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$\mathbb{G}_a$-actions of fiber type on affine $\mathbb{T}$-varieties

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ABSTRACT

Let X be a normal affine $\mathbb{T}$-variety, where $\mathbb{T}$ stands for the algebraic torus. We classify $\mathbb{G}_a$-actions on X arising from homogeneous locally nilpotent derivations of fiber type. We deduce that any variety with trivial Makar-Limanov (ML) invariant is birationally decomposable as $Y \times \mathbb{P}^2$, for some Y. Conversely, given a variety Y, there exists an affine variety X with trivial ML invariant birational to $Y \times \mathbb{P}^2$.

Finally, we introduce a new version of the ML invariant, called the FML invariant. According to our conjecture, the triviality of the FML invariant implies rationality. We confirm this conjecture in dimension at most 3.

Introduction

The paper is devoted mainly to a birational characterization of normal affine algebraic varieties with trivial Makar-Limanov invariant. Let us introduce the necessary notation and definitions.

We let $\mathbf{k}$ be an algebraically closed field of characteristic 0, M be a lattice of rank n, and $\mathbb{T}$ be the algebraic torus $\mathbb{T} = \operatorname{Spec} \mathbf{k}[M] \simeq (\mathbf{k}^*)^n$. A $\mathbb{T}$-variety X is a variety endowed with an algebraic action of $\mathbb{T}$. For an affine variety $X = \operatorname{Spec} A$, to introduce a $\mathbb{T}$-action on X is the same as to endow A with an M-grading. There are well-known combinatorial descriptions of normal $\mathbb{T}$-varieties. We send the reader to [3] and [10, Ch. 1] for the case of toric varieties, to [10, Ch. 2 and 4] and [16] for the complexity 1 case, where $\dim X = \dim \mathbb{T} + 1$, and to [1,2] for the general case.

We let $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$, where $N = \operatorname{Hom}(M, \mathbb{Z})$ is the dual lattice of M. Any affine toric variety can be described via a polyhedral cone $\sigma \subseteq N_{\mathbb{Q}}$. Similarly, the description of a normal affine $\mathbb{T}$-varieties X due to Altmann and Hausen [1] involves the data $(Y, \sigma, \mathfrak{D})$ where Y is a normal semiprojective

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variety, $\sigma \subseteq N_{\mathbb{Q}}$ is a polyhedral cone, and $\mathfrak{D}$ is a divisor on Y whose coefficients are polyhedra in $N_{\mathbb{Q}}$ that can be decomposed as the Minkowski sum of a bounded polyhedron and σ.

To introduce a $\mathbb{G}_a$-action on an affine variety X is equivalent to fix a locally nilpotent derivation (LND) on its structure ring A [7, §1.5]. Any LND on A can be extended to a derivation on $K = \operatorname{Frac} A$ by the Leibniz rule. If an LND of A is homogeneous with respect to the M-grading on A we say that the associated $\mathbb{G}_a$-action on X is *compatible* with the $\mathbb{T}$-action. Furthermore, we say that a homogeneous LND ∂ (or, equivalently, the associated $\mathbb{G}_a$-action) is of *fiber type* if $\partial(K^{\mathbb{T}}) = 0$ and of *horizontal type* otherwise [6,12].

In [6] Flenner and Zaidenberg gave a classification of compatible $\mathbb{G}_a$-actions on normal affine $\mathbf{k}^*$-surfaces. Generalizing this construction, in [12] a classification of $\mathbb{G}_a$-actions on normal affine $\mathbb{T}$-varieties of complexity 1 was given. In Theorem 2.4 below, we extend this classification to $\mathbb{G}_a$-actions of fiber type on normal affine $\mathbb{T}$-varieties of arbitrary complexity.

The Makar-Limanov (ML) invariant [9] showed to be an important tool for affine geometry. In particular, it allows to distinguish certain varieties from the affine space. For an algebra A, this invariant is defined as the intersection of the kernels of all locally nilpotent derivations on A. Nevertheless, this invariant is far form being optimal. Indeed, the ML invariant of the affine space $\mathbb{A}^n$ is trivial i.e., $\mathrm{ML}(\mathbb{A}^n) = \mathbf{k}$. However, it can also be trivial for a non-rational affine variety [12]. In Theorem 4.2 we give a birational characterization of normal affine varieties with trivial ML invariant.

To avoid such a pathology, we introduce a new invariant called the FML invariant. This is defined as the intersection of the fields of fractions of the kernels of all locally nilpotent derivations on A. For an affine variety X, we conjecture that $\mathrm{FML}(X) = \mathbf{k}$ implies that X is rational. We confirm this conjecture for dimensions up to 3, see Theorem 5.6.

The content of the paper is as follows. In Section 1 we recall some generalities about $\mathbb{T}$-actions and $\mathbb{G}_a$-actions. In Section 2 we obtain the announced classification of LNDs of fiber type. In Section 3 we introduce the homogeneous ML invariant and show some of its limitations. In Section 4 we establish our principal result concerning the birational characterization. Finally, in Section 5 we introduce the FML invariant, investigate the aforementioned conjecture, and give a comparison with the classical ML invariant.

In the entire paper, unless stated otherwise, the term variety means a normal integral scheme of finite type over a field $\mathbf{k}$ of characteristic 0, not necessarily algebraically closed.

1. Preliminaries

1.1. Combinatorial description of $\mathbb{T}$-varieties

Let N be a lattice of rank n and $M = \operatorname{Hom}(N, \mathbb{Z})$ be its dual lattice. We let as before $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$, $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$, and we consider the natural duality pairing $M_{\mathbb{Q}} \times N_{\mathbb{Q}} \to \mathbb{Q}$, $(m, p) \mapsto \langle m, p\rangle$.

Let $\mathbb{T} = \operatorname{Spec} \mathbf{k}[M]$ be the n-dimensional algebraic (split) torus associated to M and let $X = \operatorname{Spec} A$ be an affine $\mathbb{T}$-variety. The comorphism $A \to A \otimes \mathbf{k}[M]$ induces an M-grading on A and, conversely, every M-grading on A arises in this way. The $\mathbb{T}$-action on X is effective if and only if the corresponding M-grading is effective.

In [1], a combinatorial description of normal affine $\mathbb{T}$-varieties is given. In what follows we recall the main features of this description. Let σ be a pointed polyhedral cone in $N_{\mathbb{Q}}$. We define $\operatorname{Pol}_\sigma(N_{\mathbb{Q}})$ to be the set of all polyhedra in $N_{\mathbb{Q}}$ that can be decomposed as the Minkowski sum of a bounded polyhedron and σ.

To a polyhedron $\Delta \in \operatorname{Pol}_\sigma(N_{\mathbb{Q}})$ we associate its support function $h_\Delta : \sigma^\vee \to \mathbb{Q}$ defined by

$$h_\Delta(m) = \min\langle m, \Delta\rangle = \min_{p\in\Delta}\langle m, p\rangle.$$

Clearly, this function h_Δ is piecewise linear on $\sigma^\vee$. Furthermore, h_Δ is concave and positively homogeneous, i.e.

$$h_\Delta(m + m') \geqslant h_\Delta(m) + h_\Delta(m'), \quad \text{and} \quad h_\Delta(\lambda m) = \lambda h_\Delta(m), \quad \forall m, m' \in \sigma^\vee,\ \forall \lambda \in \mathbb{Q}_{\geqslant 0}.$$

Definition 1.1. A variety Y is called semiprojective if it is projective over an affine variety. A *σ-polyhedral divisor* on Y is a formal sum $\mathfrak{D} = \sum_H \Delta_H \cdot H$, where H runs over all prime divisors on Y, $\Delta_H \in \mathrm{Pol}_\sigma(N_\mathbb{Q})$, and $\Delta_H = \sigma$ for all but finitely many values of H.

For $m \in \sigma_M^\vee := \sigma^\vee \cap M$ we can evaluate $\mathfrak{D}$ in m by letting $\mathfrak{D}(m)$ be the $\mathbb{Q}$-divisor

$$\mathfrak{D}(m) = \sum_{H \subseteq Y} h_H(m) \cdot H,$$

where $h_H = h_{\Delta_H}$. A σ-polyhedral divisor $\mathfrak{D}$ is called *proper* if the following hold:

(i) $\mathfrak{D}(m)$ is semiample and $\mathbb{Q}$-Cartier for all $m \in \sigma_M^\vee$, and
(ii) $\mathfrak{D}(m)$ is big for all $m \in \mathrm{rel.int}(\sigma^\vee) \cap M$.

For a $\mathbb{Q}$-divisor D on Y, $\mathcal{O}_Y(D)$ stands for the sheaf $\mathcal{O}_Y(\lfloor D \rfloor)$, where $\lfloor D \rfloor$ is the integral part of D. Recall that D is semiample if $\mathcal{O}_Y(rD)$ is globally generated for some $r > 0$, and big if $\dim H^0(Y, \mathcal{O}_Y(rD)) \geqslant c \cdot r^{\dim Y}$ for some $c > 0$ and $r \gg 1$.

The following theorem gives a combinatorial description of $\mathbb{T}$-varieties analogous to the classical combinatorial description of toric varieties.

Theorem 1.2. *(See [1].) To any proper σ-polyhedral divisor $\mathfrak{D}$ on a semiprojective variety Y one can associate a normal finitely generated effectively M-graded domain of dimension* $\mathrm{rank}\, M + \dim Y$ *given by*

$$A[Y, \mathfrak{D}] = \bigoplus_{m \in \sigma_M^\vee} A_m \chi^m, \quad \textit{where } A_m = H^0\big(Y, \mathcal{O}_Y\big(\mathfrak{D}(m)\big)\big) \subseteq \mathbf{k}(Y).$$

Conversely, if $\mathbf{k}$ *is algebraically closed then any normal finitely generated effectively M-graded domain is isomorphic to $A[Y, \mathfrak{D}]$ for some semiprojective variety Y and some proper σ-polyhedral divisor $\mathfrak{D}$ on Y.*

1.2. Locally nilpotent derivations and $\mathbb{G}_a$-actions

Let $X = \mathrm{Spec}\, A$ be an affine variety. A derivation on A is called *locally nilpotent* (LND for short) if for every $a \in A$ there exists $n \in \mathbb{Z}_{\geqslant 0}$ such that $\partial^n(a) = 0$. Given an LND ∂ on A, the map $\phi_\partial : \mathbb{G}_a \times A \to A$, $\phi_\partial(t, f) = e^{t\partial} f$ defines a $\mathbb{G}_a$-action on X, and any $\mathbb{G}_a$-action arises in this way.

In the following lemma we collect some well-known facts about LNDs over a field of characteristic 0, see e.g., [7].

Lemma 1.3. *Let A be a finitely generated normal domain over a field of characteristic* 0*. For any two LNDs ∂ and ∂' on A, the following hold.*

(i) $\ker \partial$ *is a normal subdomain of codimension* 1.
(ii) $\ker \partial$ *is factorially closed i.e.,* $ab \in \ker \partial \Rightarrow a, b \in \ker \partial$.
(iii) *If* $a \in A$ *is invertible, then* $a \in \ker \partial$.
(iv) *If* $\ker \partial = \ker \partial'$, *then there exist* $a, a' \in \ker \partial$ *such that* $a\partial = a'\partial'$.
(v) *If* $a \in \ker \partial$, *then* $a\partial$ *is again an LND.*
(vi) *If* $\partial(a) \in (a)$ *for some* $a \in A$, *then* $a \in \ker \partial$.
(vii) *The field extension* $\mathrm{Frac}(\ker \partial) \subseteq \mathrm{Frac}\, A$ *is purely transcendental of degree* 1.

Definition 1.4. We say that two LNDs ∂ and ∂' on A are *equivalent* if $\ker \partial = \ker \partial'$.

Let $\mathfrak{D}$ be a proper σ-polyhedral divisor on a semiprojective variety Y, and let $A = A[Y, \mathfrak{D}]$ be the corresponding M-graded domain. A derivation ∂ on A is called *homogeneous* if it sends homogeneous elements into homogeneous elements. Given a homogeneous LND ∂, we define its degree as $\deg\partial = \deg\partial(f) - \deg f$ for any homogeneous $f \in A \setminus \ker\partial$.

Let K_Y be the field of rational functions of Y. A homogeneous LND ∂ on A extends to a derivation on $\operatorname{Frac} A = K_Y(M)$, where $K_Y(M)$ is the field of fractions of $K_Y[M]$. The LND ∂ is said to be *of fiber type* if $\partial(K_Y) = 0$ and *of horizontal type* otherwise.

Geometrically speaking, ∂ is of fiber type if and only if the general orbits of the corresponding $\mathbb{G}_a$-action on $X = \operatorname{Spec} A$ are contained in the closures of general orbits of the $\mathbb{T}$-action given by the M-grading.

1.3. Locally nilpotent derivations on toric varieties

In this section we recall the classification of homogeneous LNDs given in [12] for toric varieties defined over a field $\mathbf{k}$ of characteristic 0. A similar description is implicit in the paper [3] devoted to complete toric varieties.

Let $\rho \in N$ and $e \in M$ be lattice vectors. We define $\partial_{\rho,e}$ as the homogeneous derivation of degree e on $\mathbf{k}[M]$ given by $\partial_{\rho,e}(\chi^m) = \langle m, \rho\rangle \cdot \chi^{m+e}$.

For a pointed polyhedral cone σ in the vector space $N_{\mathbb{Q}}$, we let

$$A = \mathbf{k}\big[\sigma_M^\vee\big] = \bigoplus_{m\in\sigma_M^\vee} \mathbf{k}\chi^m$$

be the affine semigroup algebra of the corresponding affine toric variety $X_\sigma = \operatorname{Spec} A$.

If $\sigma = \{0\}$, then A is spanned by the characters which are invertible functions. By Lemma 1.3 (iii) any LND on A is trivial. In the following, we fix a ray ρ of σ, and we let τ be the facet of $\sigma^\vee$ dual to ρ. As usual, we denote by the same letter ρ the ray and its primitive vector.

Definition 1.5. We define

$$S_\rho = \sigma_\rho^\vee \cap \big\{e \in M \;\big|\; \langle e, \rho\rangle = -1\big\},$$

where σ_ρ is the cone spanned by the rays of σ except ρ. We have $S_\rho \neq \emptyset$. Furthermore, $e + m \in S_\rho$ whenever $e \in S_\rho$ and $m \in \tau_M$.

The following theorem gives a classification of $\mathbb{T}$-compatible $\mathbb{G}_a$-actions on $\operatorname{Spec} A$, or equivalently, a classification of the homogeneous LNDs on A.

Theorem 1.6. *To any pair (ρ, e), where ρ is a ray of σ and $e \in S_\rho$, we can associate a homogeneous LND $\partial_{\rho,e}$ on $A = \mathbf{k}[\sigma_M^\vee]$ of degree e with kernel $\ker\partial_{\rho,e} = \mathbf{k}[\tau_M]$. Conversely, if $\partial \neq 0$ is a homogeneous LND on A, then $\partial = \lambda\partial_{\rho,e}$ for some ray $\rho \subseteq \sigma$, some lattice vector $e \in S_\rho$, and some $\lambda \in \mathbf{k}^*$.*

Proof. The first assertion is Lemma 2.6 in [12]. The second follows from Theorem 2.7 in [12]. □

2. Locally nilpotent derivations of fiber type

In this section we completely describe compatible $\mathbb{G}_a$-actions of fiber type on a normal affine $\mathbb{T}$-variety over an algebraically closed field of characteristic 0. The particular case of complexity 1 is done in [12, §3.1].

If the base field is algebraically closed, by Theorem 1.2 every normal finitely generated effectively M-graded domain is isomorphic to $A[Y, \mathfrak{D}]$ for some semiprojective variety Y and some proper σ-polyhedral divisor $\mathfrak{D}$ on Y.

We fix a smooth semiprojective variety Y and a proper σ-polyhedral divisor on Y

$$\mathfrak{D} = \sum_H \Delta_H \cdot H.$$

Letting K_Y be the field of rational functions on Y, we consider the affine variety $X = \operatorname{Spec} A$, where

$$A = A[Y, \mathfrak{D}] = \bigoplus_{m \in \sigma_M^\vee} A_m \chi^m, \quad \text{with } A_m = H^0\big(Y, \mathcal{O}\big(\mathfrak{D}(m)\big)\big) \subseteq K_Y.$$

We denote by h_H the support function of Δ_H so that $\mathfrak{D}(m) = \sum_{H \in Y} h_H(m) \cdot H$. We also fix a homogeneous LND ∂ of fiber type on A.

We let $\bar{A} = K_Y[\sigma_M^\vee]$ be the affine semigroup algebra over K_Y with cone $\sigma \in N_{\mathbb{Q}}$. By Lemma 1.13 in [12] ∂ can be extended to a homogeneous locally nilpotent K_Y-derivation $\bar{\partial}$ on $\bar{A}$.

If $\sigma = \{0\}$ then $\bar{\partial} = 0$ by Theorem 1.6 and so ∂ is trivial. In the sequel we assume that there is at least one ray, say ρ, of σ. Let τ be its dual facet, and let S_ρ be as defined in Definition 1.5.

Definition 2.1. For any $e \in S_\rho$, we let D_e be the $\mathbb{Q}$-divisor on Y defined by

$$D_e := \sum_H \max_{m \in \sigma_M^\vee \setminus \tau_M} \big(h_H(m) - h_H(m+e)\big) \cdot H.$$

Remark 2.2. An alternative description of D_e is as follows. Since the function h_H is concave and piecewise linear on $\sigma^\vee$, the above maximum is achieved by one of the linear pieces of h_H i.e., by one of the maximal cones in the normal quasifan $\Lambda(h_H)$.

For every prime divisor H on Y, we let $\{\delta_{1,H}, \ldots, \delta_{\ell_H,H}\}$ be the set of all maximal cones in $\Lambda(h_H)$ and $g_{r,H}$, where $r \in \{1, \ldots, \ell_H\}$, be the linear extension of $h_H|_{\delta_{r,H}}$ to $M_{\mathbb{Q}}$. Since the maximum is achieved on one of the linear pieces in $\sigma^\vee$ we have

$$\max_{m \in \sigma_M^\vee \setminus \tau_M} \big(h_H(m) - h_H(m+e)\big) = \max_{r \in \{1,\ldots,\ell_H\}} \big(-g_{r,H}(e)\big) = -\min_{r \in \{1,\ldots,\ell_H\}} g_{r,H}(e).$$

Since τ is a facet of $\sigma^\vee$, it is contained as a face in one and only one maximal cone $\delta_{r,H}$. We may assume that $\tau \subseteq \delta_{1,H}$. By the concavity of h_H we have $g_{1,H}(e) \leqslant g_{r,H}(e)$, $\forall r$ and so

$$D_e = -\sum_H g_{1,H}(e) \cdot H.$$

We need the following lemma.

Lemma 2.3. *For any $e \in S_\rho$ we define $\Phi_e = H^0(Y, \mathcal{O}_Y(-D_e))$. If $\varphi \in K_Y$ then $\varphi \in \Phi_e$ if and only if $\varphi \cdot A_m \subseteq A_{m+e}$ for any $m \in \sigma_M^\vee \setminus \tau_M$.*

Proof. If $\varphi \in \Phi_e$, then for every $m \in \sigma_M^\vee \setminus \tau_M$,

$$\operatorname{div}(\varphi) \geqslant D_e \geqslant \sum_H \big(h_z(m) - h_z(m+e)\big) \cdot H = \mathfrak{D}(m) - \mathfrak{D}(m+e).$$

If $f \in \varphi A_m$ then $\operatorname{div}(f) + \mathfrak{D}(m) \geqslant \operatorname{div}(\varphi)$ and so $\operatorname{div}(f) + \mathfrak{D}(m+e) \geqslant 0$. Thus $\varphi A_m \subseteq A_{m+e}$.

To prove the converse, we let $\varphi \in K_Y$ be such that $\varphi A_m \subseteq A_{m+e}$ for any $m \in \sigma_M^\vee \setminus \tau_M$. With the notation of Remark 2.2, we let $m \in M$ be a lattice vector such that $\mathfrak{D}(m)$ is an integral divisor, and m and $m+e$ belong to rel.int$(\delta_{1,H})$, for any prime divisor H.

For every $H \in \operatorname{Supp}\mathfrak{D}$, we let $f_H \in A_m$ be a rational function such that

$$\operatorname{ord}_H(f_H) = -h_H(m) = -g_{1,H}(m).$$

By our assumption $\varphi \cdot f_H \in A_{m+e}$ and so

$$\operatorname{ord}_H(\varphi f_H) \geqslant -h_H(m+e) = -g_{1,H}(m+e).$$

This yields $\operatorname{ord}_H(\varphi) \geqslant -g_{1,H}(m+e) + g_{1,H}(m) = -g_{1,H}(e)$, hence $\varphi \in \Phi_e$. This proves the lemma. □

The following theorem gives a classification of LNDs of fiber type on an arbitrary normal affine $\mathbb{T}$-variety. We let $\Phi_e^* = \Phi_e \setminus \{0\}$.

Theorem 2.4. *To any triple (ρ, e, φ), where ρ is a ray of σ, $e \in S_\rho$, and $\varphi \in \Phi_e^*$, we can associate a homogeneous LND $\partial_{\rho,e,\varphi}$ on $A = A[Y, \mathfrak{D}]$ of fiber type and of degree e, with kernel*

$$\ker \partial_{\rho,e,\varphi} = \bigoplus_{m \in \tau_M} A_m \chi^m.$$

Conversely, if $\mathbf{k}$ is algebraically closed then every non-trivial homogeneous LND ∂ of fiber type on A is of the form $\partial = \partial_{\rho,e,\varphi}$ for some ray $\rho \subseteq \sigma$, some lattice vector $e \in S_\rho$, and some function $\varphi \in \Phi_e^*$.

Proof. Letting $\bar{A} = K_Y[\sigma_M^\vee]$, we consider the K_Y-LND $\partial_{\rho,e}$ on $\bar{A}$ as in Theorem 1.6. Since $\varphi \in K_Y^*$, $\varphi\partial_{\rho,e}$ is again a K_Y-LND on $\bar{A}$.

We claim that $\varphi\partial_{\rho,e}$ stabilizes $A \subseteq \bar{A}$. Indeed, let $f \in A_m \subseteq K_Y$ be a homogeneous element. If $m \in \tau_M$, then $\varphi\partial_{\rho,e}(f\chi^m) = 0$. If $m \in \sigma_M^\vee \setminus \tau_M$, then

$$\varphi\partial_{\rho,e}\big(f\chi^m\big) = \varphi f \partial_{\rho,e}\big(\chi^m\big) = m_0 \varphi f \chi^{m+e},$$

where $m_0 := \langle m, \rho\rangle \in \mathbb{Z}_{>0}$. By Lemma 2.3, $m_0\varphi f\chi^{m+e} \in A_{m+e}$, proving the claim.

Finally $\partial_{\rho,e,\varphi} := \varphi\partial_{\rho,e}|_A$ is a homogeneous LND on A with kernel

$$\ker \partial_{\rho,e,\varphi} = A \cap \ker \partial_{\rho,e} = \bigoplus_{m \in \tau_M} (A_m \cap K_Y)\chi^m = \bigoplus_{m \in \tau_M} A_m \chi^m,$$

as desired.

To prove the converse, since $\mathbf{k}$ is algebraically closed we have $A = A[Y, \mathfrak{D}]$. We consider a homogeneous LND ∂ on A of fiber type. Since ∂ is of fiber type, $\partial|_{K_Y} = 0$ and so ∂ can be extended to a K_Y-LND $\bar{\partial}$ on the affine semigroup algebra $\bar{A} = K_Y[\sigma_M^\vee]$. By Theorem 1.6, $\bar{\partial} = \varphi\partial_{\rho,e}$ for some ray ρ of σ, some $e \in S_\rho$ and some $\varphi \in K_Y^*$. Since A is stable under $\varphi\partial_{\rho,e}$, by Lemma 2.3 $\varphi \in \Phi_e^*$ and so $\partial = \varphi\partial_{\rho,e}|_A = \partial_{\rho,e,\varphi}$. □

Corollary 2.5. *Let A be a normal finitely generated effectively M-graded domain, where M is a lattice of finite rank, and let ∂ be a homogeneous LND on A. If ∂ is of fiber type then* $\ker\partial$ *is finitely generated.*

Proof. Let $A = A[Y, \mathfrak{D}]$, where $\mathfrak{D}$ is a proper σ-polyhedral divisor on a semiprojective variety Y. In the notation of Theorem 2.4 we have $\partial = \partial_{\rho,e,\varphi}$, where ρ is a ray of σ. Letting $\tau \subseteq \sigma^\vee$ be the facet dual to ρ, by Theorem 2.4 we have $\ker\partial = \bigoplus_{m\in\tau_M} A_m\chi^m$.

Let $a_1, \ldots, a_r$ be a set of homogeneous generators of A. Without loss of generality, we may assume that $\deg a_i \in \tau_M$ if and only if $1 \leqslant i \leqslant s < r$. We claim that $a_1, \ldots, a_s$ generate $\ker\partial$. Indeed, let P be any polynomial such that $P(a_1, \ldots, a_r) \in \ker\partial$. Since $\tau \subseteq \sigma^\vee$ is a face, $\sum m_i \in \tau_M$ for $m_i \in \sigma_M^\vee$ implies that $m_i \in \tau$, $\forall i$. Hence all the monomials composing $P(a_1, \ldots, a_r)$ are monomials in $a_1, \ldots, a_s$, proving the claim. □

Corollary 2.6. *Let as before ∂ be a homogeneous LND of fiber type on $A = A[Y, \mathfrak{D}]$, and let $f\chi^m \in A \setminus \ker\partial$ be a homogeneous element. Then ∂ is completely determined by the image $g\chi^{m+e} := \partial(f\chi^m) \in A_{m+e}\chi^{m+e}$.*

Proof. By the previous theorem $\partial = \partial_{\rho,e,\varphi}$ for some ray ρ, some $e \in S_\rho$, and some $\varphi \in \Phi_e$. Here $e = \deg\partial$ and ρ is uniquely determined by e, see Corollary 2.8 in [12].

In the course of the proof of Theorem 2.4 it was shown that $\partial_{\rho,e,\varphi}(f\chi^m) = m_0\varphi f\chi^{m+e}$. Thus $\varphi = \frac{g}{m_0 f} \in K_0$ is also uniquely determined by our data. □

It might happen that Φ_e^* as above is empty. Given a ray $\rho \subseteq \sigma$, in the following theorem we give a criterion for the existence of $e \in S_\rho$ such that Φ_e^* is non-empty.

Theorem 2.7. *Let $A = A[Y, \mathfrak{D}]$, and let $\rho \subseteq \sigma$ be the ray dual to a codimension one face $\tau \subseteq \sigma^\vee$. Then there exists $e \in S_\rho$ such that $\dim\Phi_e$ is positive if and only if the divisor $\mathfrak{D}(m)$ is big for all lattice vectors $m \in \operatorname{rel.int}(\tau)$.*

Proof. Assuming that $\mathfrak{D}(m)$ is big for every lattice vector $m \in \operatorname{rel.int}(\tau)$, we consider the linear map

$$G : M_{\mathbb{Q}} \to \operatorname{Div}_{\mathbb{Q}}(Y), \quad m \mapsto \sum_H g_{1,H}(m) \cdot H,$$

so that $G(m) = \mathfrak{D}(m)$ for all $m \in \tau$ and $D_e = -G(e)$ for all $e \in S_\rho$. Choosing $m \in \operatorname{rel.int}(\tau) \cap (S_\rho + \mu)$ and $r \in \mathbb{Z}_{>0}$, we let $j = m - \frac{1}{r} \cdot \mu$. Let us consider the divisor

$$G(j) = G(m) - \frac{1}{r} \cdot G(\mu) = \mathfrak{D}(m) - \frac{1}{r} \cdot G(\mu).$$

Since $\mathfrak{D}(m)$ is big and the cone of big divisors is open in $\operatorname{Div}_{\mathbb{R}}(Y)$ (see [11, Def. 2.2.25]), by choosing r big enough, we may assume that $G(j)$ is big. Furthermore, after increasing r, if necessary, we may assume that $G(r \cdot j)$ has a section. Now, $r \cdot j = r \cdot m - \mu = (r-1) \cdot m + (m - \mu)$. Since $(r-1) \cdot m \in \tau_M$ and $m - \mu \in S_\rho$, we have $r \cdot j \in S_\rho$. Letting $e = r \cdot j \in S_\rho$ we obtain $D_e = -G(e)$ and so $\dim H^0(Y, \mathcal{O}_Y(-D_e))$ is positive.

Assume now that there is $m \in \operatorname{rel.int}(\tau)$ such that $\mathfrak{D}(m)$ is not big. Since the set of big divisors is an open and convex set in $\operatorname{Div}_{\mathbb{R}}(Y)$, the divisor $\mathfrak{D}(m)$ is not big whatever is $m \in \tau$. We let B be the algebra

$$B = \bigoplus_{m \in \tau_M} A_m \chi^m.$$

Under our assumption $\dim B < n + k - 1$. Since $\dim A = n + k$, by Lemma 1.3 (i) B cannot be the kernel of an LND on A. By Theorem 2.4, the latter implies that for none of the $e \in S_\rho$ the dimension $\dim\Phi_e$ is positive. □

Finally, we deduce the following corollary.

Corollary 2.8. *Two homogeneous LNDs of fiber type $\partial = \partial_{\rho,e,\varphi}$ and $\partial' = \partial_{\rho',e',\varphi'}$ on $A = A[Y, \mathfrak{D}]$ are equivalent if and only if $\rho = \rho'$. Furthermore, the equivalence classes of homogeneous LNDs of fiber type on A are in one-to-one correspondence with the rays $\rho \subseteq \sigma$ such that $\mathfrak{D}(m)$ is big $\forall m \in \operatorname{rel.int}(\tau)$, where τ is the facet dual to ρ.*

Proof. The first assertion follows from the description of $\ker \partial_{\rho,e,\varphi}$ in Theorem 2.4. The second follows from the first one due to Theorem 2.7. □

Remark 2.9. Let $X = \operatorname{Spec} A[Y, \mathfrak{D}]$ be an affine $\mathbb{T}$-variety and recall that $n = \operatorname{rank} M$. There are two types of $\mathbb{T}$-invariant divisors on X. The first type corresponds to families of n-dimensional orbit closures over a prime divisor in Y; and the second one corresponds to families of $(n-1)$-dimensional orbit closures over Y. By [15, Proposition 3.13], the equivalence classes of homogeneous LNDs of fiber type on X are also in one-to-one correspondence with the $\mathbb{T}$-invariant divisors of the second type.

3. Homogeneous Makar-Limanov invariant

Let $X = \operatorname{Spec} A$, where A is a finitely generated $\mathbf{k}$-domain, and let $\mathrm{LND}(A)$ be the set of all LNDs on A. The *Makar-Limanov invariant* (ML invariant for short) of A (or of $X = \operatorname{Spec} A$) is defined as

$$\mathrm{ML}(A) = \bigcap_{\partial \in \mathrm{LND}(A)} \ker \partial.$$

In the case where A is effectively M-graded we let $\mathrm{LND}_{\mathrm{h}}(A)$ be the set of all homogeneous LNDs on A and $\mathrm{LND}_{\mathrm{fib}}(A)$ be the set of all homogeneous LNDs of fiber type on A. Following [12], we define

$$\mathrm{ML}_{\mathrm{h}}(A) = \bigcap_{\partial \in \mathrm{LND}_{\mathrm{h}}(A)} \ker \partial \quad \text{and} \quad \mathrm{ML}_{\mathrm{fib}}(A) = \bigcap_{\partial \in \mathrm{LND}_{\mathrm{fib}}(A)} \ker \partial.$$

$\mathrm{ML}_{\mathrm{h}}(A)$ is called the *homogeneous Makar-Limanov invariant* of A. Clearly,

$$\mathrm{ML}(A) \subseteq \mathrm{ML}_{\mathrm{h}}(A) \subseteq \mathrm{ML}_{\mathrm{fib}}(A).$$

In this section we provide examples showing that, in general, these inclusions are strict and so, the homogeneous LNDs are not enough to compute the ML invariant.

Example 3.1. Let $A = \mathbf{k}[x, y]$ with the grading given by $\deg x = 0$ and $\deg y = 1$. In this case, both partial derivatives $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$ are homogeneous. Since $\ker \partial_x = \mathbf{k}[y]$ and $\ker \partial_y = \mathbf{k}[x]$ we have $\mathrm{ML}_{\mathrm{h}} = \mathbf{k}$. Furthermore, it is easy to see that there is only one equivalence class of LNDs of fiber type. A representative of this class is ∂_y (see Corollary 2.8). This yields $\mathrm{ML}_{\mathrm{fib}}(A) = \mathbf{k}[x]$. Thus $\mathrm{ML}_{\mathrm{h}}(A) \subsetneq \mathrm{ML}_{\mathrm{fib}}(A)$ in this case.

Example 3.2. To provide an example where $\mathrm{ML}(A) \subsetneq \mathrm{ML}_{\mathrm{h}}(A)$ we consider the Koras–Russell affine cubic threefold $X = \operatorname{Spec} A$, where

$$A = \mathbf{k}[x, y, z, t]/\left(x + x^2 y + z^2 + t^3\right).$$

The ML invariant was first introduced in [9] to distinguish X from $\mathbb{A}^3$. In fact $\mathrm{ML}(A) = \mathbf{k}[x]$ while $\mathrm{ML}(\mathbb{A}^3) = \mathbf{k}$. In the recent paper [5] Dubouloz shows that the cylinder over the Koras–Russell threefold has trivial ML invariant i.e., $\mathrm{ML}(A[w]) = \mathbf{k}$, where w is a new variable.

Let ∂ be a homogeneous LND on $A[w]$ graded via $\deg A = 0$ and $\deg w = 1$. If $e := \deg \partial \leqslant -1$ then $\partial(A) = 0$. By Lemma 1.3 (i) we have $\ker \partial = A$ and so ∂ is equivalent to the partial derivative $\partial/\partial w$.

If $e \geqslant 0$ then $\partial(w) = a w^{e+1}$, where $a \in A$ and so, by Lemma 1.3 (vi) $w \in \ker \partial$. Furthermore, for any $a \in A$ we have $\partial(a) = b w^e$ for a unique $b \in A$. We define a derivation $\bar{\partial} : A \to A$ by $\bar{\partial}(a) = b$. Since $\partial^r(a) = \bar{\partial}^r(a) w^{re}$ the derivation $\bar{\partial}$ is LND. This yields $\mathrm{ML}_{\mathrm{h}}(A[w]) = \mathrm{ML}(A) = \mathbf{k}[x]$ while $\mathrm{ML}(A[w]) = \mathbf{k}$.

4. Birational equivalence classes of varieties with trivial ML invariant

In this section we establish a birational characterization of normal affine varieties with trivial ML invariant over a field $\mathbf{k}$ of characteristic 0, not necessarily algebraically closed.

The following lemma was proven in [14, Lemma 16] in the case where $\mathbf{k}=\mathbb{C}$.

Lemma 4.1. *Let A be a finitely generated 2-dimensional normal $\mathbf{k}$-domain. If $\mathrm{ML}(A)=\mathbf{k}$ then $\operatorname{Frac} A$ is a purely transcendental extension of $\mathbf{k}$.*

Proof. Since $\mathrm{ML}(A)=\mathbf{k}$, there are at least two non-equivalent LNDs ∂ and ∂' on A. Let ϕ and ϕ' be the respective $\mathbb{G}_a$-actions. The general orbits of these two $\mathbb{G}_a$-actions intersect transversally on $X=\operatorname{Spec} A$. Let $\operatorname{Orb}(x')$ be a general orbit of ϕ'.

By Lemma 1.3 and the Zariski finiteness theorem [7, p. 147], $\ker\partial$ is a normal finitely generated 1-dimensional domain. Furthermore, the inclusion $\ker\partial\subseteq A$ induces a dominant morphism $X\to\operatorname{Spec}(\ker\partial)$. The composition $\mathbb{A}^1_{\mathbf{k}}\simeq\operatorname{Orb}(x')\hookrightarrow X\to\operatorname{Spec}(\ker\partial)$ is not constant. Therefore, $\ker\partial=\mathbf{k}'[t]$ for some $t\in\ker\partial$ and some field $\mathbf{k}'$ algebraic over $\mathbf{k}$. By Lemma 1.3 (iii) $\mathbf{k}'=\mathbf{k}$ and so $\ker\partial\simeq\mathbf{k}[t]$. Now the result follows from Lemma 1.3 (vii). □

The following theorem is the main result of this section.

Theorem 4.2. *Let $X=\operatorname{Spec} A$ be an affine variety of dimension $n\geqslant 2$ over $\mathbf{k}$. If $\mathrm{ML}(X)=\mathbf{k}$ then $X\simeq_{\mathrm{bir}} Y\times\mathbb{P}^2$ for some variety Y. Conversely, in any birational class $Y\times\mathbb{P}^2$ there is an affine variety X with $\mathrm{ML}(X)=\mathbf{k}$.*

Proof. Let $K=\operatorname{Frac} A$ be the field of rational functions on X so that $\operatorname{tr.deg}_{\mathbf{k}}(K)=n$. As usual $\operatorname{tr.deg}_{\mathbf{k}}(K)$ denotes the transcendence degree of the field extension $\mathbf{k}\subseteq K$.

Since $\mathrm{ML}(X)=\mathbf{k}$, there exist at least two non-equivalent LNDs ∂_1 and $\partial_2: A\to A$. We let $L_i=\operatorname{Frac}(\ker\partial_i)\subseteq K$, for $i=1,2$. By Lemma 1.3 (vii), $L_i\subseteq K$ is a purely transcendental extension of degree 1, for $i=1,2$.

We let $L=L_1\cap L_2$. By an inclusion–exclusion argument we have $\operatorname{tr.deg}_L(K)=2$. We consider the 2-dimensional algebra $\bar{A}=A\otimes_{\mathbf{k}} L$ over L. Since $\operatorname{Frac}\bar{A}=\operatorname{Frac} A=K$ and $L\subseteq\ker\partial_i$ for $i=1,2$, the LND ∂_i extends to a locally nilpotent L-derivation $\bar{\partial}_i$ by setting

$$\bar{\partial}_i(a\otimes l)=\partial_i(a)\otimes l,\quad \text{where } a\in A \text{ and } l\in L.$$

Furthermore, $\ker\bar{\partial}_i=\bar{A}\cap L_i$, for $i=1,2$ and so

$$\ker\bar{\partial}_1\cap\ker\bar{\partial}_2=\bar{A}\cap L_1\cap L_2=L.$$

Thus the Makar-Limanov invariant of $\bar{A}$ is trivial. By Lemma 4.1, $K=\operatorname{Frac}\bar{A}$ is a purely transcendental extension of L of degree 2. Thus $X\simeq_{\mathrm{bir}} Y\times\mathbb{P}^2$, where Y is any model having L as the field of rational functions.

The second assertion follows from Lemma 4.4 bellow. This completes the proof. □

Remark 4.3. The previous proof depends only on the fact that X has at least two non-equivalent LNDs. An alternative proof of the first assertion of Theorem 4.2 can be obtained adapting the argument of Theorem 2.5 in [4].

The following lemma provides examples of affine varieties with trivial ML invariant in any birational class $Y\times\mathbb{P}^n$, $n\geqslant 2$. It generalizes the results in Section 4.3 in [12]. Let us introduce the necessary notation.

As before, we let N be a lattice of rank $n \geqslant 2$ and M be its dual lattice. We let $\sigma \subseteq N_{\mathbb{Q}}$ be a pointed polyhedral cone of full dimension. We fix $p \in \text{rel.int}(\sigma) \cap M$. We let $\Delta = p + \sigma$ and $h = h_\Delta$ so that

$$h(m) = \langle p, m\rangle > 0, \quad \text{for all } m \in \sigma_M^\vee \setminus \{0\}.$$

Letting Y be a projective variety and H be a semiample and big Cartier $\mathbb{Z}$-divisor on Y, we let $A = A[Y, \mathfrak{D}]$, where $\mathfrak{D}$ is the proper σ-polyhedral divisor $\mathfrak{D} = \Delta \cdot H$, so that

$$\mathfrak{D}(m) = \langle p, m\rangle \cdot H, \quad \text{for all } m \in \sigma_M^\vee.$$

Recall that $\operatorname{Frac} A = K_Y(M)$ so that $\operatorname{Spec} A \simeq_{\text{bir}} Y \times \mathbb{P}^n$.

Lemma 4.4. *With the notation as above, the affine variety $X = \operatorname{Spec} A[Y, \mathfrak{D}]$ has trivial ML invariant.*

Proof. Let $\{\rho_i\}_i$ be the set of all rays of σ and $\{\tau_i\}_i$ the set of the corresponding dual facets of $\sigma^\vee$. Since rH is big for all $r > 0$, by Theorem 2.7, there exists $e_i \in S_{\rho_i}$ such that $\dim \Phi_{e_i}$ is positive. So we can chose a non-zero element $\varphi_i \in \Phi_{e_i}$. By Theorem 2.4 there exists a non-trivial locally nilpotent derivation $\partial_{\rho_i, e_i, \varphi_i}$, with

$$\ker \partial_{\rho_i, e_i, \varphi_i} = \bigoplus_{m \in \tau_i \cap M} A_m \chi^m.$$

Since the cone σ is pointed and has full dimension, the same holds for $\sigma^\vee$. Thus, the intersection of all facets reduces to one point $\bigcap_i \tau_i = \{0\}$ and so

$$\bigcap_i \ker \partial_{\rho_i, e_i, \varphi_i} \subseteq A_0 = H^0(Y, \mathcal{O}_Y) = \mathbf{k}.$$

This yields the equalities

$$\mathrm{ML}(A) = \mathrm{ML}_{\mathrm{h}}(A) = \mathrm{ML}_{\mathrm{fib}}(A) = \mathbf{k}. \qquad \square$$

Example 4.5. Let us provide yet another explicit construction of a class of normal affine $\mathbb{T}$-varieties with trivial ML invariant. With the notation as in the proof of Lemma 4.4, we fix isomorphisms $M \simeq \mathbb{Z}^n$ and $N \simeq \mathbb{Z}^n$ such that the standard bases $\{\mu_1, \ldots, \mu_n\}$ and $\{\nu_1, \ldots, \nu_n\}$ for $M_{\mathbb{Q}}$ and $N_{\mathbb{Q}}$, respectively, are mutually dual. We let σ be the first quadrant in $N_{\mathbb{Q}}$, and $p = \sum_i \nu_i$, so that

$$h(m) = \sum_i m_i, \quad \text{and} \quad \mathfrak{D}(m) = \sum_i m_i \cdot H, \quad \text{where } m = (m_1, \ldots, m_n), \text{ and } m_i \in \mathbb{Q}_{\geqslant 0}.$$

We let $\rho_i \subseteq \sigma$ be the ray spanned by the vector ν_i, and let τ_i be its dual facet. In this setting, $S_{\rho_i} = (\tau_i - \mu_i) \cap M$. Furthermore, letting $e_{i,j} = -\mu_i + \mu_j$ (where $j \neq i$) yields

$$h(m) = h(m + e_{i,j}), \quad \text{so that} \quad D_{e_{i,j}} = 0, \quad \text{and} \quad \Phi_{e_{i,j}} = H^0(Y, \mathcal{O}_Y) = \mathbf{k}.$$

Recall that ∂_{ν_i} are the partial derivatives defined in Section 1.3. Choosing $\varphi_{i,j} = 1 \in \Phi_{e_{i,j}}$ we have

$$\partial_{i,j} := \partial_{\rho_i, e_{i,j}, \varphi_{i,j}} = \chi^{\mu_j} \partial_{\nu_i}, \quad \text{where } i, j \in \{1, \ldots, n\},\ i \neq j$$

is a homogeneous LND on $A = A[Y, \mathfrak{D}]$ of degree $e_{i,j}$ and with kernel

$$\ker \partial_{i,j} = \bigoplus_{\tau_i \cap M} A_m \chi^m.$$

As in the proof of Lemma 4.4 we have

$$\bigcap_{i,j} \ker \partial_{i,j} = \mathbf{k} \quad \text{and so} \quad \mathrm{ML}(X) = \mathbf{k}.$$

We can give a geometrical description of X. Consider the $\mathcal{O}_Y$-algebra

$$\widetilde{A} = \bigoplus_{m \in \sigma_M^\vee} \mathcal{O}_Y\big(\mathfrak{D}(m)\big)\chi^m \quad \text{so that} \quad A = H^0(Y, \widetilde{A}).$$

In this case, we can write

$$\widetilde{A} = \bigoplus_{r=0}^{\infty} \bigoplus_{\sum m_i = r,\, m_i \geqslant 0} \mathcal{O}_Y(rH)\chi^m \simeq \mathrm{Sym}\left(\bigoplus_{i=1}^{n} \mathcal{O}_Y(H)\right).$$

Thus $\widetilde{X} = \mathbf{Spec}_Y \widetilde{A}$ is the vector bundle over Y associated to the locally free sheaf $\mathcal{E} = \bigoplus_{i=1}^n \mathcal{O}_Y(H)$ (see Ex. 5.18 in [8, Ch. II]). We let $\pi : \widetilde{X} \to Y$ be the corresponding affine morphism.

The morphism $\varphi : \widetilde{X} \to X$ induced by taking global sections corresponds to the contraction of the zero section to a point $\bar{0}$. We let $\theta := \pi \circ \varphi^{-1} : X \setminus \{\bar{0}\} \to Y$. The point $\bar{0}$ corresponds to the augmentation ideal $A \setminus \mathbf{k}$. This point is the only attractive fixed point of the $\mathbb{T}$-action. The orbit closures of the $\mathbb{T}$-action on X are $\Theta_y := \overline{\theta^{-1}(y)} = \theta^{-1}(y) \cup \{0\}$, $\forall y \in Y$. Let $\chi^{\mu_i} = u_i$. Then Θ_y is equivariantly isomorphic to $\operatorname{Spec} \mathbf{k}[\sigma_M^\vee] = \operatorname{Spec} \mathbf{k}[u_1, \ldots, u_n] \simeq \mathbb{A}^n$.

The $\mathbb{G}_a$-action $\phi_{i,j} : \mathbb{G}_a \times X \to X$ induced by the homogeneous LND $\partial_{i,j}$ restricts to a $\mathbb{G}_a$-action on Θ_y given by

$$\phi_{i,j}|_{\Theta_Y} : \mathbb{G}_a \times \mathbb{A}^n \to \mathbb{A}^n, \quad \text{where } u_i \mapsto u_i + tu_j,\ u_r \mapsto u_r,\ \forall r \neq i.$$

Moreover, the unique fixed point $\bar{0}$ is singular unless Y is a projective space and there is no other singular point. By Theorem 2.9 in [13] X has rational singularities if and only if $\mathcal{O}_Y$ and $\mathcal{O}_Y(H)$ are acyclic. The latter assumption can be fulfilled by taking, for instance, Y toric or Y a rational surface, and H a large enough multiple of an ample divisor.

5. FML invariant

The ML invariant serves to distinguish some varieties from the affine space. Nevertheless, this invariant is far from being optimal as we have seen in the previous section. Indeed, there is a large class of non-rational normal affine varieties with trivial ML invariant. To eliminate such a pathology, we propose below a generalization of the classical ML invariant.

Let A be a finitely generated normal domain. We define the FML invariant of A to be the subfield of $K = \operatorname{Frac} A$ given by

$$\mathrm{FML}(A) = \bigcap_{\partial \in \mathrm{LND}(A)} \mathrm{Frac}(\ker \partial).$$

In the case where A is M-graded we define $\mathrm{FML}_{\mathrm{h}}$ and $\mathrm{FML}_{\mathrm{fib}}$ in the analogous way, see Section 3.

Remark 5.1. Let $A=\mathbf{k}[x_1,\ldots,x_n]$ so that $K=\mathbf{k}(x_1,\ldots,x_n)$. For the partial derivative $\partial_i=\partial/\partial x_i$ we have $\operatorname{Frac}(\ker\partial_i)=\mathbf{k}(x_1,\ldots,\widehat{x_i},\ldots,x_n)$, where $\widehat{x_i}$ means that x_i is omitted. This yields

$$\mathrm{FML}(A)\subseteq\bigcap_{i=1}^{n}\operatorname{Frac}(\ker\partial_i)=\mathbf{k},$$

and so $\mathrm{FML}(A)=\mathbf{k}$. Thus, the FML invariant of the affine space is trivial.

For any finitely generated normal domain A there is an inclusion $\mathrm{ML}(A)\subseteq\mathrm{FML}(A)$, while still $\mathrm{FML}(\mathbb{A}^n)=\mathbf{k}$. Hence the FML invariant can be stronger than the classical one in the sense to be able to distinguish more varieties form the affine space than the classical one. In the next proposition we show how to recover the classical ML invariant from the FML invariant.

Proposition 5.2. *For a finitely generated normal domain A we have*

$$\mathrm{ML}(A)=\mathrm{FML}(A)\cap A.$$

Proof. We must show that for any LND ∂ on A,

$$\ker\partial=\operatorname{Frac}(\ker\partial)\cap A.$$

The inclusion "$\subseteq$" is trivial. To prove the converse inclusion, we fix $a\in\operatorname{Frac}(\ker\partial)\cap A$. Letting $b,c\in\ker\partial$ be such that $ac=b$, Lemma 1.3 (ii) shows that $a\in\ker\partial$. □

Let $A=A[Y,\mathfrak{D}]$ for some proper σ-polyhedral divisor $\mathfrak{D}$ on a normal semiprojective variety Y. In this case $K=\operatorname{Frac}A=K_Y(M)$, where $K_Y(M)$ corresponds to the field of fractions of the semigroup algebra $K_Y[M]$. It is a purely transcendental extension of K_Y of degree $\operatorname{rank}M$.

Let ∂ be a homogeneous LND of fiber type on A. By definition, $K_Y\subseteq\operatorname{Frac}(\ker\partial)$ and so, $K_Y\subseteq\mathrm{FML}_{\mathrm{fib}}(A)$. This shows that the pathological examples as in Lemma 4.4 where $\mathrm{ML}_{\mathrm{fib}}(A)=\mathbf{k}$ cannot occur any more for the FML invariant. Let us formulate the following conjecture.

Conjecture 5.3. *Let X be an affine variety. If $\mathrm{FML}(X)=\mathbf{k}$ then X is (uni)rational.*

The following lemma proves Conjecture 5.3 in the particular case where $X\simeq_{\mathrm{bir}}C\times\mathbb{P}^n$, with C a curve.

Lemma 5.4. *Let $X=\operatorname{Spec}A$ be an affine variety such that $X\simeq_{\mathrm{bir}}C\times\mathbb{P}^n$, where C is a curve. Denote by L the field of rational functions on C. If C has positive genus then $L\subseteq\mathrm{FML}(X)$. In particular, if $\mathrm{FML}(X)=\mathbf{k}$ then C is rational.*

Proof. Assume that C has positive genus. We have $K=\operatorname{Frac}A=L(x_1,\ldots,x_n)$ for some $x_1,\ldots,x_n\in K$, and L is not a rational field.

Let ∂ be an LND on A. We claim that $L\subseteq\operatorname{Frac}(\ker\partial)$. Indeed, let $f,g\in L\setminus\mathbf{k}$. Since $\operatorname{tr.deg}_{\mathbf{k}}(L)=1$, there exists a polynomial $P\in\mathbf{k}[x,y]\setminus\mathbf{k}$ such that $P(f,g)=0$. Applying the derivation $\partial:K\to K$ to $P(f,g)$ we obtain

$$\frac{\partial P}{\partial x}(f,g)\cdot\partial(f)+\frac{\partial P}{\partial y}(f,g)\cdot\partial(g)=0.$$

Since f and g are not constant we may suppose that $\frac{\partial P}{\partial x}(f,g)\neq 0$ and $\frac{\partial P}{\partial y}(f,g)\neq 0$. Hence $\partial(f)=0$ if and only if $\partial(g)=0$. This shows that one of the two following possibilities occurs:

$$L\subseteq\operatorname{Frac}(\ker\partial)\quad\text{or}\quad L\cap\operatorname{Frac}(\ker\partial)=\mathbf{k}.$$

Assume first that $L \cap \mathrm{Frac}(\ker\partial) = \mathbf{k}$. Then, by Lemma 1.3 (i) $\mathrm{Frac}(\ker\partial) = \mathbf{k}(x_1, \ldots, x_n)$ and so the field extension $\mathrm{Frac}(\ker\partial) \subseteq K$ is not purely transcendental. This contradits Lemma 1.3 (vii). Thus $L \subseteq \mathrm{Frac}(\ker\partial)$ proving the claim and the lemma. □

Remark 5.5. We can apply Lemma 5.4 to show that the FML invariant carries more information than usual ML invariant. Indeed, let, in the notation of Lemma 4.4, Y be a smooth projective curve of positive genus. Lemma 4.4 shows that $\mathrm{ML}(A[Y,\mathfrak{D}]) = \mathbf{k}$. While by Lemma 5.4, $\mathrm{FML}(A[Y,\mathfrak{D}]) \supseteq K_Y$.

In the following theorem we prove Conjecture 5.3 in dimension at most 3.

Theorem 5.6. *Let X be an affine variety of dimension at most 3. If $\mathrm{FML}(X) = \mathbf{k}$ then X is rational.*

Proof. Since $\mathrm{FML}(X)$ is trivial, the same holds for $\mathrm{ML}(X)$. If $\dim X \leqslant 2$ then X is rational by virtue of Lemma 4.1. Assume that $\dim X = 3$. Theorem 4.2 implies that $X \simeq_{\mathrm{bir}} C \times \mathbb{P}^2$ for some curve C. While by Lemma 5.4, C is a rational curve. □

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