A uniform construction of smooth integral models over an arbitrary local field and a recipe for computing local densities

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Outline

1. **Local densities**
   - Notations
   - Local density
   - Group scheme
2. **A uniform construction of smooth integral models**
   - Main theorem
   - A recipe for computing local densities

Notations

- A, the ring of integers of a local field F, π its uniformizer, and κ its residue field
- B, the ring of integers (a maximal A-order) in K where K is one of
  - $K = F$;
  - a separable quadratic extension of F;
  - the quaternion algebra over F.
- q, the cardinality of κ
- (L, h), (anti)-hermitian B-lattice (including quadratic A-lattice when $K = F$)
- the dual lattice $L^{\#} = \{x \in L \otimes_A F : h(x, L) \subset B\}$.

Definition

The local density of (L, h) is

$$\beta_L = \frac{1}{[G : G^\circ]} \cdot \lim_{N \to \infty} q^{-N \dim G} \# \underline{G}'(A/\pi^N A).$$

Here, $\underline{G}'(K) = \mathrm{Aut}_{B \otimes_A K}(L \otimes_A K, h \otimes_A K)$ for any commutative A-algebra K.

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Definition (another definition)

$$\beta_L = \frac{1}{[G : G^\circ]} \int_{\mathrm{Aut}_B(L,h)} |\omega^{\mathrm{ld}}|.$$

Here, ω^{ld} is a certain volume form associated to $\underline{G}'$.

Theorem (Raynaud)

*Let A be a discrete valuation ring. Let $\underline{G}'$ be an affine group scheme of finite type over A with the smooth generic fiber G. Then, there exists a **unique smooth** affine group scheme (called smooth integral model) $\underline{G}$ over A such that $\underline{G}$ and $\underline{G}'$ have the same generic fiber G and*

$$\underline{G}(R) = \underline{G}'(R)$$

for any étale A-algebra R.

Theorem (Raynaud)

Let A be a discrete valuation ring. Let $\underline{G}'$ be an affine group scheme of finite type over A with the smooth generic fiber G. Then, there exists a ***unique smooth*** *affine group scheme (called smooth integral model) $\underline{G}$ over A such that $\underline{G}$ and $\underline{G}'$ have the same generic fiber G and*

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The following integral is well known:

$$\int_{\mathrm{Aut}_B(L,h)} |\omega^{\mathrm{can}}| = q^{-\mathrm{dim} G} \cdot \#\underline{G}(\kappa).$$

Here ω^{can} is a volume form associated to $\underline{G}$.

Summary

- $\beta_L = \frac{1}{2} \int_{\mathrm{Aut}_B(L,h)} |\omega^{\mathrm{ld}}|$.
- $\int_{\mathrm{Aut}_B(L,h)} |\omega^{\mathrm{can}}| = q^{-\dim G} \cdot \#\underline{G}(\kappa)$.

In order to obtain an explicit formula for the local density, it suffices to

- determine the special fiber of $\underline{G}$, especially its maximal reductive quotient;
- relate the volume forms ω^{ld} and ω^{can}.

Theorem (-, 2013)

There exist suitable, canonical, inclusions (easily and explicitly constructed)

$$\cdots \subseteq T_1^n \subseteq \cdots \subseteq T_1^1 \subseteq T_1^0$$

of representable sheaves on the small flat site over A such that a morphism $\rho : \underline{M}^* \longrightarrow \underline{H}$, defined by $\rho(m) = h \circ m$, is smooth and the desired smooth integral model $\underline{G} = \rho^{-1}(h)$.

Theorem (-, 2013)

Let $\tilde{G}$ be the special fiber of $\underline{G}$. Let $M' = \mathrm{End}_B(L)$ and $H' = \{f : f$ is a quadratic (or hermitian) form on $L\}$. Let

$$q^N = \frac{\#(H'/\widetilde{T_2}(A))}{\#(M'/\widetilde{T_1}(A))}$$

for an integer N. Then the local density of (L, h) is

$$\beta_L = \frac{1}{[G : G^o]} q^N \cdot q^{-\dim G} \cdot \#\tilde{G}(\kappa).$$