Transformation Groups, Vol. 15, No. 2, 2010, pp. 389–425

AFFINE $\mathbb{T}$-VARIETIES OF COMPLEXITY ONE AND LOCALLY NILPOTENT DERIVATIONS

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Abstract. Let $X = \operatorname{Spec} A$ be a normal affine variety over an algebraically closed field $\mathbf{k}$ of characteristic 0 endowed with an effective action of a torus $\mathbb{T}$ of dimension n. Let also ∂ be a homogeneous locally nilpotent derivation on the normal affine $\mathbb{Z}^n$-graded domain A, so that ∂ generates a $\mathbf{k}_+$-action on X that is normalized by the $\mathbb{T}$-action.

We provide a complete classification of pairs (X, ∂) in two cases: for toric varieties ($n = \dim X$) and in the case where $n = \dim X - 1$. This generalizes previously known results for surfaces due to Flenner and Zaidenberg. As an application we compute the homogeneous Makar-Limanov invariant of such varieties. In particular, we exhibit a family of nonrational varieties with trivial Makar-Limanov invariant.

Introduction

Let $\mathbf{k}$ be an algebraically closed field of characteristic 0. For an algebraic torus $\mathbb{T} \simeq (\mathbf{k}^*)^n$ acting on an algebraic variety X, the complexity of this action is the codimension of the general orbit. Without loss of generality, we restrict ourselves to effective $\mathbb{T}$-actions, so the complexity is $\dim X - \dim \mathbb{T}$. In particular, a $\mathbb{T}$-variety of complexity 0 has an open orbit and is thus a toric variety. It is well-known that a $\mathbb{T}$-action on $X = \operatorname{Spec} A$ gives rise to an M-grading on A, where M is a lattice of rank n.

More generally, let $A = \bigoplus_{m \in M} \widetilde{A}_m$ be a finitely generated effectively M-graded domain and let $K = \operatorname{Frac} A$. For any $m \in M$ we let

$$K_m = \{f/g \in K \mid f \in \widetilde{A}_{m+e}, g \in \widetilde{A}_e\}.$$

Then $\widetilde{A}_m \subseteq K_m$, and $\mathbf{k} \subseteq K_0 \subseteq K$ are field extensions. Letting $\{\mu_1, \ldots, \mu_n\}$ be a basis of M we fix, for every $i = 1, \ldots, n$, an element $\chi^{\mu_i} \in K_{\mu_i}$. For every $m = \sum_i a_i \mu_i$ we have $K_m = \chi^m K_0$, where $\chi^m = \prod_i (\chi^{\mu_i})^{a_i}$. Thus, without loss of generality, we assume in the sequel that

$$A = \bigoplus_{m \in M} A_m \chi^m \subseteq K_0[M], \qquad \text{where} \quad A_m \subseteq K_0,$$

DOI: 10.1007/s00031-010-9089-2
Received July 10, 2009. Accepted March 8, 2010. Published online April 9, 2010.

and $K_0[M]$ denotes the semigroup K_0-algebra of M. In this setting the complexity of the $\mathbb{T}$-action equals the transcendence degree of K_0 over $\mathbf{k}$. In particular, for a toric variety X, $K_0 = \mathbf{k}$, and χ^m is just a character of $\mathbb{T}$ regarded as a rational function on X.

There are well-known combinatorial descriptions of normal $\mathbb{T}$-varieties. For toric varieties see, e.g., [De], Chapter 1 in [KKMS], and [Od]. For complexity 1 case, see Chapters 2 and 4 in [KKMS] and, more generally, [Ti_1], [Ti_2]. Finally, for arbitrary complexity, see [AlHa], [AHS][1].

We let $N = \mathrm{Hom}(M, \mathbb{Z})$, $M_\mathbb{Q} = M \otimes \mathbb{Q}$, and $N_\mathbb{Q} = N \otimes \mathbb{Q}$. Any affine toric variety can be described via the weight cone $\sigma^\vee \subseteq M_\mathbb{Q}$ spanned over $\mathbb{Q}_{\geq 0}$ by all $m \in M$ such that $A_m \neq \{0\}$ or, alternatively, via the dual cone $\sigma \subseteq N_\mathbb{Q}$. Similarly, the description of normal affine $\mathbb{T}$-varieties of complexity 1 due to Altmann and Hausen deals with a polyhedral cone $\sigma \subseteq N_\mathbb{Q}$ (dual to the weight cone $\sigma^\vee \subseteq M_\mathbb{Q}$), a smooth curve C, and a divisor $\mathfrak{D}$ on C whose coefficients are polyhedra in $N_\mathbb{Q}$ having tail cone σ. The degree $\deg \mathfrak{D}$ is defined as the Minkowski sum of the coefficients of $\mathfrak{D}$ (see Subsection 1.1 for precise definitions).

For affine surfaces with a $\mathbb{C}^*$-action an alternative description[2] was proposed in [FlZa_1]. This description was used in [FlZa_2] in order to classify all $\mathbb{C}_+$-actions on normal $\mathbb{C}^*$-surfaces. Generalizing this construction, in the present paper we use the description in [AlHa] to classify normal affine $\mathbb{T}$-varieties of complexity 0 or 1 endowed with a $\mathbf{k}_+$-action.

A $\mathbf{k}_+$-action gives rise to a locally nilpotent derivation (LND) on A. To any LND on A we can associate a homogeneous LND which maps homogeneous elements into homogeneous elements, see Lemma 1.10. A homogeneous LND ∂ on $A = \bigoplus_{m\in M} A_m\chi^m \subseteq K_0[M]$ can be extended to a derivation on $K_0[M]$. We say that ∂ is of *fiber type* if $\partial(K_0) = 0$ and of *horizontal type* otherwise. In geometric terms, the fact that the LND ∂ is homogeneous means that the corresponding $\mathbf{k}_+$-action on $X = \operatorname{Spec} A$ is normalized by the torus $\mathbb{T}$.

In Theorem 2.7 we obtain a classification of homogeneous LNDs on toric varieties. For $\mathbb{T}$-varieties of complexity 1, such a classification is given in Theorems 3.8 (for fiber type) and 3.28 (for horizontal type). These theorems are the main results of the paper. In [Li_1] this classification of homogeneous LNDs of fiber type is generalized to arbitrary complexity.

We show as a corollary that the equivalence classes of homogeneous LNDs on the toric variety defined by the cone $\sigma \subseteq N_\mathbb{Q}$ are in one-to-one correspondence with the extremal rays of σ (see Corollary 2.10). This is also true for normal affine $\mathbb{T}$-varieties of complexity 1 over an affine curve C. Over a projective curve C, these classes are in one-to-one correspondence with the extremal rays of σ disjoint from the polyhedron $\deg \mathfrak{D}$ (see Remark 3.14). The classification of homogeneous LNDs of horizontal type is more involved, see Corollary 3.31.

The Makar-Limanov invariant [ML] is an important tool which allows us, in particular, to distinguish certain varieties from the affine space. For an algebra A, this invariant is defined as the intersection of the kernels of all locally nilpotent

[1]In the case of complexity 1, the descriptions in [AlHa] and [Ti_2] are equivalent and agree with the one in [KKMS, Chaps. 2 and 4], see [Ti_2, Sect. 6] and [Vo].

[2]Which is actually equivalent, see Example 3.5 in [AlHa].

derivations on A. For graded algebras, we introduce a homogeneous version of the Makar-Limanov invariant. For $\mathbb{T}$-varieties of complexity 0 and 1 we give an explicit expression of the latter invariant. The triviality of the homogeneous Makar-Limanov invariant implies that of the usual one.

As an application we exhibit in Subsection 4.2 a family of nonrational singular varieties with a trivial Makar-Limanov invariant. These examples (in a preliminary version of our paper) attracted the attention of V. L. Popov, who proposed in a recent preprint [Po] yet another family of affine varieties with these same properties, this time in addition smooth. It is worth mentioning that generalizing the methods in Subsection 4.2 we obtained a birational characterization of normal affine varieties with trivial Makar-Limanov invariant [Li_1].

The content of the paper is as follows. In Section 1 we recall the combinatorial description of $\mathbb{T}$-varieties due to Altmann and Hausen, and also some generalities on locally nilpotent derivations and $\mathbf{k}_+$-actions. In Sections 2 and 3 we obtain our principal classification results for toric varieties and for $\mathbb{T}$-varieties of complexity 1, respectively. The comparison with previously known results in the surface case is given in Subsection 3.3. Finally, in Section 4 we provide the applications to the Makar-Limanov invariant.

In the entire paper $\mathbf{k}$ is an algebraically closed field of characteristic 0, except in Section 2, where $\mathbf{k}$ is not necessarily algebraically closed.

Acknowledgments. The author is grateful to Mikhail Zaidenberg for posing the problem and for permanent encouragement, and to Dimitri Timashev for useful discussions. We also thank Vladimir Popov for kindly communicating to us his preprint [Po].

1. Preliminaries

1.1. Combinatorial description of $\mathbb{T}$-varieties

Let N be a lattice of rank n and let $M = \mathrm{Hom}(N, \mathbb{Z})$ be its dual lattice. We also let $N_\mathbb{Q} = N \otimes \mathbb{Q}$, $M_\mathbb{Q} = M \otimes \mathbb{Q}$, and we consider the natural duality $M_\mathbb{Q} \times N_\mathbb{Q} \to \mathbb{Q}$, $(m, p) \mapsto \langle m, p \rangle$.

Let $\mathbb{T} = \mathrm{Spec}\, \mathbf{k}[M]$ be the corresponding n-dimensional algebraic torus associated to M. Thus M is the character lattice of $\mathbb{T}$ and N is the lattice of one-parameter subgroups. It is customary to write the character associated to a lattice vector $m \in M$ as χ^m, so that χ^m is the comorphism of the morphism $\mathbf{k}[t] \to \mathbf{k}[M]$, $t \mapsto m$ [Od].

Let $X = \mathrm{Spec}\, A$ be an affine $\mathbb{T}$-variety. It is well-known that the morphism $A \to A \otimes \mathbf{k}[M]$ induces an M-grading on A and, conversely, every M-grading on A arises in this way. Furthermore, a $\mathbb{T}$-action is effective if and only if the corresponding M-grading is effective[3].

Let $A = \bigoplus_{m \in M} A_m \chi^m$ be a finitely generated effectively M-graded domain. The *weight cone* of A is the cone $\sigma^\vee \subseteq M_\mathbb{Q}$ spanned by all the lattice vectors $m \in M$ such that $A_m \neq \{0\}$. In the sequel, for a cone $\sigma^\vee \subseteq M_\mathbb{Q}$, we let $\sigma^\vee_M = \sigma^\vee \cap M$

[3]We say that an M-graded algebra A is effectively graded by M if the set $\{m \in M \mid A_m \neq 0\}$ is not contained in a proper sublattice of M.

denote the set of lattice points in $\sigma^\vee$, so that

$$A = \bigoplus_{m\in\sigma_M^\vee} A_m\chi^m.$$

Since A is finitely generated, the cone $\sigma^\vee$ is polyhedral and since the grading is effective, $\sigma^\vee$ is of full dimension or, equivalently, σ is pointed[4].

An affine $\mathbb{T}$-variety of complexity 0 is a toric variety. There is a well-known way of describing affine toric varieties in terms of pointed polyhedral cones in $N_\mathbb{Q}$. To any such cone $\sigma \subseteq N_\mathbb{Q}$ we associate an affine semigroup algebra $\mathbf{k}[\sigma_M^\vee] := \bigoplus_{m\in\sigma_M^\vee} \mathbf{k}\chi^m$ and an affine toric variety $X = \operatorname{Spec}\mathbf{k}[\sigma_M^\vee]$. Conversely, for an affine toric variety, the corresponding cone σ is the dual of the weight cone $\sigma^\vee$. We note that in this particular case, $\sigma^\vee \subseteq M_\mathbb{Q}$ is the cone spanned by all lattice vectors $m \in M$ such that the character $\chi^m : \mathbb{T} \to \mathbf{k}^*$ extends to a regular function on X.

In [AlHa] a combinatorial description of affine $\mathbb{T}$-varieties of arbitrary complexity is given. In what follows we recall the main features of this description specialized to the case of complexity 1 torus actions. In [Ti$_1$] a combinatorial description of complexity 1 actions of reductive groups is given and in [Ti$_2$] it is specialized for torus actions. For torus actions of complexity 1, the descriptions in [AlHa] and [Ti$_1$] are equivalent and agree with the one given earlier (in a slightly more restrictive setting) by Mumford [KKMS, Chaps. 2 and 4], cf. [Ti$_2$] and [Vo].

Definition 1.1.

(i) Let σ be a pointed cone in $N_\mathbb{Q}$. We define $\operatorname{Pol}_\sigma(N_\mathbb{Q})$ to be the set of all σ-tailed polyhedra, i.e., polyhedral domains in $N_\mathbb{Q}$ which can be decomposed as the Minkowski sum of a compact polyhedron and σ. The set $\operatorname{Pol}_\sigma(N_\mathbb{Q})$ equipped with the Minkowski sum forms an abelian semigroup with neutral element σ.

(ii) We also let $\operatorname{CPL}_\mathbb{Q}(\sigma^\vee)$ denote the set of all piecewise linear $\mathbb{Q}$-valued functions $h : \sigma^\vee \to \mathbb{Q}$ which are upper convex and positively homogeneous, i.e.,

$$h(m+m') \geq h(m) + h(m') \quad \text{and} \quad h(\lambda m) = \lambda h(m), \quad \forall m, m' \in \sigma^\vee,\ \forall \lambda \in \mathbb{Q}_{\geq 0}.$$

The set $\operatorname{CPL}_\mathbb{Q}(\sigma^\vee)$ with the usual addition forms an abelian semigroup with neutral element 0.

For a polyhedron $\Delta \in \operatorname{Pol}_\sigma(N_\mathbb{Q})$ we define its support function

$$h_\Delta : \sigma^\vee \to \mathbb{Q}, \qquad m \mapsto \min\langle m, \Delta\rangle.$$

Clearly, $h_\Delta \in \operatorname{CPL}_\mathbb{Q}(\sigma^\vee)$. The map $\operatorname{Pol}_\sigma(N_\mathbb{Q}) \to \operatorname{CPL}_\mathbb{Q}(\sigma^\vee)$ given by $\Delta \mapsto h_\Delta$ is an isomorphism of abelian semigroups.

For the following definition we refer to [AlHa].

[4]A cone in a vector space is called pointed if it contains no subspaces of positive dimension.

Definition 1.2. Let C be a smooth curve. A *σ-polyhedral divisor* on C is a formal sum $\mathfrak{D} = \sum_{z\in C} \Delta_z \cdot z$, where $\Delta_z \in \mathrm{Pol}_\sigma(N_\mathbb{Q})$ and $\Delta_z = \sigma$ for all but finitely many values of z. For $m \in \sigma^\vee$ we can evaluate $\mathfrak{D}$ in m by letting $\mathfrak{D}(m)$ be the $\mathbb{Q}$-divisor on C,

$$\mathfrak{D}(m) = \sum_{z\in C} h_{\Delta_z}(m) \cdot z.$$

A σ-polyhedral divisor is called *proper* if either C is affine, or C is projective and the following two conditions hold:

(1) The polyhedron $\deg \mathfrak{D} := \sum_{z\in C} \Delta_z$ is a proper subset of the cone σ.
(2) If $h_{\deg \mathfrak{D}}(m) = 0$, then m is contained in the boundary of $\sigma^\vee$ and a multiple of $\mathfrak{D}(m)$ is principal.

These two assumptions are counterparts of the conditions that $\mathfrak{D}(m)$ is semi-ample for all $m \in \sigma^\vee_M$ and big for all m contained in the relative interior of $\sigma^\vee$, cf. [AlHa, Def. 2.7]. They are automatically fulfilled if C is affine.

Definition 1.3. A fan which defines a toric variety consists of pointed cones. We need to consider more generally objects which we call *quasifans*. These satisfy the usual definition of a fan except that their cones are not necessarily pointed.

As usual, for a function $h \in \mathrm{CPL}_\mathbb{Q}(\sigma^\vee)$, we define its *normal quasifan* $\Lambda(h)$ as the coarsest refinement of the quasifan of $\sigma^\vee$ such that h is linear in each cone $\delta \in \Lambda(h)$. For a σ-polyhedral divisor $\mathfrak{D}$ on C, we define its normal quasifan $\Lambda(\mathfrak{D})$ as the coarsest common refinement of all $\Lambda(h_{\Delta_z})\, \forall z \in C$. We have $\Lambda(\mathfrak{D}) = \Lambda(h_{\deg \mathfrak{D}})$.

The following theorem gives a combinatorial description of $\mathbb{T}$-varieties of complexity 1 analogous to the classical combinatorial description of toric varieties. This is a specialization of results in [AlHa] to torus actions of complexity 1. Alternatively, a direct proof is given in [Ti$_2$] for (1) and (2), while (3) is straightforward from loc. cit. See also Theorem 4.3 in [FlZa$_1$] for the particular case of $\mathbb{C}^*$-surfaces.

Theorem 1.4.

(1) *To any proper σ-polyhedral divisor $\mathfrak{D}$ on a smooth curve C one can associate a normal finitely generated effectively M-graded domain of dimension $n+1$, where $n = \mathrm{rank}(M)$, given by*[5]

$$A[C,\mathfrak{D}] = \bigoplus_{m\in\sigma^\vee_M} A_m\chi^m, \qquad \textit{where} \quad A_m = H^0(C, \mathcal{O}_C(\lfloor \mathfrak{D}(m) \rfloor)).$$

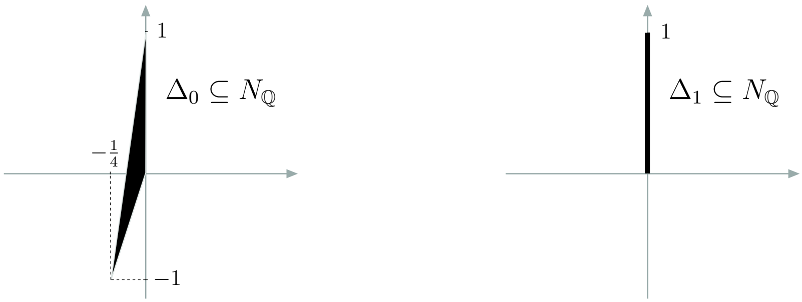
(2) *Conversely, any normal finitely generated effectively M-graded domain of dimension $n+1$ is isomorphic to $A[C,\mathfrak{D}]$ for some smooth curve C and some proper σ-polyhedral divisor $\mathfrak{D}$ on C.*
(3) *Moreover, the M-graded domains $A[C,\mathfrak{D}]$ and $A[C',\mathfrak{D}']$ are isomorphic if and only if $C \simeq C'$ and, under this identification, $\mathfrak{D}(m) - \mathfrak{D}'(m)$ is principal for all $m \in \sigma^\vee_M$ and depends linearly on m.*

[5]For a $\mathbb{Q}$-divisor D, $\lfloor D \rfloor$ stands for the integral part and $\{D\}$ for the fractional part of D.

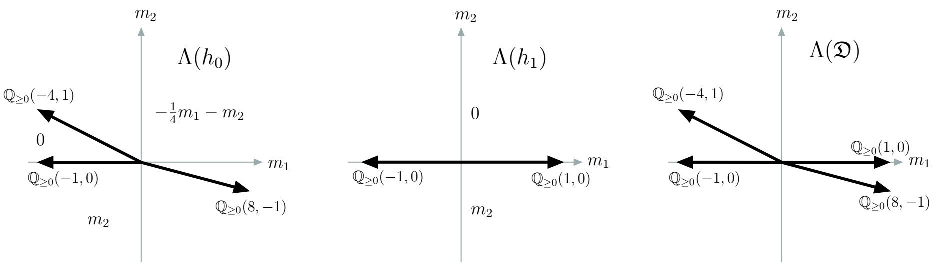
In [KaFi] (see also [FlZa$_1$]), all $\mathbb{C}^*$-surfaces are divided into three types: elliptic, parabolic, and hyperbolic. In the general case, we will use the following terminology.

An M-graded domain $A = A[C, \mathfrak{D}]$ (or, equivalently, a $\mathbb{T}$-variety X) will be called *elliptic* if C is projective. A nonelliptic $\mathbb{T}$-variety will be called *parabolic* if σ is of full dimension and *hyperbolic* if $\sigma = \{0\}$. If $\dim X \geq 3$, this does not exhaust all the possibilities.

Example 1.5. Letting $N = \mathbb{Z}^2$ and $\sigma = \{(0,0)\}$, in $N_\mathbb{Q} = \mathbb{Q}^2$ we consider the triangle Δ_0 with vertices $(0,0)$, $(0,1)$, and $(-\frac{1}{4}, -1)$ and the segment $\Delta_1 = \{0\} \times [0,1]$.



Let $C = \operatorname{Spec} \mathbf{k}[t]$ and let $\mathfrak{D} = \Delta_0 \cdot [0] + \Delta_1 \cdot [1]$. In the following picture, for the normal quasifans $\Lambda(h_{\Delta_0})$, $\Lambda(h_{\Delta_1})$ and $\Lambda(\mathfrak{D})$ in $M_\mathbb{Q} = \mathbb{Q}^2$, for $i = 0,1$ we show the values of $h_i = h_{\Delta_i}$ on each maximal cone.



We let $A = A[C, \mathfrak{D}]$ as in Theorem 1.4 and we let $X = \operatorname{Spec} A$. The torus $\mathbb{T} = (\mathbf{k}^*)^2$ acts on X. Since C is affine and $\sigma = \{(0,0)\}$, X is hyperbolic as a $\mathbb{T}$-variety. By Theorem 1.4 we have

$$A_{(4,0)} = t\mathbf{k}[t], \quad A_{(-1,0)} = \mathbf{k}[t], \quad A_{(-4,1)} = \mathbf{k}[t], \quad \text{and} \quad A_{(8,-1)} = t(t-1)\mathbf{k}[t].$$

An easy calculation shows that the elements

$$u_1 = -t\chi^{(4,0)}, \quad u_2 = \chi^{(-1,0)}, \quad u_3 = -\chi^{(-4,1)}, \quad \text{and} \quad u_4 = t(t-1)\chi^{(8,-1)},$$

generate A as an algebra. Furthermore, they satisfy the irreducible relation $u_1 + u_1^2 u_2^4 + u_3 u_4 = 0$, and so

$$A \simeq \mathbf{k}[x_1, x_2, x_3, x_4]/(x_1 + x_1^2 x_2^4 + x_3 x_4). \tag{1}$$

The $\mathbb{Z}^2$-grading on A is given by $\deg x_1 = (4,0)$, $\deg x_2 = (-1,0)$, $\deg x_3 = (-4,1)$, and $\deg x_4 = (8,-1)$. The curve C and the proper polyhedral divisor $\mathfrak{D}$ can be recovered from this description of A following the recipe in [AlHa, Sect. 11].

We let K_0 denote the function field of C. There is a natural embedding of M-graded algebras $A \hookrightarrow K_0[M]$. If C is affine, then A_m is a locally free A_0-module of rank 1 for every $m \in \sigma_M^\vee$.

Following [FlZa$_1$, Prop. 4.12], in the next lemma we show the way in which our combinatorial description is affected when passing to a certain cyclic covering.

Lemma 1.6. *Let $A = A[C,\mathfrak{D}]$, where C is a smooth curve with function field K_0 and $\mathfrak{D}$ is a proper σ-polyhedral divisor on C. Consider the normalization A' of the cyclic ring extension $A[s\chi^e]$, where $e \in M$, $s^d = f \in A_{de} \subseteq K_0$, and $d > 0$. Then $A' = A[C',\mathfrak{D}']$, where C' and $\mathfrak{D}'$ are defined as follows:*

(i) *If A is elliptic, then A' is also elliptic and C' is the smooth projective curve with function field $K_0[s]$.*

(ii) *If A is nonelliptic, then A' is also nonelliptic and $C = \operatorname{Spec} A_0'$, where A_0' is the normalization of A_0 in $K_0[s]$.*

(iii) *In both cases $\mathfrak{D}' = \sum_{z\in C} \Delta_z \cdot p^*(z)$, where $p : C' \to C$ is the projection.*

Proof. The normalization A' admits a natural M-grading. The latter is defined by the M-grading on A and by letting $\deg s\chi^e = e$. Let $K = \operatorname{Frac} A$. Since $(s\chi^e)^d - f\chi^{de} = 0$, A' is the normalization of A in the function field $K' := K[s\chi^e]$. But $\chi^{-e} \in K$, so $K' = K[s]$. Moreover, $K[s] = K_0[s] \otimes \operatorname{Frac} \mathbf{k}[M]$, so the function field of C' is $K_0[s]$, and A_0' is the normalization of A_0 in the field $K_0[s]$. This proves (i) and (ii).

For every $m \in M$ we have $\mathfrak{D}'(m) = \sum_{z\in C} h_z(m) p^*(z) = p^*(\mathfrak{D}(m))$. Therefore, for every $f \in K_0$, there are equivalences:

$$\operatorname{div}_C(f) + \mathfrak{D}(m) \ge 0 \;\Leftrightarrow\; \operatorname{div}_{C'}(p^*f) + p^*(\mathfrak{D}(m)) \ge 0 \;\Leftrightarrow\; \operatorname{div}_{C'}(f) + \mathfrak{D}'(m) \ge 0.$$

Let $m \in \sigma_M^\vee$ and let $r > 0$ be such that $\mathfrak{D}(rd\cdot m)$ is integral. Then

$$\begin{aligned} g \in A_m' \;&\Leftrightarrow\; g^{rd} \in A_{rdm} \;\Leftrightarrow\; \operatorname{div}_C(g^{rd}) + \mathfrak{D}(rd\cdot m) \ge 0 \\ &\Leftrightarrow\; \operatorname{div}_{C'}(g^{rd}) + \mathfrak{D}'(rd\cdot m) \ge 0 \;\Leftrightarrow\; \operatorname{div}_{C'}(g) + \mathfrak{D}'(m) \ge 0, \end{aligned}$$

which proves (iii). ☐

1.2. Locally nilpotent derivations and $\mathbf{k}_+$-actions

Let A be a commutative ring. A derivation on A is called *locally nilpotent* (LND for short) if for every $a \in A$ there exists $n \in \mathbb{Z}_{\ge 0}$ such that $\partial^n(a) = 0$.

Let $X = \operatorname{Spec} A$ be an affine variety. Given an LND ∂ on A, the map $\phi_\partial : \mathbf{k}_+ \times A \to A$, $\phi_\partial(t,f) = e^{t\partial} f$ defines a $\mathbf{k}_+$-action on X, and any $\mathbf{k}_+$-action arises in this way. In the following lemma we collect some well-known facts about LNDs over a field of characteristic 0, not necessarily algebraically closed, needed for later purposes, see, e.g., [Fr$_2$], [ML].

Lemma 1.7. *Let A be a finitely generated normal domain over a field of characteristic 0. If ∂ and ∂' are two LNDs on A, then the following hold*

(i) $\ker\partial$ *is a normal subdomain of codimension* 1.
(ii) $\ker\partial$ *is factorially closed, i.e.,* $ab\in\ker\partial\Rightarrow a,b\in\ker\partial$.
(iii) *If* $a\in A$ *is invertible, then* $a\in\ker\partial$.
(iv) *If* $\ker\partial=\ker\partial'$*, then there exist* $f,f'\in\ker\partial$ *such that* $f'\partial=f\partial'$.
(v) *For* $a\in A$, $\partial a\in(a)\Rightarrow\partial a=0$.
(vi) *If* $a\in\ker\partial$*, then* $a\partial$ *is again an LND.*

Definition 1.8. We say that two LNDs ∂ and ∂' on A are *equivalent* if $\ker\partial=\ker\partial'$. Geometrically, this means that the generic orbits of the associated $\mathbf{k}_+$-actions coincide, cf. also Lemma 1.7(iv).

With dual lattices M and N as in Subsection 1.1, for a field extension $\mathbf{k}\subseteq K_0$ we consider a finitely generated effectively M-graded domain

$$A=\bigoplus_{m\in\sigma_M^\vee}A_m\chi^m,\qquad\text{where}\qquad A_m\subseteq K_0,$$

(we keep our convention from the Introduction regarding M-graded algebras).

A derivation ∂ on A is called *homogeneous* if it sends homogeneous elements into homogeneous elements. Hence, ∂ sends homogeneous pieces of A into homogeneous pieces.

Let $M_\partial=\{m\in\sigma_M^\vee\mid\partial(A_m\chi^m)\neq0\}$. The action of ∂ on homogeneous pieces of A defines a map $\partial_M:M_\partial\to\sigma_M^\vee$, i.e., $\partial(A_m)\subseteq A_{\partial_M(m)}$. By the Leibniz rule, for homogeneous elements $f\in A_m\setminus\ker\partial$ and $g\in A_{m'}\setminus\ker\partial$, we have

$$\partial(fg)=f\partial(g)+g\partial(f)\in A_{\partial_M(m+m')},\qquad\partial_M(m+m')=m+\partial_M(m')=m'+\partial_M(m).$$

Thus $\partial_M(m)-m\in M$ is a constant function on M_∂. This leads to the following definition.

Definition 1.9. Let ∂ be a nonzero homogeneous derivation on A. The *degree* of ∂ is the lattice vector $\deg\partial$ defined by $\deg\partial=\deg\partial(f)-\deg(f)$ for any homogeneous element $f\notin\ker\partial$. With this notation the map $\partial_M:M_\partial\to\sigma_M^\vee$ is just the translation by the vector $\deg\partial$.

We also say that an LND ∂ on A is *negative* if $\deg\partial\notin\sigma_M^\vee$, *nonnegative* if $\deg\partial\in\sigma_M^\vee$, and *positive* if ∂ is nonnegative and $\deg\partial\neq0$.

It is well-known that any LND on A decomposes into a sum of homogeneous derivations, some of which are locally nilpotent. Short of a good reference, in the next lemma we provide a short argument.

Lemma 1.10. *Let A be a finitely generated normal M-graded domain. For any derivation ∂ on A there is a decomposition $\partial=\sum_{e\in M}\partial_e$, where ∂_e is a homogeneous derivation of degree e. Moreover, let $\Delta(\partial)$ be the convex hull in $M_\mathbb{Q}$ of the set $\{e\in M\mid\partial_e\neq0\}$. Then $\Delta(\partial)$ is a bounded polyhedron and for every vertex e of $\Delta(\partial)$, ∂_e is locally nilpotent if ∂ is.*

Proof. Letting $a_1, \ldots, a_r$ be a set of homogeneous generators of A we have $A \simeq \mathbf{k}^{[r]}/I$, where $\mathbf{k}^{[r]} = \mathbf{k}[x_1, \ldots, x_r]$ and I denotes the ideal of relations of $a_1, \ldots, a_r$. The M-grading and the derivation ∂ can be lifted to an M-grading and a derivation ∂' on $\mathbf{k}^{[r]}$, respectively.

The proof of Proposition 3.4 in [Fr$_2$] can be applied to an M-grading, proving that $\partial' = \sum_{e\in M} \partial'_e$, where ∂'_e is a homogeneous derivation on $\mathbf{k}^{[r]}$. Furthermore, since $\partial'(I) \subseteq I$ and I is homogeneous, we have $\partial'_e(I) \subseteq I$. Hence ∂'_e induces a homogeneous derivation ∂_e on A of degree e, proving the first assertion.

The algebra A being finitely generated, the set $\{e \in M \mid \partial_e \neq 0\}$ is finite and so $\Delta(\partial)$ is a bounded polyhedron. Let e be a vertex of $\Delta(\partial)$ and let $n \geq 1$. If $ne = \sum_{i=1}^{n} m_i$ with $m_i \in \Delta(\partial) \cap M$, then $m_i = e\ \forall i$. For $a \in A_m\chi^m$ this yields $\partial_e^n(a) = (\partial^n(a))_{m+ne}$, where $(\partial^n(a))_{m+ne}$ stands for the summand of degree $m+ne$ in the homogeneous decomposition of $\partial^n(a)$. Hence ∂_e is locally nilpotent if ∂ is so. □

In the following lemma we extend Lemma 1.8 in [FlZa$_2$] to more general gradings. This lemma shows that any LND ∂ on a normal domain can be extended as an LND to a cyclic ring extension defined by an element of $\ker\partial$. Actually (i) is contained in loc. cit. while the proof of (ii) is similar and so we omit it.

Lemma 1.11.

(i) *Let A be a finitely generated normal domain and let ∂ be an LND on A. Given a nonzero element $v \in \ker\partial$ and $d > 0$, we let A' denote the normalization of the cyclic ring extension $A[u] \supseteq A$ in its fraction field, where $u^d = v$. Then ∂ extends in a unique way to an LND ∂' on A'.*

(ii) *Moreover, if A is M-graded and ∂ and v are homogeneous, with* $\deg v = dm$ *for some $m \in M$, then A' is M-graded as well, and u and ∂' are homogeneous with* $\deg u = m$ *and* $\deg\partial' = \deg\partial$.

1.12. Recall that $A = \bigoplus_{m\in\sigma_M^\vee} A_m\chi^m$, where $A_m \subseteq K_0$, K_0 is a field containing $\mathbf{k}$, and $\operatorname{Frac} A = K_0(M)$[6]. The following lemma provides some useful extension of a homogeneous LND ∂ on A.

Lemma 1.13. *For any homogeneous LND ∂ on A, the following hold:*

(i) *The derivation ∂ extends in a unique way to a homogeneous $\mathbf{k}$-derivation on $K_0[M]$.*

(ii) *If $\partial(K_0) = 0$, then the extension of ∂ as in* (i) *restricts to a homogeneous locally nilpotent K_0-derivation on $K_0[\sigma_M^\vee]$.*

Proof. The first assertion is evident. Let $\operatorname{Nil}(\partial)$ be the subalgebra of $K_0[M]$ where ∂ acts in a nilpotent way. To show (ii), suppose that $\partial(K_0) = 0$. Assuming that $f\chi^m \in K_0[\sigma_M^\vee]$, we consider $r > 0$ such that $A_{rm} \neq 0$. Letting $g \in A_{rm}$, we have $f^r\chi^{rm} = f'g\chi^{rm}$ for some $f' \in K_0$. Thus $f^r\chi^{rm} \in \operatorname{Nil}(\partial)$ and so $f\chi^m \in \operatorname{Nil}(\partial)$. □

In the setting as in the previous lemma, the extension of ∂ to $K_0[M]$ will still be denoted by ∂. Following [FlZa$_2$] we use the next definition.

[6]For a field K_0 and a lattice M, $K_0(M)$ denotes the function field of $K_0[M]$.

Definition 1.14. With A as in paragraph 1.12, a homogeneous LND ∂ on A is said to be *of fiber type* if $\partial(K_0) = 0$ and *of horizontal type* if $\partial(K_0) \neq 0$.

Let A be a finitely generated domain and let $X = \operatorname{Spec} A$. In this setting the fact that ∂ is homogeneous means that the corresponding $\mathbf{k}_+$-action on X is normalized by the $\mathbb{T}$-action given by the M-grading. Furthermore, ∂ is of fiber type if and only if the general orbits of the corresponding $\mathbf{k}_+$-action are contained in the closures of general orbits of the $\mathbb{T}$-action. Otherwise, ∂ is of horizontal type.

2. Locally nilpotent derivations on toric varieties

In this section we consider more generally toric varieties defined over a field $\mathbf{k}$ of characteristic 0, not necessarily algebraically closed. This will be important in Section 3 below.

Let M and N be lattices as in Subsection 1.1. We also let $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$, $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$, and we consider the natural duality $M_{\mathbb{Q}} \times N_{\mathbb{Q}} \to \mathbb{Q}$, $(m, p) \mapsto \langle m, p\rangle$.

Notation 2.1. Let $\rho \in N$ and $e \in M$ be lattice vectors. We define $\partial_{\rho,e}$ as the homogeneous derivation of degree e on $\mathbf{k}[M]$ given by $\partial_{\rho,e}(\chi^m) = \langle m, \rho\rangle \cdot \chi^{m+e}$.

An easy computation shows that $\partial_{\rho,e}$ is indeed a derivation. Let H_ρ be the subspace of $M_{\mathbb{Q}}$ orthogonal to ρ, and let H_ρ^+ be the half-space of $M_{\mathbb{Q}}$ given by $\langle \cdot, \rho\rangle \geq 0$. The kernel $\ker \partial_{\rho,e}$ is spanned by all characters χ^m with $m \in M$ orthogonal to ρ, i.e., $\ker \partial_{\rho,e} = \mathbf{k}[H_\rho \cap M]$.

Let $\operatorname{Nil}(\partial_{\rho,e})$ be the subalgebra of $\mathbf{k}[M]$ where $\partial_{\rho,e}$ acts in a nilpotent way. Assume that $\langle e, \rho\rangle = -1$. For every $m \in H_\rho^+ \cap M$, the character $\chi^m \in \operatorname{Nil}(\partial_{\rho,e})$ since $\partial_{\rho,e}^\ell(\chi^m) = 0$, where $\ell = \langle m, \rho\rangle + 1$. Thus, the derivation $\partial_{\rho,e}$ restricted to the subalgebra $\mathbf{k}[H_\rho^+ \cap M]$ is a homogeneous LND. On the other hand, $\partial_{\rho,e}$ is not locally nilpotent in $\mathbf{k}[M]$, in fact, for every $m \notin H_\rho^+ \cap M$, the character $\chi^m \notin \operatorname{Nil}(\partial_{\rho,e})$ is not nilpotent.

Remark 2.2. If $\partial_{\rho,e}$ stabilizes a subalgebra $A \subseteq \mathbf{k}[H_\rho^+ \cap M]$, then $\partial_{\rho,e}|_A$ is also a homogeneous LND on A of degree e and $\ker(\partial_{\rho,e}|_A) = A \cap \mathbf{k}[H_\rho \cap M]$.

For the rest of this section we let σ be a pointed polyhedral cone in the vector space $N_{\mathbb{Q}}$, and we let

$$A = \mathbf{k}[\sigma_M^\vee] = \bigoplus_{m \in \sigma_M^\vee} \mathbf{k}\chi^m$$

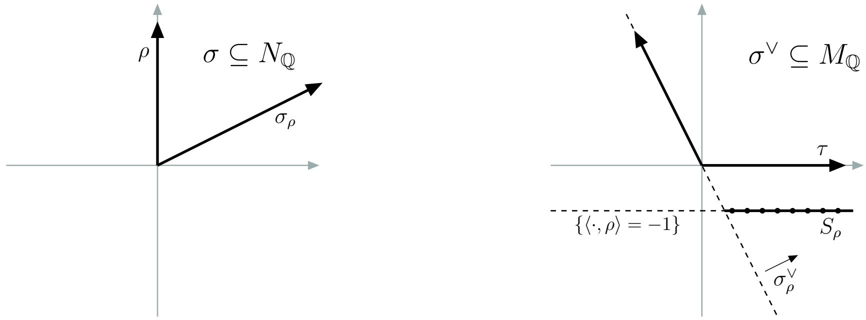
be the affine semigroup algebra of σ with the corresponding affine toric variety $X = \operatorname{Spec} A$. Since the cone σ is pointed, $\sigma^\vee$ is of full dimension and the subalgebra $A \subseteq \mathbf{k}[M]$ is effectively graded by M.

To every extremal ray $\rho \subseteq \sigma$ we can associate a codimension 1 face $\tau \subseteq \sigma^\vee$ given by $\tau = \sigma^\vee \cap \rho^\perp$. As usual, we denote an extremal ray and its primitive vector by the same letter ρ. Thus $\sigma^\vee \subseteq H_\rho^+$ and $\tau \subseteq H_\rho$.

Definition 2.3. Let σ_ρ be the cone spanned by all the extremal rays of σ except ρ, so that $\sigma^\vee = \sigma_\rho^\vee \cap H_\rho^+$. We also let

$$S_\rho = \sigma_\rho^\vee \cap \{e \in M \mid \langle e, \rho\rangle = -1\}.$$

This definition can be illustrated in the following picture, where $\rho \subseteq N_{\mathbb{Q}}$ is pointing upwards.



Lemma 2.4. *Let $e \in M$. Then $e \in S_\rho$ if and only if:*

(i) $e \notin \sigma_M^\vee$*; and*
(ii) $m + e \in \sigma_M^\vee$, $\forall m \in \sigma_M^\vee \setminus \tau_M$.

Proof. Assume first that $e \in S_\rho$. Then (i) is evident. To show (ii) we let $m \in \sigma_M^\vee \setminus \tau_M$. Then $m + e \in H_\rho^+$ because $\langle m+e, \rho\rangle = \langle m, \rho\rangle - 1$. Also $m \in \sigma^\vee \subseteq \sigma_\rho^\vee$ yielding $m + e \in \sigma_\rho^\vee$. Thus $m + e \in \sigma^\vee = \sigma_\rho^\vee \cap H_\rho^+$.

To show the converse, we let $e \in M$ be such that (i) and (ii) hold. Letting ρ_i, $i = 1, \ldots, \ell$, be all the extremal rays of σ except ρ, for $m \in \sigma_M^\vee \setminus \tau_M$ we have

$$\langle m + e, \rho_i\rangle = \langle m, \rho_i\rangle + \langle e, \rho_i\rangle \geq 0, \qquad \forall i \in \{1, \cdots, \ell\}.$$

If $m \in \rho_i^\perp \cap \sigma_M^\vee$, then $\langle m, \rho_i\rangle = 0$ and so $\langle e, \rho_i\rangle \geq 0$ $\forall i$. Thus $e \in \sigma_\rho^\vee$. Since $e \in \sigma_\rho^\vee \setminus \sigma^\vee$, $\langle e, \rho\rangle$ is negative. We have $\langle e, \rho\rangle = -1$, otherwise $m + e \notin \sigma^\vee$ for any $m \in \sigma_M^\vee$ such that $\langle m, \rho\rangle = 1$. This yields $e \in S_\rho$. ☐

Remark 2.5. Since $\rho \notin \sigma_\rho$ we have $S_\rho \neq \varnothing$. Furthermore, by the previous lemma, $e + m \in S_\rho$ whenever $e \in S_\rho$ and $m \in \tau_M$.

In the following lemma we provide a translation of Lemma 2.4 from the language of convex geometry to that of affine semigroup algebras.

Lemma 2.6. *For every pair (ρ, e), where ρ is an extremal ray in σ and e is a lattice vector in S_ρ, the homogeneous derivation $\partial_{\rho,e}$ restricts to an LND on $A = \mathbf{k}[\sigma_M^\vee]$ with kernel $\ker \partial_{\rho,e} = \mathbf{k}[\tau_M]$ and $\deg \partial_{\rho,e} = e$.*

Proof. If $\sigma = \{0\}$, then σ has no extremal rays, so the statement is trivial. We assume in the sequel that σ has at least one extremal ray ρ. By Lemma 2.4, $\partial_{\rho,e}$ stabilizes A. Hence, by Remark 2.2, $\partial_{\rho,e}$ is a homogeneous LND on A with kernel $\mathbf{k}[\tau_M]$ and of degree e. ☐

The following theorem completes our classification, cf. [De, Prop. 11] and [Od, Sect. 3.4].

Theorem 2.7. *If $\partial \neq 0$ is a homogeneous LND on A, then $\partial = \lambda \partial_{\rho,e}$ for some extremal ray ρ on σ, some lattice vector $e \in S_\rho$, and some $\lambda \in \mathbf{k}^*$.*

Proof. The kernel $\ker\partial$ is a subsemigroup subalgebra of A of codimension 1. Since $\ker\partial$ is factorially closed (see Lemma 1.7), it follows that $\ker\partial=\mathbf{k}[\sigma_M^\vee\cap H]$ for a certain codimension 1 subspace H of $M_\mathbb{Q}$.

If $\sigma^\vee\cap H$ is not a codimension 1 face of $\sigma^\vee$, then H divides the cone $\sigma^\vee$ into two pieces. Since the action of ∂ on characters is a translation by a constant vector $\deg\partial$, only the characters from one of these pieces can reach H in a finite number of iterations of ∂, which contradicts the fact that ∂ is locally nilpotent.

In the case where $\sigma^\vee\cap H=\tau$ is a codimension 1 face of $\sigma^\vee$, we let ρ be the extremal ray dual to τ. Since ∂ is a homogeneous LND, the translation by $e=\deg\partial$ maps $(\sigma_M^\vee\setminus\tau_M)$ into $\sigma_M^\vee$. So, by Lemma 2.4, $e\in S_\rho$ and $\partial=\lambda\partial_{\rho,e}$, as required. □

From our classification we obtain the following corollaries.

Corollary 2.8. *A homogeneous LND ∂ on a toric variety is uniquely determined, up to a constant factor, by its degree.*

Proof. By Theorem 2.7 we have $\partial=\lambda\partial_{\rho,e}$ where $e=\deg\partial$. We claim that the ρ is uniquely determined by e. Indeed, the sets S_ρ and $S_{\rho'}$ are disjoint for $\rho\neq\rho'$. □

Corollary 2.9. *Every homogeneous LND ∂ on a toric variety X is of fiber type and negative*[7].

Proof. The first claim is evident because $\mathbb{T}$ acts with an open orbit. By Theorem 2.7, any LND on a toric variety is of the form $\lambda\partial_{\rho,e}$. Its degree is $\deg\partial_{\rho,e}=e\in S_\rho$ and $S_\rho\cap\sigma^\vee=\varnothing$, so ∂ is negative. □

Corollary 2.10. *Two homogeneous LNDs $\partial=\lambda\partial_{\rho,e}$ and $\partial'=\lambda'\partial_{\rho',e'}$ on A are equivalent if and only if $\rho=\rho'$. In particular, there is only a finite number of pairwise nonequivalent homogeneous LNDs on A.*

Proof. The first assertion follows from the description of $\ker\partial_{\rho,e}$ in Lemma 2.6 and the second one from the fact that σ, being polyhedral, has only a finite number of extremal rays. □

The following corollary shows that the kernel of a homogeneous LND on a semigroup algebra is finitely generated. Since toric varieties are rational, this is also a consequence of Theorem 1.2 in [Ku].

Corollary 2.11. *Let $X=\operatorname{Spec}A$ be a toric variety. If $\partial:A\to A$ is a homogeneous LND, then* $\ker\partial$ *is finitely generated as a* $\mathbf{k}$*-algebra.*

Proof. The corollary follows directly from the description of $\ker\partial$ in Lemma 2.6. □

Example 2.12. With $N=\mathbb{Z}^3$ we let σ be the cone in $N_\mathbb{Q}$ having extremal rays $\rho_1=(1,0,0)$, $\rho_2=(0,1,0)$, $\rho_3=(1,0,1)$, and $\rho_4=(0,1,1)$. The dual cone $\sigma^\vee\subseteq M_\mathbb{Q}=\mathbb{Q}^3$ is spanned by the lattice vectors $u_1=(1,0,0)$, $u_2=(0,1,0)$, $u_3=(0,0,1)$, and $u_4=(1,1,-1)$. Furthermore, these elements satisfy the relation

[7] See Definitions 1.9 and 1.14.

$u_1+u_2=u_3+u_4$ and the algebra $A=\mathbf{k}[\sigma_M^\vee]$ is generated by $x_i=\chi^{u_i}$, $i=1,\dots,4$. Thus,

$$A\simeq \mathbf{k}[x_1,x_2,x_3,x_4]/(x_1x_2-x_3x_4). \tag{2}$$

Corollary 2.10 shows that there are four nonequivalent homogeneous LNDs on A corresponding to the extremal rays $\rho_i\subseteq\sigma$. By a routine calculation we obtain

$$S_{\rho_1}=\{(-1,b,c)\in M\mid b\geq 0, c\geq 1\},\qquad S_{\rho_2}=\{(a,-1,c)\in M\mid a\geq 0, c\geq 1\},$$
$$S_{\rho_3}=\{(a,b,c)\in M\mid a\geq 0, b+c\geq 0, a+c=-1\},\ \text{and}$$
$$S_{\rho_4}=\{(a,b,c)\in M\mid b\geq 0, a+c\geq 0, b+c=-1\}.$$

Letting $e_1=(-1,0,1)$, $e_2=(0,-1,1)$, $e_2=(0,1,-1)$, $e_4=(1,0,-1)$, $\partial_i=\partial_{\rho_i,e_i}$, and $m=(m_1,m_2,m_3)$, we have

$$\partial_1(\chi^m)=m_1\cdot\chi^{m+e_1},\qquad \partial_2(\chi^m)=m_2\cdot\chi^{m+e_2},$$
$$\partial_3(\chi^m)=(m_1+m_3)\cdot\chi^{m+e_3},\quad\text{and}\quad \partial_4(\chi^m)=(m_2+m_3)\cdot\chi^{m+e_4}.$$

Finally, under the isomorphism of (2), the four homogeneous LNDs on A are given by

$$\partial_1=x_3\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_4},\qquad \partial_2=x_3\frac{\partial}{\partial x_2}+x_1\frac{\partial}{\partial x_4},$$
$$\partial_3=x_4\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_3},\quad\text{and}\quad \partial_4=x_4\frac{\partial}{\partial x_2}+x_1\frac{\partial}{\partial x_3}.$$

3. Locally nilpotent derivations on 𝕋-varieties of complexity 1

In this section we give a complete classification of homogeneous LNDs on 𝕋-varieties of complexity 1 over an algebraically closed field k of characteristic 0. In the first part we treat the case of homogeneous LNDs of fiber type, while in the second one we deal with the more delicate case of homogeneous LNDs of horizontal type.

We fix the n-dimensional torus $\mathbb{T}$, a smooth curve C, and a proper σ-polyhedral divisor $\mathfrak{D}=\sum_{z\in C}\Delta_z\cdot z$ on C. Letting K_0 be the function field of C, we consider the affine variety $X=\operatorname{Spec}A$, where

$$A=A[C,\mathfrak{D}]=\bigoplus_{m\in\sigma_M^\vee}A_m\chi^m,\qquad\text{with}\quad A_m=H^0\left(C,\mathcal{O}(\lfloor\mathfrak{D}(m)\rfloor)\right)\subseteq K_0.$$

We denote by $h_z=h_{\Delta_z}$ the support function of Δ_z so that $\mathfrak{D}(m)=\sum_{z\in C}h_z(m)\cdot z$. We also fix a homogeneous LND ∂ on A.

In this context, we can interpret Definitions 1.9 and 1.14 as follows.

Lemma 3.1. *With the notation as above, let ∂ be a homogeneous LND on A. Then the following hold:*

(i) *If ∂ is of fiber type, then ∂ is negative and* $\ker\partial = \bigoplus_{m\in\tau_M} A_m\chi^m$ *where τ is a codimension* 1 *face of $\sigma^\vee$.*
(ii) *Assuming further that A is nonelliptic, ∂ is of fiber type if and only if ∂ is negative.*

Proof. To prove (i) we let ∂ be a homogeneous LND of fiber type on A. By Lemma 1.13 we can extend ∂ to a homogeneous LND $\bar\partial$ on $\bar A = K_0[\sigma_M^\vee]$ which is an affine semigroup algebra over K_0. Since $\partial(K_0)=0$, $\bar\partial$ is a locally nilpotent K_0-derivation. It follows from Corollary 2.9 that $\deg\partial = \deg\bar\partial \notin \sigma_M^\vee$, so ∂ is negative.

Furthermore, Lemma 2.6 and Theorem 2.7 show that $\ker\bar\partial = K_0[\tau_M]$ where τ is a codimension 1 face of $\sigma^\vee$. Thus

$$\ker\partial = A\cap\ker\bar\partial = \bigoplus_{m\in\tau_M}(A_m\cap K_0)\chi^m = \bigoplus_{m\in\tau_M} A_m\chi^m,$$

which proves (i).

To prove (ii) we assume further that A is nonelliptic. Let ∂ be a negative homogeneous LND on A. Let $\bar\partial$ be the extension of ∂ to $K_0[M]$ as in Lemma 1.13. Since ∂ is negative, $\partial(A_0)\subseteq A_{\deg\partial}=0$. Since A is nonelliptic we have $K_0=\operatorname{Frac}A_0$, so $\bar\partial(K_0)=0$ and ∂ is of fiber type. ☐

Remark 3.2. In the elliptic case, the second assertion in Lemma 3.1 does not hold, in general. Consider for instance the elliptic $\mathbf{k}$-domain $A=\mathbf{k}[x,y]$ graded via $\deg x=\deg y=1$. Then the partial derivative ∂_x is a negative homogeneous LND of horizontal type on A.

3.1. Homogeneous LNDs of fiber type

In this subsection we consider a homogeneous LND ∂ on A of fiber type. Let, as before, $\bar A = K_0[\sigma_M^\vee]$ be the affine semigroup K_0-algebra with cone $\sigma\in N_{\mathbb{Q}}$ over the field K_0 of rational functions of C. By Lemma 1.13, ∂ can be extended to a homogeneous locally nilpotent K_0-derivation on $\bar A$. To classify homogeneous LNDs of fiber type, we will rely on the classification of homogeneous LNDs on affine semigroup algebras from the previous section.

If σ has no extremal ray, then $\sigma=0$ and $\sigma^\vee=M_{\mathbb{Q}}$. By Lemma 3.1 in this case there are no homogeneous LND of fiber type. So we may assume in the sequel that σ has at least one extremal ray, say ρ. Let τ be its dual codimension 1 face and let S_ρ be as defined in Lemma 2.4.

Lemma 3.3. *For any $e\in S_\rho$,*

$$D_e := \sum_{z\in C}\max_{m\in\sigma_M^\vee\setminus\tau_M}(h_z(m)-h_z(m+e))\cdot z$$

is a well-defined $\mathbb{Q}$-divisor on C.

Proof. By Lemma 2.4, for all $m\in\sigma_M^\vee\setminus\tau_M$, $m+e$ is contained in $\sigma_M^\vee$ and thus $h_z(m)$ and $h_z(m+e)$ are well defined. Recall that for any $z\in C$, the function h_z is upper convex and piecewise linear on $\sigma^\vee$. Thus the above maximum is achieved by one of the linear pieces of h_z, i.e., by one of the maximal cones in the normal quasifan $\Lambda(h_z)$ (see Definition 1.3).

For every $z \in C$, we let $\{\delta_{1,z}, \dots, \delta_{\ell_z,z}\}$ be the set of all maximal cones in $\Lambda(h_z)$ and we let $g_{r,z},\ r \in \{1, \dots, \ell_z\}$ be the linear extension of $h_z|_{\delta_{r,z}}$ to $M_{\mathbb{Q}}$. Since the maximum is achieved by one of the linear pieces we have

$$\max_{m\in\sigma_M^\vee\setminus\tau_M} (h_z(m) - h_z(m+e)) = \max_{r\in\{1,\cdots,\ell_z\}} (-g_{r,z}(e)).$$

Since $g_{r,z}(e) \in \mathbb{Q}\ \forall (r, z)$, D_e is indeed a $\mathbb{Q}$-divisor. $\square$

Remark 3.4. With the notation as in the preceding proof we can provide a better description of D_e. Since τ is a codimension 1 face of $\sigma^\vee$, it is contained as a face in one and only one maximal cone $\delta_{r,z}$. We may assume that $\tau \subseteq \delta_{1,z}$. By the upper convexity of h_z we have $g_{1,z}(e) \leq g_{r,z}(e)\ \forall r$ and so $D_e = -\sum_{z\in C} g_{1,z}(e) \cdot z$.

Notation 3.5. We let $\Phi_e = H^0(C, \mathcal{O}_C(\lfloor -D_e \rfloor))$. Thus, for any $\varphi \in \Phi_e$ and any $m \in \sigma_M^\vee \setminus \tau_M$, we have

$$\operatorname{div}(\varphi) \geq \lceil D_e \rceil \geq D_e \geq \sum_{z\in C} (h_z(m) - h_z(m+e)) \cdot z = \mathfrak{D}(m) - \mathfrak{D}(m+e).$$

There is a natural way to associate to a nonzero function $\varphi \in \Phi_e$ a homogeneous LND of fiber type on A. More precisely, we have the following lemma.

Lemma 3.6. *To any triple (ρ, e, φ), where ρ is an extremal ray of σ, $e \in S_\rho$ is a lattice vector, and $\varphi \in \Phi_e$ is a nonzero function, we can associate a homogeneous LND $\partial_{\rho,e,\varphi}$ on $A = A[C, \mathfrak{D}]$ with kernel*

$$\ker \partial_{\rho,e,\varphi} = \bigoplus_{m\in\tau_M} A_m\chi^m \qquad \textit{and} \qquad \deg \partial_{\rho,e,\varphi} = e.$$

Proof. Letting $\bar{A} = K_0[\sigma_M^\vee]$, we consider the K_0-LND $\partial_{\rho,e}$ on $\bar{A}$ as in Lemma 2.6. Since $\varphi \in K_0$, $\varphi\partial_{\rho,e}$ is again a K_0-LND on $\bar{A}$.

We claim that $\varphi\partial_{\rho,e}$ stabilizes $A \subseteq \bar{A}$. Indeed, let $f \in A_m \subseteq K_0$ be a homogeneous element so that $\operatorname{div} f + \lfloor \mathfrak{D}(m) \rfloor \geq 0$. If $m \in \tau_M$, then $\varphi\partial_{\rho,e}(f\chi^m) = 0$. If $m \in \sigma_M^\vee \setminus \tau_M$, then

$$\varphi\partial_{\rho,e}(f\chi^m) = \varphi f \partial_{\rho,e}(\chi^m) = m_0\varphi f \chi^{m+e},$$

where $m_0 := \langle m, \rho \rangle \in \mathbb{Z}_{>0}$. Moreover, by virtue of Notation 3.5,

$$\begin{aligned}\operatorname{div}(m_0\varphi f) + \lfloor \mathfrak{D}(m+e) \rfloor &= \operatorname{div} \varphi + \operatorname{div} f + \lfloor \mathfrak{D}(m+e) \rfloor \\ &\geq \mathfrak{D}(m) - \mathfrak{D}(m+e) - \lfloor \mathfrak{D}(m) \rfloor + \lfloor \mathfrak{D}(m+e) \rfloor \\ &= \{\mathfrak{D}(m)\} - \{\mathfrak{D}(m+e)\}.\end{aligned}$$

Since the divisor $\operatorname{div}(m_0\varphi f) + \lfloor \mathfrak{D}(m+e) \rfloor$ is integral and all the values of the divisor $\{\mathfrak{D}(m)\} - \{\mathfrak{D}(m+e)\}$ are in the interval $]-1, 1[$ we have

$$\operatorname{div}(m_0\varphi f) + \lfloor \mathfrak{D}(m+e) \rfloor \geq 0 \qquad \text{and so} \qquad m_0\varphi f \in A_{m+e},$$

yielding the claim. Finally, $\partial_{\rho,e,\varphi} := \varphi\partial_{\rho,e}|_A$ is a homogeneous LND on A with kernel $\ker \partial_{\rho,e,\varphi} = \bigoplus_{m\in\tau_M} A_m\chi^m$, as desired. $\square$

Remark 3.7. We have shown actually that for every $\varphi\in\Phi_e$, $\varphi A_m\subseteq A_{m+e}$ for all $m\in\sigma_M^\vee\setminus\tau_M$. It is easily seen from the construction of the divisor D_e that all the functions $\varphi\in K_0$ satisfying this property are contained in Φ_e.

The following theorem gives the converse of Lemma 3.6 and so completes our classification of homogeneous LNDs of fiber type on $\mathbb{T}$-varieties.

Theorem 3.8. *Every nonzero homogeneous LND ∂ of fiber type on $A=A[C,\mathfrak{D}]$ is of the form $\partial=\partial_{\rho,e,\varphi}$ for some extremal ray $\rho\subseteq\sigma$, some lattice vector $e\in S_\rho$, and some function $\varphi\in\Phi_e$.*

Proof. Since ∂ is of fiber type, $\partial|_{K_0}=0$ and so ∂ can be extended to a K_0-LND $\bar{\partial}$ on the affine semigroup algebra $\bar{A}=K_0[\sigma_M^\vee]$. By Theorem 2.7 we have $\bar{\partial}=\varphi\partial_{\rho,e}$ for some extremal ray ρ of σ, some $e\in S_\rho$ and some $\varphi\in K_0$. Since A is stable under $\varphi\partial_{\rho,e}$, by Remark 3.7, $\varphi\in\Phi_e$ and so $\partial=\varphi\partial_{\rho,e}|_A=\partial_{\rho,e,\varphi}$. □

Corollary 3.9. *Let as before $X=\operatorname{Spec}A$ be a $\mathbb{T}$-variety of complexity 1, let ∂ be a homogeneous LND of fiber type on A, and let $f\chi^m\in A\setminus\ker\partial$ be a homogeneous element. Then ∂ is completely determined by the image $g\chi^{m+e}:=\partial(f\chi^m)\in A_{m+e}\chi^{m+e}$.*

Proof. By the previous theorem $\partial=\partial_{\rho,e,\varphi}$ for some extremal ray ρ, some $e\in S_\rho$, and some $\varphi\in\Phi_e$, where $e=\deg\partial$ and ρ is uniquely determined by e, see Corollary 2.8.

In the proof of Lemma 3.6 it was shown that $\partial_{\rho,e,\varphi}(f\chi^m)=m_0\varphi f\chi^{m+e}$. Thus $\varphi=g/m_0f\in K_0$ is also uniquely determined by our data. □

Corollary 3.10. *Two homogeneous LNDs $\partial=\partial_{\rho,e,\varphi}$ and $\partial'=\partial_{\rho',e',\varphi'}$ of fiber type on A are equivalent if and only if $\rho=\rho'$. In particular, there is a finite number of pairwise nonequivalent LNDs of fiber type on A.*

Proof. The first assertion follows from the description of $\ker\partial_{\rho,e,\varphi}$ in Lemma 3.6. The second one follows from the fact that σ has a finite number of extremal rays. □

In the following proposition we show that the kernel of a homogeneous LND of fiber type is finitely generated.

Proposition 3.11. *Let ∂ be a homogeneous LND on $A=A[C,\mathfrak{D}]$, where $\mathfrak{D}$ is a proper polyhedral σ-divisor on a smooth curve C. If ∂ is of fiber type, then $\ker\partial$ is finitely generated.*

Proof. In the notation of Theorem 3.8, we have $\partial=\partial_{\rho,e,\varphi}$ where $\rho\subseteq\sigma$ is an extremal ray. Letting $\tau\subseteq\sigma^\vee$ be the codimension 1 face dual to ρ, Lemma 3.6 shows that $\ker\partial=\bigoplus_{m\in\tau_M}A_m\chi^m$.

Let $a_1,\ldots,a_r$ be a set of homogeneous generators of A. Without loss of generality, we assume further that $\deg a_i\in\tau_M$ if and only if $1\leq i\leq s<r$. We claim that $a_1,\ldots,a_s$ generate $\ker\partial$. Indeed, let P be any polynomial such that $P(a_1,\ldots,a_r)\in\ker\partial$. Since $\tau\subseteq\sigma^\vee$ is a face, $\sum m_i\in\tau_M$ for $m_i\in\sigma_M^\vee$ implies that $m_i\in\tau\ \forall i$. Hence all the monomials composing $P(a_1,\ldots,a_r)$ are monomials in $a_1,\ldots,a_s$, proving the claim. □

Given an extremal ray $\rho \subseteq \sigma$ and $e \in S_\rho$, it might happen that $\dim \Phi_e = h^0(C, \mathcal{O}_C(\lfloor -D_e \rfloor)) = 0$, so that there exist no homogeneous LNDs ∂ of fiber type on A with $\deg \partial = e$ and $\ker \partial = \bigoplus_{m \in \tau_M} A_m \chi^m$. In the following lemma we give a criterion for the existence of $e \in S_\rho$ such that $\dim \Phi_e$ is nonzero.

Lemma 3.12. *Let $A = A[C, \mathfrak{D}]$ and let $\rho \subseteq \sigma$ be an extremal ray dual to a codimension 1 face $\tau \subseteq \sigma^\vee$. There exists $e \in S_\rho$ such that $\dim \Phi_e$ is positive if and only if the curve C is affine or C is projective and $h_{\deg \mathfrak{D}}|_\tau \not\equiv 0$.*

Proof. If C is affine, then for any $\mathbb{Z}$-divisor D the sheaf $\mathcal{O}_C(D)$ is generated by the global sections. It follows in this case that $\dim \Phi_e > 0$.

Further let C be a projective curve of genus g. If $\deg \lfloor -D_e \rfloor < 0$, then $\dim \Phi_e = 0$. On the other hand, by the Riemann–Roch theorem, $\dim \Phi_e > 0$ if $\deg \lfloor -D_e \rfloor \geq g$ (see Lemma 1.2 in [Ha, Chap. IV]).

Letting $h = h_{\deg \mathfrak{D}} = \sum_{z \in C} h_z$, with the notation of Remark 3.4 we have $h|_\tau = \sum_{z \in C} g_{1,z}$ and $\deg(-D_e) = \sum_{z \in C} g_{1,z}(e)$. By the definition of a proper σ-polyhedral divisor, $h(m) > 0$ for any m in the relative interior of $\sigma^\vee$.

If $h|_\tau \equiv 0$, then by the linearity of $g_{1,z}$ we obtain that $\deg(-D_e) < 0$, so $\deg \lfloor -D_e \rfloor < 0$ and $\dim \Phi_e = 0$.

If $h|_\tau \not\equiv 0$, then by the upper convexity of h, $h(m) > 0$ for all m in the relative interior of τ. By Remark 3.4, $\deg(-D_e)$ is linear on e and so, according to Remark 2.5, we can choose a suitable $e \in S_\rho$ so that $\deg \lfloor -D_e \rfloor \geq g$. Hence $\dim \Phi_e > 0$. □

We can now deduce the following corollary.

Corollary 3.13. *Let $A = A[C, \mathfrak{D}]$ and let $\rho \subseteq \sigma$ be an extremal ray dual to a codimension 1 face $\tau \subseteq \sigma^\vee$. There exists a homogeneous LND of fiber type ∂ on A such that $\ker \partial = \bigoplus_{m \in \tau_M} A_m \chi^m$ if and only if C is affine or C is projective and $\rho \cap \deg \mathfrak{D} = \varnothing$.*

Proof. Since $\rho \cap \deg \mathfrak{D} = \varnothing$ is equivalent to $h_{\deg \mathfrak{D}}|_\tau \not\equiv 0$, the corollary follows from Theorem 3.8 and Lemma 3.12. □

Remark 3.14. By Corollaries 3.10 and 3.13, the equivalence classes of LNDs of fiber type on $A = A[C, \mathfrak{D}]$ are in one-to-one correspondence with the extremal rays $\rho \subseteq \sigma$ if C is affine and with extremal rays $\rho \subseteq \sigma$ such that $\rho \cap \deg \mathfrak{D} = \varnothing$ if C is projective.

Remark 3.15. In the recent preprint [Li_1] we generalize the methods of this section to give a classification of LNDs of fiber type in arbitrary complexity.

3.2. Homogeneous LNDs of horizontal type

Let $A = A[C, \mathfrak{D}]$ where $\mathfrak{D}$ is a proper σ-polyhedral divisor on a smooth curve C. We consider a homogeneous LND ∂ of horizontal type on A. We also denote by ∂ its extension to a homogeneous $\mathbf{k}$-derivation on $K_0[M]$, where K_0 is the field of rational functions of C (see Lemma 1.13(i)).

The existence of a homogeneous LND of horizontal type imposes strong restrictions on C, as we show in the next lemma.

Lemma 3.16. *If there exists a homogeneous LND ∂ of horizontal type on $A = A[C,\mathfrak{D}]$, then $C \simeq \mathbb{P}^1$ in the case where A is elliptic and $C \simeq \mathbb{A}^1$ in the case where A is nonelliptic. In the latter case A_m is a free A_0-module of rank 1 for every $m \in \sigma_M^\vee$ and so*

$$A_m = \varphi_m A_0 \quad \textit{for some} \quad \varphi_m \in A_m \quad \textit{such that} \quad \operatorname{div}(\varphi_m) + \lfloor \mathfrak{D}(m) \rfloor = 0.$$

Proof. Let $\pi : X = \operatorname{Spec} A \dashrightarrow C$ be the rational quotient for the $\mathbb{T}$-action given by the inclusion $\pi^* : K_0 \hookrightarrow K = \operatorname{Frac} A$. Since X is normal, the indeterminacy locus X_0 of π has codimension greater than 1, and so the general orbits of the $\mathbf{k}_+$-action corresponding to ∂ are contained in $X \setminus X_0$.

Since $\partial|_{K_0} \neq 0$, the general orbits of the $\mathbf{k}_+$-action on X are not contained in the fibers of π, so map dominantly onto C. Hence, C being dominated by $\mathbb{A}^1$ we have $C \simeq \mathbb{P}^1$ in the elliptic case and $C \simeq \mathbb{A}^1$ in the nonelliptic case.

Thus, if C is affine, then $A_0 = \mathbf{k}[t]$ and so A_m is a locally free A_0-module of rank 1 for any $m \in \sigma_M^\vee$. By the primary decomposition, any locally free module over a principal ring is free and so $A_m \simeq A_0$ as a module (see also Chapter VII, §4, Corollary 2 in [Bu]). Now the last assertion follows easily. □

3.17. For the rest of this section we let $K_0 = \mathbf{k}(t)$, $C = \mathbb{P}^1$ in the elliptic case, and $C = \mathbb{A}^1$ otherwise. We also let S_∂ be the set of all lattice vectors $m \in M$ such that $\ker \partial \cap A_m\chi^m \neq \{0\}$, we let $L(\partial) \subseteq M$ be the sublattice spanned by S_∂, and we let $\omega^\vee(\partial)$ be the cone spanned by S_∂ in $M_\mathbb{Q}$. We write L and $\omega^\vee$ instead of $L(\partial)$ and $\omega^\vee(\partial)$ whenever ∂ is clear from the context.

Lemma 3.18. *Let $A = A[C,\mathfrak{D}]$, where $\mathfrak{D}$ is a proper σ-polyhedral divisor on C, and let ∂ be a homogeneous LND of horizontal type on A. With the notation as above, the following hold*:

(1) *The kernel* $\ker \partial$ *is a semigroup algebra given by* $\ker \partial = \bigoplus_{m \in \omega_L^\vee} \mathbf{k}\varphi_m\chi^m$ *where $\varphi_m \in A_m$.*
(2) *For all $m \in \omega_L^\vee$, in the nonelliptic case* $\operatorname{div}(\varphi_m) + \mathfrak{D}(m) = 0$, *while in the elliptic case* $\operatorname{div}(\varphi_m) + \mathfrak{D}(m) = \lambda \cdot [z_\infty]$ *for some $z_\infty \in \mathbb{P}^1$ and some positive $\lambda \in \mathbb{Q}$.*
(3) *The cone $\omega^\vee$ is a maximal cone of the quasifan $\Lambda(\mathfrak{D})$ in the nonelliptic case, and of the quasifan $\Lambda(\mathfrak{D}|_{\mathbb{P}^1 \setminus \{z_\infty\}})$ in the elliptic case. In particular,* $\operatorname{rank}(L) = n$.
(4) *M is spanned by* $\deg \partial$ *and L. More precisely, any $m \in M$ can be uniquely written as $m = l + r \deg \partial$ for some $l \in L$ and some $r \in \mathbb{Z}$ with $0 \leq r < d$, where $d > 0$ is the smallest integer such that $d \deg \partial \in L$.*

Proof. Since $\mathbf{k} \subseteq \ker \partial$ we have $0 \in S_\partial$. If $m, m' \in S_\partial$, then $m + m' \in S_\partial$ and so S_∂ is a subsemigroup of $\sigma_M^\vee$.

For any $f \in K_0 = \mathbf{k}(t)$ we have $\partial(f) = f'(t)\partial(t)$, where $\partial(t) \neq 0$ since ∂ is of horizontal type. Thus $\partial(f) = 0$ if and only if f is constant. Let us fix $m \in S_\partial$. If $\varphi_m, \varphi'_m \in \ker \partial \cap A_m\chi^m$ are nonzero, then $\varphi_m/\varphi'_m \in \ker \partial \cap K_0 = \mathbf{k}$ and so $\varphi'_m = \lambda\varphi_m$ for some $\lambda \in \mathbf{k}^*$.

Hence, $\ker \partial = \bigoplus_{m \in S_\partial} \mathbf{k}\varphi_m\chi^m$ and $\ker \partial$ is a semigroup algebra. Since $\ker \partial$ is normal, S_∂ is saturated, and so $S_\partial = \omega_L^\vee$, which proves (1).

To prove (2), we assume first that C is affine. Given $m \in \omega_L^\vee$, we let φ_m be as in Lemma 3.16. Since $\ker\partial$ is factorially closed, if $f\varphi_m\chi^m \in \ker\partial \cap A_m\chi^m$ for some $f \in A_0$, then $f \in \ker\partial \cap A_0 = \mathbf{k}$ and $\varphi_m\chi^m \in \ker\partial \cap A_m\chi^m$. The latter implies that $\varphi_m^r\chi^{rm} \in \ker\partial \cap A_{rm}\chi^{rm}\ \forall r \geq 1$, and so $r\lfloor\mathfrak{D}(m)\rfloor = \lfloor r\mathfrak{D}(m)\rfloor\ \forall r \geq 1$. Hence, $\mathfrak{D}(m)$ is an integral divisor, which yields (2) in the nonelliptic case.

In the case where $C = \mathbb{P}^1$, we may suppose that $z_\infty = \infty$. Given $m \in \omega_L^\vee$, let us assume that $\operatorname{div}(\varphi_m) + \lfloor\mathfrak{D}(m)\rfloor \geq [0] + [\infty]$ so that $t\varphi_m \in A_m$ and $t^{-1}\varphi_m \in A_m$. We have $(t\varphi_m\chi^m)(t^{-1}\varphi_m\chi^m) = (\varphi_m\chi^m)^2 \in \ker\partial$. Thus $t\varphi_m\chi^m \in \ker\partial$, which contradicts (1). Henceforth, $\operatorname{div}(\varphi_m) + \lfloor\mathfrak{D}(m)\rfloor = \lambda\cdot[z_\infty]$, $\lambda \in \mathbb{Z}_{\geq 0}$. An argument similar to that employed in the nonelliptic case, yields $\operatorname{div}(\varphi_m) + \mathfrak{D}(m) = \lambda\cdot[z_\infty]$ for some positive $\lambda \in \mathbb{Q}$, proving (2).

We have $\dim\ker\partial = \dim\omega^\vee$. Since ∂ is an LND, $\ker\partial$ has codimension 1 in A. Hence $\omega^\vee$ is of full dimension in $M_\mathbb{Q}$. Furthermore, in the nonelliptic case (2) shows that $h_z|_{\omega^\vee}$ is linear $\forall z \in \mathbb{A}^1$, so that $\omega^\vee$ is contained in a maximal cone δ in $\Lambda(\mathfrak{D})$.

Assume that $\omega^\vee \subsetneq \delta$. Let $m \in \delta\setminus\omega^\vee$ and let $\varphi_m \in \mathbf{k}(t)$ be such that $\mathfrak{D}(m)$ is integral and $\operatorname{div}(\varphi_m) + \mathfrak{D}(m) = 0$. Letting $m' \in \omega_L^\vee$ be such that $m + m' \in \omega_L^\vee$, the linearity of $\mathfrak{D}$ implies $\varphi_m\chi^m\varphi_{m'}\chi^{m'} = \varphi_{m+m'}\chi^{m+m'} \in \ker\partial$. Hence $\varphi_m\chi^m \in \ker\partial$ which is a contradiction, proving (3) in the nonelliptic case. In the elliptic case, a similar argument (with $z \in \mathbb{P}^1\setminus\{z_\infty\}$) provides the result.

Finally, since $\sigma_M^\vee$ spans M as a lattice and ∂ is a homogeneous LND, for any $m \in M$ we have $m + r\deg\partial \in L$ for some $r \in \mathbb{Z}$. Thus, for $0 \geq r > -d$, the decomposition as in (4) is unique because of the minimality of d. $\square$

The following corollary shows that the kernel of a homogeneous LND on A is a semigroup algebra and so the kernel is finitely generated. Since, by Lemma 3.16, the function field of A is rational over $\mathbf{k}$, this is also a consequence of Theorem 1.2 in [Ku].

Corollary 3.19. *In the notation of Lemma* 3.18*, by* (3) $\omega \subseteq N_\mathbb{Q}$ *is a pointed polyhedral cone and, by* (1)*,*

$$\ker\partial = \bigoplus_{m\in\omega_L^\vee}\mathbf{k}\varphi_m\chi^m \simeq \mathbf{k}[\omega_L^\vee]$$

is an affine semigroup algebra, in particular, $\ker\partial$ *is finitely generated.*

Let us consider two basic examples, one with a nonelliptic $\mathbb{T}$-action and the other with an elliptic $\mathbb{T}$-action. They are universal in the sense of Lemma 3.23 below. We use both examples in our final classification, cf. Lemma 3.26 and Theorem 3.28.

Starting with an affine toric variety X and a homogeneous LND ∂ of fiber type (see Corollary 2.9), we can restrict the big torus action to an appropriate codimension 1 subtorus $\mathbb{T}$ so that ∂ becomes of horizontal type for the $\mathbb{T}$-action of complexity 1 on X. This is actually the case in our examples.

Example 3.20. Letting $A = A[C,\mathfrak{D}]$, where $C = \mathbb{A}^1$, $p \in N_\mathbb{Q}$, and $\mathfrak{D} = (p+\sigma)\cdot[0]$, we have that $h_0 : \sigma^\vee \to \mathbb{Q}$, $m \mapsto \langle m,p\rangle$ is linear and $h_z = 0\ \forall z \in \mathbf{k}^*$. Denoting by

$h: M_{\mathbb{Q}} \to \mathbb{Q}$ the linear extension of h_0 to the whole $M_{\mathbb{Q}}$, for $m \in \sigma_M^\vee$ we obtain

$$A_m = t^{-\lfloor h(m) \rfloor}\mathbf{k}[t] = \bigoplus_{r \geq -h(m)} \mathbf{k}t^r.$$

Letting $\widehat{N} = N \times \mathbb{Z}$, $\widehat{M} = M \times \mathbb{Z}$, and letting $\widehat{\sigma}$ be the cone in $\widehat{N}_{\mathbb{Q}}$ spanned by $(\sigma, 0)$ and $(p, 1)$, a vector $(m, r) \in \widehat{M}_{\mathbb{Q}}$ belongs to the dual cone $\widehat{\sigma}^\vee$ if and only if $m \in \sigma^\vee$ and $r \geq -h(m)$. By identifying $\chi^{(0,1)}$ with t we obtain

$$A = \bigoplus_{(m,r) \in \widehat{\sigma}^\vee_{\widehat{M}}} \mathbf{k}t^r\chi^m = \bigoplus_{(m,r) \in \widehat{\sigma}^\vee_{\widehat{M}}} \mathbf{k}\chi^{(m,r)} = \mathbf{k}[\widehat{\sigma}_{\widehat{M}}].$$

Hence, A is an affine semigroup algebra and so we can apply the results of the previous section.

Since A_0 is spanned as an affine semigroup algebra by the character $\chi^{(0,1)}$, the only codimension 1 face of $\widehat{\sigma}^\vee$, not containing the lattice vectors $(0, 1)$, is

$$\tau = \{(m, r) \in \widehat{M}_{\mathbb{Q}} \mid m \in \sigma^\vee,\ r = -h(m)\}.$$

This is the face of $\widehat{\sigma}^\vee$ dual to the extremal ray ρ spanned by $(p, 1)$ in $\widehat{N}_{\mathbb{Q}}$.

In the notation of Lemma 2.4, picking $e' \in S_\rho$ and $\lambda \in \mathbf{k}^*$ we let $\partial = \lambda\partial_{\rho,e'}$ be the homogeneous LND with respect to the $\widehat{M}$-grading described in Lemma 2.6. Since $(0, 1) \notin \tau$, ∂ is of horizontal type with respect to the M-grading on A. Let $\deg_M$ stand for the corresponding degree function.

For any $e' = (e, s) \in M \times \mathbb{Z}$ we have $\deg_M \partial = e$ and $\ker \partial = \mathbf{k}[\tau_{\widehat{M}}]$. Therefore, in the notation of Lemma 3.18, $\omega^\vee = \sigma^\vee$ and $L = \{m \in M \mid h(m) \in \mathbb{Z}\}$.

To be more concrete, we let $d > 0$ be the smallest integer such that $d \cdot p \in N$. Then $d \cdot h$ is an integer valued function on $\sigma_M^\vee$. Letting $m_1 \in M$ be a lattice vector such that $\{h(m_1)\} = \{1/d\}$, by a routine calculation we obtain

$$S_\rho = \left\{(e, s) \in \widehat{M} \;\middle|\; e \in L - m_1,\ s = -h(e) - \frac{1}{d}\right\} \cap \sigma_\rho^\vee, \tag{3}$$

and

$$\partial(\chi^m \cdot t^r) = \lambda\,(r + h(m)) \cdot \chi^{m+e} \cdot t^{r-h(e)-1/d}, \qquad \forall\,(m, r) \in \widehat{M}, \tag{4}$$

where $\sigma_\rho \subseteq \widehat{N}_{\mathbb{Q}}$ is as defined in Lemma 2.4, $\lambda \in \mathbf{k}^*$, and ∂_t is the partial derivative with respect to t. Moreover, in this case $\sigma_\rho = \sigma \times \{0\}$ and so

$$S_\rho = \left\{(e, s) \in \widehat{M} \;\middle|\; e \in \sigma^\vee \cap (L - m_1),\ s = -h(e) - \frac{1}{d}\right\}.$$

Example 3.21. Let $C = \mathbb{P}^1$, $p \in N_\mathbb{Q}$. Let Δ_∞ be a σ-tailed polyhedron (see Definition 1.1(i)), and let $\mathfrak{D} = (p+\sigma)\cdot[0] + \Delta_\infty \cdot [\infty]$. Under these assumptions $h_0 : \sigma^\vee \to \mathbb{Q}$, $m \mapsto \langle m, p\rangle$ is linear, and $h_z = 0\ \forall z \in \mathbf{k}^*$. As before let $h : M_\mathbb{Q} \to \mathbb{Q}$ denote the linear extension of h_0 to the whole $M_\mathbb{Q}$. We also suppose that $p + \Delta_\infty \subsetneq \sigma$ and so the sum $h_0 + h_\infty \geq 0$ is not identically 0. Under these assumptions the σ-polyhedral divisor $\mathfrak{D}$ is proper in the sense of Definition 1.2. Letting $A = A[C, \mathfrak{D}]$, for any $m \in \sigma^\vee_M$ we have

$$A_m = \bigoplus_{-h_0(m) \leq r \leq h_\infty(m)} \mathbf{k}t^r.$$

Let $\widehat{N} = N \times \mathbb{Z}$, $\widehat{M} = M \times \mathbb{Z}$, and let $\widehat{\sigma}$ be the cone in $\widehat{N}_\mathbb{Q}$ spanned by $(\sigma, 0)$, $(p, 1)$, and $(\Delta_\infty, -1)$. A vector $(m, r) \in \widehat{M}_\mathbb{Q}$ belongs to the dual cone $\widehat{\sigma}^\vee$ if and only if $m \in \sigma^\vee$, $r \geq -h_0(m)$, and $r \leq h_\infty(m)$. Thus, by identifying $\chi^{(0,1)}$ with t, we obtain

$$A = \bigoplus_{(m,r)\in \widehat{\sigma}^\vee_{\widehat{M}}} \mathbf{k}t^r\chi^m = \bigoplus_{(m,r)\in \widehat{\sigma}^\vee_{\widehat{M}}} \mathbf{k}\chi^{(m,r)} = \mathbf{k}[\widehat{\sigma}_{\widehat{M}}].$$

Hence, A is again an affine semigroup algebra, and so the results in the previous section can be applied.

We let as before $\rho \subseteq \widehat{\sigma}$ be the extremal ray spanned by $(p, 1)$. The codimension 1 face dual to ρ is

$$\tau = \{(m, r) \in \widehat{M}_\mathbb{Q} \mid m \in \sigma^\vee,\ r = -h(m)\}.$$

In the notation of Lemma 2.4, picking $e' \in S_\rho$ and $\lambda \in \mathbf{k}^*$ we let $\partial = \lambda\partial_{\rho,e'}$ be the homogeneous LND with respect to the $\widehat{M}$-grading described in Lemma 2.6. Again ∂ is of horizontal type with respect to the M-grading on A.

Furthermore, for any $e' = (e, s) \in M \times \mathbb{Z}$ we have $\deg_M \partial = e$ and $\ker \partial = \mathbf{k}[\tau_{\widehat{M}}]$. Therefore, in the notation of Lemma 3.18, $\omega^\vee = \sigma^\vee$ and $L = \{m \in M \mid h(m) \in \mathbb{Z}\}$.

To be more concrete, we let d and m_1 be as in the previous example. By a routine calculation we obtain that S_ρ is as in (3) and ∂ is as in (4).

Remark 3.22. (i) In both examples, the homogeneous LND ∂ extends to a derivation on $K_0[M]$ given by (4).

(ii) With the same formula (4), ∂ extends to a homogeneous LND on

$$A_M := \bigoplus_{m\in M} t^{-\lfloor h(m)\rfloor}\mathbf{k}[t]\chi^m, \qquad \text{where} \quad A \subseteq A_M \subseteq K_0[M].$$

(iii) In particular, if $p = 0$, then ρ is the extremal ray spanned by $(0, 1)$, $d = 1$, and $L = M$. Furthermore, we can choose $m_1 = 0$ so that $S_\rho = (M \times \{-1\}) \cap \sigma_1^\vee$, and the homogeneous LND ∂ of horizontal type on A is given by $\partial = \lambda\chi^e\partial_t$, where $(e, -1) \in S_\rho$.

We return now to the general case. We recall that

$$A = A[C, \mathfrak{D}], \qquad \text{where} \quad \mathfrak{D} = \sum_{z\in C} \Delta_z \cdot z$$

is a proper σ-polyhedral divisor on $C = \mathbb{A}^1$ or $C = \mathbb{P}^1$, h_z is the support function of Δ_z, and ∂ is a homogeneous LND of horizontal type on A.

In the next lemma we show that the subalgebra A_ω of A, generated by the homogeneous elements whose degrees are contained in $\omega^\vee$, is as in the previous examples.

Lemma 3.23. *With the notation of Lemma* 3.18*, we let* $A_\omega = \bigoplus_{m\in\omega_M^\vee} A_m\chi^m$. *Then* $A_\omega \simeq A[C, \mathfrak{D}_\omega]$ *as* M*-graded algebras, where*:

(i) $\mathfrak{D}_\omega = (p+\omega)\cdot[0]$ *for some* $p \in N_\mathbb{Q}$*, in the case where* $C = \mathbb{A}^1$*; and*
(ii) $\mathfrak{D}_\omega = (p+\omega)\cdot[0] + \Delta_\infty\cdot[\infty]$ *for some* $p \in N_\mathbb{Q}$ *and some* $\Delta_\infty \in \mathrm{Pol}_\sigma(N_\mathbb{Q})$ *with* $p + \Delta_\infty \subsetneq \sigma$*, in the case where* $C = \mathbb{P}^1$.

Proof. By Lemma 3.18(3), the support functions h_z restricted to $\omega^\vee$ are linear for all $z \in \mathbb{A}^1$ in the nonelliptic case and for all $z \in \mathbb{P}^1 \setminus \{z_\infty\}$ in the elliptic case. In the nonelliptic case this shows that $\mathfrak{D}_\omega = \sum_{z\in C}(p_z+\omega)\cdot z$, where $p_z \in N_\mathbb{Q}$. In the elliptic case, we may suppose that $z_\infty = \infty$ and so $\mathfrak{D}_\omega = \sum_{z\in\mathbb{A}^1}(p_z+\omega)\cdot z + \Delta_\infty\cdot[\infty]$, where $\Delta_\infty \in \mathrm{Pol}_\sigma(N_\mathbb{Q})$ and $p_z \in N_\mathbb{Q}$ $\forall z \in \mathbb{A}^1$.

By Lemma 1.7(vi), without loss of generality we may assume that $\deg\partial \in \omega_M^\vee$. Letting $e = \deg\partial$ we consider the two-dimensional finitely generated normal $\mathbb{Z}_{\geq 0}$-graded domain

$$B_e = \bigoplus_{r\in\mathbb{Z}_{\geq 0}} A_{re}\chi^{re}.$$

If C is affine, then $(B_e, \partial|_{B_e})$ is a parabolic pair in the sense of Definition 3.1 in [FlZa$_2$]. Now Corollary 3.19 in loc. cit. shows that, for any $r \in \mathbb{Z}_{\geq 0}$, the fractional part $\{\mathfrak{D}_\omega(re)\}$ is supported in at most one point [8]. While for C projective, $(B_e, \partial|_{B_e})$ is an elliptic pair in the sense of *loc. cit.* Then Theorem 3.3 in loc. cit. shows that B_e is an affine semigroup algebra. According to Example 5.1 in [Ti$_2$], for any $r \in \mathbb{Z}_{\geq 0}$, the fractional part $\{\mathfrak{D}_\omega(re)\}$ is supported in at most two points.

Given $m \in L$, the derivation $\varphi_m\chi^m\partial$ on A with φ_m as in Lemma 3.18(1) is again locally nilpotent. Applying the previous analysis to this LND shows that, for any $r \in \mathbb{Z}_{\geq 0}$, the fractional part $\{\mathfrak{D}_\omega(r\cdot(e+m))\}$ is supported in at most one point in the nonelliptic case and in at most two points in the elliptic case. By Lemma 3.18(4) L and e span M. So the functions $h_z|_{\omega^\vee}$ are integral except for at most one value of z in the nonelliptic case and at most two values of z in the elliptic case. Furthermore, in the elliptic case one of the two values of $z \in \mathbb{P}^1$ such that h_z is not integral corresponds to $z = \infty$.

Without loss of generality, in both cases we may suppose that $z = 0$ is an exceptional value in $\mathbb{A}^1$, provided there is one. In particular, $p_z \in N$ is a lattice vector for any $z \in \mathbf{k}^*$. Since any integral divisor on $\mathbb{A}^1$ and any integral divisor of degree 0 on $\mathbb{P}^1$ are principal, Theorem 1.4 shows that $\mathfrak{D}_\omega$ can always be chosen so that $p_z = 0$ $\forall z \in \mathbf{k}^*$. Now the result follows. □

[8]The classification results in [FlZa$_2$] are stated for surfaces over the field $\mathbb{C}$ but they are valid over any algebraically closed field of characteristic 0 with the same proofs.

Remark 3.24. (1) By Examples 3.20 and 3.21, the previous lemma shows that A_ω is an affine semigroup algebra or, equivalently, $\operatorname{Spec} A_\omega$ is a toric variety. Hence, $\operatorname{Spec} A_\omega$ is a toric variety containing $X = \operatorname{Spec} A$ as an open subset.

(2) In the notation of Lemma 3.23, let $h(m) = \langle m, p\rangle$. By virtue of Lemma 3.18(1) and (2), $L = \{m \in M \mid h(m) \in \mathbb{Z}\}$.

Remark 3.25. For every isomorphism $A \simeq A[C,\mathfrak{D}]$, the proof of the previous lemma implies the following:

1. (1) If $C = \mathbb{A}^1$, then all $h_z|_{\omega^\vee}$ are linear and all but possibly one of them are integral.
2. (2) If $C = \mathbb{P}^1$, then all but possibly one of $h_z|_{\omega^\vee}$ are linear and all but possibly two of them are integral.
3. (3) By virtue of Theorem 1.4, we may suppose, in both cases, that $h_z|_{\omega^\vee} = 0$ $\forall z \in \mathbf{k}^*$ and that $h_0|_{\omega^\vee}$ is linear.

The following lemma provides the main ingredient in our classification of the homogeneous LNDs of horizontal type on $A = A[C,\mathfrak{D}]$.

Lemma 3.26. *Let $\mathfrak{D}$ be a proper σ-polyhedral divisor on $C = \mathbb{A}^1$ or $C = \mathbb{P}^1$. Let $\omega^\vee$ be a maximal cone in the quasifan $\Delta(\mathfrak{D})$ or $\Delta(\mathfrak{D}|_{\mathbb{A}^1})$, respectively, such that $h_z|_{\omega^\vee} = 0$ $\forall z \in \mathbf{k}^*$. Let ∂ be the derivation of degree e given by formula (4). Then ∂ extends to a homogeneous LND on $A = A[C,\mathfrak{D}]$ if and only if, for every $m \in \sigma_M^\vee$ such that $m + e \in \sigma_M^\vee$, the following hold:*

- (i) *If $h_z(m+e) \neq 0$, then $\lfloor h_z(m+e)\rfloor - \lfloor h_z(m)\rfloor \geq 1$ $\forall z \in \mathbf{k}^*$.*
- (ii) *If $h_0(m+e) \neq h(m+e)$, then $\lfloor dh_0(m+e)\rfloor - \lfloor dh_0(m)\rfloor \geq 1 + dh(e)$.*
- (iii) *If $C = \mathbb{P}^1$, then $\lfloor dh_\infty(m+e)\rfloor - \lfloor dh_\infty(m)\rfloor \geq -1 - dh(e)$.*

Here h is the linear extension of $h_0|_{\omega^\vee}$ and $d > 0$ is the smallest integer such that dh is integral.

Proof. Similarly, as in Example 3.20, $h(m) = \langle m, p\rangle$ for some $p \in N_\mathbb{Q}$. Since each h_z is upper convex (see Definition 1.1(ii)), $h_z(m) \leq 0$ for $z \in \mathbf{k}^*$ and $h_0(m) \leq h(m)$. Letting $A_M = \bigoplus_{m\in M} \varphi_m \mathbf{k}[t]\chi^m$, where $\varphi_m = t^{-\lfloor h(m)\rfloor}$ (see Remark 3.22), we have $A \subseteq A_M$. By virtue of this remark ∂ extends to a homogeneous LND on A_M. We still denote by ∂ this extension. Thus ∂ extends to a homogeneous LND on A if and only if ∂ stabilizes A.

To show that ∂ stabilizes A, let us start with the simplest case where $h = 0$.

Case $h = 0$. In this case, Remark 3.22(3) shows that $L = M$, $d = 1$, and $r = -1$, and so $\partial = \lambda\chi^e\partial_t$. Furthermore, $h_z \leq 0$ $\forall z \in \mathbb{A}^1$ and in the elliptic case $h_\infty \geq 0$. For any $m \in \sigma_M^\vee$ such that $m + e \in \sigma_M^\vee$, the conditions in the lemma can be reduced to:

- (i′) If $h_z(m+e) \neq 0$, then $\lfloor h_z(m+e)\rfloor - \lfloor h_z(m)\rfloor \geq 1$ $\forall z \in \mathbb{A}^1$.
- (iii′) If $C = \mathbb{P}^1$, then $\lfloor h_\infty(m+e)\rfloor - \lfloor h_\infty(m)\rfloor \geq -1$ $\forall m \in \sigma_M^\vee$.

In this case, $A_m = H^0(C, \mathcal{O}(\lfloor\mathfrak{D}(m)\rfloor)) \subseteq \mathbf{k}[t]$ and ∂ stabilizes A if and only if

$$f(t) \in A_m \;\Rightarrow\; f'(t) \in A_{m+e}, \qquad \forall m \in \sigma_M^\vee,$$

or, equivalently,

$$\operatorname{div} f + \lfloor\mathfrak{D}(m)\rfloor \geq 0 \;\Rightarrow\; \operatorname{div} f' + \lfloor\mathfrak{D}(m+e)\rfloor \geq 0, \qquad \forall m \in \sigma_M^\vee,$$

or else

$$\mathrm{ord}_z(f)+\lfloor h_z(m)\rfloor\geq 0 \;\Rightarrow\; \mathrm{ord}_z(f')+\lfloor h_z(m+e)\rfloor\geq 0, \quad \forall m\in\sigma_M^\vee \text{ and } \forall z\in C. \quad (5)$$

Next we show that (i′) and (iii′) hold if and only if (5) holds.

Let $z \in \mathbb{A}^1$ and let $m \in \sigma_M^\vee$ be such that $m+e \in \sigma_M^\vee$. If $h_z(m+e) = 0$ condition (5) holds since $f \in \mathbf{k}[t]$.

Assume $h_z(m + e) \neq 0$. Since $h_z \leq 0$ is upper convex, if $h_z(m) = 0$ then $h_z(m + re) \neq 0$ $\forall r > 1$ contradicting the fact that ∂ is an LND. Hence we may assume that $h_z(m) \neq 0$ so that $f \in (t - z)\mathbf{k}[t]$. In this setting, $\mathrm{ord}_z(f') = \mathrm{ord}_z(f) - 1$ and so

$$\mathrm{ord}_z(f') + \lfloor h_z(m+e)\rfloor = \mathrm{ord}_z(f)+\lfloor h_z(m)\rfloor+(\lfloor h_z(m + e)\rfloor-\lfloor h_z(m)\rfloor-1). \quad (6)$$

Therefore (i′) implies (5).

To show the converse, let us suppose that (5) holds. Assuming that C is affine, for every $m \in \sigma_M^\vee$ we consider φ_m as in Lemma 3.18. Since by this lemma $\mathrm{ord}_z(\varphi_m) + \lfloor h_z(m)\rfloor = 0$, applying (5) and (6) to φ_m we obtain

$$\mathrm{ord}_z(\varphi_m)+\lfloor h_z(m)\rfloor+(\lfloor h_z(m + e)\rfloor - \lfloor h_z(m)\rfloor - 1)=\lfloor h_z(m+e)\rfloor-\lfloor h_z(m)\rfloor-1\geq 0,$$

proving (i′) when C is affine. If C is projective, then for any $z \in \mathbb{A}^1$ and any $m \in \sigma_M^\vee$ we can still find $\varphi_{m,z} \in A_m$ such that $\mathrm{ord}_z(\varphi_{m,z}) + \lfloor h_z(m)\rfloor = 0$. Thus again, the previous argument applies.

In the elliptic case, we let $z = \infty$ and we fix $m \in \sigma_M^\vee$. If f is constant, then (5) holds because $h_\infty(m) \geq 0$. Otherwise, $\mathrm{ord}_\infty(f') = \mathrm{ord}_\infty(f) + 1$ and so

$$\begin{aligned}\mathrm{ord}_\infty(f')+&\lfloor h_\infty(m + e)\rfloor \\ &= \mathrm{ord}_\infty(f) + \lfloor h_\infty(m)\rfloor + (\lfloor h_\infty(m + e)\rfloor - \lfloor h_\infty(m)\rfloor + 1)\,. \end{aligned} \quad (7)$$

Therefore (iii′) implies (5).

To show the converse, as before we let $\varphi_{m,\infty} \in A_m$ be such that $\mathrm{ord}_\infty(\varphi_{m,\infty}) + \lfloor h_\infty(m)\rfloor = 0$. Applying (5) and (7) to $\varphi_{m,\infty}$ we obtain

$$\begin{aligned}\mathrm{ord}_\infty(\varphi_{m,\infty}) + \lfloor h_\infty(m)\rfloor + (\lfloor h_\infty(m + e)\rfloor - \lfloor h_\infty(m)\rfloor + 1)& \\ = \lfloor h_\infty(m + e)\rfloor - \lfloor h_\infty(m)\rfloor + 1 \geq 0&,\end{aligned}$$

proving (iii′).

Next we assume that h is integral.

Case h integral. In this case we still have $d = 1$. We recall that $h(m) = \langle m, p\rangle$. Letting $\mathfrak{D}' = \mathfrak{D} - (p+\sigma)\cdot[0]$ if C is affine and $\mathfrak{D}' = \mathfrak{D} - (p+\sigma)\cdot[0] + (p+\sigma)\cdot[\infty]$ if C is projective, by Theorem 1.4(iii) $A \simeq A[C, \mathfrak{D}']$. In this setting $A[C, \mathfrak{D}']$ is as in the previous case with $h'_0 = h_0 - h$, $h'_\infty = h_\infty + h$ and $h'_z = h_z$ $\forall z \in \mathbf{k}^*$.

This consideration shows that ∂ stabilizes A if and only if (i′) and (iii′) hold for $h'_z(m)$ $\forall z \in C$. For any $z \in \mathbf{k}^*$, (i′) is equivalent to (i) in the lemma. Since

$$\lfloor h'_0(m + e)\rfloor - \lfloor h'_0(m)\rfloor - 1 = \lfloor h_0(m + e)\rfloor - \lfloor h_0(m)\rfloor - 1 - h(e),$$

condition (i$'$) for $z=0$ is equivalent to (ii).

Similarly, if C is projective

$$\lfloor h'_\infty(m+e)\rfloor - \lfloor h'_\infty(m)\rfloor + 1 = \lfloor h_\infty(m+e)\rfloor - \lfloor h_\infty(m)\rfloor + 1 + h(e),$$

and so (iii$'$) is equivalent to (iii).

Now we turn to the general case.

General case. We may assume that h is not integral, i.e., $d>1$. We consider the normalization A' of $A[\sqrt[d]{\varphi_{de}}\chi^e]$, where $\varphi_{de} := t^{-h(de)}$ so that $A\subseteq A'$ is a cyclic extension. With the notation of Lemma 1.6 we have $A' = A[C',\mathfrak{D}']$ and $K_0' = K_0[\sqrt[d]{\varphi_{de}}]$.

By the minimality of d we deduce that $\gcd(h(de),d)=1$ and so $\sqrt[d]{\varphi_{de}} = t^{a+b/d}$, where $\gcd(b,d)=1$. So $K_0' = \mathbf{k}(s)$, where $s^d=t$. Thus $C'\simeq \mathbb{A}^1$ if A is nonelliptic and $C'\simeq\mathbb{P}^1$ if A is elliptic. Let $p: C'\to C$, $z'\mapsto z'^d = z$ be the projection induced by the morphism $K_0\hookrightarrow K_0'$, $t\mapsto t=s^d$. By Lemma 1.6 we have

$$\mathfrak{D}' = d\cdot\Delta_0\cdot[0] + \sum_{z'\in\mathbf{k}^*}\Delta_z\cdot z' \qquad \text{if} \quad C=\mathbb{A}^1,$$

and

$$\mathfrak{D}' = d\cdot\Delta_0\cdot[0] + d\cdot\Delta_\infty\cdot[\infty] + \sum_{z'\in\mathbf{k}^*}\Delta_z\cdot z' \qquad \text{if} \quad C=\mathbb{P}^1.$$

So $h_0' = dh_0$, $h'_\infty = dh_\infty$, and $h'_{z'} = h_z$. Moreover, $h_0'|_{\omega^\vee}$ is integral and A' is as in the previous case.

Recall that $A_M = \bigoplus_{m\in M}\varphi_m\mathbf{k}[t]\chi^m$, where $\varphi_m = t^{-\lfloor h(m)\rfloor}$. We define further

$$A'_M = \bigoplus_{m\in M}\varphi'_m\mathbf{k}[s]\chi^m, \qquad \text{where} \qquad \varphi'_m = -s^{dh(m)}.$$

Since $A_M\subseteq A'_M$ is a cyclic extension, by Lemma 1.11, $\partial: A_M\to A_M$ extends to a homogeneous LND $\partial': A'_M\to A'_M$.

We claim that ∂ stabilizes A if and only if ∂' stabilizes A'. In fact the "only if" direction is a consequence of Lemma 1.11. If ∂' stabilizes A' then $\partial'(A)=\partial(A)\subseteq A_M\cap A' = A$, proving the claim.

We let h' be the linear extension of $h_0'|_{\omega^\vee}$. Clearly $h'=dh$. The previous case shows that ∂' stabilizes A' if and only if, for any $m\in\sigma_M^\vee$ such that $m+e\in\sigma_M^\vee$, the following conditions hold:

(i$''$) If $h'_{z'}(m+e)\neq 0$, then $\lfloor h'_{z'}(m+e)\rfloor - \lfloor h'_{z'}(m)\rfloor \geq 1$ $\forall z'\in\mathbf{k}^*$.
(ii$''$) If $h_0'(m+e)\neq h'(m+e)$, then $\lfloor h_0'(m+e)\rfloor - \lfloor h_0'(m)\rfloor \geq 1+h'(e)$.
(iii$''$) If $C=\mathbb{P}^1$, then $\lfloor h'_\infty(m+e)\rfloor - \lfloor h'_\infty(m)\rfloor \geq -1-h'(e)$.

Replacing in (i$''$)–(iii$''$) h' by dh, h_0' by dh_0, h'_∞ by dh_∞, and $h'_{z'}$ by h_z for $z\in\mathbf{k}^*$, shows that ∂ stabilizes A if and only if (i)-(iii) of the lemma hold. Now the proof is completed. $\square$

Remark 3.27. In the elliptic case, if $e \in \omega_M^\vee$, then (iii) in Lemma 3.26 holds. In fact,

$$\begin{aligned}\lfloor dh_\infty(m+e)\rfloor - \lfloor dh_\infty(m)\rfloor &\geq dh_\infty(m+e) - 1 - dh_\infty(m)\\ &\geq dh_\infty(e) - 1 \geq -dh(e) - 1.\end{aligned}$$

In the following theorem we describe all the homogeneous LND of horizontal type on a $\mathbb{T}$-variety of complexity 1. It is our main classification result which summarizes the previous ones.

Theorem 3.28. *Let $\mathfrak{D}$ be a proper σ-polyhedral divisor on $C = \mathbb{A}^1$ or $C = \mathbb{P}^1$, and let $A = A[C, \mathfrak{D}]$. Let $\omega^\vee \subseteq M_\mathbb{Q}$ be a polyhedral cone, and let $e \in M$ be a lattice vector. Then there exists a homogeneous LND $\partial : A \to A$ of horizontal type with* $\deg \partial = e$ *and $\omega^\vee(\partial) = \omega^\vee$ if and only if the following conditions* (i)–(v) *hold:*

- (i) *If $C = \mathbb{A}^1$, then $\omega^\vee$ is a maximal cone in the quasifan $\Lambda(\mathfrak{D})$, and there exists $z_0 \in C$ such that $h_z|_{\omega^\vee}$ is integral $\forall z \in C \setminus \{z_0\}$.*
- (i′) *If $C = \mathbb{P}^1$, then there exists $z_\infty \in \mathbb{P}^1$ such that* (i) *holds for $C_0 := \mathbb{P}^1 \setminus \{z_\infty\}$.*

Without loss of generality, we may suppose that $z_0 = 0$, $z_\infty = \infty$ in the elliptic case, and that $h_z(m)|_{\omega^\vee} = 0$ $\forall z \in \mathbf{k}^*$. Let h and d be as in Lemma 3.26*, let m_1 be as in Example* 3.20*, and let L be as in Remark* 3.24(2)*.*

- (ii) *The lattice vector $(e, -1/d - h(e))$ belongs to S_ρ as defined in* (3)*.*

For any $m \in \sigma_M^\vee$ such that $m + e \in \sigma_M^\vee$, the following hold:

- (iii) *If $h_z(m+e) \neq 0$, then $\lfloor h_z(m+e)\rfloor - \lfloor h_z(m)\rfloor \geq 1$ $\forall z \in \mathbf{k}^*$.*
- (iv) *If $h_0(m+e) \neq h(m+e)$, then $\lfloor dh_0(m+e)\rfloor - \lfloor dh_0(m)\rfloor \geq 1 + dh(e)$.*
- (v) *If $C = \mathbb{P}^1$, then $\lfloor dh_\infty(m+e)\rfloor - \lfloor dh_\infty(m)\rfloor \geq -1 - dh(e)$.*

Moreover,

$$\ker \partial = \bigoplus_{m \in \omega_L^\vee} \mathbf{k}\varphi_m \chi^m,$$

where $\varphi_m \in A_m$ satisfy the relation

$$\operatorname{div}(\varphi_m) + \mathfrak{D}(m) = 0 \quad \textit{if} \quad C = \mathbb{A}^1 \quad \textit{or} \quad \operatorname{div}(\varphi_m)|_{C_0} + \mathfrak{D}(m)|_{C_0} = 0 \quad \textit{if} \quad C = \mathbb{P}^1.$$

Proof. Let ∂ be a homogeneous LND of horizontal type on A with $\deg \partial = e$ and $\omega^\vee(\partial) = \omega^\vee$. Lemma 3.18(3) and Remark 3.25 show that (i) and (i′) hold. Lemma 3.23 and Examples 3.20 and 3.21 show that (ii) holds. To conclude, Lemma 3.26 shows that (iii)–(v) hold.

To show the converse, assume that (i), (i′), and (ii)–(v) are fulfilled. By Theorem 1.4, (i) and (i′) imply that $A_\omega \simeq A[C, \mathfrak{D}_\omega]$ with $\mathfrak{D}_\omega$ as in Lemma 3.23. By Examples 3.20 and 3.21 and Remark 3.22(2), (ii) shows that there exists a homogeneous LND $\partial : A_M \to A_M$ with $\deg \partial = e$. By Lemma 3.26 and its proof, (iii)–(v) imply that ∂ restricts to a homogeneous LND on A. Finally, by Lemma 3.18(3), (i) and (i′) imply that $\omega^\vee(\partial) = \omega^\vee$.

Moreover, Lemma 3.18 (1) and (2) give the desired description of $\ker \partial$. □

Remark 3.29. The maximal cones in the quasifan $\Lambda(\mathfrak{D})$ are in one-to-one correspondence with the vertices of the σ-polyhedron $\deg \mathfrak{D}$.

Corollary 3.30. *In the notation of Theorem* 3.28*, A admits a homogeneous LND ∂ of horizontal type such that $\omega^\vee(\partial) = \omega^\vee$ if and only if* (i) *and* (i′) *in the theorem hold.*

Proof. The "only if" part follows directly from Lemma 3.26.

Assume that (i) and (i′) hold. By Lemma 3.26 and Examples 3.20 and 3.21, we only need to show that there exists $e \in M$ such that $(e, -1/d - h(e)) \in S_\rho$ and (iii)–(v) hold.

Let $(e', r') \in S_\rho$ (by Remark 2.5, this set is nonempty). By this remark $e = e' + m\ \forall m \in \omega_L^\vee$ is such that $(e, r' - h(m)) \in S_\rho$. In particular, we can assume that e belongs to the relative interior of $\omega^\vee$. In this setting, Remark 3.27 shows that (v) holds.

As in the proof of Lemma 3.3, for every $z \in \mathbb{A}^1$, we let $\{\delta_{0,z}, \cdots, \delta_{\ell_z,z}\}$ denote the set of all maximal cones in $\Lambda(h_z)$ and we let $g_{r,z},\ r \in \{0, \cdots, \ell_z\}$, be the linear extension of $h_z|_{\delta_{r,z}}$ to $M_\mathbb{Q}$. We assume further that $\omega^\vee \subseteq \delta_{0,z}\, \forall z \in \mathbb{A}^1$.

Since the functions h_z are upper convex, the inequalities in (iii) and (iv) hold if they hold in every maximal cone on $\Lambda(h_z)$ except $\delta_{0,z}$, i.e.,

(iii′) $\lfloor g_{r,z}(m + e)\rfloor - \lfloor g_z(m)\rfloor \geq 1\ \forall z \in \mathbf{k}^*,\ \forall r \in \{1, \ldots, \ell_z\}$ and $\forall m \in \delta_{r,z} \cap M$.
(iv′) $\lfloor dg_{r,0}(m+e)\rfloor - \lfloor dg_{r,0}(m)\rfloor \geq 1 + dh(e)\ \forall r \in \{1, \ldots, \ell_0\}$ and $\forall m \in \delta_{r,0} \cap M$.

These inequalities are fulfilled if

$$\begin{aligned} &g_{r,z}(e) \geq 1, \qquad \forall z \in \mathbf{k}^* \text{ and } \forall r \in \{1, \ldots, \ell_z\}, \quad \text{and} \\ &g_{r,0}(e) \geq 1/d + \lceil h(e)\rceil, \qquad \forall r \in \{1, \ldots, \ell_0\}. \end{aligned} \tag{8}$$

Since e belongs to the relative interior of $\omega^\vee$, we have $g_{r,z}(e) > g_{0,z}(e)\ \forall z \in \mathbb{A}^1$, $g_{0,0}(e) = h(e)$, and $g_{0,z} = 0\ \forall z \in \mathbf{k}^*$. By the linearity of the functions $g_{r,z}$ we can choose e such that (8) holds, proving the corollary. □

Corollary 3.31. *In the notation on Theorem* 3.28*, two homogeneous LNDs ∂ and ∂' of horizontal type on A are equivalent if and only if $\omega^\vee(\partial) = \omega^\vee(\partial')$ and, in the elliptic case, $z_\infty(\partial) = z_\infty(\partial')$.*

Proof. Indeed, the description of $\ker \partial$ given in Theorem 3.28 depends only on $\omega^\vee$ in the nonelliptic case and on $\omega^\vee$ and $z_\infty \in C$ in the elliptic case. □

Corollary 3.32. *The number of pairwise nonequivalent homogeneous LNDs of horizontal type on $A = A[C, \mathfrak{D}]$ is finite except in the case where A is elliptic and there exists a maximal cone $\omega^\vee$ of $\Lambda(\mathfrak{D})$ such that all but possibly one $h_z|_{\omega^\vee}$ are integral.*

Proof. Since $\Lambda(\mathfrak{D})$ has only a finite number of maximal cones, Corollary 3.31 gives the result in the case where A is nonelliptic. Furthermore, in the elliptic case by this corollary there is an infinite number of pairwise nonequivalent LNDs on A if and only if in Theorem 3.28(i′) we can choose $z_\infty \in \mathbb{P}^1$ arbitrarily. However, the latter is indeed possible under the assumptions of the corollary. □

Example 3.33. A combinatorial description of $\mathbf{k}^{[2]} = \mathbf{k}[x,y]$ with the grading induced by $\deg x = \deg y = 1$ is given by the proper σ-polyhedral divisor $\mathfrak{D} = (1+\sigma)\cdot[0]$ on $\mathbb{P}^1$, where $\sigma = \mathbb{Q}_{\geq 0} \subseteq N_{\mathbb{Q}} \simeq \mathbb{Q}$. By Corollary 3.32 there exists an infinite number of pairwise nonequivalent LNDs on $\mathbf{k}^{[2]}$ homogeneous with respect to the given grading. Indeed, the derivations on the family

$$\partial_\lambda = \lambda\frac{\partial}{\partial x} + (1-\lambda)\frac{\partial}{\partial y}$$

are homogeneous and pairwise nonequivalent for different values of λ.

In contrast, a combinatorial description of $\mathbf{k}^{[2]}$ with the grading induced by $\deg x = -\deg y = 1$ is given by the proper σ-polyhedral divisor $\mathfrak{D} = [0,1]\cdot[0]$ on $\mathbb{A}^1$. By Corollary 3.32 there exists a finite number of pairwise nonequivalent LNDs homogeneous with respect to this grading. Indeed, by Corollary 3.30 the only such LNDs are the partial derivatives.

Remark 3.34. Let A be a normal finitely generated effectively M-graded algebra, such that the complexity of the corresponding $\mathbb{T}$-action on $\operatorname{Spec} A$ is 0 or 1. In Corollaries 2.11 and 3.19 and Proposition 3.11 we have shown that the kernel of a homogeneous LND on A is finitely generated.

On the other hand, there are examples of homogeneous LNDs on $\mathbb{A}^r$ for $r \geq 5$, whose kernel is not finitely generated, see [Ro], [Fr_1], and [DaFr]. For instance, Daigle and Freudenburg showed in [DaFr] that $\ker\partial$ is not finitely generated for the LND

$$\partial = x_1^3\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4} + x_1^2\frac{\partial}{\partial x_5}$$

on $\mathbf{k}^{[5]} = \mathbf{k}[x_1,\ldots,x_5]$. Furthermore, it is easy to see that ∂ is homogeneous of degree $(0,-1)$ under the effective $\mathbb{Z}^2$-grading on $\mathbf{k}^{[5]}$ given by

$$\deg x_1 = (1,0), \qquad \deg x_2 = (3,1), \qquad \deg x_3 = (3,2),$$
$$\deg x_4 = (3,3), \quad \text{and} \quad \deg x_5 = (2,1).$$

The corresponding $\mathbb{T}$-action on $\mathbb{A}^5$ is of complexity 3.

In the following example we study the existence of homogeneous LNDs on the M-graded algebra A of Example 1.5.

Example 3.35. Let the notation be as in Example 1.5. Since $\sigma = \{0\}$, Lemma 3.1 shows that there is no homogeneous LND of fiber type on A. In contrast, let us show that there exist exactly four pairwise nonequivalent homogeneous LNDs on A.

Indeed, since h_0 is the only support function which is nonintegral, Corollaries 3.30 and 3.31 show that there are four nonequivalent homogeneous LNDs of horizontal type on A corresponding to the four maximal cones in $\Lambda(\mathfrak{D})$,

$$\delta_1 = \operatorname{cone}((1,0),(-4,1)), \qquad \delta_2 = \operatorname{cone}((-4,1),(-1,0)),$$
$$\delta_3 = \operatorname{cone}((-1,0),(8,-1)), \qquad \delta_4 = \operatorname{cone}((8,-1),(1,0)).$$

For the cones δ_1 and δ_2 the hypotheses of Lemma 3.26 are fulfilled, i.e., $h_z|_{\delta_i} = 0$ $\forall z \in \mathbf{k}^*$ for $i = 1, 2$. Moreover, $e_1 = (-3, 1)$ and $e_2 = (-8, 1)$ satisfy conditions (i)-(iii) in this lemma for δ_1 and δ_2, respectively.

We let ∂_1 and ∂_2 be the respective LNDs defined in (4). Letting $m = (m_1, m_2) \in M$, by a routine calculation we obtain

$$\partial_1(\chi^m t^r) = (r - \tfrac{1}{4}m_1 - m_2) \cdot \chi^{m+e_1} t^r \qquad \text{and} \qquad \partial_2(\chi^m t^r) = r \cdot \chi^{m+e_2} t^r.$$

Furthermore, under the isomorphism (1) in Example 1.5, ∂_1 and ∂_2 can be extended to $\mathbf{k}^{[4]} = \mathbf{k}[x_1, x_2, x_3, x_4]$ as LNDs

$$\partial_1 = -\tfrac{1}{4}x_3\frac{\partial}{\partial x_2} + x_1^2x_2^3\frac{\partial}{\partial x_4} \qquad \text{and} \qquad \partial_2 = x_3\frac{\partial}{\partial x_1} - (2x_1x_2^4 + 1)\frac{\partial}{\partial x_4}.$$

To obtain the derivations corresponding to δ_3 and δ_4 we let $C' = \operatorname{Spec} \mathbf{k}[s]$, $\Delta_1' = \{0\} \times [-1, 0]$, and $\mathfrak{D}' = \Delta_0 \cdot [0] + \Delta_1' \cdot [1]$. Theorem 1.4(3) shows that $A \simeq A[C', \mathfrak{D}']$. Under this new combinatorial description we have

$$u_1 = -s\chi^{(4,0)}, \qquad u_2 = \chi^{(-1,0)}, \qquad u_3 = (1-s)\chi^{(-4,1)}, \qquad \text{and} \qquad u_4 = s\chi^{(8,-1)}.$$

Now the assumptions of Lemma 3.26 are satisfied for δ_3 and δ_4. Moreover, $e_3 = (4, -1)$ and $e_4 = (9, -1)$ satisfy conditions (i)–(iii) in this lemma for δ_3 and δ_4, respectively.

We let ∂_3 and ∂_4 be the respective LNDs defined by (4). By a simple computation we obtain

$$\partial_3(\chi^m s^r) = (r + m_2) \cdot \chi^{m+e_3} s^r \quad \text{and} \quad \partial_4(\chi^m s^r) = \left(r - \tfrac{1}{4}m_1 - m_2\right) \cdot \chi^{m+e_4} s^{r+1}.$$

Furthermore, under the isomorphism (1), ∂_3 and ∂_4 are induced by the LNDs

$$\partial_3 = -x_4\frac{\partial}{\partial x_1} + (2x_1x_2^4 + 1)\frac{\partial}{\partial x_3} \qquad \text{and} \qquad \partial_4 = \tfrac{1}{4}x_4\frac{\partial}{\partial x_2} - x_1^2x_2^3\frac{\partial}{\partial x_3}$$

on $\mathbf{k}^{[4]}$.

3.3. The surface case

A description of $\mathbb{C}^*$-surfaces was given in [FlZa$_1$] in terms of the DPD (Dolgachev-Pinkham-Demazure) presentation. In [FlZa$_2$] this description was applied to classify the homogeneous LNDs on normal affine $\mathbb{C}^*$-surfaces (of both horizontal and fiber type). Here we relate both descriptions. Besides, we stress the difference that appears in higher dimensions.

In the case of dimension 2 the lattice N has rank 1, which makes things quite explicit (cf., e.g., [Su]).

We treat the elliptic case first. In this case σ is of full dimension, and so we can assume that $\sigma = \mathbb{Q}_{\geq 0} \subseteq N_{\mathbb{Q}} = \mathbb{Q}$. Let $A = A[C, \mathfrak{D}]$, where $\mathfrak{D}$ is a proper σ-polyhedral divisor on a smooth projective curve C. In this setting, $\mathfrak{D}$ is uniquely determined by the $\mathbb{Q}$-divisor $\mathfrak{D}(1)$ on C. Here $(C, \mathfrak{D}(1))$ coincides with the DPD presentation data. Since the only extremal ray of σ is σ itself and $\deg \mathfrak{D}$ is σ-tailed (see Definition 1.1), by Corollary 3.13 there is no homogeneous LND of fiber type on A.

Furthermore, if there is a homogeneous LND ∂ of horizontal type on A, then $\omega^\vee(\partial)=\sigma^\vee$ and so, by Remark 3.24(1), $A=A_\omega$ is an affine semigroup algebra, i.e., $\operatorname{Spec} A$ is an affine toric surface. This corresponds to Theorem 3.3 in loc. cit.

Next we consider a nonelliptic algebra A so that C is an affine curve. In loc. cit. this case is further divided into two subcases, the parabolic one which corresponds to $\sigma=\mathbb{Q}_{\geq 0}$, and the hyperbolic one which corresponds to $\sigma=\{0\}$.

In the parabolic case, the DPD presentation data are the same as in the elliptic case. In this case there is again just one extremal ray $\rho=\sigma$ and $S_\rho=\{-1\}$. Moreover, since the support functions h_z are positively homogeneous on $\sigma^\vee=\mathbb{Q}_{\geq 0}$, they are linear and so $D_{-1}=\mathfrak{D}(1)$ (see Lemma 3.3). By Theorem 3.8 the homogeneous LNDs of fiber type on A are in one-to-one correspondence with the rational functions

$$\varphi\in H^0(C,\mathcal{O}_C(\lfloor -\mathfrak{D}(1)\rfloor)).$$

This corresponds to Theorem 3.12 in loc. cit.

If a graded parabolic two-dimensional algebra A admits a homogeneous LND of horizontal type, then $\operatorname{Spec} A$ is a toric variety by the same argument as in the elliptic case. This yields Theorem 3.16 and Corollary 3.19 in loc. cit.

In the hyperbolic case, $\mathfrak{D}$ is uniquely determined by the pair of $\mathbb{Q}$-divisors $(\mathfrak{D}(1),\mathfrak{D}(-1))$ which correspond to the pair (D_+,D_-) in the DPD presentation data. According to our Definition 1.1(ii), this pair satisfies $\mathfrak{D}(1)+\mathfrak{D}(-1)\leq 0$. In this case, by Lemma 3.1, there is no homogeneous LND of fiber type on A since $\sigma=\{0\}$. This corresponds to Lemma 3.20 in loc. cit.

The homogeneous LNDs of horizontal type are classified in Theorem 3.28 above. Specializing this classification to dimension 2 gives Theorem 3.22 in loc. cit. More precisely, conditions (i) and (ii) of Theorem 3.28 lead to (i) of Theorem 3.22 in loc. cit. while (iii) and (iv) in Theorem 3.28 lead to (ii) in Theorem 3.22 in loc. cit.

In contrast, in dimension 3 new phenomena appear. For instance, there exist nontoric threefolds with an elliptic $\mathbb{T}$-action and a homogeneous LND of horizontal or fiber type, see Subsection 4.2 for an example of fiber type. With the notation as in Subsection 4.2, considering $C=\mathbb{P}^1$ and $\mathfrak{D}=\frac{1}{2}\Delta\cdot[0]+\frac{1}{2}\Delta\cdot[1]+\Delta'\cdot[\infty]$, where $\Delta'=\sigma\cap\{\langle(1,1),\cdot\rangle\geq 1\}\subseteq N_{\mathbb{Q}}$ gives a nontoric example with two equivalence classes of homogeneous LNDs fiber type and four equivalence classes of homogeneous LNDs of horizontal type.

4. Applications

4.1. The Makar-Limanov invariant

Let A be a finitely generated normal domain and let $\mathrm{LND}(A)$ be the set of all LNDs on A. The *Makar-Limanov invariant* of A is defined as

$$\mathrm{ML}(A)=\bigcap_{\partial\in\mathrm{LND}(A)}\ker\partial.$$

Similarly, if A is effectively M-graded we let $\mathrm{LND}_h(A)$ be the set of all homogeneous LNDs on A, and we call

$$\mathrm{ML}_h(A) = \bigcap_{\partial \in \mathrm{LND}_h(A)} \ker \partial$$

the *homogeneous Makar-Limanov invariant* of A. Clearly, $\mathrm{ML}(A) \subseteq \mathrm{ML}_h(A)$.

In the sequel we apply the results in Sections 2 and 3 in order to compute $\mathrm{ML}_h(A)$ in the case where the complexity of the $\mathbb{T}$-action on $\operatorname{Spec} A$ is 0 or 1. We also give some partial results for the usual invariant $\mathrm{ML}(A)$ in this particular case.

Remark 4.1. Since two equivalent LNDs (see Definition 1.8) have the same kernel, to compute $\mathrm{ML}(A)$ or $\mathrm{ML}_h(A)$ it is sufficient to consider pairwise nonequivalent LNDs on A. The pairwise nonequivalent homogeneous LNDs on A are classified in Corollary 2.10 for the complexity 0 case, and in Corollaries 3.10 and 3.31 for the complexity 1 case.

We treat first the case of complexity 0, i.e., the case of affine toric varieties. Let $\sigma \subseteq N_{\mathbb{Q}}$ be a pointed polyhedral cone.

Proposition 4.2. *Let $A = \mathbf{k}[\sigma^\vee_M]$ be an affine semigroup algebra so that $X = \operatorname{Spec} A$ is a toric variety. Then*

$$\mathrm{ML}(A) = \mathrm{ML}_h(A) = \mathbf{k}[\theta_M],$$

where $\theta \subseteq M_{\mathbb{Q}}$ is the maximal linear subspace contained in $\sigma^\vee$. In particular, $\mathrm{ML}(A) = \mathbf{k}$ if and only if σ is of complete dimension, i.e., if and only if there is no torus factor in X.

Proof. By Corollary 2.10 and Theorem 2.7, the pairwise nonequivalent homogeneous LNDs on A are in one-to-one correspondence with the extremal rays of σ. For any extremal ray $\rho \subseteq \sigma$ and any $e \in S_\rho$ as in Lemma 2.4, the kernel of the corresponding homogeneous LND is $\ker \partial_{\rho,e} = \mathbf{k}[\tau_M]$, where $\tau \subseteq \sigma^\vee$ is the codimension 1 face dual to ρ.

Since $\theta \subseteq \sigma^\vee$ is the intersection of all codimension 1 faces, we have $\mathrm{ML}_h(A) = \mathbf{k}[\theta_M]$. Furthermore, the characters in $\mathbf{k}[\theta_M] \subseteq A$ are invertible functions on A and so, by Lemma 1.7(iii), $\partial(\mathbf{k}[\theta_M]) = 0\ \forall \partial \in \mathrm{LND}(A)$. Hence $\mathbf{k}[\theta_M] \subseteq \mathrm{ML}(A)$, proving the lemma. ☐

For the rest of this section we let $A = A[C, \mathfrak{D}]$, where $\mathfrak{D}$ is a proper σ-polyhedral divisor on a smooth curve C. We also let $\mathrm{ML}_{\mathrm{fib}}(A)$ and $\mathrm{ML}_{\mathrm{hor}}(A)$ be the intersection of the kernels of all homogeneous LNDs of fiber type and of horizontal type, respectively, so that

$$\mathrm{ML}_h(A) = \mathrm{ML}_{\mathrm{fib}}(A) \cap \mathrm{ML}_{\mathrm{hor}}(A). \tag{9}$$

We first compute $\mathrm{ML}_{\mathrm{fib}}(A)$. If A is nonelliptic (elliptic, respectively) we let $\{\rho_i\}$ be the set of all extremal rays of $\sigma^\vee$ (of all extremal rays of $\sigma^\vee$ such that $\rho \cap \deg \mathfrak{D} = \varnothing$, respectively). In both cases, we let $\tau_i \subseteq M_{\mathbb{Q}}$ denote the codimension 1 face dual to ρ_i and $\theta = \bigcap \tau_i$.

Lemma 4.3. *With the notation as above,*

$$\mathrm{ML}_{\mathrm{fib}}(A)=\bigoplus_{m\in\theta_M} A_m\chi^m.$$

Proof. By Corollary 3.13, for every extremal ray ρ_i, there is a homogeneous LND ∂_i of fiber type with kernel $\ker\partial_i=\bigoplus_{m\in\tau_i\cap M}A_m\chi^m$. By Corollary 3.10 any homogeneous LND of fiber type on A is equivalent to one of the ∂_i. Finally, taking the intersection $\bigcap_i\ker\partial_i$ gives the desired description of $\mathrm{ML}_{\mathrm{fib}}(A)$. □

Remark 4.4. If A is nonelliptic, then $\theta\subseteq M_{\mathbb{Q}}$ is the maximal linear subspace contained in $\sigma^\vee$, as in the toric case. In particular, if A is parabolic, then $\theta=\{0\}$ and $\mathrm{ML}_{\mathrm{fib}}(A)=A_0$, and if A is hyperbolic, then $\theta=M_{\mathbb{Q}}$ and $\mathrm{ML}_{\mathrm{fib}}(A)=A$.

If there is no LND of horizontal type on A, then $\mathrm{ML}_{\mathrm{hor}}(A)=A$ and $\mathrm{ML}_h(A)=\mathrm{ML}_{\mathrm{fib}}(A)$. In the sequel we assume that A admits a homogeneous LND of horizontal type.

If A is nonelliptic, we let $\{\delta_i\}$ be the set of all cones in $M_{\mathbb{Q}}$ satisfying (i) in Theorem 3.28, and $\delta=\bigcap_i\delta_i$. If A is elliptic, we let $\{\delta_{i,z}\}$ be the set of all cones in $M_{\mathbb{Q}}$ satisfying (i′) in Theorem 3.28 with $z_\infty=z$. We also let $B=\{m\in\sigma^\vee\mid h_{\deg\mathfrak{D}}=0\}$ and $\delta=\bigcap_{i,z}\delta_{i,z}\cap B$.

Lemma 4.5. *With the notation as before, if ∂ is a homogeneous LND on A of horizontal type, then*

$$\mathrm{ML}_{\mathrm{hor}}(A)=\bigoplus_{m\in\delta_L}\mathbf{k}\varphi_m\chi^m,$$

where $L=L(\partial)$ and $\varphi_m\in A_m$ satisfy the relation $\mathrm{div}(\varphi_m)+\mathfrak{D}(m)=0$.

Proof. We treat first the nonelliptic case. By Corollary 3.30 for every δ_i there is a homogeneous LND ∂_i of horizontal type with kernel

$$\ker\partial_i=\bigoplus_{m\in\delta_i\cap L_i}\mathbf{k}\varphi_m\chi^m,$$

where $L_i=L(\partial_i)$ and $\varphi_m\in A_m$ is such that $\mathrm{div}(\varphi_m)+\mathfrak{D}(m)=0$. By Corollary 3.31, any homogeneous LND of horizontal type on A is equivalent to one of the ∂_i. Taking the intersection of all $\ker\partial_i$ gives the lemma in this case.

Further let A be elliptic and let ∂ be a homogeneous LND of horizontal type on A. Let $z_0,z_\infty\in\mathbb{P}_1$ and let $\omega^\vee$ and L be as in Theorem 3.28 so that

$$\ker\partial=\bigoplus_{m\in\omega^\vee_L}\mathbf{k}\varphi_m\chi^m,$$

where $\varphi_m\in A_m$ satisfies $\mathrm{div}(\varphi_m)+\mathfrak{D}(m)=\lambda[z_\infty]$ for some positive $\lambda\in\mathbf{k}$.

By permuting the roles of z_0 and z_∞ in Theorem 3.28 we obtain another LND ∂' on A. The description of $\ker\partial$ and $\ker\partial'$ shows that

$$\ker\partial\cap\ker\partial'=\bigoplus_{m\in\omega^\vee_L\cap B}\mathbf{k}\varphi_m\chi^m,$$

where $\varphi_m\in A_m$ is such that $\mathrm{div}(\varphi_m)+\mathfrak{D}(m)=0$.

Now the lemma follows by an argument similar to that in the nonelliptic case. □

Theorem 4.6. *In the notation of Lemmas 4.3 and 4.5, if there is no homogeneous LND of horizontal type on A, then*

$$\mathrm{ML}_h(A) = \bigoplus_{m \in \theta_M} A_m \chi^m.$$

If ∂ is a homogeneous LND of horizontal type on A, then

$$\mathrm{ML}_h(A) = \bigoplus_{m \in \theta \cap \delta_L} \mathbf{k} \varphi_m \chi^m,$$

where $L = L(\partial)$ and $\varphi_m \in A_m$ is such that $\operatorname{div}(\varphi_m) + \mathfrak{D}(m) = 0$.

Proof. The assertions follow immediately by virtue of (9) and Lemmas 4.3 and 4.5. □

In the following corollary we give a criterion of triviality of the homogeneous Makar-Limanov invariant $\mathrm{ML}_h(A)$.

Corollary 4.7. *With the notation as above,* $\mathrm{ML}_h(A) = \mathbf{k}$ *if and only if one of the following conditions hold*:

(i) *A is elliptic,* $\operatorname{rank}(M) \geq 2$, *and* $\deg \mathfrak{D}$ *does not intersect any extremal ray of σ.*
(ii) *A admits a homogeneous LND of horizontal type and $\theta \cap \delta = \{0\}$.*

In particular, in both cases $\mathrm{ML}(A) = \mathbf{k}$.

Proof. By Lemma 4.3, (i) holds if and only if $\mathrm{ML}_{\mathrm{hor}}(A) = \mathbf{k}$. By Theorem 4.6, (ii) holds if and only if there is a homogeneous LND of horizontal type and $\mathrm{ML}_h(A) = \mathbf{k}$. □

Example 4.8. It easily seen that $\mathrm{ML}_h(A) = \mathbf{k}$ for A as in Example 3.35.

4.2. A nonrational threefold with trivial Makar-Limanov invariant

To exhibit such an example, we let σ be a pointed polyhedral cone in $M_{\mathbb{Q}}$, where $\operatorname{rank}(M) = n \geq 2$. We let as before $A = A[C, \mathfrak{D}]$, where $\mathfrak{D}$ is a proper σ-polyhedral divisor on a smooth curve C. By Subsection 1.12. $\operatorname{Frac} A = K_0(M)$ and so $\operatorname{Spec} A$ is birational to $C \times \mathbb{P}^n$ (cf. Corollary 3 in [Ti$_2$]).

By Corollary 4.7, if A is nonelliptic and $\mathrm{ML}(A) = \mathbf{k}$, then A admits a homogeneous LND of horizontal type. So $C \simeq \mathbb{A}^1$ and $\operatorname{Spec} A$ is rational. On the other hand, the curve C does not participate in the assumptions of Corollary 4.7(i). So if (i) is fulfilled, then $\mathrm{ML}(A) = \mathbf{k}$ while $\operatorname{Spec} A$ is birational to $C \times \mathbb{P}^n$. This leads to the following result.

Proposition 4.9. *Let $A = A[C, \mathfrak{D}]$, where $\mathfrak{D}$ is a proper σ-polyhedral divisor on a smooth projective curve C of positive genus. Suppose further that* $\deg \mathfrak{D}$ *does not intersect any extremal ray of σ. Then* $\mathrm{ML}(A) = \mathbf{k}$ *whereas* $X = \operatorname{Spec} A$ *is nonrational.*

Remark 4.10. It is evident that X in Proposition 4.9 is in fact stably nonrational, i.e., $X \times \mathbb{P}^\ell$ is nonrational for all $\ell \geq 0$, cf. [Po, Example 1.22].

In the rest of this section we give a simple geometric example illustrating this proposition.

Letting $N = \mathbb{Z}^2$ and $M = \mathbb{Z}^2$ with the canonical bases and duality, we let $\sigma \subseteq N_\mathbb{Q}$ be the first quadrant, $\Delta = (1,1)+\sigma$, and $h = h_\Delta$ so that $h(m_1, m_2) = m_1 + m_2$. Furthermore, we let $A = A[C, \mathfrak{D}]$, where $C \subseteq \mathbb{P}^2$ is the elliptic curve with affine equation $s^2 - t^3 + t = 0$, and $\mathfrak{D} = \Delta \cdot P$ is the proper σ-polyhedral divisor on C with P being the point at infinity of C.

Since $C \not\simeq \mathbb{P}^1$ and $\deg \mathfrak{D} = \Delta$, A satisfies the assumptions of Corollary 4.9. Letting K_0 be the function field of C, by Theorem 1.4 we obtain

$$A_{(m_1,m_2)} = H^0(C, \mathcal{O}_C((m_1 + m_2)P)) \subseteq K_0.$$

The functions $t, s \in K_0$ are regular in the affine part of C, and have poles of order 2 and 3 on P, respectively. By the Riemann–Roch theorem $\dim H^0(C, \mathcal{O}(rP)) = r$ $\forall r > 0$. Hence the functions $\{t^i, t^j s \mid 2i \leq r \text{ and } 2j + 3 \leq r\}$ form a basis of $H^0(C, \mathcal{O}(rP))$ (see [Ha, Chap. IV, Prop. 4.6]).

In this setting the first gradded pieces are the $\mathbf{k}$-modules

$$\begin{gathered}
A_{(0,0)} = A_{(1,0)} = A_{(0,1)} = \mathbf{k}, \\
A_{(2,0)} = A_{(1,1)} = A_{(0,2)} = \mathbf{k} \oplus \mathbf{k}t, \\
A_{(3,0)} = A_{(2,1)} = A_{(1,2)} = A_{(0,3)} = \mathbf{k} \oplus \mathbf{k}t \oplus \mathbf{k}s, \\
A_{(4,0)} = A_{(3,1)} = A_{(2,2)} = A_{(1,3)} = A_{(0,4)} = \mathbf{k} \oplus \mathbf{k}t \oplus \mathbf{k}t^2 \oplus \mathbf{k}s.
\end{gathered}$$

Remark 4.11. Let $\mathcal{E}$ be the locally free sheaf of rank 2, $\mathcal{O}_C(P) \oplus \mathcal{O}_C(P)$. The variety $\operatorname{Spec} A$ corresponds to the contraction of the zero section of the vector bundle associated to $\mathcal{E}$.

It is easy to see that A admits the following set of generators.

$$\begin{gathered}
u_1 = \chi^{(1,0)}, \qquad u_2 = \chi^{(0,1)}, \qquad u_3 = t\chi^{(2,0)}, \qquad u_4 = t\chi^{(1,1)}, \qquad u_5 = t\chi^{(0,2)}, \\
u_6 = s\chi^{(3,0)}, \qquad u_7 = s\chi^{(2,1)}, \qquad u_8 = s\chi^{(1,2)}, \qquad u_9 = s\chi^{(0,3)}.
\end{gathered}$$

So $A \simeq \mathbf{k}^{[9]}/I$, where $\mathbf{k}^{[9]} = \mathbf{k}[x_1, \ldots, x_9]$, and I is the ideal of relations of u_i $(i = 1, \ldots, 9)$[9].

Furthermore, $A_m \subseteq \mathbf{k}[s,t]/(s^2 - t^3 + t)$ $\forall m \in \sigma_M^\vee$ since $\mathfrak{D}$ is supported at the point at infinity P. The semigroup $\sigma_M^\vee$ is spanned by $(1,0)$ and $(0,1)$, so letting $v = \chi^{(1,0)}$ and $w = \chi^{(0,1)}$ we obtain

$$A = \mathbf{k}[v, w, tv^2, tvw, tw^2, sv^3, sv^2w, svw^2, sw^3] \subseteq \mathbf{k}[s,t,v,w]/(s^2 - t^3 + t).$$

Thus $\operatorname{Spec} A$ is birationally dominated by $C_0 \times \mathbb{A}^2$ where $C_0 = C \setminus \{P\}$.

Since $C \not\simeq \mathbb{P}^1$, by Lemma 3.16 there is no homogeneous LND of horizontal type on A. There are two extremal rays $\rho_i \subseteq \sigma$ spanned by the vectors $(1,0)$ and $(0,1)$.

[9]Using a software for elimination theory, we were able to find a minimal generating set of I consisting of 22 polynomials.

Since $\deg \mathfrak{D} = \Delta$ is contained in the relative interior of σ, Corollaries 3.10 and 3.13 imply that there are exactly two pairwise nonequivalent homogeneous LNDs ∂_i of fiber type which correspond to the extremal rays ρ_i, $i = 1, 2$, respectively.

The codimension 1 face τ_1 dual to ρ_1 is spanned by $(0, 1)$ and, in the notation of Lemma 3.6, $S_{\rho_1} = \{(-1, r) \mid r \geq 0\}$. Letting $e_1 = (-1, 1)$ yields $D_{e_1} = 0$ and so $\Phi_{e_1} = \mathbf{k}$. We fix $\varphi_1 = 1 \in \Phi_{e_1}$. By the same lemma we can choose $\partial_1 = \partial_{\rho_1,e_1,\varphi_1}$ as

$$\partial_1\big(\chi^{(m_1,m_2)}\big) = m_1 \cdot \chi^{(m_1-1,m_2+1)} \qquad \text{for all} \quad (m_1, m_2) \in \sigma^\vee_M.$$

Likewise, the codimension 1 face τ_2 dual to ρ_2 is spanned by $(1, 0)$ and, in the notation of Lemma 3.6, $S_{\rho_2} = \{(r, -1) \mid r \geq 0\}$. Letting $e_2 = (1, -1)$ yields $D_{e_2} = 0$ and so $\Phi_{e_2} = \mathbf{k}$. We fix $\varphi_2 = 1 \in \Phi_{e_2}$. By Lemma 3.6 we can choose $\partial_2 = \partial_{\rho_2,e_2,\varphi_2}$ as

$$\partial_2\big(\chi^{(m_1,m_2)}\big) = m_2 \cdot \chi^{(m_1+1,m_2-1)} \qquad \text{for all} \quad (m_1, m_2) \in \sigma^\vee_M.$$

The kernels of ∂_1 and ∂_2 are given by

$$\ker \partial_1 = \bigoplus_{m\in\tau_1\cap M} A_m\chi^m \qquad \text{and} \qquad \ker \partial_2 = \bigoplus_{m\in\tau_2\cap M} A_m\chi^m.$$

Since $\tau_1 \cap \tau_2 = \{0\}$ we have

$$\mathrm{ML}(A) = \ker \partial_1 \cap \ker \partial_2 = A_{(0,0)} = \mathbf{k}.$$

This agrees with Corollary 4.9.

The LNDs ∂_i are induced, under the isomorphism $A \simeq \mathbf{k}^{[9]}/I$, by the following LNDs on $\mathbf{k}^{[9]}$:

$$\partial_1 = x_2\frac{\partial}{\partial x_1} + 2x_4\frac{\partial}{\partial x_3} + x_5\frac{\partial}{\partial x_4} + 3x_7\frac{\partial}{\partial x_6} + 2x_8\frac{\partial}{\partial x_7} + x_9\frac{\partial}{\partial x_8},$$

and

$$\partial_2 = x_1\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_4} + 2x_4\frac{\partial}{\partial x_5} + x_6\frac{\partial}{\partial x_7} + 2x_7\frac{\partial}{\partial x_8} + 3x_8\frac{\partial}{\partial x_9},$$

respectively.

Below we let $X = \operatorname{Spec} A$ and we let $\pi : X \dashrightarrow C$ be the rational quotient for the $\mathbb{T}$-action on X. The comorphism of π is given by the inclusion $\pi^* : K_0 \hookrightarrow \operatorname{Frac} A = K_0(u_1, u_2)$.

The orbit closure $\Theta = \overline{\pi^{-1}(0, 0)}$ over $(0, 0) \in C$ is general and is isomorphic to $\mathbb{A}^2 = \operatorname{Spec} \mathbf{k}[x_1, x_2]$. The restrictions to Θ of the $\mathbf{k}_+$-actions ϕ_i corresponding to ∂_i, $i = 1, 2$, respectively, are given by

$$\phi_1|_\Theta : (t, (x_1, x_2)) \mapsto (x_1 + tx_2, x_2) \qquad \text{and} \qquad \phi_2|_\Theta : (t, (x_1, x_2)) \mapsto (x_1, x_2 + tx_1).$$

Furthermore, there is a unique singular point $\bar{0} \in X$ corresponding to the fixed point of the $\mathbb{T}$-action on X. The point $\bar{0}$ is given by the augmentation ideal

$$A_+ = \bigoplus_{\sigma^\vee_M\setminus\{0\}} A_m\chi^m.$$

On the other hand, let $A = A[C, \mathfrak{D}]$, where $\mathfrak{D}$ is a proper σ-polyhedral divisor on a smooth projective curve C. By Theorem 2.5 in [KaRu], if $\operatorname{Spec} A$ is smooth, then $\operatorname{Spec} A \simeq \mathbb{A}^{n+1}$ (see also Proposition 3.1 in [Su]). In particular, $\operatorname{Spec} A$ is rational.

Remark 4.12. (i) In [Li_1] generalizing the methods of this section we obtain a birational characterization of normal affine varieties with trivial ML-invariant.

(ii) In [Li_2] we studied singularities of $\mathbb{T}$-varieties. In particular, we showed that the singularities of the $X = \operatorname{Spec} A[C, \mathfrak{D}]$ are not Cohen–Macaulay. On the other hand, in the recent preprint [Po] a new family of examples of nonrational affine varieties with trivial ML-invariant is given. This time, these varieties are smooth.

References

[AlHa] K. Altmann, J. Hausen, *Polyhedral divisors and algebraic torus actions*, Math. Ann. **334** (2006), 557–607.

[AHS] K. Altmann, J. Hausen, H. Süß, *Gluing affine torus actions via divisorial fans*, Transform. Groups **13** (2008), 215–242.

[Bu] N. Bourbaki, *Éléments de mathématiques. Algèbre*. Chapitres 4 à 7, Lecture Notes in Mathematics, Vol. 864, Masson, Paris, 1981.

[DaFr] D. Daigle, G. Freudenburg, *A counterexample to Hilbert's Fourteenth Problem in dimension five*, J. Algebra **221** (1999), 528–535.

[De] M. Demazure, *Sous-groupes algébriques de rang maximum du groupe de Cremona*, Ann. Sci. École Norm. Sup. (4) **3** (1970), 507–588.

[FlZa_1] H. Flenner, M. Zaidenberg, *Normal affine surfaces with $\mathbb{C}^*$-actions*, Osaka J. Math. **40** (2003), 981–1009.

[FlZa_2] H. Flenner, M. Zaidenberg, *Locally nilpotent derivations on affine surfaces with a $\mathbb{C}^*$-action*, Osaka J. Math. **42** (2005), 931–974.

[Fr_1] G. Freudenburg, *A counterexample to Hilbert's fourteenth problem in dimension six*, Transform. Groups **5** (2000), 61–71.

[Fr_2] G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*, Encyclopaedia of Mathematical Sciences, Vol. 136, *Invariant Theory and Algebraic Transformation Groups*, Vol. 7, Springer, Berlin, 2006.

[Ha] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, Vol. 52, Springer-Verlag, New York, 1977. Russian transl.: Р. Хартсхорн, *Алгебраическая геометрия*, Мир, М, 1981.

[KaRu] T. Kambayashi, P. Russell, *On linearizing algebraic torus actions*, J. Pure Appl. Algebra **23** (1982), 243–250.

[KaFi] L. Kaup, K-H. Fieseler, *Hyperbolic $\mathbb{C}^*$-actions on affine algebraic surfaces*, in: *Complex Analysis*, Proc. Int. Workshop Ded. H. Grauert, Wuppertal, 1991, Aspects Math., Vol. E 17, Vieweg, Braunschweig, 1991, pp. 160–168.

[KKMS] G. Kempf, F. Knudson, D. Mumford, B. Saint-Donat, *Toroidal Embeddings* I, Lecture Notes in Mathematics, Vol. 339, Springer-Verlag, New York, 1973.

[Ku] S. Kuroda, *A condition for finite generation of the kernel of a derivation*, J. Algebra **262**, (2003), 391–400.

[Li_1] A. Liendo, *$\mathbb{G}_a$-actions of fiber type on affine $\mathbb{T}$-varieties*, `arXiv:0911.1110v1` [math.AG] (2009), 12 pp.

[Li_2] A. Liendo, *Rational singularities of normal $\mathbb{T}$-varieties*, `arXiv:0909.4134v1` [math.AG] (2009), 12 pp.

[ML] L. Makar-Limanov, *Locally nilpotent derivations, a new ring invariant and applications*, 60 pp., available at: `http://www.math.wayne.edu/~lml/`.

[Od] T. Oda, *Convex Bodies and Algebraic Geometry. An Introduction to the Theory of Toric Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **3**, Folge, Bd. 15, Springer-Verlag, Berlin, 1985.

[Po] V. L. Popov, *On the Makar-Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties*, `arXiv:1001.1311v2` [math.AG](2010), 22 pp.

[Ro] P. Roberts, *An infinitely generated symbolic blow-up in a power series ring and a new counterexample to Hilbert's fourteenth problem*, J. Algebra **132** (1990), 461–473.

[Su] H. Süß, *Canonical divisors on $\mathbb{T}$-varieties*, `arXiv:0811.0626v1`[math.AG](2008), 22 pp.

[Ti_1] Д. А. Тимашев, *Классификация G-многообразий сложности* 1, Изв. РАН, сер. мат. **61(2)** (1997), 127–162. Engl. transl.: D. A. Timashev, *Classification of G-varieties of complexity* 1, Izv. Math. **61(2)** (1997), 363–397.

[Ti_2] D. Timashev, *Torus actions of complexity one*, in: *Toric Topology*, Osaka, Japan, May 28–June 3, 2006, Contemp. Math., Vol. 460, Amer. Math. Soc., Providence, RI, 2008, pp. 349–364.

[Vo] R. Vollmert, *Toroidal embeddings and polyhedral divisors*, `arXiv:0707.0917v1` [math.AG] (2007), 5 pp.