Quadratic and cubic form invariants of certain algebras with involution

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Decomposability for algebras with involution

Let F be a field and (A, σ) an F-algebra with involution.

Question

Is (A, σ) totally decomposable, i.e. isomorphic to a tensor product of quaternion algebras with involution?

Question

Does $\mathrm{Sym}(\sigma)$ contain a quadratic extension K/F?

Question

Given such K/F, is $K \subseteq Q$ for an F-quaternion algebra $Q \subseteq A$ with $\sigma(Q) = Q$?

Type and capacity

We define the capacity of (A, σ) as

$$\mathrm{cap}(A,\sigma) = \begin{cases} \deg(A) & \text{if } \sigma \text{ is orthogonal,} \\ \deg(A) & \text{if } \sigma \text{ is unitary,} \\ \frac{1}{2}\deg(A) & \text{if } \sigma \text{ is symplectic.} \end{cases}$$

Proposition

$$\mathrm{cap}(A,\sigma) = \max\{[F[x] : F] \mid x \in \mathrm{Symd}(A,\sigma)\}$$

If $\mathrm{cap}(A,\sigma) = 1$, then σ is the unique involution of its type on A.

If $\mathrm{cap}(A,\sigma) = 2$, then (A,σ) is totally decomposable.

Here, we study the case where $\mathrm{cap}(A,\sigma) = 4$.

Successive decomposition

Let (A, σ) be an F-algebra with involution.

Proposition

Let $Q \subseteq A$ be an F-quaternion algebra such that $\sigma(Q) = Q$ and let $C = C_A(Q)$. Then

$$(A, \sigma) \simeq (Q, \sigma|_Q) \otimes (C, \sigma|_C) .$$

If $\sigma|_Q$ is orthogonal, then $\sigma|_C$ is of same type as σ and

$$\mathrm{cap}(C, \sigma|_C) = \tfrac{1}{2}\mathrm{cap}(A, \sigma) .$$

Hence, if $\mathrm{cap}(A, \sigma) = 4$ and A contains a σ-stable F-quaternion algebra, then (A, σ) is totally decomposable.

Capacity 2

Assume from now that σ is not orthogonal whenever $\mathrm{char}(F) = 2$.

Assume that $\mathrm{cap}(A, \sigma) = 2$ and let $V = \mathrm{Symd}(\sigma)$. Then

$$\dim(V) = \begin{cases} 3 & \text{if } \sigma \text{ is orthogonal,} \\ 4 & \text{if } \sigma \text{ is unitary,} \\ 6 & \text{if } \sigma \text{ is symplectic.} \end{cases}$$

There exists a natural symmetry

$$V \longrightarrow V, x \mapsto \overline{x}$$

such that $x + \overline{x}, x\overline{x} \in F$ for all $x \in V$.

For $x \in V$ we have $[F[x] : F] \leq 2$, as $x^2 - (x + \overline{x})x + x\overline{x} = 0$.

Moreover, $q : V \longrightarrow F, x \mapsto x\overline{x}$ is a regular quadratic form.

Let's call (V, q) the symmetrizer form of (A, σ).

Symmetric quadratic extensions

Let (A, σ) be an F-algebra with involution with $\mathrm{cap}(A, \sigma) = 4$.

Let K/F be a quadratic étale extension with $K \subseteq \mathrm{Symd}(\sigma)$ and $C = C_A(K)$ satisfying $\dim_F(C) = \frac{1}{2}\dim_F(A)$. Then:

$(C, \sigma|_C)$ is a K-algebra with involution with $\mathrm{cap}(C, \sigma) = 2$.

There exist (many) biquadratic étale L/F with $K \subseteq L \subseteq \mathrm{Symd}(\sigma)$.

We take the symmetrizer form of (C, σ) and apply the Scharlau transfer from K to F to obtain a regular quadratic form over F.

In this form L is a 4-dimensional hyperbolic subspace. We take its orthogonal complement and denote it π^K. Then

$$\dim(\pi^K) = \begin{cases} 2 & \text{if } \sigma \text{ is orthogonal} \\ 4 & \text{if } \sigma \text{ is unitary} \\ 8 & \text{if } \sigma \text{ is symplectic} \end{cases}$$

What else can we say about the form π^K?

Decomposability and isotropy

Let (A, σ) be an F-algebra with involution with $\mathrm{cap}(A, \sigma) = 4$.

Consider quadratic étale K/F as before, with $K \subseteq \mathrm{Symd}(\sigma)$.

Proposition

The form π^K is isotropic if and only if there exists an F-quaternion algebra $Q \subseteq A$ with $K \subseteq Q$, $\sigma(Q) = Q$ and $\sigma|_Q$ orthogonal.

Theorem

The form π^K is either anisotropic or hyperbolic, and it is independent of the choice of K/F.

Corollary

If (A, σ) is totally decomposable, then there is a decomposition where K is contained in one quaternion factor.

The decomposability form

Let (A, σ) be an F-algebra with involution with $\mathrm{cap}(A, \sigma) = 4$.

Assume that σ is not orthogonal if $\mathrm{char}(F) = 2$.

We have almost shown the following:

Theorem

To (A, σ) there is associated an r-fold Pfister form π where

$$r = \begin{cases} 1 & \text{if } \sigma \text{ is orthogonal} \\ 2 & \text{if } \sigma \text{ is unitary} \\ 3 & \text{if } \sigma \text{ is symplectic} \end{cases}$$

For any field extension L/F we have that $(A, \sigma)_L$ is totally decomposable if and only if π_L is hyperbolic.

However, we assumed the existence of a convenient quadratic extension K/F contained in $\mathrm{Symd}(\sigma)$!

This is a challenge when A is a division algebra and σ is symplectic.

The Pfaffian polynomial

Let (A, σ) be an F-algebra with symplectic involution and $\deg(A) = 8$.

The elements $x \in \mathrm{Symd}(\sigma)$ satisfy an equation

$$x^4 - c_1(x)x^3 + c_2(x)x^2 - c_3(x)x + c_4(x) = 0$$

where $c_i : \mathrm{Symd}(\sigma) \longrightarrow F$ is a form of degree i over F $(i \leq 4)$.

Proposition

There exists $x \in \mathrm{Symd}(\sigma) \setminus \{0\}$ with $c_1(x) = c_3(x) = 0$ and in particular $[F(x^2) : F] \leq 2$.

Springer's Theorem for cubic forms

Proposition

There exists $x \in \mathrm{Symd}(\sigma) \setminus \{0\}$ *with* $c_1(x) = c_3(x) = 0$ *and in particular* $[F(x^2) : F] \leq 2$.

Consider the cubic form $\gamma = (V, c_3)$ of dimension 27 over F where

$$V = \{x \in \mathrm{Symd}(\sigma) \mid c_1(x) = 0\}.$$

Claim: γ is isotropic.

This is true if A is split, so in particular if F is quadratically closed.

Hence, γ_L is isotropic over a 2-extension L/F.

Theorem

Let L/F *be a 2-extension and let* γ *be a cubic form over* F.
Then γ *is isotropic over* L *if and only if* γ *is isotropic over* F.

Rowen's Theorem

Let A be a central simple algebra of exponent 2 and degree 8.

Theorem (Garibaldi-Parimala-Tignol for $\mathrm{char}(F) \neq 2$)

For any symplectic involution σ on A, there exists a quadratic étale extension K/F contained in $\mathrm{Symd}(A, \sigma)$.

Corollary (Rowen)

The algebra A contains a triquadratic étale extension of F.

The new proof used:

Theorem

Any central simple algebra of exponent 2 is split by a 2-extension.

This follows from Merkurjev's Theorem, but a direct elementary proof can be given.