

Automorphisms of prime order of smooth cubic n -folds

VÍCTOR GONZÁLEZ-AGUILERA AND ALVARO LIENDO

Abstract. In this paper we give an effective criterion as to when a prime number p is the order of an automorphism of a smooth cubic hypersurface of \mathbb{P}^{n+1} , for a fixed $n \geq 2$. We also provide a computational method to classify all such hypersurfaces that admit an automorphism of prime order p . In particular, we show that $p < 2^{n+1}$ and that any such hypersurface admitting an automorphism of order $p > 2^n$ is isomorphic to the Klein n -fold. We apply our method to compute exhaustive lists of automorphisms of prime order of smooth cubic threefolds and fourfolds. Finally, we provide an application to the moduli space of principally polarized abelian varieties.

Mathematics Subject Classification (2010). Primary 14J40; Secondary 14J30.

Keywords. Cubic n -folds, Automorphisms of hypersurfaces, Principally polarized abelian varieties.

1. Introduction. The smooth cubic hypersurfaces of the projective space \mathbb{P}^{n+1} (or cubic n -fold for short), $n \geq 2$ are classical objects in algebraic geometry. Their groups of regular automorphisms are finite and induced by linear automorphisms of \mathbb{P}^{n+1} [9]. In the case $n = 2$ they correspond to the classical cubic surfaces. Segre in 1942 used the geometry of its 27 lines to produce a list of cubic surfaces admitting a non-trivial automorphism group [12], see also [4].

Let $n \geq 2$ be a fixed integer. The main result of this paper is the following criterion for the order of an automorphism of a cubic n -fold: a prime number p is the order of an automorphism of a smooth cubic hypersurface of \mathbb{P}^{n+1} if and only if $p = 2$ or there exists $\ell \in \{1, \dots, n+2\}$ such that $(-2)^\ell \equiv 1 \pmod{p}$. See Theorem 2.6 in Section 2.

Victor González-Aguilera was partially supported by the Fondecyt project 1080030 and the DGIP of the UTFSM.

In Corollary 1.8 we apply this criterion to show that if a prime number p is the order of an automorphism of a cubic n -fold, then $p < 2^{n+1}$. This bound is sharper than the general bound in [11] specialized to cubic hypersurfaces. Moreover, we show that any smooth cubic n -fold admitting an automorphism of prime order $p > 2^n$ is isomorphic to the Klein n -fold given by the cubic form $F = x_0^2 x_1 + x_1^2 x_2 + \cdots + x_n^2 x_{n+1} + x_{n+1}^2 x_0$.

In Section 3 we develop a computational method to classify all smooth cubic n -folds admitting an automorphism of order p prime. We illustrate our method in the cases of threefolds and fourfolds, see Theorems 3.5 and 3.8.

Finally, in Section 4 we apply a theorem in [5] to show that the intermediate jacobian of the Klein 5-fold (the only intermediate jacobian admitting an automorphism of order 43) is not isolated in the singular locus of the moduli space of principally polarized abelian varieties of dimension 21. An analogous result for the Klein threefold was proved in [5].

2. Admissible prime orders of automorphisms of cubic n -folds. In this section we give a criterion as to when a prime number p appears as the order of an automorphism of a smooth cubic n -fold in \mathbb{P}^{n+1} , $n \geq 2$. We also give a simple bound for the order of any such automorphism.

Let V be a vector space over \mathbb{C} of dimension $n + 2$, $n \geq 2$ with a fixed basis, and let $\mathbb{P}^{n+1} = \mathbb{P}(V)$ be the corresponding projective space. We also let $\{x_0, \dots, x_{n+1}\}$ be the dual basis of the linear forms on V so that $\{x_i x_j x_k : 0 \leq i \leq j \leq k \leq n + 1\}$ is a basis of the vector space $S^3(V^*)$ of cubic forms on V . The dimension of $S^3(V^*)$ is $\binom{n+4}{3}$.

For a cubic form $F \in S^3(V^*)$, we denote by $X = V(F) \subseteq \mathbb{P}^{n+1}$ the cubic hypersurface of dimension n (cubic n -fold for short) associated to F . We denote by $\text{Aut}(X)$ the group of regular automorphisms of X and by $\text{Lin}(X)$ the subgroup of $\text{Aut}(X)$ that extends to automorphisms of \mathbb{P}^{n+1} . By Theorems 1 and 2 in [9] if X is smooth, then

$$\text{Aut}(X) = \text{Lin}(X) \quad \text{and} \quad |\text{Aut}(X)| < \infty.$$

In this setting $\text{Aut}(X) < \text{PGL}(V)$ and for any automorphism in $\text{Aut}(X)$ we can choose a representative in $\text{GL}(V)$. This automorphism induces an automorphism of $S^3(V^*)$ such that $\varphi(F) = \lambda F$, $\lambda \in \mathbb{C}^*$. These three automorphisms will be denoted by the same letter φ .

In this paper we consider automorphisms of order p prime. In this case, multiplying by an appropriate constant, we can assume that $\varphi^p = \text{Id}_V$, so that φ is also a linear automorphism of finite order p of V and $\varphi(F) = \xi^a F$ where ξ is a p -th root of the unity. Furthermore, we can apply a linear change of coordinates on V to diagonalize φ , so that

$$\varphi : V \rightarrow V, \quad (x_0, \dots, x_{n+1}) \mapsto (\xi^{\sigma_0} x_0, \dots, \xi^{\sigma_{n+1}} x_{n+1}), \quad 0 \leq \sigma_i < p.$$

Definition 2.1. We define the signature σ of an automorphism φ as above by

$$\sigma = (\sigma_0, \dots, \sigma_{n+1}) \in \mathbb{F}_p^{n+2},$$

where we identify σ_i with its class in the field \mathbb{F}_p . We also denote $\varphi = \text{diag}(\sigma)$ and we say that φ is a diagonal automorphism.

Remark 2.2. Let σ be the signature of an automorphism φ . The signature of φ^a is $a \cdot \sigma$. Changing the representative of φ in $\text{GL}(V)$ by $\xi^a \varphi$, corresponds to change the signature by $\sigma + a \cdot \mathbb{1}$, where $\mathbb{1} = (1, \dots, 1)$. Finally, the natural action of the symmetric group S_{n+2} by permutation of the basis of V corresponds to permutation of the signature σ .

The following simple lemma is a key ingredient in our classification of automorphisms of prime order of smooth cubic n -folds.

Lemma 2.3. *Let X be a cubic hypersurface of \mathbb{P}^{n+1} , given by the homogeneous form $F \in S^3(V^*)$. If the degree of F is smaller than 2 in some variable x_i , $i \in \{0, \dots, n+1\}$, then X is singular.*

Proof. Without loss of generality, we may assume that the degree of F is smaller than 2 in the variable x_0 . A direct computation shows that the point $(1 : 0 : \dots : 0)$ is a singular point of X . \square

Remark 2.4. It is easy to see that for any $n \geq 2$, there exist at least one smooth cubic n -fold admitting an automorphism of order 2 and 3. For instance the Fermat cubic n -fold $X = V(F)$, where

$$F = x_0^3 + x_1^3 + \dots + x_n^3 + x_{n+1}^3,$$

is smooth and admits the action of the symmetric group S_{n+2} by permutation of the coordinates. Any transposition gives an automorphism of order 2 and any cycle of length 3 gives an automorphism of order 3.

Definition 2.5. We say that a prime number p is *admissible in dimension n* if either $p = 2$ or there exists $\ell \in \{1, \dots, n+2\}$ such that

$$(-2)^\ell \equiv 1 \pmod{p}.$$

This definition is justified by the following Theorem which shows that the admissible primes in dimension n are exactly those that are the order of an automorphism of a smooth cubic n -fold.

Theorem 2.6. *Let $n \geq 2$. A prime number p is the order of an automorphism of a smooth cubic n -fold if and only if p is admissible in dimension n .*

Proof. The theorem holds for the admissible primes 2 and 3 since for all $n \geq 2$, the Fermat n -fold admits automorphisms of order 2 and 3. In the following we assume that $p > 3$.

To prove the “only if” part, suppose that $F \in S^3(V^*)$ is a cubic form such that the n -fold $X = V(F) \subseteq \mathbb{P}^{n+1}$ is smooth and admits an automorphism φ of order p . Assume that φ is diagonal and let $\sigma = (\sigma_0, \dots, \sigma_{n+1}) \in \mathbb{F}_p^{n+2}$ be its signature.

We have $\varphi(F) = \xi^a F$. Let b be such that $3b \equiv -a \pmod{p}$ and consider the automorphism $\psi = \xi^b \varphi$ of $\text{GL}(V)$. Clearly, ψ and φ induce the same automorphism in \mathbb{P}^{n+1} . Furthermore, for the cubic form F we have $\psi(F) = \xi^{3b} \varphi(F) = \xi^{3b+a} F = F$. Hence, replacing φ by ψ , we may and will assume that $\varphi(F) = F$.

Let now k_0 be such that $\sigma_{k_0} \not\equiv 0 \pmod{p}$. By Lemma 2.3, F contains a monomial $x_{k_0}^2 x_{k_1}$ for some $k_1 \in \{0, \dots, n+1\}$ (not necessarily with coefficient 1).

Since $p \neq 3$ we have $k_1 \neq k_0$ and since $p \neq 2$ we have $\sigma_{k_1} \not\equiv 0 \pmod p$. Furthermore, F is invariant by the diagonal automorphism φ so the monomial $x_{k_0}^2 x_{k_1}$ is invariant by φ , i.e., $2\sigma_{k_0} + \sigma_{k_1} \equiv 0 \pmod p$, and so

$$\sigma_{k_1} \equiv -2\sigma_{k_0} \pmod p. \tag{2.1}$$

Applying the above argument with k_0 replaced by k_1 , we let k_2 be such that the monomial $x_{k_1}^2 x_{k_2}$ is invariant by φ and is contained in F (not necessarily with coefficient 1). Iterating this process, for all $i \in \{3, \dots, n+2\}$ we let $k_i \in \{0, \dots, n+1\}$ be such that $x_{k_{i-1}}^2 x_{k_i}$ is a monomial in F (not necessarily with coefficient 1) invariant by φ .

By (2.1), we have

$$\sigma_{k_i} \equiv -2\sigma_{k_{i-1}} \equiv (-2)^2 \sigma_{k_{i-2}} \equiv \dots \equiv (-2)^i \sigma_{k_0} \pmod p, \quad \forall i \in \{2, \dots, n+2\}, \tag{2.2}$$

and all of the σ_{k_i} are non-zero.

Since $k_i \in \{0, \dots, n+1\}$ there are at least two $i, j \in \{0, \dots, n+2\}$, $i > j$ such that $k_i = k_j$. Thus $\sigma_{k_i} = \sigma_{k_j}$, and since $\sigma_{k_i} \equiv (-2)^i \sigma_{k_0} \pmod p$ and $\sigma_{k_j} \equiv (-2)^j \sigma_{k_0} \pmod p$, we have

$$(-2)^{i-j} \equiv 1 \pmod p,$$

and the prime number p is admissible in dimension n .

To prove the converse statement, let $p > 3$ be an admissible prime in dimension n . We let $\ell \in \{1, \dots, n+2\}$ be such that $(-2)^\ell \equiv 1 \pmod p$ and consider the cubic form

$$F = \sum_{i=1}^{\ell-1} x_{i-1}^2 x_i + x_{\ell-1}^2 x_0 + \sum_{i=\ell}^{n+1} x_i^3.$$

By construction, the cubic F form admits the automorphism $\varphi = \text{diag}(\sigma)$, where

$$\sigma = \left(1, -2, (-2)^2, \dots, (-2)^{\ell-1}, \overbrace{0, \dots, 0}^{n+2-\ell} \right).$$

A routine computation shows that $X = V(F)$ is smooth, proving the theorem. □

Remark 2.7. Let $\varphi = \text{diag}(\sigma)$ be an automorphism of order $p > 3$ prime of the smooth cubic n -fold $X = V(F)$. As in the proof of Theorem 2.6, assume that $\varphi(F) = F$ and let ℓ be as in Definition 2.5. If $\sigma_0 \neq 0$ is a component of the signature σ , then by (2.2) we have that $(-2)^i \sigma_0$ is also a component of σ , $\forall i < \ell$. Furthermore, replacing φ by φ^a , where $a \in \mathbb{F}_p$ is such that $a \cdot \sigma_0 \equiv 1$, we can assume that $\sigma_0 = 1$, see Remark 2.2.

Theorem 2.6 allows us to give, in the following corollary, a bound for the prime numbers that appear as the order of an automorphism of a smooth cubic n -fold.

Corollary 2.8. *If a prime number $p > 3$ is the order of an automorphism of a smooth cubic n -fold, then $p < 2^{n+1}$.*

Proof. Suppose that $p > 2^{n+1}$. By Theorem 2.6, p is admissible in dimension n , and so

$$(-2)^{n+1} \equiv 1 \pmod{p}, \quad \text{or} \quad (-2)^{n+2} \equiv 1 \pmod{p}.$$

This yields

$$p = (-2)^{n+1} - 1, \quad \text{or} \quad p = (-2)^{n+2} - 1, \quad \text{or} \quad 2p = (-2)^{n+2} - 1.$$

Since $-2 \equiv 1 \pmod{3}$, we have that p is divisible by 3, which provides a contradiction. \square

In [11] a general bound is given for the order of a linear automorphism of an n -dimensional projective variety of degree d . Restricted to the case of cubic n -folds this bound is $p \leq 3^{n+1}$. Thus in this particular case our bound above is sharper.

Definition 2.9. For any $n \geq 2$, we define the Klein cubic n -fold as $X = V(F) \in \mathbb{P}^{n+1}$, where

$$F = x_0^2x_1 + x_1^2x_2 + \cdots + x_n^2x_{n+1} + x_{n+1}^2x_0.$$

The group of automorphisms of the Klein cubic threefold X was first studied by Klein who showed that $\text{PSL}(2, \mathbb{F}_{11}) < \text{Aut}(X)$ [6]. Later, Adler [1] showed that $\text{Aut}(X) = \text{PSL}(2, \mathbb{F}_{11})$. In the following theorem we show that if a smooth cubic n -fold X admits an automorphism of prime order greater 2^n , then X is isomorphic to the Klein n -fold.

Theorem 2.10. *Let $X = V(F)$ be a smooth cubic n -fold, $n \geq 2$, admitting an automorphism φ of order p prime. If $p > 2^n$ then X is isomorphic to the Klein cubic n -fold.*

Proof. Since $p > 2^n$, by Corollary 2.8 p is not admissible in dimension $n - 1$ and so

$$(-2)^{n+2} \equiv 1 \pmod{p}. \tag{2.3}$$

By Remark 2.7, we can assume that $\varphi(F) = F$ and $\varphi = \text{diag}(\sigma)$, where

$$\sigma = (\sigma_0, \dots, \sigma_{n+1}) = (1, -2, 4, \dots, (-2)^{n+1}).$$

Let $\mathcal{E} \subset S^3(V^*)$ be the eigenspace associated to the eigenvalue 1 of the linear automorphism $\varphi : S^3(V^*) \rightarrow S^3(V^*)$, so that $F \in \mathcal{E}$. In the following we compute a basis for \mathcal{E} . For a monomial $x_i x_j x_k$, $0 \leq i \leq j \leq k \leq n + 1$, in the basis of $S^3(V^*)$ we have

$$x_i x_j x_k \in \mathcal{E} \Leftrightarrow \sigma_i + \sigma_j + \sigma_k \equiv 0 \pmod{p} \Leftrightarrow (-2)^i + (-2)^j + (-2)^k \equiv 0 \pmod{p}.$$

Clearly the only monomials with $i = j$ or $j = k$ contained in \mathcal{E} are $x_{n+1}^2 x_0$ and $x_i^2 x_{i+1}$, $\forall i \in \{0, \dots, n\}$. In the following we assume that $i < j < k$. Multiplying by $(-2)^{-i}$ we obtain

$$x_i x_j x_k \in \mathcal{E} \Leftrightarrow 1 + (-2)^{j-i} + (-2)^{k-i} \equiv 0 \pmod{p}.$$

Let $j' = j - i$ and $k' = k - i$. If $k' < n$ then $0 < |1 + (-2)^{j'} + (-2)^{k'}| < 2^n < p$ and $x_i x_j x_k \notin \mathcal{E}$. If $k' = n + 1$ then $j' \leq n$ and by (2.3)

$$1 + (-2)^{j'} + (-2)^{k'} \equiv 0 \pmod{p} \Leftrightarrow -2 + (-2)^{j'+1} + 1 \equiv 0 \pmod{p}.$$

This is only possible if $(-2)^{j'+1} - 1 = p$ or $(-2)^{j'+1} - 1 = 2p$, but as in the proof of Corollary 2.8, $(-2)^k - 1$ is divisible by 3 for any $k \in \mathbb{Z}$, so $x_i x_j x_k \notin \mathcal{E}$.

If $k' = n$ then $j' \leq n - 1$ and by (2.3)

$$1 + (-2)^{j'} + (-2)^{k'} \equiv 0 \pmod{p} \Leftrightarrow 4 + (-2)^{j'+2} + 1 \equiv 0 \pmod{p}.$$

Again, this is only possible if $(-2)^{j'+2} + 5 = p$ or $(-2)^{j'+2} + 5 = 2p$. The same argument as before gives $x_i x_j x_k \notin \mathcal{E}$.

We have shown that $\mathcal{E} = \langle x_{n+1}^2 x_0, x_i^2 x_{i+1}, \forall i \in \{0, \dots, n\} \rangle$, and so

$$F = \sum_{i=0}^n a_i x_i^2 x_{i+1} + a_{n+1} x_{n+1}^2 x_0, \quad a_i \in \mathbb{C}.$$

Since X is smooth, by Lemma 2.3 all of the a_i above are non-zero and applying a linear change of coordinates we can put

$$F = \sum_{i=0}^n x_i^2 x_{i+1} + x_{n+1}^2 x_0.$$

□

The particular case of $n = 3$ in Theorem 2.10, was shown by Roulleau [10]. In that article it is shown that the Klein threefold is the only cubic threefold admitting an automorphism of order 11.

The criterion in Theorem 2.6 is easily computable. In the following table we give the list of admissible prime numbers for $n \leq 10$.

n	admissible primes
2	2, 3, 5
3	2, 3, 5, 11
4	2, 3, 5, 7, 11
5	2, 3, 5, 7, 11, 43
6	2, 3, 5, 7, 11, 17, 43
7	2, 3, 5, 7, 11, 17, 19, 43
8	2, 3, 5, 7, 11, 17, 19, 31, 43
9	2, 3, 5, 7, 11, 17, 19, 31, 43, 683
10	2, 3, 5, 7, 11, 13, 17, 19, 31, 43, 683

In the following table, we give the maximal admissible prime number p for $11 \leq n \leq 19$.

n	11	12	13	14	15	16	17	18	19
p	2,731	2,731	2,731	2,731	43,691	43,691	174,763	174,763	174,763

Remark 2.11. By the proof of Theorem 2.6, whenever p is admissible for dimension n and not for $n - 1$, the Klein n -fold X admits an automorphism of order p . Furthermore, If $p > 2^n$, then by Theorem 2.10 X is the only smooth cubic n -fold admitting such an automorphism. This is the case for the maximal admissible prime for $n = 2, 3, 5, 9, 11, 15, 17$.

3. Classification of automorphisms of prime order. In this section we apply the results in Section 2 to provide a computational method to classify all smooth cubic n -folds admitting an automorphism of order p prime, or what is the same, n -folds that admits the action of a cyclic group of order p . A previous version of the results in this section appeared first in the master thesis of the second author [8].

The classification of automorphism of smooth cubic surfaces was done by Segre [12] and later revisited by Dolgachev and Verra [4] with more sophisticated methods, in this case all the computations can be carried out by hand. As an example, we apply our method to the cases of threefolds and fourfolds since in greater dimension the lists are rather long.

All the possible signatures for an automorphism of order p prime of a cubic n -fold, $n \geq 2$ are given by \mathbb{F}_p^{n+2} , but many of them represent identical or conjugated cyclic groups \mathbb{Z}_p in $\text{PGL}(V)$.

- (i) Two diagonal automorphism $\text{diag}(\sigma), \text{diag}(\sigma') \in \text{GL}(V)$ are conjugated if and only if there exists a permutation $\pi \in S_{n+2}$ such that $\sigma' = \pi(\sigma)$.
- (ii) By Remark 2.2, two diagonal automorphism $\text{diag}(\sigma), \text{diag}(\sigma') \in \text{PGL}(V)$ span the same cyclic group if and only if $\sigma' = a \cdot \sigma + b \cdot \mathbb{1}$, for some $a \in \mathbb{F}_p^*$ and some $b \in \mathbb{F}_p$.

Definition 3.1. We define an equivalence relation \sim in \mathbb{F}_p^{n+2} by $\sigma \sim \sigma'$ if and only if $\sigma' = a \cdot \pi(\sigma) + b \cdot \mathbb{1}$, for some $\pi \in S_{n+2}$, some $a \in \mathbb{F}_p^*$, and some $b \in \mathbb{F}_p$.

Remark 3.2. (i) Each class in $(\mathbb{F}_p^{n+2} / \sim)$ represent a conjugacy class of cyclic groups of order p in $\text{PGL}(V)$. The class of zero $\bar{0}$ represents the identity of \mathbb{P}^{n+1} .

- (ii) The map $\sigma \mapsto a \cdot \pi(\sigma) + b \cdot \mathbb{1}$, for some $\pi \in S_{n+2}$, some $a \in \mathbb{F}_p^*$, and some $b \in \mathbb{F}_p$ is an automorphism of \mathbb{F}_p^{n+2} regarded as the affine space over the field \mathbb{F}_p . Let $G < \text{Aut}(\mathbb{F}_p^{n+2})$ be the finite group spanned by all such automorphisms. The equivalence classes of the relation \sim are given by the orbits of the action of G in \mathbb{F}_p^{n+2} . Hence,

$$(\mathbb{F}_p^{n+2} / \sim) = \mathbb{F}_p^{n+2} // G,$$

where $//$ denotes the algebraic quotient.

For any signature $\sigma \in \mathbb{F}_p^{n+2}$ we let $\mathcal{E}_\sigma < S^3(V^*)$ be the eigenspace associated to the eigenvalue 1 of the automorphism $\varphi : S^3(V^*) \rightarrow S^3(V^*)$.

Remark 3.3. Let $X = V(F)$ be a smooth cubic n -fold admitting an automorphism $\varphi = \text{diag}(\sigma)$ of order p prime. Whenever $p \neq 3$, as in the proof of Theorem 2.6, we can assume that $\varphi(F) = F$, i.e., $F \in \mathcal{E}_\sigma$.

Computational method. For any admissible prime $p \neq 3$ in dimension n ,

- (i) Compute the set of signatures $R' \subset \mathbb{F}_p^{n+2}$ such that there exists a cubic form $F \in \mathcal{E}_\sigma$ with $X = V(F)$ smooth.
- (ii) Let $\tilde{R} = R' / \sim$ and compute a set R of representatives of $\tilde{R} \setminus \{\bar{0}\}$.

By construction, our method provides a set of signatures R such that for every smooth cubic n -fold $X = V(F)$ admitting an automorphism φ order p ,

there exists one and only one $\sigma \in R$ such that after a linear change of coordinates $F \in \mathcal{E}_\sigma$ and $\text{diag}(\sigma)$ is a generator of $\langle \varphi \rangle$.

Remark 3.3 does not hold for $p = 3$. In order to apply the same method, in the first step we let R' be the set of signatures $\sigma \in \mathbb{F}_3^{n+2}$ such that there exists a cubic form $F \in \mathcal{E}_\sigma$ with $X = V(F)$ smooth or there exists a cubic form F in the eigenspace associated to eigenvalue ξ with $X = V(F)$ smooth, where ξ is a primitive cubic root of the unity.

All of this procedure can be implemented on a software such as Maple. Lemma 2.3 is used to discard most of the signatures σ whose \mathcal{E}_σ does not contain a smooth cubic n -fold, the remaining signatures that does not contain a smooth n -fold can be easily eliminated by hand.

Given a signature $\sigma \in R$, we let $\varphi = \text{diag}(\sigma)$. For a generic cubic form $F \in \mathcal{E}_\sigma$ the cubic n -fold $X = V(F)$ is smooth. The dimension of the family of cubic n -folds given by \mathcal{E}_σ in the moduli space \mathfrak{H}_n of smooth cubic n -folds is

$$D = \dim \mathcal{E}_\sigma - \dim N_{\text{GL}(V)}(\langle \varphi \rangle). \tag{3.1}$$

To compute the dimension of the normalizer $N_{\text{GL}(V)}(\langle \varphi \rangle)$ we have the following simple lemma.

Lemma 3.4. *Let $\sigma \in \mathbb{F}_p^{n+2}$ and let $\varphi = \text{diag}(\sigma)$. If n_j is the number of times $j \in \mathbb{F}_p$ appears in σ , then*

$$\dim N_{\text{GL}(V)}(\langle \varphi \rangle) = n_0^2 + \dots + n_{p-1}^2.$$

Our method applied to cubic surfaces gives for every prime number p the corresponding result contained in the lists of Segre [12], and Dolgachev and Verra [4]. Theorems 3.5 and 3.8 contains the results of the method described in this section applied to threefolds and fourfolds, respectively.

Theorem 3.5. *Let $X = V(F)$ be a smooth cubic threefold in \mathbb{P}^4 that admits an automorphism φ of order p prime, then after a linear change of coordinates that diagonalizes φ , F is given in the following list, a generator of $\langle \varphi \rangle$ is given by*

$$\text{diag}(\sigma) : \mathbb{P}^4 \rightarrow \mathbb{P}^4, \quad (x_0 : \dots : x_4) \mapsto (\xi^{\sigma_0} x_0 : \dots : \xi^{\sigma_4} x_4),$$

and D is the dimension of the family of smooth cubic threefold that admits this automorphism.

$$\mathcal{T}_2^1. \quad p = 2, \quad \sigma = (0, 0, 0, 0, 1), \quad D = 7,$$

$$F = x_4^2 L_1(x_0, x_1, x_2, x_3) + L_3(x_0, x_1, x_2, x_3)$$

$$\mathcal{T}_2^2. \quad p = 2, \quad \sigma = (0, 0, 0, 1, 1), \quad D = 6,$$

$$F = x_0 L_2(x_3, x_4) + x_1 M_2(x_3, x_4) + x_2 N_2(x_3, x_4) + L_3(x_0, x_1, x_2)$$

$$\mathcal{T}_3^1. \quad p = 3, \quad \sigma = (0, 0, 0, 0, 1), \quad D = 4,$$

$$F = L_3(x_0, x_1, x_2, x_3) + x_4^3$$

$$\mathcal{T}_3^2. \quad p = 3, \quad \sigma = (0, 0, 0, 1, 1), \quad D = 1,$$

$$F = L_3(x_0, x_1, x_2) + M_3(x_3, x_4)$$

$$\mathcal{T}_3^3. \quad p = 3, \sigma = (0, 0, 0, 1, 2), \quad D = 4,$$

$$F = L_3(x_0, x_1, x_2) + x_3x_4L_1(x_0, x_1, x_2) + x_3^3 + x_4^3$$

$$\mathcal{T}_3^4. \quad p = 3, \sigma = (0, 0, 1, 1, 2), \quad D = 2,$$

$$F = L_3(x_0, x_1) + M_3(x_2, x_3) + x_4^3 + L_1(x_0, x_1)M_1(x_2, x_3)x_4$$

$$\mathcal{T}_5^1. \quad p = 5, \sigma = (0, 1, 2, 3, 4), \quad D = 2,$$

$$F = a_1x_0^3 + a_2x_0x_1x_4 + a_3x_0x_2x_3 + a_4x_1^2x_3 + a_5x_1x_2^2 + a_6x_2x_4^2 + a_7x_3^2x_4$$

$$\mathcal{T}_{11}^1. \quad p = 11, \sigma = (1, 3, 4, 5, 9), \quad D = 0,$$

$$F = x_0^2x_4 + x_3^2x_0 + x_1^2x_3 + x_1x_2^2 + x_2x_4^2$$

Here the L_i, M_i , and N_i are forms of degree i .

Proposition 3.6. *The Fermat cubic threefold belongs to all the above families except \mathcal{T}_{11}^1 . The Klein cubic threefold belongs to the families $\mathcal{T}_2^2, \mathcal{T}_3^4, \mathcal{T}_5^1$ and \mathcal{T}_{11}^1 .*

Proof. The automorphisms group $\text{Aut}(X)$ of the Fermat threefold X is isomorphic to an S_5 -extension of \mathbb{Z}_3^4 , where S_5 acts in \mathbb{P}^4 by permutation of the coordinates and \mathbb{Z}_3^4 acts in \mathbb{P}^4 by multiplication in each coordinate by a cubic root of the unit [7].

It is easy to produce matrices in $\text{Aut}(X)$ with the same spectrum of the automorphism of all the above families except \mathcal{T}_{11}^1 . Furthermore, since this group does not contain any element of order 11, it follows that X does not belong to the family \mathcal{T}_{11}^1 .

The automorphisms group $\text{Aut}(X)$ of the Klein threefold X is isomorphic to $\text{PSL}(2, \mathbb{F}_{11})$ [1]. The conjugacy classes of element of prime order in $\text{PSL}(2, \mathbb{F}_{11})$ are as follows: two conjugacy classes of elements of order 11; two conjugacy classes of elements of order 5; one conjugacy class of elements of order 3; and one conjugacy class of elements of order 2.

It follows immediately that X belongs to the families \mathcal{T}_5^1 and \mathcal{T}_{11}^1 . Since there is only one conjugacy class of orders 2 and 3, respectively, then X belongs to one and only one of the families with automorphisms of order 2 and 3, respectively. An easy computation shows that the Klein threefold belongs to \mathcal{T}_2^2 and \mathcal{T}_3^4 . □

Remark 3.7. The above proposition shows, in particular, that a generic element in all the families in Theorem 3.5 is smooth.

Theorem 3.8. *Let $X = V(F)$ be a smooth cubic fourfold in \mathbb{P}^5 that admits an automorphism φ of order p prime, then after a linear change of coordinates that diagonalizes φ , F is given in the following list, a generator of $\langle \varphi \rangle$ is given by*

$$\text{diag}(\sigma) : \mathbb{P}^5 \rightarrow \mathbb{P}^5, \quad (x_0 : \dots : x_5) \mapsto (\xi^{\sigma_0}x_0 : \dots : \xi^{\sigma_5}x_5),$$

and D is the dimension of the family of smooth cubic fourfolds that admits this automorphism.

$$\mathcal{F}_2^1. \quad p = 2, \quad \sigma = (0, 0, 0, 0, 0, 1), \quad D = 14,$$

$$F = L_3(x_0, x_1, x_2, x_3, x_4) + x_5^2 L_1(x_0, \dots, x_4).$$

$$\mathcal{F}_2^2. \quad p = 2, \quad \sigma = (0, 0, 0, 0, 1, 1), \quad D = 12,$$

$$F = L_3(x_0, x_1, x_2, x_3) + x_4^2 L_1(x_0, x_1, x_2, x_3) \\ + x_4 x_5 M_1(x_0, x_1, x_2, x_3) + x_5^2 N_1(x_0, x_1, x_2, x_3).$$

$$\mathcal{F}_2^3. \quad p = 2, \quad \sigma = (0, 0, 0, 1, 1, 1), \quad D = 10,$$

$$F = L_3(x_0, x_1, x_2) + x_0 L_2(x_3, x_4, x_5) \\ + x_1 M_2(x_3, x_4, x_5) + x_2 N_2(x_3, x_4, x_5).$$

$$\mathcal{F}_3^1. \quad p = 3, \quad \sigma = (0, 0, 0, 0, 0, 1), \quad D = 10,$$

$$F = L_3(x_0, x_1, x_2, x_3, x_4) + x_5^3.$$

$$\mathcal{F}_3^2. \quad p = 3, \quad \sigma = (0, 0, 0, 0, 1, 1), \quad D = 4,$$

$$F = L_3(x_0, x_1, x_2, x_3) + M_3(x_4, x_5).$$

$$\mathcal{F}_3^3. \quad p = 3, \quad \sigma = (0, 0, 0, 0, 1, 2), \quad D = 8,$$

$$F = L_3(x_0, x_1, x_2, x_3) + x_4^3 + x_5^3 + x_4 x_5 L_1(x_0, x_1, x_2, x_3).$$

$$\mathcal{F}_3^4. \quad p = 3, \quad \sigma = (0, 0, 0, 1, 1, 1), \quad D = 2,$$

$$F = L_3(x_0, x_1, x_2) + M_3(x_3, x_4, x_5).$$

$$\mathcal{F}_3^5. \quad p = 3, \quad \sigma = (0, 0, 0, 1, 1, 2), \quad D = 7,$$

$$F = L_3(x_0, x_1, x_2) + M_3(x_3, x_4) + x_5^3 \\ + x_3 x_5 L_1(x_0, x_1, x_2) + x_4 x_5 M_1(x_0, x_1, x_2).$$

$$\mathcal{F}_3^6. \quad p = 3, \quad \sigma = (0, 0, 1, 1, 2, 2), \quad D = 8,$$

$$F = L_3(x_0, x_1) + M_3(x_2, x_3) + N_3(x_4, x_5) \\ + \sum_{i=1,2; j=3,4; k=5,6} a_{ijk} x_i x_j x_k.$$

$$\mathcal{F}_3^7. \quad p = 3, \quad \sigma = (0, 0, 1, 1, 2, 2), \quad D = 6,$$

$$F = x_2 L_2(x_0, x_1) + x_3 M_2(x_0, x_1) + x_4^2 L_1(x_0, x_1) \\ + x_4 x_5 M_1(x_0, x_1) + x_5^2 N_1(x_0, x_1) + x_4 N_2(x_2, x_3) + x_5 O_2(x_2, x_3).$$

$$\mathcal{F}_5^1. \quad p = 5, \quad \sigma = (0, 0, 1, 2, 3, 4), \quad D = 4,$$

$$F = L_3(x_0, x_1) + x_2 x_5 L_1(x_0, x_1) + x_3 x_4 M_1(x_0, x_1) \\ + x_2^2 x_4 + x_2 x_3^2 + x_3 x_5^2 + x_4^2 x_5.$$

$$\mathcal{F}_5^2. \quad p = 5, \quad \sigma = (1, 1, 2, 2, 3, 4), \quad D = 2,$$

$$F = x_0 L_2(x_2, x_3) + x_1 M_2(x_2, x_3) + x_4 N_2(x_0, x_1) + x_5^2 M_1(x_2, x_3) + x_4^2 x_5.$$

$$\mathcal{F}_7^1. \quad p = 7, \quad \sigma = (1, 2, 3, 4, 5, 6), \quad D = 2,$$

$$F = x_0^2 x_4 + x_1^2 x_2 + x_0 x_2^2 + x_3^2 x_5 + x_3 x_4^2 + x_1 x_5^2 + a x_0 x_1 x_3 + b x_2 x_4 x_5.$$

$$\mathcal{F}_{11}^1. \quad p = 11, \quad \sigma = (0, 1, 3, 4, 5, 9), \quad D = 0,$$

$$F = x_0^3 + x_1^2x_5 + x_2^2x_4 + x_2x_3^2 + x_1x_4^2 + x_3x_5^2.$$

Here L_i, M_i, N_i and O_i are forms of degree i .

Proposition 3.9. *The Fermat cubic fourfold belongs to all the above families except $\mathcal{F}_3^7, \mathcal{F}_5^2, \mathcal{F}_7^1$, and \mathcal{F}_{11}^1 .*

Proof. The automorphisms group $\text{Aut}(X)$ of the Fermat fourfold X given by $F = x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3$ is isomorphic to an S_6 -extension of \mathbb{Z}_3^5 , where S_6 acts in \mathbb{P}^5 by permutation of the coordinates and \mathbb{Z}_3^5 acts in \mathbb{P}^5 by multiplication in each coordinate by a cubic root of the unit [7].

It is easy to produce matrices in $\text{Aut}(X)$ with the same spectrum of the automorphism of all the above families except $\mathcal{F}_3^7, \mathcal{F}_5^2, \mathcal{F}_7^1$, and \mathcal{F}_{11}^1 . Furthermore, X does not admit automorphisms of order 7 or 11, thus X does not belong to \mathcal{F}_7^1 or \mathcal{F}_{11}^1 .

For the remaining two cases we have: X does not belong to the family \mathcal{F}_5^2 since there is only one conjugacy class of elements of order 5 in S_6 and its spectrum corresponds to the family \mathcal{F}_5^1 ; and X does not belong to \mathcal{F}_3^7 since every automorphism of order 3 of X leaves F invariant and this is not the case if the family \mathcal{F}_3^7 . □

Remark 3.10. To obtain a smooth fourfold in the remaining 4 families we proceed as follows: by Remark 2.11 the Klein fourfold belongs to \mathcal{F}_7^1 ; the family \mathcal{F}_{11}^1 is reduced one fourfold, a triple covering of the Klein threefold, which is smooth; and for the remaining two families we can check with a software such as Maple that the fourfold obtained by putting all the parameters equal to one is smooth.

4. An application to abelian varieties. Let X be a smooth compact Kahler manifold of dimension n and let $q \leq n$ be a positive integer. We denote by $\mathcal{J}_q(X)$ its q -intermediate Griffiths jacobian, which is always a complex torus. If $n = 2q - 1$ then $\mathcal{J}_q(X) := \mathcal{J}(X)$ is called the intermediate jacobian of X . Furthermore, if X is a smooth hypersurface of degree d of \mathbb{P}^{n+1} it is known [3, p. 22] that $\mathcal{J}(X)$ is a non-trivial principally polarized abelian variety (p.p.a.v.) if and only if $n = 1$ and $d \geq 3$, $n = 3$ and $d = 3, 4$, or $n = 5$ and $d = 3$.

Let \mathcal{A}_g be the moduli space of p.p.a.v.'s of dimension g . It follows that $\mathcal{J}(X_3) \in \mathcal{A}_5$ and $\mathcal{J}(X_5) \in \mathcal{A}_{21}$, where X_3 and X_5 are the Klein cubic threefold and 5-fold, respectively (see Definition 2.9).

The Klein 3-fold X_3 admits an automorphism of prime order $p = 11$, see Theorem 3.5. By Theorems 2.6 and 2.10, the Klein 5-fold X_5 is the only 5-fold admitting an automorphism of prime order $p = 43$ given by

$$\varphi(x_0, \dots, x_6) = (\xi x_0, \xi^{41} x_1, \xi^4 x_2, \xi^{35} x_3, \xi^{16} x_4, \xi^{11} x_5, \xi^{21} x_6).$$

The intermediate jacobians $\mathcal{J}(X_3)$ and $\mathcal{J}(X_5)$ are p.p.a.v. of complex multiplication type [2] and they are zero-dimensional components of the singular locus of \mathcal{A}_5 and \mathcal{A}_{21} , respectively [5, Theorem 1].

It was proved in [5, Theorem 2] that $\mathcal{J}(X_3)$ is a non-isolated singular point of \mathcal{A}_5 . The automorphism φ of X_5 induces an automorphism $\tilde{\varphi}$ of the tangent space $T_0\mathcal{J}(X_5)$. A routine computation shows that the spectrum of $\tilde{\varphi}$ is

$$\{\xi^2, \xi^3, \xi^5, \xi^8, \xi^9, \xi^{12}, \xi^{13}, \xi^{14}, \xi^{15}, \xi^{17}, \xi^{19}, \xi^{20}, \xi^{22}, \xi^{25}, \xi^{27}, \xi^{32}, \xi^{33}, \xi^{36}, \xi^{37}, \xi^{39}, \xi^{42}\}.$$

The spectrum of $\tilde{\varphi}$ is stabilized by the map

$$\psi : \mathbb{F}_{43}^* \rightarrow \mathbb{F}_{43}^*, \quad a \mapsto \psi(a) = a^{11}.$$

It follows by [5, Theorem 1] that $\mathcal{J}(X_5)$ is also a non-isolated singular point of \mathcal{A}_{21} . Indeed, $\mathcal{J}(X_5)$ is contained in a 33-dimensional component of the singular locus of \mathcal{A}_{21} admitting an automorphism of order 7.

References

- [1] A. ADLER, On the automorphism group of a certain cubic threefold, *Amer. J. Math.* **100** (1978), 1275–1280.
- [2] H. BENNAMA AND J. BERTIN, Remarque sur les variétés abéliennes avec un automorphisme d'ordre premier, *Manuscripta Math.* **94** (1997), 409–425.
- [3] P. DELIGNE, Cohomologie des intersections complètes, *Groupes de Monodromie en Géométrie Algébrique. SGA 7 II, Exposé XI*, Springer Lectures Notes **340**, 1973.
- [4] I. V. DOLGACHEV AND A. VERRA, Topics in classical algebraic geometry, preprint, 2004.
- [5] V. GONZÁLEZ-AGUILERA, J. M. MUÑOZ-PORRAS, AND A.G.R. ZAMORA, On the 0-dimensional irreducible components of the singular locus of \mathcal{A}_g , *Arch. Math.* **84** (2005), 298–303.
- [6] F. KLEIN, Über die Ausflösung gewisser Gleichungen vom siebenten und achten Grade, *Math. Ann.* **15** (1879), 251–282.
- [7] A. KONTOGEOORGIS, Automorphisms of Fermat-like varieties, *Manuscripta math.* **107** (2002), 187–205.
- [8] A. LIENDO, Sobre hipersuperficies cúbicas de \mathbb{P}^n , Master thesis at the Universidad Técnica Federico Santa María, 2005.
- [9] H. MATSUMURA AND P. MONSKY, On the automorphisms of hypersurfaces, *J. Math. Kyoto Univ.* **3** (1964), 347–361.
- [10] X. ROULLEAU, The Fano surface of the Klein cubic threefold, *J. Math. Kyoto Univ.* **49** (2009), 113–129.
- [11] E. SZABO, Bounding automorphism groups, *Math. Ann.* **304** (1996), 801–811.
- [12] B. SEGRE, The non-singular cubic surfaces, Oxford at the Clarendon Press (1942).

VÍCTOR GONZÁLEZ-AGUILERA
Departamento de Matemáticas,
Universidad Técnica Federico Santa María,
Casilla 110-V,
Valparaíso,
Chile
e-mail: victor.gonzalez@usm.cl

ALVARO LIENDO
Mathematisches Institut,
Universität Basel,
Rheinsprung 21,
CH-4051 Basel,
Switzerland
e-mail: alvaro.liendo@unibas.ch

Received: 9 September 2010

Revised: 3 February 2011