries".[2] The equations of motion provide *sufficient but not necessary* conditions for these symmetries to hold; the consistency conditions (*cf.* , $\tilde{\Upsilon}=0$ in section 2), provide *necessary conditions* for the invariance of the action and they can be therefore called *symmetry constraints*. The symmetry constraints are, in general, less restrictive than the equations of motion and we shall refer to the symmetries that arise when these constraints hold as *conditional symmetries*.

The MacDowell-Mansouri approach [MM77] of pure SUGRA shows clearly this pattern. Their supergravity action principle is a quadratic form of the gauge curvature for a $osp(4|1)$-valued connection, however, this bilinear explicitly breaks the $OSp(4|1)$ symmetry leaving unbroken only the Lorentz subgroup. The translation symmetry is broken and the corresponding "dual" symmetry constraint holds on the surface of the torsion constraint. Supersymmetry is also broken and the dual symmetry constraint appears as a product of the torsion and the fermion curvature, which is therefore automatically satisfied also imposing the torsion constraint.

More precisely, let $\mathfrak{g}$ be a super algebra, $\mathcal{A}\in\mathfrak{g}$ the gauge potential, and $S_{pt}[\mathcal{A}]$ the corresponding action principle. The variation of the action with respect to the $\mathcal{A}$ reads,

$$\delta S_{pt}=\int \delta\mathcal{A}^{\mathbf{M}}\Upsilon_{\mathbf{M}}+\text{b.t.}, \tag{1.1}$$

where $\mathbf{M}$ labels the (super)algebra generators and the differential operator $\Upsilon_{\mathbf{M}}=\Upsilon_{\mathbf{M}}[\mathcal{A}]$ is "dual" to $\delta\mathcal{A}$. The action is invariant under the proposed variation if $\Upsilon_{\mathbf{M}}=0$, which defines the field equations of the system. In what follows we shall often omit the $\wedge$-product of differential forms, and assume that the (anti)commutator $[\,,\,]$ is graded with respect the form degree and statistics of the fields, consistently with the Lie (super)algebra under consideration.

For gauge transformations the transformation parameter takes the particular form $\delta\mathcal{A}^{M}=(\mathcal{D}\lambda)^{\mathbf{M}}$, where $\mathcal{D}$ is the covariant derivative. Clearly the action remains invariant for "Killing vectors" parameters λ, $\mathcal{D}\lambda=0$, or when λ is in the kernel of $\tilde{\Upsilon}$. When this is not the case, upon partial integration (1.1) yields,

$$\delta_{\lambda}S_{pt}=\int \lambda^{\mathbf{M}}\tilde{\Upsilon}_{\mathbf{M}}+\text{b.t.}, \tag{1.2}$$

where the dual differential operator $\tilde{\Upsilon}_{\mathbf{M}}\cong(\mathcal{D}\Upsilon[\mathcal{A}])_{\mathbf{M}}$ is dual to the parameter $\lambda^{\mathbf{M}}$. It turns out that $\tilde{\Upsilon}=0$ is an integrability condition for the equation of motion $\Upsilon=0$ and, at the same time, an indicator of whether the parameter $\lambda^{\mathbf{M}}$ generates a symmetry or not.

Consistently with our previous definition, an *off-shell symmetry* is the one for which $\tilde{\Upsilon}_{\mathbf{M}}\equiv 0$ is an identity. Reciprocally, a *conditional gauge symmetry* is one for which $\tilde{\Upsilon}_{\mathbf{M}}$ does

[2] The expression "on-shell symmetry" appears to us as vacuous, since any transformation of a field $\delta\mathcal{A}$ leaves invariant the action on-shell. Instead of on- and off-shell symmetries, we prefer to refer to them as unconditional or conditional symmetries, respectively.

not vanish identically, but needs to be imposed as a constraint $\tilde{\Upsilon}_{\mathbf{M}} = 0$. Thus, the index $\mathbf{M}$ in (1.2) can be restricted to run over the off-shell broken symmetry generators only.

Geometrically, the distinction between off-shell and conditional symmetries can be understood as follow. Let $\mathfrak{g} \ni \mathcal{A}$ be the Lie (super)algebra generating the a Lie (super)group G—which combines off-shell and conditional symmetries—and let $\mathfrak{h}$ be the algebra of off-shell symmetries generating the subgroup $H \subset G$. The broken gauge symmetries are those in the coset G/H, spanned locally by the vector subspace $\mathfrak{f} \subset \mathfrak{g}$. Denoting by $\mathfrak{h}$ the Lie subalgebra corresponding to the subgroup H, we can decompose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{f}$, where the direct sum is a vector space direct sum. The differential operators $\tilde{\Upsilon}_{\mathbf{M}}$ dual to the generators of $\mathfrak{h}$ vanish identically whilst those dual to the generators of $\mathfrak{f}$ do not.

Since $[\mathfrak{h}, \mathfrak{f}]$ is a subset of $\mathfrak{f}$ the gauge fields valued in the algebra elements $\mathfrak{f}$ transform as vectors under the endomorphisms generated by the unbroken symmetry algebra $\mathfrak{h}$. Effectively, the gauge fields components in $\mathfrak{f}$ can be regarded as fiber bundle sections. The group H is the real structure group of the fiber bundle. From this perspective, the group G puts together the structure group H and fiber sections, unifying the H-principal bundle and the associated fiber bundle with sections in $\mathfrak{f}$. For a more rigorous exposition of these subjects see *e.g.* [Ede20; Ede21].

In order to illustrate these aspects we shall consider briefly Yang-Mills theories and the MacDowell-Mansouri approach [MM77] for $\mathcal{N} = 1$ (super)gravity. In the first case, the action reads,

$$S_{\mathrm{YM}} = \int \mathtt{tr}\ \mathcal{F} * \mathcal{F}\,, \tag{1.3}$$

where $\mathcal{F} = d\mathcal{A} + \mathcal{A}^2$ is the 2-form field strength of the gauge connection one form $\mathcal{A} \in \mathfrak{g}$ and $*$ is the Hodge dual operator. The variation of the gauge field $\delta\mathcal{A}$ and the gauge transformation $\delta\mathcal{A} = \mathcal{D}\lambda$ yields respectively (1.1) and (1.2) with dual differential operators,

$$\Upsilon = D * \mathcal{F}\,, \qquad \tilde{\Upsilon} = [\mathcal{F}, *\mathcal{F}]\,. \tag{1.4}$$

For general Yang-Mills theories $[\mathcal{F}, *\mathcal{F}] \equiv 0$ and the whole symmetry group G is preserved off-shell.

Next, the MacDowell-Mansouri action is given by

$$S_{\mathrm{MM}} = \int \mathcal{F}^{\mathbf{M}} Q_{\mathbf{MN}} \mathcal{F}^{\mathbf{N}}\,, \tag{1.5}$$

where now the field strength $\mathcal{F}$ is valued in the algebra $\mathfrak{g} = so(3,2)$ for first order gravity, or in $\mathfrak{g} = osp(4|1)$ for pure supergravity. Appealing to the standard nomenclature, in the first case, $G = SO(3,2)$ and $\mathcal{A} = \frac{1}{2}\omega^{ab}\mathbb{B}_{ab} + e^a\mathbb{B}_a$; in the second case, $G = OSp(4|1)$ and $\mathcal{A} = \frac{1}{2}\omega^{ab}\,\mathbb{B}_{ab} + e^a\,\mathbb{B}_a + \overline{\mathbb{Q}}_\alpha\,\psi^\alpha$. In both cases $Q_{\mathbf{MN}}$ is a Lorentz invariant tensor but it breaks explicitly the transvection symmetry generated by $\mathbb{B}_a$, and it also breaks the supersymmetry transformations generated by $\overline{\mathbb{Q}}$. It can be shown that for the Lorentz transformation parameters the dual constraint vanishes identically, whilst for supersymmetry the dual constraints (see *e.g.* [Nie04]),

$$\tilde{\Upsilon} \cong \mathcal{F}^a \gamma_a D\psi = 0\,, \tag{1.6}$$

where $\mathcal{F}^a$ is the transvection-valued component of the field strength, γ is a Dirac matrix and $D\psi$ is the covariant derivative for the $so(3,2)$ connection acting on the gravitino field in the Majorana representation. Although (1.6) admits different solutions, it is often solved using the more restrictive field equations. Indeed, the equation obtained by extremizing the action with respect to the spin connection yields,

$$\Upsilon_{cd} \cong \epsilon_{abcd}\, \mathcal{F}^a\, e^b = 0\,, \tag{1.7}$$

implying, for invertible vierbeins (e^b_μ), the *torsion constraint*

$$\mathcal{F}^a = 0\,. \tag{1.8}$$

Here $\mathcal{F}^a$ consists of the vierbein torsion in pure gravity, and in the case of supergravity it also contains an additional 2-form fermion-current. Hence on the surface of the constraint (1.7) the action is invariant under local supersymmetry transformations. Alternatively, when the gravitino field strength $D\psi$ vanishes, or when it is in the kernel of $\mathcal{F}\gamma$, supersymmetry also holds.

The constraint dual to the transvection transformation parameters is

$$\tilde{\Upsilon} \cong \frac{1}{2}\epsilon_{abcd}\, \mathcal{F}^{bc}\, \mathcal{F}^d - D\overline{\psi}\,\gamma_a\, i\gamma_5 D\psi = 0\,, \tag{1.9}$$

which, using the torsion constraint (1.8), reduces to

$$D\overline{\psi}\,\gamma_a\, i\gamma_5 D\psi = 0\,. \tag{1.10}$$

This constraint can be satisfied for example if

$$i\gamma_5 D\psi = \varphi D\psi + \varphi' * D\psi, \tag{1.11}$$

where φ and φ' are scalar fields. This is because $(C\gamma^a)_{\alpha\beta}$ is symmetric in its spinor indices, where C is the conjugation matrix, whilst the products $(D\psi)^\alpha (D\psi)^\beta$ and $(D\psi)^\alpha * (D\psi)^\beta$ are antisymmetric. Conditions (1.11) can be fulfilled by configurations that are not necessarily solutions of the field equations, but it can be checked that on-shell configurations do satisfy the constraint (1.10). Indeed, the Rarita-Schwinger equation obtained by varying with respect to the gravitino,

$$\not{e}\, D\psi = 0\,, \qquad \text{where } \not{e} := \gamma_a e^a, \tag{1.12}$$

implies that (1.10) holds. In order to prove this, we can use the equivalent form of the Rarita-Schwinger equation in four dimensions (see (4.18)),

$$(i\gamma_5 - *)D\psi = 0\,, \tag{1.13}$$

which is in the class of (1.11) for $\varphi = 0$ and $\varphi' = 1$. Hence, both local supersymmetry and transvection invariance are conditional symmetries of $\mathcal{N} = 1$ supergravity.

2. Notations

In the following sections we consider an $\mathcal{N}=2$ supergravity model following the same pattern, unifying the MacDowell-Mansouri supergravity and the non-abelian $U(1)\times SU(2)$ Yang-Mills theory. This model has been presented in [Alv+21b]. Our action principle can be expressed in the Yang-Mills fashion,

$$S := -\int \mathtt{str}\ \mathcal{F} \circledast \mathcal{F}\,, \tag{2.1}$$

(*cf.* [Wis10] in the pure gravity case) where $\mathcal{F}=d\mathcal{A}+\mathcal{A}^2 \in \mathfrak{g}=su(2,2|2)$, $\mathtt{str}$ is the supertrace and $\circledast$ combines the standard Hodge operator and an involution of the superalgebra $su(2,2|2)$. The explicit form of $\circledast$ is given in Eq. (3.3).

The field equations and corresponding consistency conditions that follow from (2.23) can be written as

$$\Upsilon \cong \mathcal{D}\circledast\mathcal{F}^+ = 0\,, \qquad \tilde{\Upsilon} = \mathcal{D}^2\circledast\mathcal{F}^+ = [\mathcal{F}\,,\circledast\mathcal{F}^+] = 0\,, \tag{2.2}$$

where $\mathcal{D}$ is the $su(2,2|2)$ covariant derivative and $\mathcal{F}^+$ is the $su(2,2|2)$ curvature with the terms along transvection generators removed. The removal of the transvection terms is prompted by the $\circledast$ operator, which is necessary in order to recover the pure (super)gravity sector.

The gauge transformation of the action, (1.2), takes the form

$$\delta S := -\int \mathtt{str}\ \lambda\,[\mathcal{F},\circledast\mathcal{F}^+]\ + \text{b.t.}\,, \tag{2.3}$$

hence the $su(2,2|2)$ symmetry holds on the surface of the non-trivial components of the integrability condition $\tilde{\Upsilon}$ in (2.2). As we shall see $[\mathcal{F},\circledast\mathcal{F}^+]$ vanishes trivially except for the terms along transvection generators and supercharges, analogously to $\mathcal{N}=1$ supergravity. Thus, the group $G=SU(2,2|2)$ breaks into $H=SO(3,1)\times\mathbb{R}\times U(1)\times SU(2)$ off-shell symmetries while transvections and supersymmetry are conditional symmetries.

2.1. Superalgebra representation.

In this section, we consider $su(2,2|2)$ as spanned by

$$\{\underbrace{\mathbb{B}_{ab}\,,\mathbb{B}_a\,,\tilde{\mathbb{B}}_a\,,\mathbb{B}_5}_{so(4,2)};\ \underbrace{\mathbb{B}_6\,,\mathbb{B}_I}_{u(1)\oplus su(2)}\ ;\quad \underbrace{\mathbb{Q}^{\alpha}_i\,,\overline{\mathbb{Q}}^i_{\alpha}}_{\text{supercharges}}\quad\}\,. \tag{2.4}$$

This representation will allow us to handle complex gravitino fields charged under $U(1)\times SU(2)$ interactions

Here the $so(4,2)$ generators are labeled by spacetime indices (a,b) in the range $0,1,2,3$, $su(2)$ indices (I) in the range $7,8,9$, and spinorial labels (α) in the range $1,\dots,4$, whilst there is single $u(1)$ generator with the label 6. Hence the whole set of internal symmetry generators are labeled by Latin letters, $r,s,\dots$, in the range $6,7,8,9$, $\mathbb{B}_r\in u(1)\oplus su(2)$. The supercharges *isospin* labels $i=1,2$, transform in the fundamental representation of $su(2)\oplus u(1)$.

The adjoint action of the bosonic generators, denoted $\mathbb{B}$, onto the fermionic generators

$$[\mathbb{Q}^{\alpha}_i,\mathbb{B}_M] = (B_M)^{\alpha\,j}_{i\,\beta}\ \mathbb{Q}^{\beta}_j\,, \qquad [\mathbb{B}_M,\overline{\mathbb{Q}}^i_{\alpha}] = \overline{\mathbb{Q}}^j_{\beta}\ (B_M)^{\beta\,i}_{j\,\alpha}\,, \tag{2.5}$$

provides the fundamental representations of the spacetime symmetry algebra $so(4,2)$ and the internal symmetry algebra $u(1)\oplus su(2)$, by means of the structure constants B_M, where the indices $M\in\{a,[ab],r\}$ label Lorentz vectors/tensors and internal symmetry generators.

The constant of structures B_M can be expressed in terms of tensor products involving 4×4 spinor representations for spacetime symmetry generators,

$$so(4,2): \qquad (B_M)^{\alpha j}_{i\beta}=\delta_i{}^j\times\left\{\frac{1}{2}(\gamma_{ab})^\alpha{}_\beta,\quad \frac{1}{2}(\gamma_a)^\alpha{}_\beta,\quad \frac{1}{2}(\gamma_a\gamma_5)^\alpha{}_\beta,\quad \frac{1}{2}(\gamma_5)^\alpha{}_\beta\right\}, \tag{2.6}$$

or from 2×2 matrices for internal symmetries

$$u(1)\oplus su(2): \qquad (B_M)^{\alpha j}_{i\beta}=\delta^\alpha{}_\beta\times\left\{-i(\mathbb{1}_{2\times 2})_i{}^j,\quad -\frac{i}{2}(\sigma_I)_i{}^j\right\}, \tag{2.7}$$

where γ's are Dirac gamma matrices and σ_I are the Pauli matrices. We shall denote the adjoint representation

$$\rho(\mathbb{B}_M)=B_M\,, \tag{2.8}$$

simply by ρ-representation.

Introducing the Killing form $\mathcal{K}_{MN}$ normalized by,

$$\mathtt{str}\,(\mathbb{B}_M\mathbb{B}_N)=\mathcal{K}_{MN}\,, \tag{2.9}$$

the anti-commutator $[\mathbb{Q},\overline{\mathbb{Q}}]_+$ can be cast in a compact form using the representation ρ of the bosonic subalgebra and the inverse Killing form $\mathcal{K}^{MN}$,

$$[\mathbb{Q},\overline{\mathbb{Q}}]_+=\mathcal{K}^{MN}B_M\,\mathbb{B}_N\,, \tag{2.10}$$

where the B_Ms are given in (2.6)-(2.7). Note that the $su(2,2|2)$ contains the subalgebras,

$$so(3,2)=\{\mathbb{B}_{ab},\mathbb{B}_a\}\ \cong\ sp(4)\,, \tag{2.11}$$

$$so(4,1)=\{\mathbb{B}_{ab},\tilde{\mathbb{B}}_a\}\,, \tag{2.12}$$

$$iso(3,1)_\pm=\{\mathbb{B}_{ab},\mathbb{B}_{\pm a}\}\,,\qquad \mathbb{B}_{\pm a}:=\frac{1}{2}(\mathbb{B}_a\pm\tilde{\mathbb{B}}_a)\,, \tag{2.13}$$

which are isometries correspondingly of anti-de Sitter, de Sitter and Minkowski spacetimes. In the latter case we have two options, $iso(3,1)_+$ or $iso(3,1)_-$, for Poincaré subalgebras.

2.2. The gauge potential.

The gauge potential (connection) is of a one-form valued in the superalgebra (2.4),

$$\mathcal{A}:=\mathbb{A}+\Psi-\overline{\Psi}\quad\in\quad su(2,2|2)\,, \tag{2.14}$$

$$\mathbb{A}:=A^M\mathbb{B}_M\,, \tag{2.15}$$

which we have decomposed in its fermion sector containing the gravitino supercharge-valued field, $\Psi:=\psi^\alpha_i\,\overline{\mathbb{Q}}^i_\alpha$ and its conjugate $\overline{\Psi}:=\overline{\psi}^i_\alpha\,\mathbb{Q}^\alpha_i$, and the bosonic sector containing spacetime

($\mathbb{W}$) and internal ($\mathbb{U}$) symmetry components,

$$\mathbb{A} = \mathbb{W} + \mathbb{U}\,, \tag{2.16}$$

$$\mathbb{W} = \frac{1}{2}\omega^{ab}\mathbb{B}_{ab} + p^a\,\mathbb{B}_a + \tilde{p}^{\,a}\,\tilde{\mathbb{B}}_a + h\,\mathbb{B}_5 \qquad \in \quad so(4,2)\,, \tag{2.17}$$

$$\mathbb{U} = U^r\,\mathbb{B}_r \qquad \in \quad u(1)\oplus su(2)\,. \tag{2.18}$$

Here ω^{ab} is the Lorentz connection, p^a and $\tilde{p}^{\,a}$ are respectively AdS_4- and dS_4- type transvection gauge fields (*cf.* respectively (2.11) and (2.12)), h is the dilation gauge field, U^6 is the $u(1)$ *electromagnetic* gauge field and U^I are $SU(2)$ gauge fields.

Using the adjoint representation ρ (2.8), which does not affect the field coefficients, we can map the bosonic gauge connection $\mathbb{A}$ to its adjoint action $A := \rho(\mathbb{A})$ upon the gravitino fields,

$$A = W + U\,, \qquad W = \rho(\mathbb{W})\,, \qquad U = \rho(\mathbb{U})\,,$$

$$W = \Omega + P + \tilde{P} + H\,, \qquad U = U^r B_r\,, \tag{2.19}$$

where

$$\Omega = \frac{1}{2}\omega^{ab}B_{ab}\,, \quad P = p^a B_a\,, \quad \tilde{P} = \tilde{p}^{\,a}\tilde{B}_a\,, \quad H = hB_5, \quad U^r B_r = U^6 B_6 + U^I B_I\,. \tag{2.20}$$

2.3. The field strength.

The covariant derivative associated to the gauge connection (2.14) acts on $su(2,2|2)$-valued differential forms as

$$\mathcal{D}\Phi := d\Phi + [\mathcal{A}, \Phi]\,. \tag{2.21}$$

We also introduce the covariant derivative, with respect to the bosonic gauge connection (2.16),

$$\mathbb{D} := d + \mathbb{A}\,. \tag{2.22}$$

The $su(2,2|2)$ field strength, $\mathcal{F} := d\mathcal{A} + \mathcal{A}\mathcal{A}$, has components

$$\mathcal{F} = \frac{1}{2}\mathcal{F}^{ab}\mathbb{B}_{ab} + \mathcal{F}^{\,a}\mathbb{B}_a + \tilde{\mathcal{F}}^{\,a}\tilde{\mathbb{B}}_a + \mathcal{F}^5\mathbb{B}_5 + \mathcal{F}^r\,\mathbb{B}_r + \overline{\mathbb{Q}}{}^i_\alpha\,\mathcal{X}^\alpha_i - \overline{\mathcal{X}}{}^i_\alpha\,\mathbb{Q}^\alpha_i\,, \tag{2.23}$$

where

$$\mathcal{X}^\alpha_i = (D\psi)^\alpha_i\,, \quad \overline{\mathcal{X}}{}^i_\alpha = (D\overline{\psi})^i_\alpha\,, \tag{2.24}$$

$$D\psi = d\psi + (W+U)\psi\,, \qquad D\overline{\psi} = d\overline{\psi} + \overline{\psi}(W+U) \tag{2.25}$$

The covariant derivative $D = d + W + U$ is induced by the action of (2.22) on supercharge-valued gauge fields $\mathbb{X} = \mathbb{D}\Psi - \mathbb{D}\overline{\Psi}$, where $\mathbb{D}\Psi = \overline{\mathbb{Q}}D\psi$ and $\mathbb{D}\overline{\Psi} = (D\overline{\psi})\,\mathbb{Q}$.

We identify three main sectors of the gauge curvature:

$$\mathcal{F} = \mathbb{F} - \mathbb{I} + \mathbb{X}\,, \tag{2.26}$$

$$\mathbb{F} := \mathbb{D}^2 = F^M\,\mathbb{B}_M\,, \qquad \mathbb{I} := [\overline{\Psi}, \Psi] = I^M\mathbb{B}_M\,, \qquad \mathbb{X} := \overline{\mathbb{Q}}\,\mathcal{X} - \overline{\mathcal{X}}\mathbb{Q}\,, \tag{2.27}$$

respectively the bosonic gauge field strength, the gravitino (bosonic) 2-form current, where

$$I^M = \mathcal{K}^{MN}\overline{\psi}B_N\psi, \tag{2.28}$$

$$I^{ab} = -\frac{1}{2}\overline{\psi}B^{ab}\psi\,, \tag{2.29}$$

$$I^a = \overline{\psi}B^a\psi\,, \qquad \tilde{I}^a = -\overline{\psi}\tilde{B}^a\psi\,, \tag{2.30}$$

$$I^5 = \overline{\psi}B_5\psi\,, \tag{2.31}$$

$$I^6 = \frac{1}{4}\overline{\psi}B_6\psi\,, \qquad I^I = 2\overline{\psi}B_I\psi\,. \tag{2.32}$$

Thus, the boson and fermion components of the curvature are respectively,

$$\mathcal{F}|_{\mathtt{BOS}} = \mathcal{F}^M\mathbb{B}_M = \mathbb{F}-\mathbb{I}, \qquad \mathcal{F}|_{\mathtt{FER}} = \mathbb{X}. \tag{2.33}$$

For future reference, we shall use the "evaluate" symbol to project the superalgebra-valued differential forms on particular elements of the algebra, namely, $|_{\mathtt{BOS}}$ and $|_{\mathtt{FER}}$ to be the projections onto the bosonic and the fermionic sectors, $|_{\mathtt{ST}}$ and $|_{\mathtt{INT}}$ the projections onto the spacetime and internal generators, $|_{\mathtt{L}}$, $|_{\mathtt{T}}$ and $|_{\mathtt{D}}$ the projections onto Lorentz, transvection and dilation generators.

Thus the boson components of $\mathcal{F}|_{\mathtt{BOS}}$ can be subdivided in their spacetime and internal type of components, $\mathbb{F}|_{\mathtt{ST}}$ and $\mathbb{F}|_{\mathtt{INT}}$, respectively given by,

$$\mathbb{F} := \mathbb{F}|_{\mathtt{ST}} + \mathbb{F}|_{\mathtt{INT}}\,, \qquad \mathbb{F}|_{\mathtt{ST}} = d\mathbb{W}+\mathbb{W}\mathbb{W}\,, \quad \mathbb{F}|_{\mathtt{INT}} = d\mathbb{U}+\mathbb{U}\mathbb{U}\,. \tag{2.34}$$

The 2-form current $\mathbb{I} = \mathbb{I}|_{\mathtt{ST}} + \mathbb{I}|_{\mathtt{INT}}$ is decomposed similarly.

In more detail we have,

$$\mathbb{F}|_{\mathtt{L}} = \frac{1}{2}F^{ab}\mathbb{B}_{ab}\,, \quad \mathbb{F}|_{\mathtt{T}} = F^a\mathbb{B}_a + \tilde{F}^a\tilde{\mathbb{B}}_a\,, \quad \mathbb{F}|_{\mathtt{D}} = F^5\,\mathbb{B}_5\,, \qquad \mathbb{F}|_{\mathtt{INT}} = G^6\mathbb{B}_6 + G^I\mathbb{B}_I\,, \tag{2.35}$$

where

$$F^{ab} = R^{ab}(w) + p^a p^b - \tilde{p}^a\tilde{p}^b\,, \qquad R^{ab}(w) := dw^{ab} + w^{ac}w_c{}^b\,, \tag{2.36}$$

$$F^a = D_\Omega p^a - h\tilde{p}^a\,, \qquad \tilde{F}^a = D_\Omega\tilde{p}^a - hp^a\,, \tag{2.37}$$

$$F^5 = dh + p^a\tilde{p}_a\,, \tag{2.38}$$

$$G^6 = dU^6\,, \qquad G^I = dU^I + U^J U^K\epsilon_{JK}{}^I\,, \tag{2.39}$$

and for the gravitino currents,

$$\mathbb{I}|_{\mathtt{L}} = \frac{1}{2}\,I^{ab}\,\mathbb{B}_{ab}\,, \quad \mathbb{I}|_{\mathtt{T}} = I^a\,\mathbb{B}_a + \tilde{I}^a\,\tilde{\mathbb{B}}_a\,, \quad \mathbb{I}|_{\mathtt{D}} = I^5\,\mathbb{B}_5\,, \qquad \mathbb{I}|_{\mathtt{INT}} = I^6\,\mathbb{B}_6 + I^I\,\mathbb{B}_I\,. \tag{2.40}$$

In the adjoint representation (2.8) we write, from (2.35)

$$F := \rho(\mathbb{F}) = F^M B_M\,, \qquad I := \rho(\mathbb{I}) = I^M B_M \tag{2.41}$$

In what follows we shall also use gauge-field symbols as labels in the covariant derivative in order to specify the gauge connection being used,

$$D_W = d + W\,, \quad D_U = d + U\,, \quad D_\Omega = d + \Omega\,, \quad D_{\Omega+H} = d + \Omega + H\,.$$

2.4. Γ–gradding.

The matrix

$$\Gamma = \left[\begin{array}{c|c} \gamma_5 & 0_{4\times 2} \\ \hline 0_{2\times 4} & 0_{2\times 2} \end{array}\right]. \tag{2.42}$$

induces a natural graded structure on the bosonic generators of $su(2,2|2)$,

$$[\mathbb{B}^-_M, \Gamma]_+ = 0\,, \qquad [\mathbb{B}^+_M, \Gamma] = 0\,, \tag{2.43}$$

where $\mathbb{B}^- = \mathbb{B}_a\,, \tilde{\mathbb{B}}_a$, and $\mathbb{B}^+ = \mathbb{B}_{ab}\,, \tilde{\mathbb{B}}_5\,, \mathbb{B}_r\,.$

The grading (2.43) of the bosonic component of any differential form $\Theta \in \mathfrak{g}$ is preserved in the representation ρ. Since $\rho(\Gamma) = \gamma_5$, we have $[\,\rho(\Theta|^-_{\mathtt{BOS}})\,, \gamma_5]_+ = 0$, $[\,\rho(\Theta|^+_{\mathtt{BOS}})\,, \gamma_5] = 0$. Thus we can also decompose the differential forms valued in fundamental representations of the bosonic gauge algebra accordingly. In particular, for future reference, the gauge connection decomposition reads,

$$A^- = W^- = P + \tilde{P}\,, \qquad A^+ = W^+ + U = \Omega + H + U\,. \tag{2.44}$$

Henceforth all differential form $\Theta \in \mathfrak{g}$ can be decomposed as follows,

$$\Theta = \Theta^+ + \Theta^-\,, \tag{2.45}$$

$$\Theta^- = \Theta|_{\mathtt{T}} = \Theta^a \mathbb{B}_a + \tilde{\Theta}^a \tilde{\mathbb{B}}_a\,, \quad \Theta^+ = \Theta|^+_{\mathtt{BOS}} + \Theta|_{\mathtt{FER}}\,, \quad \Theta|^+_{\mathtt{BOS}} = \Theta|_{\mathtt{L}} + \Theta|_{\mathtt{D}} + \Theta|_{\mathtt{INT}}\,.$$

Hence the denoted $^+$–component contains all the generators of the corresponding gauge algebra excluding transvection generators. The $^-$–components refer therefore only to the transvection components. In particular, the transvection term of the field strength is given by,

$$\mathcal{F}^- = (F^a - I^a)\,\mathbb{B}_a + (\tilde{F}^a - \tilde{I}^a)\,\tilde{\mathbb{B}}_a\,, \qquad F^- = (F^a - I^a)\,B_a + (\tilde{F}^a - \tilde{I}^a)\,\tilde{B}_a\,. \tag{2.46}$$

3. Lagrangian, dynamics and symmetries

3.1. Construction the Lagrangian and the ⊛ operator.

With the necessary ingredients at hand, we can proceed with the generalized Yang-Mills action (2.23). The corresponding Lagrangian density, built from the field strength $\mathcal{F}$ (2.26), is given by,

$$\mathcal{L} := -\ \mathtt{str}\left(\mathcal{F} \circledast \mathcal{F}\right). \tag{3.1}$$

We would like that this construction follows the Yang-Mills spirit of gauge theories, hence that the ⊛ operator extend the traditional Hodge operator used in Yang-Mills theories. Hence, ⊛ must produce an authomorphism of the complexified $su(2,2|2)$ two-form curvature: $\circledast\mathcal{F} \in$

$sl(4|2,\mathbb{C})$. In order to operate similarly to the regular Hodge dual in Lorentzian signature, we shall also require $\circledast^2 = -\mathbb{1}$ on the 2-forms. These restrictions still allow for a large variety (at least too many for us) of possibilities. As we want our model to extend the standard model of theoretical physics, we add the following restrictions:

i) $\mathtt{str}\ (\mathbb{F}|_{\mathtt{FER}} \circledast \mathbb{F}|_{\mathtt{FER}})$ contains Rarita-Schwinger terms.
ii) $\mathtt{str}\ (\mathbb{F}|_{\mathtt{ST}} \circledast \mathbb{F}|_{\mathtt{ST}})$ contains Einstein-Hilbert terms.
iii) $\mathtt{str}\ (\mathbb{F}|_{\mathtt{INT}} \circledast \mathbb{F}|_{\mathtt{INT}})$ contains the Yang-Mills term.
iv) The action does not contain torsion kinetic terms.

Inspecting these terms we observe that the goal is achieved with the following actions of the generalized Hodge operator, specified on the different sectors of the field strength:

1) $\circledast\mathbb{X} = i\Gamma\mathbb{X} + \mathbb{X}i\Gamma = \overline{\mathbb{Q}}i\gamma_5\mathcal{X} - \overline{\mathcal{X}}i\gamma_5\mathbb{Q}$.
2) $\circledast\mathbb{F}|_{\mathtt{L}} = i\Gamma\mathbb{F}|_{\mathtt{L}}$.
3) $\circledast\mathbb{F}|_{\mathtt{INT}} = *\,\mathbb{F}|_{\mathtt{INT}}$.
4) $\circledast\mathbb{F}|_{\mathtt{T}} = i\Gamma\mathbb{F}|_{\mathtt{T}} = -i\tilde{F}^a\mathbb{B}_a - iF^a\tilde{\mathbb{B}}_a$. Here, even though $\circledast$ produces imaginary factors, in the Lagrangian these terms will cancel out.
5) In addition we choose $\circledast\mathbb{F}|_{\mathtt{D}} = *\,\mathbb{F}|_{\mathtt{D}}$ upon the dilation sector. The option $\circledast\mathbb{F}|_{\mathtt{D}} = i\Gamma\mathbb{F}|_{\mathtt{D}}$ does not belong to the algebra hence it is discarded. The option $\circledast\mathbb{F}|_{\mathtt{D}} = i\mathbb{F}|_{\mathtt{D}}$ yields an imaginary term in the Lagrangian, hence we avoid it.

The $\circledast$ operator will act in the same way on any $\mathfrak{g}$-valued 2-form, in agreement with their $su(2,2|2)$. The requirements above are satisfied by the choice

$$\begin{aligned}\circledast\mathcal{F} =& \frac{1}{2}(\circledast\mathcal{F}^{ab})\mathbb{B}_{ab} + (\circledast\mathcal{F}^{a})\mathbb{B}_a + \circledast(\tilde{\mathcal{F}}^a)\tilde{\mathbb{B}}_a + (*\mathcal{F}^5)\mathbb{B}_5 \\ &+ (*\mathcal{F}^r)\mathbb{B}_r + \overline{\mathbb{Q}} \circledast \mathcal{X} - (\circledast\overline{\mathcal{X}})\,\mathbb{Q}\,,\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}\circledast\mathcal{F}^{ab} = \frac{1}{2}\epsilon^{ab}{}_{cd}\mathcal{F}^{cd}, \quad \circledast\mathcal{F}^a = -i\tilde{\mathcal{F}}^a, \quad \circledast\tilde{\mathcal{F}}^a = -i\mathcal{F}^a, \quad \circledast\mathcal{F}^5 = *\mathcal{F}^5, \\ \circledast\mathcal{F}^r = *\mathcal{F}^r, \quad \circledast\mathcal{X} = i\gamma_5\mathcal{X}, \quad \circledast\overline{\mathcal{X}} = \overline{\mathcal{X}}i\gamma_5\,.\end{aligned} \tag{3.3}$$

Variations of the signs of the $\circledast$ operator on particular sectors of the curvature that may lead to different models. We shall discuss briefly two additional cases in sections 4.4 and 4.5. For example the option used in (4.49) also fulfills the requirements.

Collecting only bosonic terms in $\mathcal{L}_{\mathtt{bos}}$ and fermion terms in $\mathcal{L}_{\mathtt{fer}}$, is composed as,

$$\mathcal{L} = \mathcal{L}_{\mathtt{bos}} + \mathcal{L}_{\mathtt{fer}} \tag{3.4}$$

$$\mathcal{L}_{\mathtt{fer}} = 4\overline{\psi}\left(i\gamma_5 W^- D + \frac{1}{2}(* - i\gamma_5)(F|_{\mathtt{D}} + F|_{\mathtt{INT}}) - i\gamma_5\frac{1}{2}F^- - \frac{1}{4}\circledast I^+\right)\psi\,. \tag{3.5}$$

In components the bosonic component of the Lagrangian reads,

$$\begin{aligned}\mathcal{L}_{\mathtt{bos}} :=& \frac{1}{2}R^{ab}(w)p^{cd}\epsilon_{abcd} + \frac{1}{4}p^{ab}p^{cd}\epsilon_{abcd} + \frac{1}{4}R^{ab}(w)R^{cd}(w)\epsilon_{abcd} \\ &- dh * dh - 2dh * p^a\tilde{p}_a - p^a\tilde{p}_a * p^b\tilde{p}_b - \frac{1}{2}F^I * F^I - 4dU^0 * dU^0\,,\end{aligned} \tag{3.6}$$

where

$$p^{ab} := p^a p^b - \tilde{p}^a\tilde{p}^b\,. \tag{3.7}$$

The Lagrangian contains boson and fermion kinetic terms at exception of the transvection gauge fields, couplings of fermion-currents and field strengths (Pauli couplings) and four-fermion self-interactions. It can be shown that the terms containing the transvection-like component of the curvature (2.46) cancel out from the Lagrangian (3.1) as a consequence of the $\circledast$ action (3.3) along these terms. Hence the absence of $F|_{\mathrm{T}}$ Pauli couplings and Kinetic terms is natural. The absence of $F|_{\mathrm{L}}$ Pauli couplings in (3.5) is consequence of a cancelation of the identical terms provided by the boson gauge curvatures and the fermion gauge curvatures.

As a last remark on this construction, we observe that since the $\circledast$ operator removes the transvection type of terms from the Lagrangian, (3.1) can be alternatively written as,

$$\begin{aligned}\mathcal{L} &= -\,\mathtt{str}\ \mathcal{F}^{+}\circledast\mathcal{F}^{+}\\ &= -\,\mathtt{str}\ (\mathbb{F}^{+}-\mathbb{I}^{+})\circledast(\mathbb{F}^{+}-\mathbb{I}^{+}) - \mathtt{str}\ \mathbb{X}\circledast\mathbb{X}.\end{aligned} \tag{3.8}$$

Here we can see why the imaginary components of $\circledast$, on transvections, do not produce imaginary terms in the Lagrangian. Since the supertraces of the bosonic generators (2.9) produce the Killing form of the bosonic subalgebra, and since $\circledast\mathbb{F}$ is in the algebra by construction (see (3.3)), the bosonic Lagrangian is equivalent to,

$$\mathcal{L}_{\mathtt{bos}} := -(F^{+})_{N}(\circledast F^{+})^{N}, \tag{3.9}$$

where $(F^{+})_{N} := (F^{+})^{M}\mathcal{K}_{MN}$. For the fermionic component we get,

$$\mathcal{L}_{\mathtt{fer}} = 4\overline{\psi}\left(i\gamma_5 W^{-}D + \frac{1}{2}\left(\circledast F^{+} - i\gamma_5 F\right) - \frac{1}{4}\circledast I^{+}\right)\psi - d\left(2i\overline{\psi}\gamma_5 D\psi\right). \tag{3.10}$$

Noticing that $i\gamma_5 F = i\gamma_5 F^{+} + i\gamma_5 F^{-}$ and that $\circledast F^{+} = i\gamma_5 F|_{\mathrm{L}} + *F|_{\mathtt{D}} + *F|_{\mathtt{INT}}$, the Lorentz components of the Pauli terms in (3.10) cancel out,

$$\circledast F^{+} - i\gamma_5 F = (\circledast - i\gamma_5)F^{+} - i\gamma_5 F^{-} = (* - i\gamma_5)(F|_{\mathtt{D}} + F|_{\mathtt{INT}}) - i\gamma_5 F^{-}. \tag{3.11}$$

Thus we can write $\mathcal{L}_{\mathtt{fer}}$ as in (3.5).

3.2. Field equations.

The equations of motion are given by the vanishing condition of the variation of the action with respect to the gauge connection $\mathcal{A}$,

$$\delta\mathcal{L} = -2\ \mathtt{str}\ \left(\delta\mathcal{A}\,\mathcal{D}\circledast\mathcal{F}^{+}\right) - \mathtt{str}\ d(\delta\mathcal{A}\circledast\mathcal{F}^{+}), \tag{3.12}$$

where $\mathcal{F}^{+} = \mathbb{F}^{+} - \mathbb{I}^{+} + \mathbb{X}$ from definition (2.45).

From (3.12) the equations of motion reads

$$\mathcal{D}\circledast\mathcal{F}^{+} = 0. \tag{3.13}$$

In an extended form the equations of motion (3.13) are given by:

δw :

$$\epsilon_{cdab}\left[(D_\Omega p^a - I^a)p^b - (D_\Omega \tilde{p}^a - \tilde{I}^a)\tilde{p}^b\right] = 0\,. \tag{3.14}$$

This equation is equivalent to

$$\epsilon_{cdab}\left[(F^a - I^a)p^b - (\tilde{F}^a - \tilde{I}^a)\tilde{p}^b\right] = 0\,, \tag{3.15}$$

since the components hp and $h\tilde{p}$ in the definitions (2.37) cancel.

δh :

$$d\left(*F^5 - \frac{i}{2}\overline{\psi}\psi\right) = 0\,. \tag{3.16}$$

δp :

$$\frac{1}{2}\epsilon_{abcd}\,p^b(F^{cd} - I^{cd}) - \tilde{p}_a * (F^5 - I^5) - D\overline{\psi}\,i\gamma_5 B_a\psi + \overline{\psi}B_a i\gamma_5 D\psi = 0\,. \tag{3.17}$$

$\delta \tilde{p}$:

$$\frac{1}{2}\epsilon_{abcd}\,\tilde{p}^b(F^{cd} - I^{cd}) - p_a * (F^5 - I^5) + D\overline{\psi}\,i\gamma_5 \tilde{B}_a\psi - \overline{\psi}\tilde{B}_a i\gamma_5 D\psi = 0\,. \tag{3.18}$$

δU^r :

$$\mathbb{D}_{\mathbb{U}}\left(*\mathbb{F}|_{\mathtt{INT}} + \mathbb{I}'|_{\mathtt{INT}} - *\mathbb{I}|_{\mathtt{INT}}\right) = 0\,, \tag{3.19}$$

where $\mathbb{I}'|_{\mathtt{INT}} := \overline{\psi}i\gamma_5 B^r\psi\,\mathbb{B}_r = \frac{1}{4}\overline{\psi}i\gamma_5 B_6\psi\,\mathbb{B}_6 + 2\overline{\psi}i\gamma_5 B_I\psi\,\mathbb{B}_I$.

$\delta\overline{\psi}$:

$$\left(i\gamma_5 W^- D + \frac{1}{2}(\circledast - i\gamma_5)(F^+ - I^+) - \frac{1}{2}i\gamma_5(F^- - I^-) - \frac{1}{2}i\gamma_5 I\right)\psi = 0\,, \tag{3.20}$$

or alternatively,

$$\begin{aligned}\Big(i\gamma_5 W^- D_{\Omega+U} + i\gamma_5(W^-)^2 + \frac{1}{2}(* - i\gamma_5)((F-I)|_{\mathtt{D}} + (F-I)|_{\mathtt{INT}})\\ -\frac{1}{2}i\gamma_5(D_\Omega W^- - I^-) - \frac{1}{2}i\gamma_5 I\Big)\psi = 0\,,\end{aligned} \tag{3.21}$$

where I is given in (2.41).

$\delta\psi$: Similarly,

$$D\overline{\psi}W^-\,i\gamma_5 + \frac{1}{2}\overline{\psi}(\circledast - i\gamma_5)(F^+ - I^+) - \frac{1}{2}\overline{\psi}(F^- - I^-)\,i\gamma_5 - \frac{1}{2}\overline{\psi}I\,i\gamma_5 = 0\,, \tag{3.22}$$

or alternatively,

$$\begin{aligned}(D_{\Omega+U}\overline{\psi})W^- i\gamma_5 + \overline{\psi}P^2 i\gamma_5 + \frac{1}{2}\overline{\psi}(* - i\gamma_5)\Big((F-I)|_{\mathtt{D}} + (F-I)|_{\mathtt{INT}}\Big)\\ -\frac{1}{2}\overline{\psi}(D_\Omega W^- - I^-)\,i\gamma_5 - \frac{1}{2}\overline{\psi}I\,i\gamma_5 = 0\,.\end{aligned} \tag{3.23}$$

In (3.21) and (3.23) we observe that the terms including the gauge field H in the covariant derivative $D\Psi$ and in $F^- = dW^- + [\Omega + H, W^-]$ cancel each other.

3.3. Integrability conditions and conditional symmetries.

Acting once again with the operator $\mathcal{D}$ on (3.13) we obtain theintegrability condition

$$[\mathcal{F}\,,\circledast\mathcal{F}^{+}]=0\,. \tag{3.24}$$

Note that more generally, the system of equations

$$\mathcal{D}\mathcal{B}=0\,,\qquad [\mathcal{F},\mathcal{B}]=0\,, \tag{3.25}$$

where $\mathcal{B}$ is a generic differential form and $\mathcal{F}$ is the curvature for the connection in $\mathcal{D}$, is self-consistent by virtue of the Bianchi identity, $\mathcal{D}\mathcal{F}\equiv 0$. In fact, acting once more with the covariant derivative produces no new constraints on $\mathcal{B}$. In the same sense, the equations of motion (3.13) and their integrability conditions (3.24) are also self-consistent.

It can be verified that all the components of the commutator (3.24) along the subalgebra

$$\mathfrak{h}=so(3,1)\oplus\mathbb{R}\oplus u(1)\oplus su(2)\,, \tag{3.26}$$

vanish identically,

$$[\mathcal{F}\,,\circledast\mathcal{F}^{+}]|_{\mathrm{L}}\equiv 0\,,\qquad [\mathcal{F}\,,\circledast\mathcal{F}^{+}]|_{\mathrm{D}}\equiv 0\,,\qquad [\mathcal{F}\,,\circledast\mathcal{F}^{+}]|_{\mathtt{INT}}\equiv 0\,. \tag{3.27}$$

Hence, the non-trivial components of (3.24) are along transvections and supercharge generators;

$$[\mathcal{F}\,,\circledast\mathcal{F}^{+}]\equiv[\mathcal{F}\,,\circledast\mathcal{F}^{+}]|_{\mathrm{T}}+[\mathcal{F}\,,\circledast\mathcal{F}^{+}]|_{\mathtt{FER}}\,, \tag{3.28}$$

where

$$[\mathcal{F}\,,\circledast\mathcal{F}^{+}]|_{\mathrm{T}}=[\mathcal{F}^{-}\,,\circledast\mathcal{F}^{+}]+[\mathbb{X}\,,\circledast\mathbb{X}]\,, \tag{3.29}$$

$$[\mathcal{F}\,,\circledast\mathcal{F}^{+}]|_{\mathtt{FER}}=\overline{\mathbb{Q}}\,\rho(\mathcal{F}|_{\mathtt{BOS}}\,i\Gamma-\circledast\mathcal{F}^{+})D\psi+D\overline{\psi}\,\rho(\,i\Gamma\,\mathcal{F}|_{\mathtt{BOS}}-\circledast\mathcal{F}^{+})\mathbb{Q}\,. \tag{3.30}$$

Therefore, (3.24) is equivalent to the system

$$\begin{aligned}&\Big(\mathcal{F}^{b}\,(\circledast\mathcal{F})_{b}{}^{a}+\tilde{\mathcal{F}}^{a}*\mathcal{F}^{5}+2\,D\overline{\psi}\,i\gamma_{5}B^{a}\,D\psi\Big)\mathbb{B}_{a}\\&+\Big(\tilde{\mathcal{F}}^{b}\,(\circledast\mathcal{F})_{b}{}^{a}+\mathcal{F}^{a}*\mathcal{F}^{5}-2\,D\overline{\psi}\,i\gamma_{5}\tilde{B}^{a}\,D\psi\Big)\tilde{\mathbb{B}}_{a}=0\,,\end{aligned} \tag{3.31}$$

$$\begin{aligned}&\overline{\mathbb{Q}}\,\rho\Big(\{\mathcal{F}|_{\mathrm{D}}+\mathcal{F}|_{\mathtt{INT}}\}\,(i\Gamma-*)-i\Gamma\,\mathcal{F}^{-}\Big)D\psi\\&+D\overline{\psi}\,\rho\Big((i\Gamma-*)\,\{\mathcal{F}|_{\mathrm{D}}+\mathcal{F}|_{\mathtt{INT}}\}+i\Gamma\mathcal{F}^{-}\Big)\,\mathbb{Q}=0\,.\end{aligned} \tag{3.32}$$

Alternatively, the supercharge-valued constraint can be expressed as

$$\begin{aligned}&\overline{\mathbb{Q}}\,\Big(\{\,(F-I)|_{\mathrm{D}}+\,(F-I)|_{\mathtt{INT}}\,\}\,(i\gamma_{5}-*)D\psi+(F^{-}-I^{-})\,i\gamma_{5}D\psi\Big)\\&+\Big(D\overline{\psi}\,(i\gamma_{5}-*)\,\{\,(F-I)|_{\mathrm{D}}+\,(F-I)|_{\mathtt{INT}}\,\}+D\overline{\psi}\,i\gamma_{5}\,(F-I)|_{\mathtt{INT}}\Big)\,\mathbb{Q}=0\,.\end{aligned} \tag{3.33}$$

An $su(2,2|2)$ transformation of the connection gauge field ($\delta\mathcal{A}=\mathcal{D}\lambda$) and its curvature ($\delta\mathcal{F}=[\mathcal{F}\,,\lambda]$), implies that the Lagrangian changes as

$$\delta_{\lambda}\mathcal{L}=2\ \mathtt{str}\ \Big(\lambda\,[\mathcal{F}\,,\circledast\mathcal{F}^{+}]\Big)\ +\ \text{b.t.} \tag{3.34}$$

From (3.27) we know that (3.34) vanishes identically for $\lambda \in \mathfrak{h}$, hence (3.26) is a genuine gauge (off-shell) symmetry of the system. As for transvections and supersymmetry, they are *conditional symmetries*, *i.e.* subjected to their dual symmetry constraints (3.31) and (3.32) respectively.

4. Ground states and effective theories

We have not yet established the relation between the symmetric tensor $g_{\mu\nu}$, used to build the Hodge dual necessary for the Yang-Mills action, and the transvection gauge fields in the $W^- = P + \tilde{P}$ component of the gauge connection.

So far, we have assumed, as in Yang-Mills theories, that the symmetric tensor $g_{\mu\nu}$ is a prescribed function, like a fixed parameter of the action, not dynamical field. It is therefore not varied in the computation of field equations and the symmetry transformations of the Lagrangian. In this picture, the expected correspondence of the type $e^a_\mu \sim p^a_\mu$, $e^a_\mu \sim \tilde{p}^{\,a}_\mu$, so that $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, should be established *a posteriori*, as part of the solutions around a ground state.

In order to avoid the emergence of new fields related to the basis-change matrices,

$$\frac{\delta p^a}{\delta e^b}\,, \qquad \frac{\delta \tilde{p}^a}{\delta e^b}\,, \tag{4.1}$$

that could spoil Lorentz invariance, they must be proportional to the only available invariant tensor of rank 2, the Kronecker delta. Hence, following [Tow77; AVJ20], we consider a ground state sector in which the transvection fields are chosen as,

$$p^a = \alpha_+ e^a\,, \qquad \tilde{p}^{\,a} = \alpha_- e^a\,, \tag{4.2}$$

with constants $\alpha_\pm$.

The field equation for the spin connection,

$$\epsilon_{cdab}\Big(\mathcal{F}^a p^b - \tilde{\mathcal{F}}^a \tilde{p}^{\,b}\Big) = 0\,, \tag{4.3}$$

is an algebraic equation. When the system of equations (4.3) is non-degenerate, the Lorentz connection can be solved in terms of the transvection gauge fields p^a, $\tilde{p}^a$, and the gravitino currents I^a and $\tilde{I}^a$.

In the ground state (4.2), for non degenerate (3.7)

$$p^{ab} = (\alpha_+^2 - \alpha_-^2)\, e^a e^b\,, \tag{4.4}$$

the equation (4.3) can be reduced to the torsion constraint

$$T^a = \frac{\alpha_+ I^a - \alpha_- \tilde{I}^a}{\alpha_+^2 - \alpha_-^2}\,, \qquad T^a := D_\Omega e^a\,. \tag{4.5}$$

Hence decomposing the spin connection in a torsionless component (such that $D_{\Omega(e)} e^a = 0$) and the contorsion, $\Omega = \Omega(e) + K$, we obtain the solution

$$\Omega^{ab}_\nu(e) = 2 e^{[a|\rho} \partial_{[\nu} e^{|b]}_{\rho]} - e_{c\nu} e^{[a|\mu} e^{|b]\rho} \partial_\mu e^c_\rho\,, \qquad K^{ab}_\mu = -\frac{1}{2} e^{a\nu} e^{b\rho} (T_{\mu\nu\rho} - T_{\nu\rho\mu} + T_{\rho\mu\nu})\,, \tag{4.6}$$

where $T_{\mu\nu\rho} = T^a_{\mu\nu}\, e_{a\rho}$.

4.1. $\mathcal{N}=2$ supergravity ground state.

Imposing Majorana reality conditions on the gravitino fields, and setting $\alpha_-=0$, the solution (4.6) produces

$$\mathcal{F}^a = F^a - I^a = 0\,, \qquad \tilde{\mathcal{F}}^a = \tilde{F}^a - \tilde{I}^a = 0\,. \tag{4.7}$$

Using this back in the transvection symmetry constraint (3.31), we are left with

$$D\overline{\psi}\, i\gamma_5 B^a\, D\psi = 0\,, \qquad D\overline{\psi}\, i\gamma_5 \tilde{B}^a\, D\psi = 0\,, \tag{4.8}$$

which can be alternatively written as

$$D\overline{\psi}B(i\gamma_5 - *)D\psi - (D\overline{\psi}i\gamma_5 - *D\overline{\psi})BD\psi = 0\,, \quad \text{with} \quad B = B^a, \tilde{B}^a\,. \tag{4.9}$$

Now using (4.7) in the supersymmetry constraint (3.32), we get

$$\begin{aligned} \{\,(F-I)|_{\mathtt{D}} + (F-I)|_{\mathtt{INT}}\,\}\,(i\gamma_5 - *)D\psi = 0\,, \\ (D\overline{\psi}i\gamma_5 - *D\overline{\psi})\,\{\,(F-I)|_{\mathtt{D}} + (F-I)|_{\mathtt{INT}}\,\} = 0\,. \end{aligned} \tag{4.10}$$

Since (4.9) and (4.10) can be factorized by the Rarita-Schwinger equations (4.18),

$$(i\gamma_5 - *)D\psi = 0\,, \qquad D\overline{\psi}i\gamma_5 - *D\overline{\psi} = 0\,, \tag{4.11}$$

the torsion constraints (4.7) and the Rarita-Schwinger equation (4.11) provide enough conditions for transvection symmetry and supersymmetry. We stress that (3.31) and (3.32) could be solved by more general methods, which can allow complex gravitino configurations and non-trivial field strengths.

Finally, about the Rarita-Schwinger equations, we would like to recall a result found in [DKS77]. Let B be a spinor 2-form satisfying,

$$\not{e} \wedge \Omega = 0\,. \tag{4.12}$$

Then it follows:

$$\gamma^\mu * \Omega_{\mu\nu} = 0\,, \qquad *\Omega_{\mu\nu} := \frac{1}{2} e\epsilon_{\mu\nu\lambda\rho}\Omega^{\lambda\rho}\,, \tag{4.13}$$

$$\gamma^{\mu\nu\lambda}\Omega_{\nu\lambda} = 0\,, \tag{4.14}$$

$$\gamma^\mu\Omega_{\mu\nu} = 0\,, \tag{4.15}$$

$$(i\gamma_5 - *)\Omega = 0\,. \tag{4.16}$$

From (4.12), equivalent to $\gamma_{[\mu}\Omega_{\nu\lambda]} = 0$, we demonstrate these identities performing the following operations:

- $\epsilon_{\rho\mu\nu\lambda}\gamma^{[\mu}\Omega^{\nu\lambda]} = 0 \quad \Rightarrow \quad (4.13)$
- from (4.13) using identity $i\gamma_5\gamma_{\mu\nu\lambda} = -e\epsilon_{\mu\nu\lambda\rho}\gamma^\rho \quad \Rightarrow \quad (4.14)$
- $\gamma^\mu\gamma_{[\mu}\Omega_{\nu\lambda]} = 0$ and (4.14) $\quad \Rightarrow \quad$ (4.15)
- we multiply (4.13) and (4.15) by γ_λ and $i\gamma_5\gamma_\lambda$ respectively, then we add the both terms and anti-symmetrize the 2 free indices to obtain,

$$\gamma_{[\lambda|}\gamma^\mu\Omega_{\mu|\nu]} + i\gamma_5\gamma_{[\lambda|}\gamma^\mu * \Omega_{\mu|\nu]} = \Omega_{\lambda\nu} + i\gamma_5 * \Omega_{\lambda\nu} = 0\,, \tag{4.17}$$

which is equivalent to (4.16). In particular these results are valid for $\Omega = D\psi$, hence,

$$\not{e}D\psi = 0 \quad \cong \quad (i\gamma_5 - *)D\psi = 0\,. \tag{4.18}$$

4.2. Gravitino ground state.

The supersymmetry constraint (3.33) can be fulfilled also in the gravitino vacuum configuration

$$D\psi = 0\,, \qquad D\overline{\psi} = 0\,. \tag{4.19}$$

Since $D^2\psi = F\psi$ we also need

$$F\psi = 0\,. \tag{4.20}$$

In particular, the case $F = 0$ implies that all the bosonic curvatures (2.36-2.39) must vanish:

$$0 = R^{ab}(w) + p^a p^b - \tilde{p}^a \tilde{p}^b\,, \tag{4.21}$$

$$0 = D_\Omega p^a - h\tilde{p}^a = D_\Omega \tilde{p}^a - hp^a\,, \tag{4.22}$$

$$0 = dh + p^a \tilde{p}_a\,, \tag{4.23}$$

$$0 = G^r\,. \tag{4.24}$$

From (4.21) solutions interpolating Anti de Sitter and the de Sitter spaces can be achieved with a suitable choice of the parameters α_+ and α_- in (4.2). The flat case, $R^{ab}(w) = 0$, occurs for $\alpha_+^2 = \alpha_-^2$. This case, however, is degenerate since (4.4) vanishes, which is reflected also by the fact that the Einstein Hilbert term drops out from the Lagrangian (3.6). Replacing (4.2) in the torsion-like conditions (4.22) this yields,

$$\alpha_\pm D_\Omega e^a - \alpha_\mp h e^a = 0\,, \tag{4.25}$$

which, in the non-degenerate case $\alpha_+^2 \neq \alpha_-^2$, requires $h = 0$, $D_\Omega e^a = 0$, and spacetime to be of constant curvature,

$$R^{ab}(w) + (\alpha_+^2 - \alpha_-^2) e^a e^b = 0\,. \tag{4.26}$$

4.3. Effective Lagrangian.

With the transvection fields at their ground states (4.2), the theory (3.1) yields the effective Lagrangian[3] take the form

$$\mathcal{L}^{\mathtt{eff}} = -\,\mathtt{str}\ \mathcal{F}_\circ \circledast \mathcal{F}_\circ\,, \tag{4.27}$$

with $\mathcal{F}_\circ = d\mathcal{A}_\circ + \mathcal{A}_\circ^2$ built from the 1-form

$$\mathcal{A}_\circ = \frac{1}{2}\omega^{ab}\mathbb{B}_{ab} + \alpha_+ e^a\,\mathbb{B}_a + \alpha_- e^a\,\tilde{\mathbb{B}}_a + h\,\mathbb{B}_5 + U^r\,\mathbb{B}_r + \psi_i^\alpha\,\overline{\mathbb{Q}}_\alpha^i - \overline{\psi}_\alpha^i\,\mathbb{Q}_i^\alpha\,. \tag{4.28}$$

The field equations for (4.27) are obtained from (3.14)-(3.23) taking into account the dependence on the vierbein, implicit in (4.2),

$$\delta_e \mathcal{L}^{\mathtt{eff}} = \alpha_+ \delta e^a\,\frac{\partial \mathcal{L}}{\partial p^a} + \alpha_- \delta e^a\,\frac{\partial \mathcal{L}}{\partial \tilde{p}^a} + \delta e^a\,\frac{\partial \mathcal{L}}{\partial e^a}\,. \tag{4.29}$$

[3]By effective we simply mean that the theory can be expanded around the ground state (4.2).

Here the partial derivatives indicate functional derivative w.r.t. the explicit dependency on the variables p and $\tilde{p}$. The first two terms on the right hand side of (4.29) are obtained from the sum of the field equations (3.17) and (3.18) multiplied by α_+ and α_-, respectively. The third term is obtained from the Yang-Mills terms,

$$\mathcal{L}_{\mathbb{G}} = -\ \mathtt{str}\ (\mathbb{G} * \mathbb{G}), \qquad \mathbb{G} = \mathcal{F}|_{\mathtt{D}} + \mathcal{F}|_{\mathtt{INT}}. \tag{4.30}$$

Hence

$$\delta_e \mathcal{L}_{\mathbb{G}} = -\int d^4x\, e\, \delta e^a_\mu V^\mu_a, \tag{4.31}$$

where

$$V^\mu_a := \mathtt{str}\left(\mathbb{G}_{\lambda\rho}\mathbb{G}^{\lambda\rho} e^\mu_a - \frac{1}{4}\mathbb{G}_{\lambda\rho}\mathbb{G}^{\lambda\mu} e^\rho_a\right). \tag{4.32}$$

Here e^μ_a is the inverse of the vielbein and we have also introduced the inverse metric tensor $g^{\mu\nu} = e^\mu_a e^\nu_b \eta^{ab}$ to raise the 2-form indices of $\mathbb{G}$.

With a slightly different parametrization of the linear correspondence (4.2),

$$p^a = \alpha(1-\tau)e^a, \qquad \tilde{p}^a = \alpha\tau e^a, \tag{4.33}$$

so that $W^- = \frac{1}{2}\alpha((1-\tau) - \tau\gamma_5)\, \not{e}$, the Lorentz curvature reads

$$F^{ab} = R^{ab}(w) + \alpha^2(1-2\tau)e^a e^b. \tag{4.34}$$

Thus, τ interpolates between anti de Sitter for $\tau \in (-\infty, 1/2)$ and de Sitter for $\tau \in (1/2, \infty)$, and gives the degenerate case for $\tau = 1/2$. The effective Lagrangian (4.27) reads

$$\mathcal{L}^{\mathtt{eff}}_{\mathtt{bos}} = \alpha^2\left(\frac{1}{2}-\tau\right)R^{ab}(w)e^c e^d \epsilon_{abcd} + \alpha^4\left(\frac{1}{2}-\tau\right)^2 e^a e^b e^c e^d \epsilon_{abcd} \tag{4.35}$$

$$+\frac{1}{4}R^{ab}(w)R^{cd}(w)\epsilon_{abcd} - dh * dh - \frac{1}{2}F^I * F^I - 4dU^6 * dU^6, \tag{4.36}$$

$$\mathcal{L}^{\mathtt{eff}}_{\mathtt{fer}} = 4\overline{\psi}\left[i\not{e}\alpha\left(\frac{\pi_+}{2} - \left(\frac{1}{2}-\tau\right)\pi_-\right)D_{\Omega+U} + \frac{\alpha^2}{4}\left(\frac{1}{2}-\tau\right)i\gamma_5\not{e}^2\right. \tag{4.37}$$

$$\left. + iD\not{e}\,\alpha\left(\frac{\pi_+}{2} - \left(\frac{1}{2}-\tau\right)\pi_-\right) + \frac{1}{2}(* - i\gamma_5)(F|_D + F_U) - \frac{1}{4}\circledast I^+\right]\psi. \tag{4.38}$$

In the fermionic sector, $\pi_\pm = (\mathbb{1} \pm \gamma_5)/2$ are the chiral projectors. We observe that the Lagrangian (4.38) breaks partity (asymmetric chiral terms) and it has a bi-parametric Newton constant.

Note that the fact that for $\tau = 1/2$ the gravity sector in (4.36) decouples is consistent with the fact that pure Yang-Mills theories provide a good approximate description of internal interactions at short scales, with no need of gravity.

For positive cosmological constant ($\tau > 1/2$) the Lagrangian (4.36) produces ghosts modes for gravitons, since the Einstein-Hilbert term has the opposite sign. In section 4.4 an alternative Lagrangian is proposed where this is fixed.

4.3.1. *Standard normalization of the Lagrangian.*

The standard Einstein-Hilbert and Yang-Mills Lagrangians,

$$S_{\mathtt{EH}} = \frac{1}{2\kappa^2}\int \frac{1}{2}\Big(R^{ab}(w) - \frac{\lambda}{3!}e^a e^b\Big)e^c e^d \epsilon_{abcd} = \frac{1}{2\kappa^2}\int d^4x\, e\,(R-2\lambda)\,, \tag{4.39}$$

$$S_{\mathtt{EW}} = -\frac{1}{2g^2_{SU(2)}}\int G^I * G^I - \frac{1}{2g^2_{U(1)}}\int G^6 * G^6\,, \tag{4.40}$$

are contained in the effective Lagrangians (4.36)-(4.38) for $\tau \in (-\infty, 1/2)$,

$$\frac{1}{\kappa^2} = 2\alpha^2(1-2\tau)\,, \qquad g_{SU(2)} = 1\,, \qquad g_{U(1)} = \frac{1}{2\sqrt{2}}\,. \tag{4.41}$$

We see that the coupling constant $g_{SU(2)}$ has the canonical value and the Maxwell coupling constant $g_{U(1)} < g_{SU(2)}$, which respects the electroweak hierarchy. The Gravity coupling κ is bi-parametric and from the first relation in (4.41) α has the units of the inverse of the Newton constant G_N since $\kappa^2 = 8\pi G_N$.

In addition, there is a new abelian term in (4.36) corresponding to the (non-compact symmetry) dilation gauge field h,

$$S_{\mathtt{dil}} = -\frac{1}{2g^2_{\mathtt{D}}}\int dh * dh\,, \tag{4.42}$$

hence $g_{\mathtt{D}} = 1/\sqrt{2}$ is in the hierarchy $g_{U(1)} < \tilde{g}_{\mathtt{D}} < g_{SU(2)}$. Note that the dilation gauge field h is not minimally coupled, but has a Pauli coupling to the gravitino.

For $\tau = 0$ and with a rescaled gravitino field, the standard Rarita-Schwinger action is contained in (4.38) in the form,

$$\mathcal{L}_{\mathtt{RS}} = -\frac{1}{\kappa^2}\int \overline{\zeta} i\gamma_5 \not{e} D\zeta = \frac{1}{\kappa^2}\int d^4x\, e\, \overline{\zeta}_\mu \gamma^{\mu\nu\lambda} D_\mu \zeta_\nu\,, \qquad \zeta = \frac{\psi}{\sqrt{\alpha}}\,. \tag{4.43}$$

For more general values of τ, from the presence of the chiral projectors, the gravitino field should be decomposed in its chiral sectors, which will therefore appear with different weights.

4.3.2. *Limiting chiral model.*

We can obtain fixed chirality gravitino models from the model (4.27) in the limit, $\alpha \to 0$, $\tau \to -\infty$ while

$$\alpha^2\Big(\frac{1}{2} - \tau\Big) = \frac{1}{4\kappa^2} \tag{4.44}$$

is kept fixed. Hence we obtain in the bosonic sector (4.36),

$$\mathcal{L}^{\mathtt{eff}}_{\mathtt{bos}} = \frac{1}{4\kappa^2}R^{ab}(w)e^c e^d \epsilon_{abcd} + \frac{1}{16\kappa^4}e^a e^b e^c e^d \epsilon_{abcd} \tag{4.45}$$

$$-dh * dh - \frac{1}{2}F^I * F^I - 4dU^6 * dU^6 + \frac{1}{4}R^{ab}(w)R^{cd}(w)\epsilon_{abcd}\,, \tag{4.46}$$

whilst for the fermion term (4.38), redefining $\zeta = \pi_- \psi/\sqrt{\alpha}$, we get the Rarita-Schwinger action for a chiral field with a torsion-coupling,

$$\mathcal{L}^{\mathtt{eff}}_{\mathtt{fer}} = +\frac{i}{\kappa^2}\,\overline{\zeta}\left[\not{e} D_{\Omega+U} - \frac{1}{2}D\not{e}\right]\zeta\,. \tag{4.47}$$

4.4. Alternative $\circledast_s$ operator and the de Sitter sign fix.

The wrong sign in the Einstein-Hilbert term in (4.36) for $\tau \in (1/2, \infty)$ can be fixed by redefining the action of the generalized Hodge dual operator on the Lorentz component of the field strength: $\circledast\mathcal{F}|_L \to -\circledast\mathcal{F}|_L$. This leads to the wrong sign of the Einstein-Hilbert term in the anti de Sitter sector. The Pauli-like term $\overline{\psi}F|_L\psi$ produced by the terms $\mathtt{str}\,(\mathbb{F}|_L \circledast \mathbb{I}|_L)$ will not cancel the identical term produced by the fermion sector $\mathtt{str}\,\mathbb{X}\circledast\mathbb{X}$. Hence, in order to prevent the new Lorentz-Pauli coupling we need to flip also the sign of the $\circledast$ operator on the fermionic curvatures: $\circledast\mathbb{X} \to -\circledast\mathbb{X}$.

Different choices for the operator $\circledast$ can be selected by introducing the *ad hoc* sign function,

$$s_\tau = \begin{cases} 1, & \tau \in (-\infty, 1/2] \\ -1, & \tau \in (1/2, \infty) \end{cases}, \tag{4.48}$$

such that,

$$\circledast_s\mathcal{F}|_L = s_\tau(\circledast\mathcal{F}|_L)\,,\ \circledast_s\mathbb{X} = s_\tau(\circledast\mathbb{X})\,,\ \circledast_s(\mathcal{F}|_T + \mathcal{F}|_D + \mathcal{F}|_U) = *(\mathcal{F}|_T + \mathcal{F}|_D + \mathcal{F}|_U)\,, \tag{4.49}$$

which produces the alternative Lagrangian,

$$\mathcal{L}^{\mathtt{alt}} := -\,\mathtt{str}\,\mathcal{F}^+ \circledast_s \mathcal{F}^+\,, \tag{4.50}$$

suitable for both, negative and positive curvature backgrounds. Since $(\tau - 1/2)\,s_\tau = |\tau - 1/2|$, the new bosonic and fermionic components of the Lagrangian $\mathcal{L}^{\mathtt{alt}} = \mathcal{L}^{\mathtt{alt}}_{\mathtt{bos}} + \mathcal{L}^{\mathtt{alt}}_{\mathtt{fer}}$ are given respectively by,

$$\begin{aligned} \mathcal{L}^{\mathtt{alt}}_{\mathtt{bos}} =& \alpha^2\left|\frac{1}{2} - \tau\right| R^{ab}(w)e^c e^d \epsilon_{abcd} + s_\tau\,\alpha^4\left(\frac{1}{2} - \tau\right)^2 e^a e^b e^c e^d \epsilon_{abcd} \\ &+ s_\tau\,\frac{1}{4}R^{ab}(w)R^{cd}(w)\epsilon_{abcd} - dh * dh - \frac{1}{2}F^I * F^I - 4dU^6 * dU^6\,, \end{aligned} \tag{4.51}$$

and

$$\begin{aligned} \mathcal{L}^{\mathtt{eff}}_{\mathtt{fer}} =& 4\overline{\psi}\left[s_\tau\, i\not{e}\alpha\left(\frac{\pi_+}{2} - \left(\frac{1}{2} - \tau\right)\pi_-\right) D_{\Omega+U} + \frac{\alpha^2}{4}\left|\frac{1}{2} - \tau\right| i\gamma_5 \not{e}^2 \right. \\ &\left. + s_\tau\, iD\not{e}\,\alpha\left(\frac{\pi_+}{2} - \left(\frac{1}{2} - \tau\right)\pi_-\right) + \frac{1}{2}(* - s_\tau\, i\gamma_5)(F|_D + F_U) - \frac{1}{4}\circledast_s I^+ \right]\psi\,. \end{aligned} \tag{4.52}$$

Now the Einstein-Hilbert term sign is always correct and only the cosmological term in the gravity side flips sign.

The chiral models are obtained as before in the limits $\alpha \to 0$, $\tau \to \pm\infty$, while keeping fixed the value (4.44). From (4.51) this yields

$$\mathcal{L}^{\mathtt{alt}}_{\mathtt{bos}} = \frac{1}{4\kappa^2}R^{ab}(w)e^c e^d \epsilon_{abcd} + \frac{s_{\pm\infty}}{16\kappa^4} e^a e^b e^c e^d \epsilon_{abcd} \tag{4.53}$$

$$+ \frac{s_{\pm\infty}}{4}R^{ab}(w)R^{cd}(w)\epsilon_{abcd} - dh * dh - \frac{1}{2}F^I * F^I - 4dU^6 * dU^6\,, \tag{4.54}$$

where $s_{\pm\infty} = \mp 1$, and in the fermion sector (4.52), redefining $\zeta = \pi_-\psi/\sqrt{\alpha}$ gives

$$\mathcal{L}^{\mathtt{alt}}_{\mathtt{fer}} = \frac{i}{\kappa^2}\,\overline{\zeta}\left[\not{e}D_{\Omega+U} - \frac{1}{2}D\not{e}\right]\zeta\,. \tag{4.55}$$

4.5. The matter ansatz.

In [AVJ11] a mechanism to incorporate Dirac fermion (0-forms) in a supersymmetric gauge connection was introduced, such that the corresponding 3D Chern-Simons supergravity action produced, instead of a Rarita-Schwinger term, the Dirac action minimally coupled to a Maxwell gauge field and gravity, with a torsion-dependent mass. This approach, referred to as *unconventional supersymmetry*, has been used to build several models in 3D [Alv+15; And+18], including interesting applications in condensed matter systems [And+20; IP21; Ior20; Gal21b; Gal21a]. Extensions of these ideas to four dimensions can be found in [APZ14; AVJ20].

In unconventional supersymmetry (for a review see [Alv+21a]) a spin-1/2 field is introduced directly in the supersymmetry gauge connection, not as a fundamental gravitino field but combined with the vielbein in the form,

$$\Psi := \overline{\mathbb{Q}}\,(\not{e}\xi)\,, \qquad \overline{\Psi} = \overline{(\not{e}\xi)}\,\mathbb{Q}\,, \tag{4.56}$$

where ξ is a fermion 0-form. Hence, instead the action principle for a spin-3/2 field the results is a spin-1/2 action principle. This justified to denote (4.56) as the *matter ansatz*.

In reference [AVJ20] an unconventional supersymmetry model was proposed in four dimensions based in the superalgebra $su(2,2|2)$ and a Lagrangian of the type (3.1), with the fermion sector replaced by the matter ansatz (4.56). Similarly, here when the matter ansatz (4.56) is used in (3.1), one obtains the descendant Lagrangian

$$\mathcal{L}_{\texttt{m-ans}} := -\ \texttt{str}\left(\mathcal{F}_{\texttt{m-ans}} \circledast \mathcal{F}_{\texttt{m-ans}}\right). \tag{4.57}$$

In this Lagrangian the fermion field strength (2.23) is given by $\mathcal{X} := \overline{\mathbb{Q}}^{i}_{\alpha}\, D(\not{e}\xi)^{\alpha}_{i}$. Hence one would obtain a theory for $U(1)\times SU(2)$–minimally coupled matter fermions governed by a Dirac action, with additional torsion and Pauli couplings, and four-fermion self-interactions.

The model (4.57) would differ from the one in [AVJ20] because the operator $\circledast$ used there acts on the fermions component of the curvature with the opposite sign. As a consequence, in [AVJ20] there is an additional coupling of the fermion field and the Lorentz curvature with respect to the one here.

Without further additions, the theory obtained in this way hinges on the identification between transvection and vierbein fields. Hence, considering the ground states (4.2), we can obtain a theory of fermions coupled to gravity and gauge fields in a more standard fashion applying the matter ansatz (4.56) in (4.38) and (4.52), or in the chiral-model limits (4.47) and (4.55). In particular a matter–anti-matter symmetry breaking of fermions can be fine tuned using the parameter τ in (4.38) and (4.52). Instead in (4.47)-(4.55) the chirality of the fermions is fixed. See [AVJ20] for further discussions.

CHAPTER 6

Maximally extended Chern-Simons Poincaré supergravity for all odd dimensions

1. Motivation

In [HR08], a supersymmetric extension of the Poincaré Lagrangian

$$L = \epsilon_{a_1\cdots a_{2n+1}} R^{a_1 a_2} \wedge \cdots R^{a_{2n-1} a_{2n}} \wedge e^{a_{2n+1}}, \tag{1.1}$$

was found for the $\mathcal{N} = 1$ Poincaré super-algebra (1.2) with the maximal amount of bosonic generators $\mathbf{Z}_{a_1\cdots a_p}$, arising from the anticommutator of two spinor charges of the ($\mathcal{N}$-extended) Poincaré super-algebra

$$\left\{\mathbb{Q}^i_\alpha, \mathbb{Q}^j_\beta\right\} = \left(\gamma^a\right)_{\alpha\beta} \mathbf{P}_a \mathcal{G}^{ij} + \sum_p \left(\gamma^{a_1\cdots a_p}\right)_{\alpha\beta} \mathbf{Z}^{ij}_{a_1\cdots a_p}, \tag{1.2}$$

The construction was performed in $d = 3$ mod 8, which corresponds to the odd dimensions where Majorana spinors exist. The goal of the work presented in the following sections is to further extend this construction to all other odd dimensions, and for any $\mathcal{N}$. Furthermore, we show how the symplectic Majorana condition - an alternative reality condition - can be implemented in the formalism. It may be stressed that, since the Poincaré algebras are non semi-simple, the construction of supersymmetric Poincaré Chern-Simons theories for the their maximal extension (1.2) is highly nontrivial. In order to circumvent this problem, we observe that Poincaré (super-)algebras can be obtained from the semi-simple AdS (super-)algebras by means of an expansion method inspired by Wigner-Inönü contractions. The expansion method consists in forming the algebra of Lie algebra-valued polynomial, truncate it at a given order N (i.e. quotient out the element of degree higher than N), and then extract a subalgebra from this algebra of truncated polynomial. This technique is similar to one proposed in [MM03; JO03] for generating a Lie algebra from a lower dimensional one [1].

Here, we start by considering AdS super-algebras with maximal number of bosonic (even) generators. These algebras are then expanded through the method that we briefly present in a form adapted to our specific problem. Thus, we obtain all possible Poincaré super-algebras with the maximal number of "central charges", namely the $\mathbf{Z}_{a_1\cdots a_p}$'s in (1.2). The corresponding supersymmetric Poincaré-Chern-Simons Lagrangians can be constructed from the invariant tensors of the semi-simple maximal AdS super-algebras. It can be shown that half of the $\mathbf{Z}_{a_1\ldots a_p}$, and the corresponding gauge fields associated to them, do not contribute to

[1] In [JO03], the expansion is presented in the dual form *i.e.* the expansion is done for the dual coalgebra.

the final Lagrangian, and these can be safely removed from the algebra. It is also shown that these fields could not be part of any Lorentz invariant Lagrangian built from exactly one of these fields and the Riemann curvature tensor.

2. Presentation of the Lagrangian

2.1. The Algebra.

We are interested on constructing gauge theories for some supersymmetric extensions of the Poincaré algebra (1.2) with generators given by

$$\mathbf{J}_{ab} \quad \textit{Lorentz generators,} \tag{2.1}$$

$$\mathbf{P}_a \quad \textit{Translations generators,} \tag{2.2}$$

$$\mathbf{Z}_{a_1\ldots a_n} \quad \textit{"Central" charges,} \tag{2.3}$$

$$\overline{\mathbf{Q}}_\alpha, \mathbb{Q}^\alpha \quad \textit{Super charges.} \tag{2.4}$$

There, we made two slight abuses. Firstly, the generators $\mathbb{Q}$ and $\overline{\mathbf{Q}}$ are not part of the Poincaré super-algebra but of its complexification. We do so because the fields associated to the real generators have no physical meaning, see the Appendix. Secondly the $\mathbf{Z}_{a_1\ldots a_n}$ generators are not strictly speaking central charges since they do not commute with the Lorentz generators. However they do commute with all other generators, and hence in what follows we will refer to them as "the central charges".

These central charges impose a splitting of the odd dimensions into the cases $d = 4k+1$ and $d = 4k+3$. Indeed, their possible rank n are given by a set $\mathcal{I}_d \ni n$, where

$$\mathcal{I}_{4k+1} = \{n \equiv 0, 1 \bmod 4\} \setminus \{1\}, \tag{2.5}$$

$$\mathcal{I}_{4k+3} = \{n \equiv 1, 2 \bmod 4\} \setminus \{1\}. \tag{2.6}$$

The non-vanishing (anti)commutators are

$$[\mathbf{J}_{ab}, \mathbb{Q}^\alpha] = -\frac{1}{2}(\gamma_{ab})^\alpha{}_\beta \mathbb{Q}^\beta, \qquad [\mathbf{J}_{ab}, \overline{\mathbf{Q}}_\alpha] = \overline{\mathbf{Q}}_\beta \frac{1}{2}(\gamma_{ab})^\beta{}_\alpha, \tag{2.7}$$

$$\{\mathbb{Q}^\alpha, \overline{\mathbf{Q}}_\beta\} = \frac{1}{2^{(d+1)/2}}\left(\mathbf{P}^a(\gamma_a)^\alpha{}_\beta + \sum_{n\in\mathcal{I}_d} \frac{(-1)^{n(n-1)/2}}{n!}\mathbf{Z}^{a_1\ldots a_n}(\gamma_{a_1\ldots a_n})^\alpha{}_\beta\right). \tag{2.8}$$

while the commutators between the Lorentz generators with the other bosonic generators can be read off from their tensorial characters.

2.2. Connection and Chern-Simons form.

Let us now introduce the connection for the Poincaré super-algebra $\mathbb{A}_\mathrm{P}$ as

$$\mathbb{A}_\mathrm{P} = \Omega + \Psi + \mathbf{B}, \tag{2.9}$$

and, where for convenience, we have defined

$$\Omega = \frac{1}{2}\omega^{ab}\mathbf{J}_{ab}, \tag{2.10}$$

$$\Psi = \overline{\mathbf{Q}}\psi + \overline{\psi}\mathbb{Q}, \tag{2.11}$$

$$\mathbf{B} = e^a\mathbf{P}_a + \sum_{n\in\mathcal{I}_d} b^{a_1\dots a_n}\mathbf{Z}_{a_1\dots a_n}. \tag{2.12}$$

Since the algebra is closed, the curvature 2−form $\mathbb{F} = d\mathbb{A} + \frac{1}{2}[\mathbb{A}\wedge\mathbb{A}]$ is also expanded along the generators as

$$\mathbb{F}_{\mathrm{P}} = \mathbf{R} + \mathrm{D}\Psi + \mathrm{D}\mathbf{B} + \Psi\wedge\Psi, \tag{2.13}$$

$$\mathbf{R} = \frac{1}{2}R^{ab}\mathbf{J}_{ab}, \qquad R^{ab} = \mathrm{d}\omega^{ab} + \omega^a_c\omega^{cb}, \tag{2.14}$$

$$\mathrm{D}\Psi = \overline{\mathbf{Q}}(\mathrm{d}+\omega\!\!\!/)\psi + \overline{\psi}(\overleftarrow{\mathrm{d}} + \omega\!\!\!/)\mathbb{Q}, \tag{2.15}$$

$$\mathrm{D}\mathbf{B} = \mathrm{D}e^a\mathbf{P_a} + \sum_{n\in\mathcal{I}_d}\mathrm{D}b^{a_1\dots a_n}\mathbf{Z}_{a_1\dots a_n}, \tag{2.16}$$

where D denotes the Lorentz covariant derivative, and the right exterior derivative $\overleftarrow{\mathrm{d}}$ is defined by, acting on a p-form Λ,

$$\Lambda\overleftarrow{\mathrm{d}} = (-1)^p\mathrm{d}\Lambda. \tag{2.17}$$

We also define the pure Lorentz connection

$$\mathbb{A}_{\mathrm{Lor}} = \Omega. \tag{2.18}$$

The Lagrangian is chosen to be the transgression form (denomination borrowed from [Nak91]) interpolating between the $\frac{d+1}{2}$-th Chern character of associated to the pure Lorentz connection $\mathbb{A}_{\mathrm{Lor}}$ and the full super Poincaré connection $\mathbb{A}_{\mathrm{P}}$. In details, one defines the interpolating connection and curvature

$$\mathbb{A}_t = (1-t)\mathbb{A}_{\mathrm{Lor}} + t\mathbb{A}_{\mathrm{P}} \tag{2.19}$$

$$= \Omega + t(\Psi + \mathbf{B}), \tag{2.20}$$

$$\mathbb{F}_t = \mathbf{R} + t\mathrm{D}(\Psi+\mathbf{B}) + t^2\Psi\wedge\Psi, \tag{2.21}$$

for $t \in [0,1]$. With the use of the invariant form[2] $\langle\ ,\dots,\ \rangle$, constructed in section 3, and detailed in (3.13-3.15), we are able to construct the Chern-Simons Lagrangian as

$$\mathcal{L} = \int_0^1 dt\langle\mathbb{A}_{\mathrm{P}} - \mathbb{A}_{\mathrm{Lor}}, \underbrace{\mathbb{F}_t,\dots,\mathbb{F}_t}_{\frac{d-1}{2}}\rangle \tag{2.23}$$

$$= \langle\mathbf{B}, \underbrace{\mathbf{R},\dots,\mathbf{R}}_{\frac{d-1}{2}}\rangle_1 + \frac{d-1}{4}\langle\Psi,\mathrm{D}\Psi,\underbrace{\mathbf{R},\dots,\mathbf{R}}_{\frac{d-3}{2}}\rangle_1. \tag{2.24}$$

[2]By an invariant form of a real super-algebra $\mathfrak{g}$, we mean a linear map $\langle\ ,\dots,\ \rangle : \mathfrak{g}\otimes\cdots\otimes\mathfrak{g}\to\mathbb{R}$ satisfying:

$$\forall\, h, g_1,\dots,g_n\in\mathfrak{g},\quad \sum_{i=1}^n(-1)^{|h|(|g_1|+|g_2|+\dots+|g_{i-1}|)}\langle g_1,\dots,[h,g_i],\dots,g_n\rangle = 0. \tag{2.22}$$

2.3. The main Lagrangian. In a more explicit way, the Lagrangian can be written in a very compact form as

$$\mathcal{L} = i^k \mathrm{Tr}\left[\not{R}^{(d-3)/2} \left(\not{R}(\not{e} + \sum_{n\in\mathcal{I}_d} \not{b}^n) + (\psi \mathrm{D}\overline{\psi} - \mathrm{D}\psi\overline{\psi}) \right) \right], \tag{2.25}$$

where we have defined

$$\not{R} = \frac{1}{4} R^{ab}\gamma_{ab}, \quad \not{e} = \frac{1}{2} e^a \gamma_a, \quad k \equiv \frac{d+1}{2} \bmod 2, \tag{2.26}$$

$$\not{b}^n = \begin{cases} \frac{i}{2(n!)} b^{a_1\ldots a_n}\gamma_{a_1\ldots a_n}, & n \equiv 0 \bmod 4, \\ \frac{1}{2(n!)} b^{a_1\ldots a_n}\gamma_{a_1\ldots a_n}, & n \equiv 1,2 \bmod 4. \end{cases} \tag{2.27}$$

It is interesting to note that, restricting the spinor ψ appearing in (2.11) to be a Majorana spinor (which is possible only in dimensions $d \equiv 3 \bmod 8$ for the mostly plus Lorentzian signature) [TZ99], the Lagrangian becomes (with ψ now a Majorana spinor)

$$\mathcal{L}_{\mathrm{Maj}} = \mathrm{Tr}\left[\not{R}^{(d-3)/2} \left(\not{R}(\not{e} + \sum_{n\in\mathcal{I}_3} \not{b}^n) + 2\psi \mathrm{D}\overline{\psi} \right) \right], \tag{2.28}$$

and this expression coincides with that found in [HR08]. There, the restrictions for $n \in \mathcal{I}_d$ were computed using a symmetry argument exclusive to Majorana spinors. Surprisingly, relaxing the Majorana condition did not affect those restrictions.

2.4. Higher $\mathcal{N}$ theories.

So far, our theory contains one Dirac spinor and is therefore referred as a $\mathcal{N} = 2$ theory. If we restrict the spinor to be Majorana as explained in the previous paragraph, we obtain an $\mathcal{N} = 1$ theory. However, it is possible to increase $\mathcal{N}$. To do so, we generalize the pair of odd generators $\{\mathbb{Q}, \overline{\mathbf{Q}}\}$, introducing a new index $i \in \{1, \ldots \frac{\mathcal{N}}{2}\}$[3] forming the pairs $\{\mathbb{Q}_i, \overline{\mathbf{Q}}^i\}$ with new commutation relations given by

$$[\mathbf{J}_{ab}, \mathbb{Q}_i^\alpha] = -\frac{1}{2}(\gamma_{ab})^\alpha{}_\beta \mathbb{Q}_i^\beta, \qquad [\mathbf{J}_{ab}, \overline{\mathbf{Q}}_\alpha^i] = \overline{\mathbf{Q}}_\beta^i \frac{1}{2}(\gamma_{ab})^\beta{}_\alpha, \tag{2.29}$$

$$\{\mathbb{Q}_i^\alpha, \overline{\mathbf{Q}}_\beta^j\} = \frac{1}{2^{(d+1)/2}}\delta_i^j \left(P^a(\gamma_a)^\alpha{}_\beta + \sum_{n\in\mathcal{I}_d} \frac{(-1)^{n(n-1)/2}}{n!} \mathbf{Z}_{a_1\ldots a_n}(\gamma^{a_1\ldots a_n})^\alpha{}_\beta \right).$$

For this extended Poincaré super-algebra, the resulting supersymmetric Chern-Simons Lagrangian is given

$$\mathcal{L} = i^k \mathrm{Tr}\left[\not{R}^{(d-3)/2} \left(\not{R}(\not{e} + \sum_{n\in\mathcal{I}_d} \not{b}^n) + \sum_{i=0}^{\mathcal{N}/2} (\psi_i \mathrm{D}\overline{\psi}^i - \mathrm{D}\psi_i\overline{\psi}^i) \right) \right]. \tag{2.30}$$

[3] $\mathcal{N}$ is taken to be an even integer because we are adding Dirac spinors.

2.5. Symplectic Majorana Spinors.

Our construction can also be extended in the case of symplectic Majorana spinors defined in dimensions $d \equiv 7 \bmod 8$ (for mostly plus Lorentzian signature). Indeed, when $\frac{\mathcal{N}}{2}$ being an even integer, we can ask the spinors ψ^i to satisfy a symplectic Majorana condition,

$$\overline{\psi}^i_\alpha = \psi^\beta_j C_{\beta\alpha} \varepsilon^{ji}, \tag{2.31}$$

where ε is the matrix

$$\varepsilon = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}. \tag{2.32}$$

In this case, the local supersymmetric Lagrangian obtained for the symplectic Majorana spinor reads

$$\mathcal{L} = \mathrm{Tr}\left[\not{R}^{(d-3)/2} \left(\not{R}(\not{e} + \sum_{n \in \mathcal{I}_d} \not{b}^n) + 2 \sum_{i=0}^{p} \psi_i \mathrm{D} \overline{\psi}^i \right) \right]. \tag{2.33}$$

3. Expansion of super-algebras and invariant forms

The goal of this section is to obtain the invariant form of the Poincaré super-algebra used in (2.23) to define our Lagrangian. This invariant form is computed using an expansion of the AdS super-algebra. Thus we start by briefly recalling what is the AdS super-algebra (it is reviewed in details in the appendix), then present the expansion method in its full generality before applying it concretely to obtain the desired invariant form. We close this section by discussing the more general cases of higher $\mathcal{N}$ theories.

3.1. The general expansion method.

We recall some results of section 2.1 and further develop the subject. Let $\mathfrak{g}$ be a real Lie super-algebra, and let

$$\mathbb{R}[\lambda] \otimes \mathfrak{g}, \tag{3.1}$$

be the algebra of polynomial in λ with coefficients in $\mathfrak{g}$. Here, one can view λ as an infinitesimal parameter, $\lambda \sim \frac{1}{r^{1/2}}$, and hence (3.1) can be thought as the algebra with generators $\mathbf{g}_i \in \mathfrak{g}$ together with their infinitesimal versions $\lambda^n \mathbf{g_i}$. We take the following quotient

$$\mathfrak{g}(N) = \left(\mathbb{R}[\lambda] \big/ \lambda^{N+1} \mathbb{R}[\lambda] \right) \otimes \mathfrak{g}. \tag{3.2}$$

In our picture where λ is as an infinitesimal parameter, the quotient by $\lambda^{N+1}\mathbb{R}[\lambda]$ corresponds to "keeping track of terms up to order λ^N".

Notice that $\mathfrak{g}$ is $\mathbb{Z}_2$-graded as a Lie super-algebra, and that $\mathbb{R}[\lambda]$, as well as $\mathbb{R}[\lambda] \big/ \lambda^{N+1}\mathbb{R}[\lambda]$ are also $\mathbb{Z}_2$-graded, the grading being given by powers of λ (either even or odd), thus so is $\mathfrak{g}(N)$. We denote by $\mathfrak{g}(N)_0$ the even part of $\mathfrak{g}(N)$, which is itself a Lie super-algebra. Viewing $\lambda \sim \frac{1}{r^{1/2}}$, looking at $\mathfrak{g}(N)_0$ corresponds to rescale odd generators by odd powers of λ and even generators by even ones.

Suppose we have a subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Then obviously $\mathfrak{h}(N) \subset \mathfrak{g}(N)$. Let $K < N$, and

$$\mathfrak{g}(K,N) = \left(\lambda^K \mathbb{R}[\lambda] \big/ \lambda^{N+1}\mathbb{R}[\lambda] \right) \otimes \mathfrak{g}, \tag{3.3}$$

which is the subalgebra of $\mathfrak{g}(N)$ with elements starting only at minimum λ-power K. It is easy to see that $\mathfrak{g}(K,N)$ is a subalgebra (even an ideal) of $\mathfrak{g}(N)$. The sum of two subalgebra being a subalgebra, $\mathfrak{h}(N) + \mathfrak{g}(K,N)$ is a subalgebra of $\mathfrak{g}(N)$, and we consider the even part

$$(\mathfrak{h}(N) + \mathfrak{g}(K,N))_0. \tag{3.4}$$

We note that this construction is the dual transcription of the expansion method presented in [JO03]. Indeed, in this last reference, the authors make an expansion for the dual coalgebra which, although presented differently, can be seen as taking a tensor product with $\mathbb{R}[\lambda] \big/ \lambda^{N+1}\mathbb{R}[\lambda]$ then looking at subcoalgebra and keeping only even part. It can also be generalized by taking the (Cartesian and not tensorial) product of a given algebra with a semi group. This method, called "S-expansion of algebras", has been developed in Ref. [IRS06].

Of important use for our purpose, we now show how to construct an invariant tensor of the quotient super-algebra. Keeping $\mathfrak{g}$ as a Lie super-algebra, let $\langle \quad , \dots , \quad \rangle$ denotes a p-linear invariant form over it. Then it can be extended to

$$\langle \quad , \dots , \quad \rangle_N : \mathfrak{g}(N)_0{}^{\otimes p} \to \mathbb{R}[\lambda] \big/ \lambda^{N+1}\mathbb{R}[\lambda]. \tag{3.5}$$

For each $n \leq N$, one can consider the projection

$$\mathbb{R}[\lambda] \big/ \lambda^{N+1}\mathbb{R}[\lambda] \simeq \lambda^0\mathbb{R} \oplus \lambda^1\mathbb{R} \oplus \cdots \oplus \lambda^N\mathbb{R} \xrightarrow[\mathrm{pr}_n]{} \lambda^n\mathbb{R} \simeq \mathbb{R}, \tag{3.6}$$

where $\simeq$ means isomorphism of vector spaces. The composition $\mathrm{pr}_n \circ \langle \quad , \dots , \quad \rangle_N$ defines a p-linear invariant form over $\mathfrak{g}(N)_0$. This of course, restricts to a p-linear invariant form over any of its subalgebras.

3.2. Application to the Poincaré super-algebra.

We are now in position to show that the Poincaré super-algebras of interest (2.7-2.8) and their invariant tensors can be obtained using the previous constructions applied to the AdS super-algebra $\mathfrak{su}(\frac{m}{2}, \frac{m}{2}|1)$. Let $\mathfrak{so}(d-1,1) \subset \mathfrak{su}(\frac{m}{2}, \frac{m}{2}|1)$ be the Lorentz subalgebra. In the notations of the previous paragraph, $\mathfrak{g} = \mathfrak{su}(\frac{m}{2}, \frac{m}{2}|1)$, $\mathfrak{h} = \mathfrak{so}(d-1,1)$, $N = 2$, $K = 1$. Thus we consider

$$(\mathfrak{so}(d-1,1)(2) + \mathfrak{su}(\frac{m}{2}, \frac{m}{2}|1)(1,2))_0. \tag{3.7}$$

It is almost our Poincaré super-algebra (2.7-2.8). At the level of the generators, the contact with its earlier presentation (2.1-2.4) is achieved through

$$\lambda^0 \mathbf{J}_{ab} \longrightarrow \mathbf{J}_{ab}, \tag{3.8}$$

$$\lambda^2 \mathbf{J}_{a} \longrightarrow \mathbf{P}_{a}, \tag{3.9}$$

$$\lambda^2 \mathbf{J}_{a_1\dots a_n} \longrightarrow \mathbf{Z}_{a_1\dots a_n}, \quad n \neq 1,2, \tag{3.10}$$

$$\lambda^1 \mathbb{Q} \longrightarrow \mathbb{Q}, \tag{3.11}$$

$$\lambda^1 \overline{\mathbf{Q}} \longrightarrow \overline{\mathbf{Q}}. \tag{3.12}$$

To get our Poincaré super-algebra, we still need to impose the restriction $n \in \mathcal{I}_d$ in (3.10). Let us explain why such a restriction is needed. The resulting $\frac{d+1}{2}$-linear form (obtained by taking projection onto the $\lambda^2\mathbb{R}$-subspace) is given by

$$\langle \mathbf{P}_a, \mathbf{J}_{b_1 b_2}, \dots \mathbf{J}_{b_{d-2} b_{d-1}} \rangle = \frac{1}{2^{(d+1)/2}} \epsilon_{a b_1 b_2 \dots b_{d-1}}, \tag{3.13}$$

$$\langle \mathbf{Z}_{a_1\dots a_n}, \mathbf{J}_{b_1 b_2}, \dots \mathbf{J}_{b_{d-2} b_{d-1}} \rangle = \frac{i^l}{2^{(d+1)/2}} \sigma\mathrm{Tr}(\gamma_{a_1\dots a_n} \gamma_{b_1 b_2} \cdots \gamma_{b_{d-2} b_{d-1}}), \tag{3.14}$$

$$\langle \overline{\mathbf{Q}}_\alpha, \mathbb{Q}^\beta, \mathbf{J}_{b_1 b_2}, \dots \mathbf{J}_{b_{d-4} b_{d-3}} \rangle = \frac{i^k}{2^{(d-3)/2}(d-1)} \sigma\mathrm{Tr}(E_{\alpha\beta} \gamma_{b_1 b_2} \cdots \gamma_{b_{d-4} b_{d-3}}), \tag{3.15}$$

where $l, k \in \{0,1\}$ with

$$l \equiv k + q \mod 2, \tag{3.16}$$

$$k \equiv \frac{d+1}{2} \mod 2, \tag{3.17}$$

and where σTr means symmetrized trace defined by

$$\sigma\mathrm{Tr}(M_1 M_2 \dots M_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{Tr}(M_{\sigma(1)} M_{\sigma(2)} \dots M_{\sigma(n)}).$$

As it can be seen from the expression of the invariant tensors, one is led to compute traces of the form (3.14). Note that the other trace (3.15) can be cast in the form (3.14) decomposing $E_{\alpha\beta}$ in the matrix basis formed by the $\gamma_{a_1\dots a_n}$-matrices. One can show that the trace (3.14) is proportional to

$$\mathrm{Tr}\Big(\{\gamma_{a_1\dots a_n}, \{\gamma_{b_1 b_2} \cdots \{\gamma_{b_{d-4} b_{d-3}}, \gamma_{b_{d-2} b_{d-1}}\} \dots\}\}\Big), \tag{3.18}$$

and using the following formula for the anti commutator

$$\{\gamma_{a_1\dots a_n}, \gamma_{b_1 b_2}\} = \gamma_{a_1\dots a_n b_1 b_2} - \frac{2}{n!} \sum_{\sigma \in S_n} \eta_{b_1 a_{\sigma(1)}} \eta_{b_2 a_{\sigma(2)}} \gamma_{a_{\sigma(3)} \dots a_{\sigma(n)}}, \tag{3.19}$$

it is easy to show that the trace (3.18) vanishes for $n \equiv 3 \bmod 4$ for all odd dimensions, as well as for $n \equiv 2 \bmod 4$ in dimensions $d \equiv 1 \bmod 4$ and for $n \equiv 0 \bmod 4$ in dimensions $d \equiv 3 \bmod 4$. In other words, this means that the corresponding bosonic fields will not appear in the Lagrangian (2.25). On the other hand, since any of the sub-vector spaces generated these central charges having the same rank n (*i.e.* $\mathrm{Vect}\langle\{\mathbf{Z}_{a_1\dots a_n}\}\rangle$) form an Abelian ideal, we can freely quotient the algebra by any of them. Hence, eliminating all central charges of the same rank will not in any case affect the Chern-Simons form defined previously, and this leads naturally to consider theories with central charges which are Lorentz tensor of rank n, with rank only given in $n \in \mathcal{I}_d$. Note however that taking a quotient does not, in general, preserve the invariance of (3.13-3.15). It is only because we are eliminating generators where (3.14) vanishes that we do not affect the invariant from.

3.3. Invariant forms for the (extended) $\mathcal{N}$ super-algebras.

Instead of starting with $\mathfrak{sl}(m|1,\mathbb{C})$ we could repeat the analysis with $\mathfrak{sl}(m|M,\mathbb{C})$, which allows the introduction of new fermionic generators coming in pairs, namely $\{\mathbb{Q}^\alpha_i, \overline{\mathbf{Q}}^\alpha_i\}$. This construction also brings new bosonic generators that we will denote by $\mathbf{K}_I$, forming a new $\mathfrak{sl}(M)$ algebra.

Exactly as before, we choose a real form, $\mathfrak{su}(\frac{m}{2},\frac{m}{2}|M)$, and construct the algebra

$$(\mathfrak{so}(d-1,1)(2) + \mathfrak{su}(\frac{m}{2},\frac{m}{2}|M)(1,2))_0. \tag{3.20}$$

This is the extended Poincaré super-algebra. In absence of further Majorana reduction, it has $\mathcal{N} = 2M$. The Majorana reduction achieved earlier depends uniquely on the Lorentz subalgebra $\mathfrak{so}(d-1,1)$ which has not been modified, and thus can be performed in the exact same way. In this construction, the $\lambda\mathbf{K}_I$'s are true central charges, and the invariant form (3.15) is now given by

$$\langle \overline{\mathbf{Q}}^i_\alpha, \mathbb{Q}^\beta_j, \mathbf{J}_{b_1 b_2}, \dots \mathbf{J}_{d-4 d-3}\rangle = \frac{1}{2^{(d-3)/2}(d-1)} \delta^i_j \sigma \mathrm{Tr}(E_{\alpha\beta}\gamma_{b_1 b_2}\dots\gamma_{b_{d-4} b_{d-3}}). \tag{3.21}$$

One can also compute the contribution of the central charges $\mathbf{K}_I$ whose only non trivial candidate is given by

$$\langle \lambda\mathbf{K}_I, \mathbf{J}_{a_1 a_2}, \dots \mathbf{J}_{a_{2n-1} a_{2n}}\rangle. \tag{3.22}$$

Nevertheless, computing the super-trace over the representation (3.12, 3.23), this contribution identically vanishes. As a direct consequence, we can eliminate the $\lambda\mathbf{K}_I$ generators of the extended $\mathcal{N}$ Poincaré super-algebra. For completeness, we briefly mention that one could keep the $\lambda^0\mathbf{K}_I$ generators, and the resulting theory would contain an additional $SU(N)$ gauge symmetry (but the Lagrangian will involve a bunch of new terms).

4. Maximality of the theory

In this section, we prove that the theories presented in section 2 contains the maximal number of $b^{a_1\dots a_n}$ fields allowed, *i.e.* we have all possible n for which it is possible to write a Lorentz invariant term built from one $b^{a_1\dots a_n}$ and the Riemann curvature tensor only. We had a Lagrangian with $n \equiv 1 \bmod 4$ and $n \equiv d-1 \bmod 4$. We have to show that it is impossible to build a Lagrangian respecting the conditions mentioned above if n does not follow this restriction.

4.1. To show this, we recall that any Lorentz invariant tensor is built from the Minkowski metric η_{ab} and the Levi-Civita tensor $\varepsilon_{a_1\dots a_d}$ (For example the Killing form of the Lorentz algebra can be written $\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}$). With this remark in mind, let us set up more clearly our problem. The resulting Lagrangian is a d-form, and $b^{a_1\cdots a_n}$ a 1-form. Hence we need to use $\frac{d-1}{2}$ Riemann curvature two-form R^{ab}. This gives us a total of $n+d-1$ indices, with

$n \leq \frac{d-1}{2}$, that we need to contract using $\varepsilon_{a_1\ldots a_d}$ and η_{ab}; in other words, one is force to pair these indices with some indices obtained from different tensors $\varepsilon_{a_1\ldots a_d}$ and η_{ab}.

Let us first suppose that n is even. Then we have an even number of total indices to be contracted. Thus, if we want to use the $\varepsilon_{a_1\ldots a_d}$ tensor, we need to use an even number of them. However if we use at least two, counting the number of available indices, we see that we have to contract some indices of these two tensors among themselves. But a well-known formula tells us that two Levi-Civita tensors with some of their indices contracted is proportional to a sum of antisymmetrized η_{ab}'s. We reach the conclusion that in the case n is even, we can dispose of the Levi-Civita tensor, and that our sought invariant tensor will be built using the Minkowski metric only.

Contracting indices of the curvature tensor R^{ab} using the Minkowski metric, one can form monomials without free indices as $R^a{}_b R^b{}_c R^c{}_a$ or with two free indices as $R^a{}_b R^b{}_c R^{cd}$. Note that a monomial with two free indices is symmetric in these two indices if it contains an even number of curvature tensor ($R^a{}_b R^{bc} = R^c{}_b R^{ba}$), antisymmetric if it contains an odd number of them ($R^a{}_b R^b{}_c R^{cd} = -R^d{}_b R^b{}_c R^{ca}$). A monomial without free indices, as the contraction of two monomial with two free indices, these two monomials need to have the same type of symmetry; thus a monomial with no indices necessarily contains an even number of curvature tensor. Leaving them apart, we are left with monomials with two free indices, built in total with a number of curvature tensor having the same parity as $\frac{d-1}{2}$, which is the total number of curvature tensor at our disposal.

The field $b^{a_1\cdots a_n}$ is fully antisymmetric, its indices can only be contracted with those of tensors having the same symmetry; in other words with monomials made of an odd number of curvature tensor. Suppose $\frac{d-1}{2}$ is even, then the total number of curvature tensor to be used is even, so we need an even number of these monomials. As they have two free indices, we obtain a multiple of four of indices to be contracted with those of $b^{a_1\cdots a_n}$, leading to $n \equiv 0 \bmod 4$. If $\frac{d-1}{2}$ is odd, the same argument leads to $n \equiv 2 \bmod 4$. In other words, one can construct non-vanishing Lorentz invariant Lagrangians from one $b^{a_1\cdots a_n}$ field, n even, and $d-1$ curvature tensors if and only if $n \equiv d-1 \bmod 4$.

Suppose now n is odd, then one can form $\hat{b}^{a_{n+1}\cdots a_d} = \varepsilon_{a_1\ldots a_n}{}^{a_{n+1}\cdots a_d} b^{a_1\cdots a_n}$. Forming an Lorentz invariant Lagrangian with $b^{a_1\cdots a_n}$ is equivalent to forming a Lorentz invariant Lagrangian with $\hat{b}^{a_{n+1}\cdots a_d}$, as we pass from one field to the other using a Lorentz invariant tensor. The preceding analysis tells us that this is possible only if $d-n \equiv d-1 \bmod 4$ or equivalently $n \equiv 1 \bmod 4$. This conclude our proof.

5. Further developments

The work we have presented has been done in the mostly plus Lorentzian signature. However, thanks to our formalism, the analysis in other signature will be straightforward. Indeed, one just has to change the matrix defining the Dirac conjugate (assuming a unitary representation of the γ-matrices) by:

$$i^q \gamma_0 \gamma_1 \dots \gamma_{t-1}, \tag{5.1}$$

where t denotes the number of minus signs in the metric and $q \equiv \frac{n(n+1)}{2} + 1 \bmod 2$. Working with the adequate Dirac conjugate, the representation of bosonic generators of the AdS super-algebra is modified (they will be proportional to the generators (3.13-3.15) modulo some i factors). Following the construction of the invariant form for the Poincaré super-algebra, one sees that the sporadic appearance of i in the r.h.s. of (3.14) is changed. This implies in turn a change in the definition of $\not{b}$ in (2.27). More explicitly we had an i for $n \equiv 0,(3) \bmod 4$ and no i for $n \equiv 1,2 \bmod 4$. Now there will be an i when $q \equiv 1 \bmod 2$.
Changing the signature also affects the dimensions for which the reductions to Majorana or symplectic Majorana spinors exist. Indeed, in terms of t and d, one gets

$$\text{Majorana Spinors} \begin{cases} t \equiv 1 \bmod 4 \quad \textit{and} \quad d \equiv 3 \bmod 8, \\ \quad \textit{or} \\ t \equiv 2 \bmod 4 \quad \textit{and} \quad d \equiv 5 \bmod 8. \end{cases} \tag{5.2}$$

$$\text{Symplectic Majorana Spinors} \begin{cases} t \equiv 1 \bmod 4 \quad \textit{and} \quad d \equiv 7 \bmod 8, \\ \quad \textit{or} \\ t \equiv 2 \bmod 4 \quad \textit{and} \quad d \equiv 1 \bmod 8. \end{cases} \tag{5.3}$$

It is quite striking to see that there are no Majorana spinors for the mostly minus Lorentzian signature, meaning that reversing the signature does not lead to equivalent theories. Finally let us mention that the construction of Chern-Simons forms for other invariant tensors would be possible but will probably lead to very complicated expressions to cumbersome to be manageable.

CHAPTER 7

Tensionless limit of super-strings and its Majorana condition.

1. Motivation

The first approach to tensionless strings was provided by Schild [Sch77]. Its interest has increased to concern string scattering [GM84], [GM88], AdS/CFT correspondence [Kni21] or even in Hagedorn phase transition [AW88], [PA82], [Ole85]. Since supersymmetry is a central element in string theories, there have been also some works regarding super-string tensionless limits; with the precursor work of [LST91]. More recently, tensionless limit of the super-Polyakov action have been investigated in which the spinor fields scale inhomogeneously [Bag+20], [BBP19], [Bag+18]. In these works a new action have been obtained, in which the spinor fields play a more important role. Nevertheless, it was not found any Majorana condition in their tensionless action. This is somewhat intriguing since the spinors of the super-Polyakov action are indeed Majorana spinors.

Let us remind the expression of the super-Polyakov action:

$$S = -\frac{1}{4\pi\ell}\int d^2x\sqrt{-g}g^{\mu\nu}\left[\partial_\mu X\partial_\nu X + i\bar{\psi}\gamma_\mu\partial_\nu\psi\right]. \tag{1.1}$$

The metric g is the standard Minkowski metric, diagonal in the system of coordinates (x,t), with $g_{xx}=1$ and $g_{tt}=-1$. The spinor ψ is a two-components Majorana spinor, $\bar{\psi}$ stand for its Majorana conjugate, ℓ is the string length. We consider closed strings, with periodicity T along the x-coordinates: $X(x+T,t)=X(x,t)$, $\psi(x+T,t)=\psi(x,t)$. Usually, the physical space is supposed to have several dimensions, implying that the fields X and ψ possess several coordinates (i.e. there are collections of fields $\{X^\mu\}$, $\{\psi^\mu\}$). In order to keep the present work as simple as possible, and because it doesn't play any role in the problem we are studying at the present time, we do not consider these coordinates and assume the physical world has dimension 1.

Considering only the bosonic part of this action

$$S_{\text{Bos}} = -\frac{1}{4\pi\ell}\int d^2x\sqrt{-g}g^{\mu\nu}\partial_\mu X\partial_\nu X, \tag{1.2}$$

its tensionless limit is obtained after setting

$$\ell\to\infty, \qquad \frac{t}{x}\to 0. \tag{1.3}$$

An usual way of implementing this limit it is to introduce a parameter $\lambda \in [0,1]$, producing the deformation

$$\ell \mapsto \frac{\ell}{\lambda}, \qquad t \mapsto \lambda t, \tag{1.4}$$

and then taking the limit $\lambda \to 0$.

When the full superstring action (1.1) is considered, a deformation of the spinor ψ is also necessary (it can be inferred by analyzing the physical dimension of the spinor). In the past ([BBP19], [Bag+20], [Bag+18]), this deformation has been guessed and a quite interesting tensionless limit has been obtained. However, the authors of the aforementioned articles have not been able to find any Majorana condition fulfilled by the spinor fields in their tensionless limit. We will show in this article that this is mainly due to the election of their deformation. In order to circumvent this problem, we will show how to compute the appropriate deformation for the spinor, leading to a satisfying Majorana condition, by a method we will now shortly resume. First, we observe that in the bosonic case, we can, instead of deforming the coordinate, and in a manner totally equivalent, deform the metric. Then, in the full superstring action, we will use this deformation of the metric to obtain a deformation of the Clifford algebra. Finally, invoking the fact that, similarly to the bosonic case, a deformation of the metric should be equivalent to a deformation of both the coordinates and the spinor, we transfer the deformation of the Clifford algebra to its irreducible representation.

2. Short review of superstrings

2.1. Symmetries.

It is well known that the action (1.1) is invariant under a set of transformation called "extended diffeomorphisms" (see [BLT13] for an explanation of the terminology), given by

$$\delta_\xi X = \xi^\rho \partial_\rho X, \tag{2.1}$$

$$\delta_\xi \psi = \frac{1}{2}\xi^\rho \partial_\rho \psi - \frac{1}{2}\varepsilon^\lambda{}_\rho \xi^\rho \partial_\lambda \tilde{\gamma}\psi + \frac{1}{4}\partial_\rho \xi^\rho \psi - \frac{1}{4}\varepsilon^\mu{}_\rho \partial_\mu \xi^\rho \tilde{\gamma}\psi, \tag{2.2}$$

with ξ subject to

$$g^{\mu\nu}\partial_\rho \xi^\rho - g^{\mu\rho}\partial_\rho \xi^\nu - g^{\nu\rho}\partial_\rho \xi^\mu = 0. \tag{2.3}$$

The action is also invariant under the supersymmetry transformations given by

$$\delta_\epsilon X = \bar{\epsilon}\psi, \tag{2.4}$$

$$\delta_\epsilon \psi = -i\partial_\mu X \gamma^\mu \epsilon, \tag{2.5}$$

where the spinor ϵ is subject to the condition

$$\gamma^\nu \gamma^\mu \partial_\nu \epsilon = 0. \tag{2.6}$$

2.2. Super-field formalism.

The bosonic symmetry transformations acting on the bosonic field can be represented by a differential operator, as it can be seen from (2.1). We would like to extend this feature to the whole superalgebra, and represent every (super-)symmetry transformation by a super-differential operator. For this reason, we introduce on-shell super-fields

$$Y = X + i\bar{\theta}\psi, \tag{2.7}$$

where X and ψ are on-shell fields and θ is called a super-coordinate. Here, the use of on-shell fields is required by the fact that the super-symmetry algebra only closes on-shell. In order to work with off-shell fields, it is necessary to introduce the so-called auxiliary fields. The advantage of this approach is that the addition of such fields to the theory would allow to work with an off-shell closing supersymmetry algebra. In the super-field approach, the auxiliary fields are components along $\bar{\theta}\theta$. The vanishing of this component is also part of the condition that an on-shell super-field fulfills. For more details on this topic, we refer to [Del+99]. See also [CCF10] for a both rigorous and comprehensive presentation of the mathematical nature of super-coordinates.

We define the following projectors

$$h^{\mu}_{\pm\nu} = \frac{1}{2}(g^{\mu}{}_{\nu} \mp \varepsilon^{\mu}{}_{\nu}), \tag{2.8}$$

$$\mathrm{P}_{\pm} = \frac{1}{2}(\mathbb{1} \pm \tilde{\gamma}), \tag{2.9}$$

where

$$\tilde{\gamma} = \frac{1}{2}\epsilon_{\mu\nu}\gamma^{\mu}\gamma^{\nu}, \tag{2.10}$$

as well as as the quantities

$$\psi_{\pm} = \mathrm{P}_{\pm}\psi, \tag{2.11}$$

$$\partial_{\pm\mu} = h^{\nu}_{\pm\mu}\partial_{\nu}, \tag{2.12}$$

$$\xi^{\mu}_{\pm} = h^{\mu}_{\pm\nu}\xi^{\nu}, \tag{2.13}$$

$$\bar{\epsilon}_{\pm} = \bar{\epsilon}\mathrm{P}_{\pm} = \overline{\epsilon_{\mp}}. \tag{2.14}$$

These definitions allow a clear decomposition of the symmetries.

Let us show how the super-differential operator representation s obtained by taking the example of the symmetry generated by ξ_+. Its action on Y is given by

$$\delta_{\xi_+}Y = \delta_{\xi_+}X + i\bar{\theta}\delta_{\xi_+}\psi. \tag{2.15}$$

We demand it to be of the following form

$$\delta Y = \delta x^{\mu}\partial_{\mu}Y + \delta\bar{\theta}_{\alpha}\partial_{\bar{\theta}_{\alpha}}Y. \tag{2.16}$$

Thus, we find expressions $\delta x^\mu(\xi_+)\partial_\mu$, $\delta\theta_\alpha(\xi_+)\partial_{\bar\theta_\alpha}$ and write that we call the super-differential operator representation of δ_{ξ_+}. Concerning the notations, we will write this representation as $\mathrm{L}_\pm(\xi_\pm)$, $\mathrm{Q}_\pm(\bar\epsilon_\pm)$. Explicitly, we have

$$\mathrm{L}_+(\xi_+) = \xi_+^\mu\partial_{+\mu} + \frac{1}{2}\partial_{+\mu}\xi_+^\mu\bar\theta_+\partial_{\bar\theta_+}, \tag{2.17}$$

$$\mathrm{L}_-(\xi_-) = \xi_-^\mu\partial_{-\mu} + \frac{1}{2}\partial_{-\mu}\xi_-^\mu\bar\theta_-\partial_{\bar\theta_-}, \tag{2.18}$$

$$\mathrm{Q}_+(\bar\epsilon_+) = \bar\theta_+\gamma^\mu\epsilon_-\partial_{+\mu} - i\bar\epsilon_+\partial_{\bar\theta_+}, \tag{2.19}$$

$$\mathrm{Q}_-(\bar\epsilon_-) = \bar\theta_-\gamma^\mu\epsilon_+\partial_{-\mu} - i\bar\epsilon_-\partial_{\bar\theta_-}. \tag{2.20}$$

2.3. Mode expansion.

The analysis of tensile superstring is simplified by choosing an appropriate coordinate system and an appropriate representation of the Clifford algebra. We use the light cone coordinates: $x^+ = t + x$, $x^- = t - x$, and

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.21}$$

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.22}$$

$$C = D = \gamma_0, \tag{2.23}$$

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}, \tag{2.24}$$

where, with a slight abuse of notation, we have identified ψ_+ as defined in (2.11) and its only one non-vanishing component. Using this convention, the equations of motions are

$$\partial_+\partial_- X = 0, \tag{2.25}$$

$$\partial_-\psi_+ = 0, \qquad \partial_+\psi_- = 0. \tag{2.26}$$

The solutions of these equations can be expanded in Fourier modes as

$$X = C_0 + \frac{2\pi\ell}{T}P_0 t + \frac{i\sqrt{\ell}}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\left(a_n e^{-\frac{2\pi i n x^+}{T}} + \tilde a_n e^{-\frac{2\pi i n x^-}{T}}\right), \tag{2.27}$$

$$\psi_+(x^+) = \sqrt{2\pi\ell}\sum_{n\in\mathbb{Z}} b_n^+ e^{-\frac{2\pi i n x^+}{T}}, \tag{2.28}$$

$$\psi_-(x^-) = \sqrt{2\pi\ell}\sum_{n\in\mathbb{Z}} b_n^- e^{-\frac{2\pi i n x^-}{T}}. \tag{2.29}$$

The symmetry parameters can also be expanded in Fourier modes (the symmetries have to preserve the periodicity of the fields)

$$\xi^+(x^+) = \sum_{n\in\mathbb{Z}}\xi_n^+ e^{-\frac{2\pi i n x^+}{T}}, \qquad \xi^-(x^-) = \sum_{n\in\mathbb{Z}}\xi_n^- e^{-\frac{2\pi i n x^-}{T}}, \tag{2.30}$$

$$\bar\epsilon_+(x^+) = \sum_{n\in\mathbb{Z}}\bar\epsilon_{+,n} e^{-\frac{2\pi i n x^+}{T}}, \qquad \bar\epsilon_-(x^-) = \sum_{n\in\mathbb{Z}}\bar\epsilon_{-,n} e^{-\frac{2\pi i n x^-}{T}}. \tag{2.31}$$

In the light cone coordinate system, we can identify ξ^+ (the component of ξ along x^+) with ξ_+ (defined in (2.13)). We can therefore perform a mode expansion of the operators $\mathrm{L}_\pm$, $\mathrm{Q}_\pm$ of the previous section. We write

$$\mathrm{L}_\pm(\xi_\pm) = \sum_{n\in\mathbb{Z}} \xi^\pm_n \mathrm{L}_{\pm,n}, \qquad \mathrm{Q}_\pm(\bar\epsilon_\pm) = \sum_{n\in\mathbb{Z}} \bar\epsilon_{\pm,n}\mathrm{Q}_{\pm,n}. \tag{2.32}$$

$$\mathrm{L}_{+,n} = e^{-\frac{2\pi i n x^+}{T}}\left(\partial_+ - \frac{\pi i n}{T}\bar\theta_+\partial_{\bar\theta_+}\right), \tag{2.33}$$

$$\mathrm{L}_{-,n} = e^{-\frac{2\pi i n x^-}{T}}\left(\partial_- - \frac{\pi i n}{T}\bar\theta_-\partial_{\bar\theta_-}\right), \tag{2.34}$$

$$\mathrm{Q}_{+,n} = e^{-\frac{2\pi i n x^+}{T}}\left(-i\partial_{\bar\theta_+} - 2\bar\theta_+\partial_+\right), \tag{2.35}$$

$$\mathrm{Q}_{-,n} = e^{-\frac{2\pi i n x^-}{T}}\left(-i\partial_{\bar\theta_-} - 2\bar\theta_-\partial_-\right). \tag{2.36}$$

Using their explicit representation given in the appendix, we check that the operators $\{\mathrm{L}_{+,n}, \mathrm{Q}_{+,n}\}$ and $\{\mathrm{L}_{-,n}, \mathrm{Q}_{-,n}\}$ form two independent copies of the super-Witt algebra, whose non vanishing commutation relations are given by:

$$[\mathrm{L}_{+,n}, \mathrm{L}_{+,m}] = \frac{2\pi i}{T}(m-n)\mathrm{L}_{+,n+m}, \tag{2.37}$$

$$[\mathrm{L}_{+,n}, \mathrm{Q}_{+,r}] = \frac{2\pi i}{T}(r-\frac{n}{2})Q_{+,n+r}, \tag{2.38}$$

$$\{\mathrm{Q}_{+,r}, \mathrm{Q}_{+,s}\} = 4i\mathrm{L}_{+,r+s}. \tag{2.39}$$

3. The exotic super-string action.

3.1. An exotic convention.

The tensionless super-string action is not obtained as a limit of the standard super-string action, but instead from an exotic action, as was pointed out in [Bag+20]. We briefly review this result and refer to the quoted article for more details. The exotic super-string action is obtained after replacing the traditional physicist's definition for the Clifford algebra

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}, \tag{3.1}$$

by another choice, preferred by mathematicians

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = -2g_{\mu\nu}. \tag{3.2}$$

This conventions has repercussion on the so-called Majorana representation of spinors. In full generality, the Majorana conjugation of a spinor ψ is

$$\psi^c = \psi^T C, \tag{3.3}$$

where C, called conjugation matrix, is any matrix satisfying

$$C\gamma_\mu C^{-1} = \pm{\gamma_\mu}^T, \tag{3.4}$$

In spacetime of dimension 2, the two choices are possible: there exists two matrices, let us call them C_+ and C_- depending on the sign of (3.4), that can be used to define the Majorana conjugation. Furthermore, it can be shown that these matrices can be chosen so that they satisfy

$$C_{\pm}{}^{T} = \pm C^{T}. \tag{3.5}$$

However the on-shell closure of the supersymmetry algebra requires to take $C = C_{-}$. Therefore there is no choice for the super-Polyakov action. Furthermore, the Majorana condition in nothing else than a reality condition. To keep things as clear as possible, we ask C, γ_0, γ_1 to be represented by real matrices, implying that a Majorana spinor is simply a real spinor ($\psi^* = \psi$). With all these considerations understood, when the convention (3.1) is chosen, and for real representations, it is possible to have γ_0 antisymmetric and thus $C = \gamma_0$. On the other hand, when the convention (3.2) is chosen, it is impossible to have both γ_0 real and symmetric, hence we cannot take $C = \gamma_0$.

Now, the main physical implication of such a result comes from the Dirac bracket of the spinor field

$$\{\psi^{\alpha}, \psi^{\beta}\}_{\text{D.B.}} = -2\pi i \ell \left((C\gamma^0)^{-1}\right)^{\alpha\beta}. \tag{3.6}$$

The consequence is that in the case of the standard super-Polyakov action, i.e. when convention (3.1) is chosen for the Clifford algebra, the fermionic modes, after canonical quantization, will follow the anticommutation relations of (infinitely many) standard fermionic harmonic oscillators,

$$[\frac{1}{\sqrt{n}}\hat{b}_n^{+}, \frac{1}{\sqrt{n}}\hat{b}_n^{+\dagger}]_{+} = 1. \tag{3.7}$$

On the other hand, in the case of the exotic super-Polyakov action, i.e. when the convention (3.2) is used, one of the modes, $\hat{b}_n^{+}$ say, will follow (3.7) whereas the other , $\hat{b}_n^{-}$ say, will follow

$$[\frac{1}{\sqrt{n}}\hat{b}_n^{-}, \frac{1}{\sqrt{n}}\hat{b}_n^{-\dagger}]_{+} = -1. \tag{3.8}$$

This kind of commutation relations leads to negative norm states. Differences between standard fermionic oscillators and this kind of exotic fermionic oscillator are surveyed for example in [HT94].

3.2. Symmetries of the exotic super-string action.

The appearance of the minus sign in the right-hand-side of (3.2) has repercussion in the (super-)symmetry transformations, which become

$$\delta_{\xi} X = \xi^{\rho}\partial_{\rho} X, \tag{3.9}$$

$$\delta_{\xi}\psi = \frac{1}{2}\xi^{\rho}\partial_{\rho}\psi + \frac{1}{2}\varepsilon^{\lambda}{}_{\rho}\xi^{\rho}\partial_{\lambda}\tilde{\gamma}\psi + \frac{1}{4}\partial_{\rho}\xi^{\rho}\psi + \frac{1}{4}\varepsilon^{\mu}{}_{\rho}\partial_{\mu}\xi^{\rho}\tilde{\gamma}\psi, \tag{3.10}$$

$$\delta_{\epsilon} X = -\bar{\epsilon}\psi, \tag{3.11}$$

$$\delta_{\epsilon}\psi = -i\partial_{\mu} X\gamma^{\mu}\epsilon. \tag{3.12}$$

The restrictions for the (super-)parameters (2.3), (2.6), however, are unaffected. Because we want that a clear separation (on-shell) between the "+" symmetries and the "-" ones (for example we want $\{Q_+, Q_+\} = L_+$), we have to change the definition of the projectors. Now we set

$$P_+ = \frac{1-\tilde{\gamma}}{2}, \qquad P_- = \frac{1+\tilde{\gamma}}{2}. \tag{3.13}$$

The super-differential operators representation of the bosonic symmetries $L_\pm(\xi_\pm)$ is still given by (2.17-2.18) (taking (3.13) into account), whereas for the fermionic symmetries it changes as

$$Q_+(\bar{\epsilon}_+) = \bar{\theta}_+\gamma^\mu\epsilon_-\partial_{+\mu} + i\bar{\epsilon}_+\partial_{\bar{\theta}_+}, \tag{3.14}$$

$$Q_-(\bar{\epsilon}_-) = \bar{\theta}_-\gamma^\mu\epsilon_+\partial_{+\mu} + i\bar{\epsilon}_-\partial_{\bar{\theta}_-}. \tag{3.15}$$

An appropriate representation for the gamma matrices is now given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.16}$$

$$\gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{3.17}$$

$$C = D = \gamma^1, \tag{3.18}$$

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}. \tag{3.19}$$

The mode expansion of the fields X and ψ, as well as the parameters ξ and $\bar{\epsilon}$, are given by the equations (2.27 - 2.31). The modes of the super-differential operators $L_\pm$ are given by (2.33-2.34) and the one of $Q_\pm$ are now:

$$Q_{+,n} = e^{-\frac{2\pi i n x^+}{T}}\left(i\partial_{\bar{\theta}_+} + 2\bar{\theta}_+\partial_+\right), \tag{3.20}$$

$$Q_{-,n} = e^{-\frac{2\pi i n x^-}{T}}\left(i\partial_{\bar{\theta}_-} - 2\bar{\theta}_-\partial_-\right). \tag{3.21}$$

The operators $\{L_{+,n}, Q_{+,n}\}$ and $\{L_{-,n}, Q_{-,n}\}$ still form two independent copies of the super-Witt algebra; their commutation relations are still given by (2.37-2.39), with the only exception of the anticommutator of the "-" supercharges which is now:

$$\{Q_{-,r}, Q_{-,s}\} = -4iL_{-,r+s}. \tag{3.22}$$

Although this change has mathematically speaking no importance, it should have one in the quantum theory. These consequences are beyond the scope of this article.

4. Deformation of the action

4.1. The bosonic deformed action.

In order to present some aspect of the problem in a simpler way, let us first focus on the purely bosonic theory. The tensile bosonic string action is given by

$$S_{\text{Bos}} = -\frac{1}{4\pi\ell}\int d^2x\sqrt{-g}\,g^{\mu\nu}\partial_\mu X\partial_\nu X. \tag{4.1}$$

It is straightforward to see that the two possible deformations

$$(a)\begin{cases} t\mapsto \lambda t \\ x\mapsto x \\ \ell\mapsto \frac{\ell}{\lambda}\end{cases},\qquad (b)\begin{cases} t\mapsto t \\ x\mapsto \frac{x}{\lambda} \\ \ell\mapsto \frac{\ell}{\lambda}\end{cases}, \tag{4.2}$$

lead to the same result

$$S_{\text{Bos}}(\lambda)=-\frac{1}{4\pi\ell}\int dxdt(-\dot{X}^2+\lambda^2X'^2), \tag{4.3}$$

where we denote the derivative w.r.t to t by a dot and the one w.r.t x by a prime. It is possible to, instead of deforming the coordinates like in (4.2), to deform the metric. Explicitly the deformation (4.2-a) will be replaced by

$$\begin{cases} g^{-1}=\begin{pmatrix}-1 & \\ & 1\end{pmatrix}\mapsto g^{-1}(\lambda)=\begin{pmatrix}-\frac{1}{\lambda^2} & \\ & 1\end{pmatrix} \\ \ell\mapsto\frac{\ell}{\lambda}\end{cases}, \tag{4.4}$$

and the deformation (4.2-b) by

$$\begin{cases} g^{-1}=\begin{pmatrix}-1 & \\ & 1\end{pmatrix}\mapsto g^{-1}(\lambda)=\begin{pmatrix}-1 & \\ & \lambda^2\end{pmatrix} \\ \ell\mapsto\frac{\ell}{\lambda}\end{cases}. \tag{4.5}$$

Replacing in (4.1), we obtain again (4.3), showing that all four deformations are equivalent. A final remark on this equivalence: when the deformation $x\mapsto\frac{x}{\lambda}$ is chosen, the period of the (super-)strings T has to be deformed as well by $T\mapsto\frac{T}{\lambda}$. Taking this into account, it is then straightforward to show that in both cases ($\{t\mapsto\lambda, T\mapsto T\}$ and $\{x\mapsto\frac{x}{\lambda}, T\mapsto\frac{T}{\lambda}\}$), the limit $\lambda\to 0$ of (2.27) yields

$$X=C_0+\frac{2\pi\ell}{T}P_0t+\frac{i\sqrt{\ell}}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\left(\alpha_n-\frac{2\pi in}{T}t\tilde{\alpha}_n\right)e^{-\frac{2\pi inx}{T}}, \tag{4.6}$$

with

$$\alpha_n=\frac{1}{\sqrt{\lambda}}(a_n-\tilde{a}_{-n}),\qquad \tilde{\alpha}_n=\sqrt{\lambda}(a_n+\tilde{a}_{-n}). \tag{4.7}$$

in accordance with [Bag+20].

4.2. Deformation of the Clifford algebra.

Let us expose the strategy we will use. On the one hand, by analyzing the physical dimension, we understand that a deformation of the coordinates x^μ should be accompanied with a deformation of the spinor field ψ. On the other hand, a deformation of the metric has to be accompanied with a deformation of the Clifford algebra. Computing the deformation of the Clifford algebra from the deformation of the metric is quite easy, as we will show. Hence, in order to compute the deformation of the spinor, we will assume that the equivalence between the "coordinates deformation" point of view and the "metric deformation" point of view still hold, and compute the deformation of the spinor from the deformation of the gamma matrices. We remind that we use (3.2) for the Clifford algebra.

We now turn on computing the deformation of the Clifford algebra. We consider the case where the space-space component of the metric is deformed, which should be equivalent to $x \mapsto \frac{x}{\lambda}$. This will yield to a one-parameter family of Clifford algebras, denoted $\mathcal{C}(\lambda)$, whose generators $\Gamma^\mu(\lambda)$ satisfy

$$\Gamma^\mu(\lambda)\Gamma^\nu(\lambda)+\Gamma^\nu(\lambda)\Gamma^\mu(\lambda) = -2g^{\mu\nu}(\lambda)\mathbb{1}, \tag{4.8}$$

with

$$g^{\mu\nu} = \begin{pmatrix} -1 & \\ & \lambda^2 \end{pmatrix}. \tag{4.9}$$

It is important in this construction that we work with the inverse metric $g^{\mu\nu}$, and his associated gamma matrices "with upper indices". Indeed, the inverse metric converges, in that case, to a well defined degenerate matrix, whereas the normal metric diverges as $\lambda \to 0$. Had we consider the deformation equivalent to $t \mapsto \lambda t$, we would have done the opposite, i.e. consider the normal metric and gamma matrices "with lower indices".

The central problem the sought deformation should answer is to provide a representation of $\mathcal{C}(0)$ with a Majorana condition. We will follow the strategy of [Bul13]. For this, we see the full family of $\mathcal{C}(\lambda)$ as a collection of subalgebras of the bigger Clifford algebra $\mathrm{Cl}(1,2)$, where the latter denotes the Clifford algebra associated to the three dimensional metric

$$G_{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}. \tag{4.10}$$

We will then use the fact that $\mathrm{Cl}(1,2)$ admits a Majorana representation, and restrict the Majorana condition of the representation of $\mathrm{Cl}(1,2)$ to the sub-representation of the $\mathcal{C}(\lambda)$. It is because of this injection of $\mathcal{C}(\lambda)$ into $\mathrm{Cl}(1,2)$ that we have to work with the metric deformation equivalent to $x \mapsto \frac{x}{\lambda}$. If we would try instead the metric deformation equivalent to $t \mapsto \lambda t$, we should have seen the family $\mathcal{C}(\lambda)$ as family of subalgebras of $\mathrm{Cl}(2,1)$, whose irreducible representations do not possess a Majorana condition, and we would not have obtained a reasonable Majorana condition for $\mathcal{C}(0)$.

At this point, we have to ensure that the representation of $\mathrm{Cl}(1,2)$ can be seen as an extension of the representation of $\mathcal{C}(1)$ we started with. This crucial point is doable only if we choose the Majorana conjugation matrix to satisfy $C^T = -C$. Indeed, we said that two kind of matrix, C_+, C_- could be used in spacetime of dimension equal to 2, and this is no longer true in spacetime of dimension 3, where only C_- exists. This is also what constrains us to embed our family $\mathcal{C}(\lambda)$ in a Clifford algebra of dimension 3. In higher dimension, the dimension of the representation will grow up, i.e. the spinors will have more components, which is an undesirable feature.

Explicitly, let Γ^0, Γ^1 be the generators of $\mathcal{C}(1)$ (with identification $\Gamma^\mu \equiv \Gamma^\mu(1)$), with Γ^0 the timelike generator, $(\Gamma^0)^2 = \mathbb{1}$, and Γ_1 the spacelike generator, $(\Gamma^1)^2 = -\mathbb{1}$. Let $(V, \mathcal{R}_1)$ be an irreducible representation of the starting Clifford algebra $\mathcal{C}(1)$, used for example in 1.1, with C and D the matrices defining the Majorana and Dirac conjugation.. We have $\gamma^\mu = \mathcal{R}_1(\Gamma^\mu)$. Let Ξ^0, Ξ^1 and Ξ^2 be the generators of Cl(1,2), Ξ^1 being the spacelike generator. Then $(V, \bar{\mathcal{R}})$ (with the same V), with $\bar{\mathcal{R}}(\Xi^0) = \gamma^0$, $\bar{\mathcal{R}}(\Xi^1) = \gamma^1$ and $\bar{\mathcal{R}}(\Xi^2) = \tilde{\gamma}1$ define an irreducible representation of Cl(2,1). As argued in the previous paragraph, we chose the same matrices C and D to define the Majorana and Dirac conjugation. An injection $\iota_\lambda : \mathcal{C}(\lambda) \to \mathrm{Cl}(1,2)$ is given by

$$\iota_\lambda : \qquad \Gamma^0(\lambda) \mapsto \Xi^0, \tag{4.11}$$

$$\Gamma^1(\lambda) \mapsto \frac{1+\lambda^2}{2}\Xi^1 + \frac{1-\lambda^2}{2}\Xi^2. \tag{4.12}$$

We obtain a representation $(V, \mathcal{R}_\lambda)$ of $\mathcal{C}(\lambda)$ by putting $\mathcal{R}_\lambda = \mathcal{R} \circ \iota_\lambda$. The Majorana condition on any of these $\mathcal{R}_\lambda$ is obtained by restriction of the Majorana condition defined on $\bar{\mathcal{R}}$.

4.3. Deformation of the spinor.

We finally compute the deformation of the spinors. The equivalence between the "metric deformation" and "coordinates deformation" stated before means that we are looking for a collection $\psi(\lambda)$ satisfying

$$\mathcal{S}(\lambda) = -\frac{1}{4\pi\ell(\lambda)}\int d^2x(\lambda)\sqrt{-g}\big[g^{\mu\nu}\partial_\mu(\lambda)X\partial_\mu(\lambda)X + i\bar{\psi}(\lambda)\gamma^\mu\partial_\nu(\lambda)\psi(\lambda)\big], \tag{4.13}$$

$$\doteq -\frac{1}{4\pi\ell(\lambda)}\int d^2x\sqrt{-g}(\lambda)\big[g^{\mu\nu}(\lambda)\partial_\mu X\partial_\mu X + i\bar{\psi}\gamma^\mu(\lambda)\partial_\nu\psi\big], \tag{4.14}$$

with the convention that for any quantity $\chi(\lambda)$ deformed in the l.h.s, its non-deformed counterpart in the r.h.s satisfies $\chi_{\text{r.h.s}} \doteq \chi(1)_{\text{l.h.s}}$; and reciprocally for quantities deformed in the r.h.s but not in the l.h.s.. Therefore, to compute the deformation of the spinor we first write the required equality:

$$\bar{\psi}\gamma^\mu(\lambda)\partial_\mu\psi = \bar{\psi}(\lambda)\gamma^\mu\partial_\mu(\lambda)\psi(\lambda), \tag{4.15}$$

leading to

$$\bar{\psi}\gamma^0\partial_x\psi = \bar{\psi}(\lambda)\gamma^0\partial_x\psi(\lambda), \tag{4.16}$$

$$\bar{\psi}\frac{\frac{1+\lambda^2}{2}\gamma^0 + \frac{1-\lambda^2}{2}\tilde{\gamma}}{\lambda}\partial_t\psi = \bar{\psi}(\lambda)\gamma^0\partial_t\psi(\lambda). \tag{4.17}$$

We observe that, so long $\lambda \neq 0$, the two sets of matrices $\{\gamma_0, \gamma_1\}$ and $\{\frac{\frac{1+\lambda^2}{2}\gamma_0 + \frac{1-\lambda^2}{2}\tilde{\gamma}}{\lambda}, \gamma_1\}$ obey the same Clifford relations. By uniqueness of the equivalence class of faithful irreducible representations of the Clifford algebras of even dimensions, we know that there exist an invertible matrix $P(\lambda)$ such that

$$P^{-1}(\lambda)\gamma^0 P(\lambda) = \gamma^0, \tag{4.18}$$

$$P^{-1}(\lambda)\gamma^1 P(\lambda) = \frac{\frac{1+\lambda^2}{2}\gamma^1 + \frac{1-\lambda^2}{2}\tilde{\gamma}}{\lambda}. \tag{4.19}$$

Finally, the sought deformation is

$$\psi(\lambda) = P^{-1}(\lambda)\psi. \tag{4.20}$$

A last remark: in the previous section we said that the Majorana and Dirac conjugation matrices should be conserved in order to preserve the Majorana condition. This means that searching for $P(\lambda)$, we also need to consider the two following equations:

$$P^T(\lambda) C P(\lambda) = C, \tag{4.21}$$

$$P^\dagger(\lambda) D P(\lambda) = D. \tag{4.22}$$

We insist on the fact that the matrix $P(\lambda)$ is guaranteed to exist only if $\lambda \neq 0$. In the limit $\lambda \to 0$, this matrix might become singular. However, we expect the Lagrangian to have a non singular limit.

4.4. Explicit deformation and tensionless limit.

We introduce a representation for the gamma matrices in which the equations (4.18 - 4.22) are easy to solve

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.23}$$

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = C = D. \tag{4.24}$$

The components of a spinor in this representation are $\psi = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}$. They are related to the $\psi_\pm$ by

$$\psi_u = \frac{1}{\sqrt{2}}(\psi_- + \psi_+), \qquad \psi_d = \frac{1}{\sqrt{2}}(\psi_- - \psi_+). \tag{4.25}$$

In this representation

$$P(\lambda) = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix}, \tag{4.26}$$

thus

$$\psi(\lambda) = P^{-1}(\lambda)\psi = \begin{pmatrix} \frac{1}{\sqrt{\lambda}}\psi_u \\ \sqrt{\lambda}\psi_d \end{pmatrix}. \tag{4.27}$$

From here it is possible to take the limit $\lambda \to 0$ and obtain

$$\mathcal{S}_{\text{Tensionless}} = \frac{1}{4\pi\ell}\int d^2x \left[\dot{X}^2 + i\left(\psi_d \dot{\psi}_u + \psi_u \dot{\psi}_d + \psi_u \psi'_u\right)\right]. \tag{4.28}$$

The mode expansion of the tensionless field X has been already given in (4.6). For the spinor fields, we have seen that the well defined tensionless components are ψ_u and ψ_d, whose mode expansions in the limit $\lambda \to 0$ are

$$\psi_u[\lambda](x,t) \xrightarrow[\lambda\to 0]{} \chi(x) + t\tilde{\chi}'(x), \qquad \psi_d[\lambda](x,t) \xrightarrow[\lambda\to 0]{} \tilde{\chi}(x), \tag{4.29}$$

$$\chi(x) = \sqrt{2\pi\ell}\sum_n \beta_n e^{-\frac{2\pi i n x}{T}}, \qquad \tilde{\chi}(x) = \sqrt{2\pi\ell}\sum_n \tilde{\beta}_n e^{-\frac{2\pi i n x}{T}}, \tag{4.30}$$

with

$$\beta_n = \frac{b_n + \tilde{b}_{-n}}{\lambda}, \qquad \tilde{\beta}_n = b_n - \tilde{b}_{-n}. \tag{4.31}$$

Note that it is possible to play exactly the same game starting from the standard superstring action, in which case the tensionless limit is

$$\mathcal{S}_{\text{Alternative}} = \frac{1}{4\pi\ell}\int d^2x\big[\dot{X}^2 + i\psi_d\dot{\psi}_d\big]. \tag{4.32}$$

This result explains why we have to start with the exotic action.

4.5. Deformation of the symmetries.

For $\lambda \neq 0$, the symmetries of the deformed action (4.13) are just the expressions $\mathrm{L}_\pm$, $\mathrm{Q}_\pm$ given in sections 1 and 2, but with λ dependent quantities. For example

$$\mathrm{L}_+[\lambda](\xi_+(\lambda)) = \xi_+^\mu(\lambda)\partial_{+\mu}(\lambda) + \frac{1}{2}\partial_{+\mu}(\lambda)\xi_+^\mu(\lambda)\bar{\theta}_+(\lambda)\partial_{\bar{\theta}_+}(\lambda). \tag{4.33}$$

Here the λ-dependence of $\xi_\pm$ is just through $x^\pm$ ($\xi_+(\lambda) \doteq \xi_+(x^+(\lambda))$), whereas $\theta_\pm$ need to be changed like ψ using the formula (4.27). However, at $\lambda = 0$, the $\pm$ decomposition of the symmetries do not hold anymore and is replaced by another decomposition. In other words, we have the equalities

$$\mathrm{L}[\lambda](\xi(\lambda)) = \mathrm{L}_+[\lambda](\xi_+(\lambda)) + \mathrm{L}_-[\lambda](\xi_-(\lambda)) = \mathrm{K}(f) + \mathrm{M}(g) + o(\lambda), \tag{4.34}$$

$$\mathrm{Q}[\lambda](\bar{\epsilon}(\lambda)) = \mathrm{Q}[\lambda](\bar{\epsilon}(\lambda)) + \mathrm{Q}[\lambda](\bar{\epsilon}(\lambda)) = \mathrm{G}(\bar{\zeta}) + \mathrm{H}(\bar{\rho}) + o(\lambda), \tag{4.35}$$

but it is impossible to express for example $\mathrm{K}(f)$ alone in terms of L_+ and L_-. It means that the decomposition in term of projectors $h_\pm$, $\mathrm{P}_\pm$ do not exist in the tensionless limits; or equivalently, these projectors do not possess a well defined limit when λ goes to 0. The new symmetries are

$$\mathrm{M}(g) = g\partial_t + \frac{1}{2}g'\bar{\theta}_u\partial_{\bar{\theta}_d}, \tag{4.36}$$

$$\mathrm{K}(f) = f\partial_x + \frac{1}{2}f'(\bar{\theta}_u\partial_{\bar{\theta}_u} + \bar{\theta}_d\partial_{\bar{\theta}_d}) + t\mathrm{M}(f'), \tag{4.37}$$

$$\mathrm{H}(\bar{\rho}) = \bar{\rho}(i\partial_{\bar{\theta}_d} - \bar{\theta}_u\partial_t), \tag{4.38}$$

$$\mathrm{G}(\bar{\zeta}) = \bar{\zeta}(i\partial_{\bar{\theta}_u} + \bar{\theta}_u\partial_x - \bar{\theta}_d\partial_t) - t\mathrm{H}(\bar{\zeta}'). \tag{4.39}$$

(4.40)

and f, g, ζ, ρ are related to ξ and ϵ by

$$\frac{\xi^+ - \xi^-}{2\lambda} = f, \qquad \frac{\xi^+ + \xi^-}{2} = g + tf' \tag{4.41}$$

$$\frac{\bar{\epsilon}_u}{\sqrt{\lambda}} = \bar{\zeta}, \qquad \sqrt{\lambda}\bar{\epsilon}_d = \bar{\rho} - t\bar{\zeta}'. \tag{4.42}$$

Although the symmetries $L_\pm$, $Q_\pm$ and K, M, G, H are not directly related, their modes are, by

$$K_n = \frac{L_{+,n} - L_{-,-n}}{2\lambda}, \qquad M_n = \frac{L_{+,n} - L_{-,-n}}{2}, \tag{4.43}$$

$$G_r = \frac{1}{\sqrt{2\lambda}}(Q_{-,r} - Q_{+,-r}), \qquad H_r = \sqrt{\frac{\lambda}{2}}(Q_{+,r} + Q_{-,-r}), \tag{4.44}$$

which is a (disguised) Wigner-Inönü contraction. Why the Wigner-Inönü contraction takes this form is understood by looking at (2.37-2.39), we see that T appears in the structure constants. As said earlier, the deformation $x \mapsto \frac{x}{\lambda}$ implies a deformation $T \mapsto \frac{T}{\lambda}$ and thus we get a λ-dependent algebra. In order to remove this λ-dependence, it is possible to scale $L_{\pm,n} \mapsto \frac{1}{\lambda}L_{\pm,n}$, $Q_{\pm,r} \mapsto \frac{1}{\sqrt{\lambda}}Q_{\pm,r}$, after what equations (4.43-4.44) take the form of a standard Wigner-Inönü contraction. The non-vanishing commutation relations of the new symmetry modes are

$$[K_n, K_m] = \frac{2\pi}{T} i(m-n) K_{n+m}, \tag{4.45}$$

$$[K_n, M_m] = \frac{2\pi}{T} i(m-n) M_{n+m}, \tag{4.46}$$

$$[K_n, G_r] = \frac{2\pi i}{T}(r - \frac{n}{2}) G_{n+r}, \tag{4.47}$$

$$[K_n, H_r] = \frac{2\pi i}{T}(r - \frac{n}{2}) H_{n+r}, \tag{4.48}$$

$$[M_n, G_r] = \frac{2\pi i}{T}(r - \frac{n}{2}) H_{n+r}, \tag{4.49}$$

$$\{G_r, G_s\} = 2i K_{r+s}, \tag{4.50}$$

$$\{G_r, H_s\} = 2i M_{r+s}, \tag{4.51}$$

with

$$K_n = e^{-\frac{2\pi i n x}{T}}(I_n - \frac{2\pi i n t}{T} J_n), \tag{4.52}$$

$$M_n = e^{-\frac{2\pi i n x}{T}} J_n, \tag{4.53}$$

$$G_r = e^{-\frac{2\pi i r x}{T}}(U + \frac{2\pi i r t}{T} V), \tag{4.54}$$

$$H_r = e^{-\frac{2\pi i r x}{T}} V, \tag{4.55}$$

and

$$\mathrm{I}_n = \partial_x - \frac{i\pi n}{T}(\bar{\theta}_u \partial_{\bar{\theta}_u} + \bar{\theta}_d \partial_{\bar{\theta}_d}), \tag{4.56}$$

$$\mathrm{J}_n = \partial_t - \frac{i\pi n}{T}\bar{\theta}_d \partial_{\bar{\theta}_u}, \tag{4.57}$$

$$\mathrm{U} = i\partial_{\bar{\theta}_d} + \bar{\theta}_d \partial_x - \bar{\theta}_u \partial_t, \tag{4.58}$$

$$\mathrm{V} = i\partial_{\bar{\theta}_u} - \bar{\theta}_d \partial_t. \tag{4.59}$$

We recognize the commutation relations of the super-BMS_3 algebra [AGS86]. The fact that the algebra of symmetries of the tensionless action (4.28) is the super-BMS_3 algebra, as well as its relation with the algebra of symmetries of the tensile action, was already shown in [Bag+18]. Thus, what this last paragraph shows is that the implementation of the Majorana condition haven't altered the previous results of [Bag+20], [BBP19] and [Bag+18]. Furthermore, we can now state that the generators G and H defined in (4.38-4.39) are real.

Conclusion

In this thesis we have presented three theoretical model related to supergravity. In the first one we have constructed an $\mathcal{N}=2$ supergravity model in a manner approaching a super-Yang-Mills model based on the $G = SU(2,2|2)$ symmetry. The peculiarity of this model is the Hodge-like operator $\circledast$, which acts on the form indices of the $SU(2)\times U(1)$ gauge fields as well as the "dilatation" fields, but on the Lorentz and spinor indices for the other fields forming the super-connection, mimicking the MacDowell-Mansouri approach to supergravity. This theory has to be seen as a mother theory containing interesting physical sectors, joining both gravity and electroweak interaction in a single action principle (although some care has to be taken regarding the values of the coupling constants).

In the second model we have constructed supersymmetric extensions of the so-called Poincaré invariant gravity for some extensions of the Poincaré algebras (1.2) involving Lorentz tensors of higher ranks. In order to ensure gauge invariance, the Lagrangians were chosen to be Chern-Simons forms obtained thanks to the construction of invariant tensors. The Poincaré super-algebras being non-semi simple, the construction of invariant tensors is in general an highly nontrivial task. We circumvent this difficulty by exploiting the fact that the Poincaré super-algebras (1.2) can be obtained from the semi simple AdS super-algebras by means of an expansion. Indeed, in the super AdS case, the invariant tensors can be taken as a super-trace over all the generators, and the expansion has the advantage of providing the corresponding invariant tensor for the Poincaré super-algebras. These constructions are done in any odd dimensions, and apply equally well for Majorana and symplectic Majorana spinors, as long as they exist in the given signature and dimension, and can also be extended for any $\mathcal{N}$. Furthermore, the Lagrangians we obtain are coupled to fields $b^{a_1\cdots a_n}$ for any n where it is possible to have such a coupling. Interestingly enough, our construction yields a nontrivial supersymmetric Lagrangian which can be expressed in a simple and compact form whose explicit invariance can be easily checked, either directly by means of Fierz rearrangements, or using the general theory of Chern-Simons forms.

In the third model, we have successfully given a Majorana condition for the spinors in the tensionless limit of the super-Polyakov action. This Majorana condition is obtained by carefully looking at the deformation linking the exotic super-Polyakov action to its tensionless limit; in particular we have computed the deformation of the spinor fields by preserving the equivalence between a deformation of the coordinates and a deformation of the metric. This deformation has also been constructed in a way that it preserves the Majorana condition existing in the tensile theory. It was not guaranteed at all that such deformation was possible, and a profound analysis of representation theory of Clifford algebras has been done to ensure its existence. It has been shown that it was crucial at this step to deform the spacelike coordinate x and not the timelike coordinate t, partially explaining why all previous attempts of defining a Majorana condition in the tensionless limit failed. We have also make sure that the newly computed deformation reproduce some of the most important results shown previously concerning the tensionless action. The way on how symmetries are deformed has been our main focus regarding this point; and we correctly obtain the Wigner-Inönü deformation of the two copies of the super-Witt algebra to the super-BMS_3 algebra.

Throughout the completion of the thesis, a deep attention was given to the rigor of the mathematics used. In particular, the notions of super-connection and Lie derivative of spinor fields, or even the relations between spinor fields and gravity have been studied before being used in physical models. The widely used concept of Majorana spinors have been acutely analysed which led to a simple solution of the problem of defining a Majorana condition for tensionless strings. The expansion of algebra presented in [JO03] has been carefully studied and better apprehended, leading to more systematic constructions like the maximally extended Chern-Simons Poincaré Lagrangian. Hence, although the theoretical models we have presented might find no experimental applications, their analysis have shown themselves fruitful for improving the understanding of the mathematics at the heart of fundamental models of modern physics. Thanks to many similar works, we can hope to find even more fundamental models constructing the physics of tomorrow.

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