A survey of results on the u-invariant of a rational function field

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The classical u-invariant of k, $u(k)$, is defined as the supremum of the dimensions of anisotropic quadratic forms defined over k.
If $\dim(q) > u(k)$, then q is isotropic over k.

This talk will not deal with the more general u-invariant of a field that is defined for formally real fields.

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The last result is easy to prove when char $k \neq 2$ and a bit harder to prove when char $k = 2$.

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Algebraically closed fields are $\mathcal{C}_0$-fields.
Finite fields are $\mathcal{C}_1$-fields.
It is usually very difficult to determine whether a given field is a $\mathcal{C}_i$-field for some i.

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Corollary

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We have the following.

Theorem

$2u(k) \leq 2\sup\{u(E) \mid E/k \text{ finite separable ext.}\} \leq u(k(t))$

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We now consider these questions in more detail.

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4. $u(L) \leq 2u(k)$
5. $2u(k) \leq u(k(t)) \leq 4u(k)$

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Many, but not all, of the following results hold in characteristic 2, but for simplicity we avoid this case.

Basic inequalities - algebraic extensions

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If $[E:k]=r$ *and* $u(k)=1$*, then*

$$u(E)\leq \begin{cases} 2 & \text{if } 1\leq r\leq 4, \\ \frac{r-1}{2} & \text{if } r\geq 5. \end{cases}$$

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No example is known where $u(E)>2u(k)$.
Examples are known where $u(E)=2u(k)$, $u(E)=\frac{3}{2}u(k)$, and also many other cases.

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Nothing is known in general for $r \geq 4$.

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The only known values of $u(k(t))$ are powers of two.

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If $u(E) < 2^n$, then every n-fold Pfister form defined over E is hyperbolic over E, and so $I^n(E) = 0$.

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The converse holds for $n = 1, 2$ but it does not hold for $n \geq 3$.
There are fields k with $I^3(k) = 0$ but $u(k)$ can be arbitrarily large.

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For all finite algebraic extensions, $u(E) = 1$ and so $I(E) = 0$. Therefore $I^2(k(t)) = 0$ by the Milnor exact sequence, and this implies $u(k(t)) \leq 2$.
Since $u(k(t)) \neq 1$, we have $u(k(t)) = 2$. □

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There exists a field k with $u(E) = 2$ for all finite extensions E/k and such that $u(k(t)) \geq 6$. Thus $u(k(t)) > 2\sup\{u(E) \mid E/k \text{ finite algebraic }\}$.

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The example comes from work of Colliot-Thélène and Madore. They constructed a field k with $u(E) = 2$ for all finite extensions E/k and two quadratic forms q_1, q_2 defined over k in 5 variables such that $\{q_1, q_2\}$ have no nontrivial common zero defined over k.

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Thus $u(k(t)) \geq 5$ and therefore $u(k(t)) \geq 6$.

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Assume that Q is an anisotropic quadratic form defined over $k(t)$ with $\dim(Q) = 5$, and assume that Q satisfies the following two conditions.

(1) For each monic irreducible polynomial $\pi \in k[t]$, if $\partial_\pi^2(Q) \neq 0$, then

1. $\partial_\pi^2(Q)$ is represented by a one-dimensional form over E_π,
2. $\partial_\pi^1(Q) \notin I^2(E_\pi)$.
3. $\deg(\pi)$ is a 2-power.

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3. $\deg(\pi)$ is a 2-power.

(2) If $\partial_\infty^1(Q) \neq 0$, then

1. $\partial_\infty^1(Q)$ is represented by a one-dimensional form over E_∞,
2. $\partial_\infty^2(Q) \notin I^2(E_\infty)$.

Theorem(continued)
Here E_π denotes the residue field of the valuation on $k(t)$ corresponding to π, and E_∞ is the residue field corresponding to $\frac{1}{t}$.

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Let F be a field such that either F/k is an algebraic extension with $[F : k]$ odd, or $F = k(C)$ is the function field of a conic C defined over k. Then $Q_{F(t)}$ is anisotropic over $F(t)$ and $Q_{F(t)}$ satisfies the two conditions above with F in place of k.

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Here E_π denotes the residue field of the valuation on $k(t)$ corresponding to π, and E_∞ is the residue field corresponding to $\frac{1}{t}$.

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It follows that there exists a field extension K of k such that $u(E) = 2$ for all finite extensions E/K and $u(K(t)) \geq 6$.

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I don't have an upper bound for $u(K(t))$ in this case.

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I used a theorem of Heath-Brown to give a proof valid for all p that also is valid for function fields of higher transcendence degree. (More details below.)

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Assume that k is a field that is complete with respect to a discrete valuation having residue field κ and assume that char $\kappa \neq 2$. *Then*

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With $k = \mathbf{Q}_p$, $\kappa = \mathbf{F}_p$, we have $u(\ell(t)) = 4$, and so $u(\mathbf{Q}_p(t)) = 8$.

A generalization of $\mathcal{C}_i$-fields

For $d \geq 0$, a field k satisfies property $\mathcal{C}_i(d)$ if every system of r homogeneous forms of degree d defined over k in n variables, $n > rd^i$, has a nontrivial simultaneous zero defined over k.

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If k is a $\mathcal{C}_i(2)$-field, then the case $r = 1$ shows that $u(k) \leq 2^i$.
If k is a $\mathcal{C}_i$-field, then Lang-Nagata proved that k is a $\mathcal{C}_i(d)$-field for all positive integers d.

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Thus if k is an $\mathcal{C}_i(2)$-field, then $k(t_1, \ldots, t_m)$ is an $\mathcal{C}_{i+m}(2)$-field and $u(k(t_1, \ldots, t_m)) \leq 2^{i+m}$.

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For $d \geq 0$, a field k satisfies property $\mathcal{A}_i(d)$ if every system of r homogeneous forms of degree d defined over k in n variables, $n > rd^i$, has a nontrivial simultaneous zero in an extension field over k of degree prime to d.

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If k is a $\mathcal{C}_i(d)$-field, then k is an $\mathcal{A}_i(d)$-field.

If $k \in \mathcal{A}_i(2)$, then the case $r = 1$ and Springer's theorem on odd degree extensions shows that $u(k) \leq 2^i$.

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Thus if k is an $\mathcal{A}_i(2)$-field, then $k(t_1, \ldots, t_m)$ is an $\mathcal{A}_{i+m}(2)$-field and thus $u(k(t_1, \ldots, t_m)) \leq 2^{i+m}$.

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We have $u(\mathbf{Q}_p(t_1, \ldots, t_m)) \geq 2^{m+2}$ by straightforward valuation theory. □

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We now use this theorem to prove that $\mathbf{Q}_p$ is an $\mathcal{A}_2(2)$-field.

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If K is an unramified extension of $\mathbf{Q}_p$ with residue field E and $[K : \mathbf{Q}_p] = l$, then $[E : \mathbf{F}_p] = [K : \mathbf{Q}_p] = l$ and $|E| = |\mathbf{F}_p|^l = p^l$.

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Thus there exists such a K with l odd and $|E| = p^l \geq (2r)^r$.
Then Heath-Brown's theorem implies that S is isotropic over K.
Since $[K : \mathbf{Q}_p]$ is odd, it follows that $\mathbf{Q}_p \in \mathcal{A}_2(2)$.

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Then there exist two quadratic forms q_1, q_2 defined over k such that
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That is, $Q \perp m\mathbb{H} \simeq q_1 + tq_2$ over $k(t)$ for some $m \geq 0$.*

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Theorem

$u(k(t)) = \sup_{m \geq 1}\{u_k(2, m) - 2(m - 1)\}$

Proposition

Let $n \geq 1$.

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3. $u(k(t))$ *is finite if and only if there exists an integer* N *such that* $u_k(2, m+1) = u_k(2, m) + 2$, *for all* $n \geq N$.
4. $u(k(t)) \leq N$ *if and only* $u_k(2, m) \leq N + 2(m-1)$ *for all* $m \geq 1$.

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Then

$$q_1 = x_1 L_1(x_{m+1}, \ldots, x_n) + \cdots + x_m L_m + Q_1(x_{m+1}, \ldots, x_n)$$

$$q_2 = x_1 M_1(x_{m+1}, \ldots, x_n) + \cdots + x_m M_m + Q_2(x_{m+1}, \ldots, x_n)$$

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The $2m$ linear forms $L_1(x_{m+1}, \ldots, x_n), \ldots, L_m, M_1, \ldots, M_m$ span a vector space of dimension at most $n - m$ and $n - m = N + m - 1$.
For large m, $L_1, \ldots, L_m, M_1, \ldots, M_m$ are highly linearly dependent.

I have found a way to construct spaces of zeros of q_1, q_2 where the $2m$ linear forms span a vector space whose dimension has order of magnitude equal to $\frac{3}{2}m$.

Suppose that k is an algebraically closed field, char $k \neq 2$.

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Thus $u(k(t)) = \sup_{m \geq 1}\{u_k(2, m) - 2(m-1)\} = 2$.

THANK YOU