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# Rationally integrable vector fields and rational additive group actions

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We characterize rational actions of the additive group on algebraic varieties defined over a field of characteristic zero in terms of a suitable integrability property of their associated velocity vector fields. This extends the classical correspondence between regular actions of the additive group on affine algebraic varieties and the so-called locally nilpotent derivations of their coordinate rings. Our results lead in particular to a complete characterization of regular additive group actions on semi-affine varieties in terms of their associated vector fields. Among other applications, we review properties of the rational counterpart of the Makar–Limanov invariant for affine varieties and describe the structure of rational homogeneous additive group actions on toric varieties.

*Keywords*: Rational additive group actions; rationally integrable derivations; locally nilpotent derivations.

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## 0. Introduction

During the last decades, the systematic study of regular actions of the additive group $\mathbb{G}_a$ on affine varieties has provided very useful and effective tools to understand the structure of certain of these varieties, most particularly those which are very close to complex affine spaces from a topological or differential point of view. One key feature of these actions in characteristic zero is that they are uniquely determined by their associated velocity vector fields[a] which, in turn, admit a very simple, purely algebraic characterization. Namely, a global vector field on an affine $k$-variety $X = \mathrm{Spec}(A)$ is the same as a $k$-derivation $\partial$ of $A$ into itself, and derivations corresponding to additive group actions are precisely those with the property that $A$ is the increasing union of the kernels of the iterated $k$-linear operators $\partial^n$, $n \geq 1$. Derivations $\partial$ with this property are called *locally nilpotent* and the comorphism $\mu^* : A \to A[t]$ of the corresponding $\mathbb{G}_a$-action $\mu : \mathbb{G}_a \times X \to X$ on $X$ is recovered by formally taking the exponential map

$$\exp(t\partial) : A \to A[[t]], \quad f \mapsto \sum_n \frac{\partial^n(f)}{n!} t^n,$$

and observing that the local nilpotency of $\partial$ guarantees precisely that the latter factors through the subring $A[t]$ of $A[[t]]$.

The study of affine algebraic varieties from a geometry point of view benefited a lot from the rich algebraic theory of locally nilpotent derivations and therefore, it is very desirable to push further this fruitful approach to more general settings. One possible direction consists in re-interpreting the property for a global derivation $\partial$ of a ring $A$ of being locally nilpotent as a kind of "algebraic integrability condition" through the above exponential map construction. So given an arbitrary algebraic $k$-variety $X$ with field of rational functions $K_X$ and a rational vector field $\partial$ on $X$, viewed as a $k$-derivation $\partial : K_X \to K_X$, we can again define formally the exponential map

$$\exp(t\partial) : K_X \to K_X[[t]], \quad f \mapsto \sum_n \frac{\partial^n(f)}{n!} t^n,$$

and ask for counterparts in this context of the previous integrability condition. The most natural one, which we call rational integrability (Definition 1.4), is to require that the previous map factors through the subalgebra $K_X(t) \cap K_X[[t]]$ of $K_X[[t]]$. Our first main result (Theorem 1.5) shows that rationally integrable rational vector fields on a variety $X$ are in one-to-one correspondence with rational $\mathbb{G}_a$-actions $\mathbb{G}_a \times X \dashrightarrow X$ on $X$. This notion also turns out to coincide with the abstract algebraic notion of locally nilpotent derivation of a field extension $K/k$ given by Makar–Limanov [13], with the additional advantage that rational integrability can be checked directly on generators of the field $K$ over $k$.

Being local in nature, the rational integrability condition is much more flexible than the property of being locally nilpotent, and this enables the possibility to study local and global additional conditions ensuring that a rational $\mathbb{G}_a$-action is actually regular. For instance, we obtain a complete characterization of regular $\mathbb{G}_a$-actions on semi-affine varieties $X$ in terms of their associated velocity vector fields, viewed as $k$-derivations $\tilde{\partial} : \mathcal{O}_X \to \mathcal{O}_X$ from the structure sheave of $X$ to itself. Namely, we establish (Theorem 2.1) that regular $\mathbb{G}_a$-actions

[a] This is no longer the case in positive characteristic where one has to keep track of appropriate infinite collections of higher order differential operators, see e.g. [14].

on $X$ are in one-to-one correspondence with $k$-derivations $\tilde{\partial} : \mathcal{O}_X \to \mathcal{O}_X$ for which the induced $k$-derivations $\partial : K_X \to K_X$ and $\Gamma(X, \tilde{\partial}) : \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X)$ of the field of rational functions and the ring of global regular functions on $X$ are respectively rationally integrable and locally nilpotent. In the case where $X$ is not semi-affine, these two conditions are in general no longer sufficient to characterize regular $\mathbb{G}_a$-actions. Nevertheless, they guarantee, thanks to a general construction due to Zaitsev [19], the existence of a partial completion of $X$ on which the rational $\mathbb{G}_a$-action on $X$ given by $\partial$ extends to a regular action.

The last section of the paper contains three applications of these notions. The first concerns a generalization to the rational context of the Makar–Limanov invariant [13] and of its behavior under stabilization. In our second application we give a combinatorial description of homogeneous rational $\mathbb{G}_a$-actions on toric varieties from which we derive a more conceptual proof of a characterization of regular homogeneous $\mathbb{G}_a$-actions on semi-affine toric varieties due to Demazure [4]. The last application consists of a characterization of line bundle torsors in terms of rational $\mathbb{G}_a$-actions.

## 1. Basic Results on Rational Actions of the Additive Group

In what follows, the term variety refers to a separated geometrically integral scheme of finite type over a fixed base field $k$ of characteristic zero. We denote by $\overline{k}$ an algebraic closure of $k$. An algebraic group over $k$ is a group object in the category of $k$-varieties. In particular, every algebraic group $G$ in our sense is connected. We denote by $e_G : \mathrm{Spec}(k) \to G$ the neutral element of $G$ and by $m_G : G \times G \to G$ the group law morphism.

**Definition 1.1.** A *rational action* of an algebraic group $G$ on a variety $X$ is a rational map $\alpha : G \times X \dashrightarrow X$ such that $(\mathrm{pr}_1, \alpha) : G \times X \dashrightarrow G \times X$ is dominant and such that the following diagrams of rational maps commute

$$\begin{array}{ccc} G \times G \times X & \overset{\mathrm{id}_G \times \alpha}{\dashrightarrow} & G \times X \\ {\scriptstyle m_G \times \mathrm{id}_X} \downarrow & & \downarrow {\scriptstyle \alpha} \\ G \times X & \overset{\alpha}{\dashrightarrow} & X \end{array} \qquad \begin{array}{ccc} \mathrm{Spec}(k) \times X & \xrightarrow{e_G \times \mathrm{id}_X} & G \times X \\ & {\scriptstyle \mathrm{pr}_2} \searrow & \downarrow {\scriptstyle \alpha} \\ & & X. \end{array} \tag{1.1}$$

We denote by $\mathrm{dom}(\alpha)$ the largest open subset of $G \times X$ on which $\alpha$ is defined and we say that $\alpha : G \times X \dashrightarrow X$ is defined at a point $(g, x) \in G \times X$ if the latter belongs to $\mathrm{dom}(\alpha)$. If so, we denote $\alpha(g, x)$ simply by $g \cdot x$. Remark that for every point $g \in G$, $\mathrm{dom}(\alpha) \cap X_g$ is a nonempty open subset of the fiber $X_g = X \times_G \mathrm{Spec}(\kappa(g))$ of $\mathrm{pr}_1 : G \times X \to X$ over $g$ [4, Lemme 1, p. 515]. A rational action $\alpha : G \times X \dashrightarrow X$ such that $\mathrm{dom}(\alpha) = G \times X$ is called *regular*.

The conditions above mean equivalently that if $(g, x)$ and $(g', g \cdot x)$ belong to $\mathrm{dom}(\alpha)$ then $(g'g, x)$ belongs to $\mathrm{dom}(\alpha)$ and $(g'g) \cdot x = g' \cdot (g \cdot x)$. Furthermore, if

$(e_G, x) \in \mathrm{dom}(\alpha)$ then $e_G \cdot x = x$. These can be rephrased more formally by saying that rational actions of $G$ on $X$ correspond to homomorphisms of group functors $G \to \mathrm{Bir}_k(X)$, where $\mathrm{Bir}_k(X)$ is the contravariant functor ($k$-Varieties) $\to$ (Groups) which associates to every $k$-variety $T$, the group of $T$-birational maps $X \times T \dashrightarrow X \times T$. A rational action is regular if and only if the corresponding homomorphism $G \to \mathrm{Bir}_k(X)$ factors through the automorphism group functor $\mathrm{Aut}_k(X)$ of $X$.

### 1.1. *Criterion for existence of rational* $\mathbb{G}_a$*-actions*

A rational action $\alpha : \mathbb{G}_a \times X \dashrightarrow X$ of the additive group scheme $\mathbb{G}_a = \mathbb{G}_{a,k} = \mathrm{Spec}(k[t])$ on a $k$-variety $X$ with field of rational functions $K_X$ is equivalently determined by a co-action homomorphism $\alpha^* : K_X \to K_X(t)$ of fields over $k$, which factors through the valuation ring $\mathcal{O}_{\nu_0} = \{r(t) \in K_X(t) \,|\, \mathrm{ord}_0 r(t) \geq 0\}$ of $K_X(t)$ and such that the following diagrams commute

$$\begin{array}{ccc} K_X & \xrightarrow{\alpha^*} & K_X(t) \\ \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle t\mapsto t+t'} \\ K_X(t') & \xrightarrow{\tilde{\alpha}^*} & K_X(t,t') \end{array} \qquad \begin{array}{ccc} K_X & \xrightarrow{\overline{\alpha^*}} & \mathcal{O}_{\nu_0}/t\mathcal{O}_{\nu_0} \\ & \searrow{\scriptstyle \mathrm{id}} & \uparrow \\ & & K_X, \end{array} \tag{1.2}$$

where for every $f(t') = \frac{\sum a_i (t')^i}{\sum b_j (t')^j} \in K_X(t')$,

$$\tilde{\alpha}^*(f(t')) = \frac{\sum \alpha^*(a_i)(t')^i}{\sum} \alpha^*(b_j)(t')^j \in K_X(t)(t') = K_X(t,t').$$

Indeed, the condition that $\alpha^*$ factors through $\mathcal{O}_{\nu_0}$ is ensured by the fact that $\mathrm{dom}(\alpha) \cap (\{0\} \times X)$ is a nonempty open subset of $\{0\} \times X$, and the commutativity of the two diagrams expresses the usual axioms for a co-action. The following characterization is well known.

**Proposition 1.2.** *A $k$-variety $X$ admits a nontrivial rational $\mathbb{G}_a$-action if and only if it is birationally ruled, i.e. birationally isomorphic to $Y \times \mathbb{P}^1$ for some $k$-variety $Y$.*

**Proof.** Every $k$-variety of the form $Y \times \mathbb{P}^1$ admits a regular $\mathbb{G}_a$-action by projective translation on the second factor. The converse follows for instance from Rosenlicht Theorem [18] which asserts for our purpose that a $k$-variety equipped with a rational $\mathbb{G}_a$-action is $\mathbb{G}_a$-equivariantly birationally isomorphic to $U \times \mathbb{G}_a$ on which $\mathbb{G}_a$ acts by translations on the second factor for some affine $k$-variety $U$. Nevertheless we find more enlightening to give an elementary proof borrowed from Koshevoi [10]. Suppose that $\alpha : \mathbb{G}_a \times X \dashrightarrow X$ is a nontrivial rational $\mathbb{G}_a$-action and let $K_0 = K_X^{\mathbb{G}_a} = \{h \in K_X \,|\, \alpha^* h = h\}$ be its field of invariants. It is enough to show that there exists $s \in K_X \backslash K_0$ such that $\alpha^* s = s + t$ and $K_X = K_0(s)$. Note that if such an element $s$ exists, then it is transcendental over $K_0$ for otherwise, applying

$\alpha^*$ to a nontrivial polynomial relation $P(s)=0$ for some $P\in K_0[v]$ would render the conclusion that $t\in K_X(t)$ is algebraic over $K_0(s)$ whence over $K_X$, which is absurd. Furthermore, since any two elements $s_i$, $i=1,2$, such that $\alpha^* s_i = s_i + t$ differ only by the addition of an element in $K_0$, it is enough to show that for every $f\in K_X\backslash K_0$ there exists $s\in K_X$ such that $\alpha^* s = s+t$ and $f\in K_0(s)$.

Now since $\alpha$ is nontrivial, there exists $f\in K_X\backslash K_0$, and $\alpha^* f$ can be written in the form $\alpha^*(f)=(1+b(t))^{-1}a(t)$ where $a(t)=\sum_{i=0}^n a_i t^i\in K_X[t]$ with $a_0=f$, $b(t)=\sum_{i=1}^m b_i t^i \in tK_X[t]$, and either $a(t)$ or $1+b(t)$ is nonconstant. The commutativity of the first diagram (1.2) above implies that

$$\begin{aligned}
&\left(1+\sum_{i=1}^m \alpha^*(b_i)(t')^i\right)^{-1}\left(\sum_{i=0}^n \alpha^*(a_i)(t')^i\right)\\
&\quad=\left(1+\sum_{i=1}^m b_i(t+t')^i\right)^{-1}\left(\sum_{i=0}^n a_i(t+t')^i\right)\\
&\quad=\left(1+\sum_{i=1}^m b_i t^i+\sum_{i=1}^m b_{1,i}(t)(t')^i\right)^{-1}\left(\sum_{i=0}^n a_{1,i}(t)(t')^i\right)\\
&\quad=\left(1+\sum_{i=1}^m \frac{b_{1,i}(t)}{1+\sum_{i=1}^m b_i t^i}(t')^i\right)^{-1}\left(\sum_{i=0}^n \frac{a_{1,i}(t)}{1+\sum b_i t^i}(t')^i\right),
\end{aligned}$$

where $a_{1,i}(t)=\sum_{j=i}^n \binom{j}{j-i} a_j t^{j-i}$ and $b_{1,i}(t)=\sum_{j=i}^m \binom{j}{j-i} b_j t^{j-i}$. Identifying the coefficients, we obtain

$$\alpha^*(a_j)=\frac{a_{1,j}(t)}{1+\sum_{i=1}^m b_i t^i}\quad\text{and}\quad \alpha^*(b_j)=\frac{b_{1,j}(t)}{1+\sum_{i=1}^m b_i t^i}.$$

In particular, $\alpha^*(a_n^{-1})=(a_n^{-1}+\sum_{i=1}^m a_n^{-1}b_i t^i)\in K_X[t]$ and, re-using the axioms to get the equality

$$\alpha^* a_n^{-1}+\sum_{i=1}^m \alpha^*(a_n^{-1}b_i)(t')^i = a_n^{-1}+\sum_{i=1}^m a_n^{-1}b_i(t+t')^i,$$

we deduce that $\alpha^*(a_n^{-1}b_i)=a_n^{-1}\sum_{j=i}^m \binom{j}{i} b_j t^{j-i}$ for every $i=1,\ldots,m$. Thus $a_n^{-1}b_m\in K_0$, $\alpha^*(a_n^{-1}b_{m-1})=a_n^{-1}b_{m-1}+ma_n^{-1}b_m t$ and so, letting $s=\frac{a_n^{-1}b_{m-1}}{ma_n^{-1}b_m}$ we obtain that $\alpha^* s = s+t$. We further deduce by induction that $a_n^{-1}b_i\in K_0[s]$ for every $i=1,\ldots,m$. The same argument applied to $f^{-1}$ implies that $s'=\frac{a_{n-1}b_m^{-1}}{na_n b_m^{-1}}$ and also satisfies $\alpha^* s' = s'+t$ and that $f^{-1}b_m^{-1}a_i\in K_0[s']=K_0[s]$ for every $i=1,\ldots,n$. Since $b_m^{-1}a_n\in K_0$, this shows that $f\in K_0(s)$ as desired. □

The proof above shows more precisely that for every nontrivial rational $\mathbb{G}_a$-action $\alpha:\mathbb{G}_a\times X \dashrightarrow X$ there exists a decomposition $K_X = K_X^{\mathbb{G}_a}(s)$, where $K_X^{\mathbb{G}_a}$

is the field of invariants and $s$ is an element transcendental over $K_X^{\mathbb{G}_a}$ satisfying $\alpha^* s = s + t$, for which $\alpha^*$ takes the form

$$\alpha^* = \alpha^*_{(K_X^{\mathbb{G}_a}, s)} : K_X = K_X^{\mathbb{G}_a}(s) \to K_X^{\mathbb{G}_a}(s)(t), \quad f(s) \mapsto \alpha^*_{(K_X^{\mathbb{G}_a}, s)}(f(s)) = f(s+t). \tag{1.3}$$

An element $s \in K_X$ with the above properties is called a *rational slice* for the action $\alpha$.

**Example 1.3.** A smooth curve $C$ admits a rational $\mathbb{G}_{a,k}$-action if and only it is birational to $\mathbb{P}^1_k$. Indeed, by Proposition 1.2, $C$ admits a rational $\mathbb{G}_{a,k}$-action if and only if $K_C = K_0(s)$ for some element $s$ transcendental over $K_0$. This implies that $K_0$ is an algebraic extension of $k$ and that $C$ is birational over $k$ to $\mathbb{P}^1_{K_0}$. But since by hypothesis $C$ is geometrically integral, we have $K_0 = k$ necessarily and so, $C \overset{\sim}{\dashrightarrow} \mathbb{P}^1_k$.

### 1.2. *Rational $\mathbb{G}_a$-actions and rational vector fields*

Every rational $\mathbb{G}_a$-action $\alpha : \mathbb{G}_a \times X \dashrightarrow X$ on a $k$-variety $X$ gives rise to a *rational vector field*, i.e. a $k$-derivation $\tilde{\partial} : \mathcal{O}_X \to \mathcal{K}_X$ from the structure sheaf $\mathcal{O}_X$ to the constant sheaf $\mathcal{K}_X$ of rational functions on $X$, consisting of velocity vectors along germs of general orbits. More precisely, $\alpha$ induces a rational homomorphism of sheaves

$$\eta : \alpha^*\Omega^1_{X/k} \to \Omega^1_{\mathbb{G}_a \times X/k} \to \Omega^1_{\mathbb{G}_a \times X/X}$$

on $\mathbb{G}_a \times X$, where $\Omega^1_{\mathbb{G}_a \times X/X}$ is the sheaf of relative differentials of the second projection $\mathrm{pr}_X : \mathbb{G}_a \times X \to X$. Pulling back by the zero section morphism $e_X : X \to \mathbb{G}_a \times X$, $x \mapsto (0, x)$, whose image intersects $\mathrm{dom}(\alpha)$ by definition, we obtain a global section $e_X^*\eta : e_X^*\alpha^*\Omega^1_{X/k} \simeq \Omega^1_{X/k} \to e_X^*\Omega^1_{\mathbb{G}_a \times X/X} \simeq \mathcal{O}_X$ of the sheaf $\mathcal{H}om_X(\Omega^1_{X/k}, \mathcal{O}_X) \otimes \mathcal{K}_X$, hence by composition with the canonical $k$-derivation $d : \mathcal{O}_X \to \Omega^1_{X/k}$, a $k$-derivation $\tilde{\partial} : \mathcal{O}_X \to \mathcal{K}_X$. Furthermore, we can extend this derivation via the Leibniz rule to a $k$-derivation from $\mathcal{K}_X$ to $\mathcal{K}_X$. We denote this derivation with the same symbol $\tilde{\partial} : \mathcal{K}_X \to \mathcal{K}_X$.

If the $\mathbb{G}_a$-action $\alpha$ is regular, then $\eta : \alpha^*\Omega^1_{X/k} \to \Omega^1_{\mathbb{G}_a \times X/X}$ is regular homomorphism, giving rise to global section $e_X^*\eta$ of $\mathcal{H}om_X(\Omega^1_{X/k}, \mathcal{O}_X)$, for which the corresponding derivation $\tilde{\partial} : \mathcal{O}_X \to \mathcal{K}_X$ factors through $\mathcal{O}_X$. In the case of a regular $\mathbb{G}_a$-action $\alpha : \mathbb{G}_a \times X \to X$ on an affine variety $X = \mathrm{Spec}(A)$, the $k$-derivation $\partial = \Gamma(X, \tilde{\partial}) \in \mathrm{Der}_k(A)$ deduced from $\tilde{\partial} : \mathcal{O}_X \to \mathcal{O}_X$ coincides simply with the one $\partial = \frac{d}{dt}|_{t=0} \circ \alpha^* : A \to A[t]/tA[t] \simeq A$. It is well known (see e.g. [13]) that a $k$-derivation $\partial \in \mathrm{Der}_k(A)$ arises from a regular $\mathbb{G}_a$-action on $X$ if and only if it is "algebraically integrable" in the sense that the formal exponential homomorphism $\exp(t\partial) : A \to A[[t]]$ factors through a homomorphism $\alpha^* : A \to A[t] \subset A[[t]]$. This holds precisely when $A = \bigcup_{n \geq 1} \mathrm{Ker}\, \partial^n$, and derivations with this property are called *locally nilpotent*.

Being locally nilpotent is not a local property in the Zariski topology since for instance the restriction of a locally nilpotent derivation to a non-$\mathbb{G}_a$-stable affine open subset of $X$ is no longer locally nilpotent (see Example 1.8). In contrast, the following weaker form of the algebraic integrability condition behaves well under localization.

**Definition 1.4.** A $k$-derivation $\tilde{\partial} : \mathcal{K}_X \to \mathcal{K}_X$ on a variety $X$ is called *rationally integrable* if the formal exponential homomorphism

$$\exp(t\tilde{\partial}) : \mathcal{K}_X \to \mathcal{K}_X[[t]], \quad f \mapsto \sum \frac{\tilde{\partial}^n f}{n!} t^n$$

factors through $\mathcal{K}_X(t) \cap \mathcal{K}_X[[t]]$.

By definition, every rationally integrable $k$-derivation $\tilde{\partial} : \mathcal{K}_X \to \mathcal{K}_X$ induces a global rational $k$-derivation $\partial = \Gamma(X, \tilde{\partial}) : K_X \to K_X$ which gives rise in turn to a homomorphism $\alpha^* = \exp(t\partial) : K_X \to K_X(t)$ factoring through $\mathcal{O}_{\nu_0}$ and satisfying the axioms of a rational co-action of $\mathbb{G}_a$. Conversely, for every rational $\mathbb{G}_a$-action $\alpha : \mathbb{G}_a \times X \dashrightarrow X$ with associated co-morphism $\alpha^* : K_X \to K_X(t)$, the fact that $\alpha^*$ factors through $\mathcal{O}_{\nu_0}$ guarantees that the $k$-linear homomorphism

$$\partial = \overline{\frac{d}{dt} \circ \alpha^*} : K_X \to \mathcal{O}_{\nu_0} \xrightarrow{\frac{d}{dt}} \mathcal{O}_{\nu_0} \to \mathcal{O}_{\nu_0}/t\mathcal{O}_{\nu_0} \simeq K_X \tag{1.4}$$

is well defined and the commutativity of the second diagram (1.2) implies that $\partial$ is a $k$-derivation. In fact, if we write $K_X = K_X^{\mathbb{G}_a}(s)$ for a suitable rational slice $s$ in such a way that $\alpha^*$ takes the form $\alpha^*_{(K_X^{\mathbb{G}_a}, s)}$ as in (1.3), then $\partial$ coincides with the $k$-derivation $\frac{\partial}{\partial s} : K_X^{\mathbb{G}_a}(s) \to K_X^{\mathbb{G}_a}(s)$. We deduce in turn from Taylor's formula that

$$\exp(t\partial)(f(s)) = \sum \frac{t^n}{n!} \frac{\partial^n}{\partial s^n} f(s) = f(s+t) = \alpha^*(f(s)).$$

Summing up, we obtain the following characterization.

**Theorem 1.5.** *There exists a one-to-one correspondence between rational $\mathbb{G}_a$-actions $\alpha : \mathbb{G}_a \times X \dashrightarrow X$ on a $k$-variety $X$ and rationally integrable $k$-derivations $\tilde{\partial} : \mathcal{K}_X \to \mathcal{K}_X$.*

For a rational $\mathbb{G}_a$-action $\alpha : \mathbb{G}_a \times X \dashrightarrow X$ associated with a rationally integrable $k$-derivation $\partial = \Gamma(X, \tilde{\partial}) : K_X \to K_X$, the field of invariants $K_X^{\mathbb{G}_a}$ is equal to the kernel $\operatorname{Ker} \partial$ of $\partial$ while rational slices for $\alpha$ coincide precisely with elements $s \in K_X$ such that $\partial s = 1$.

**Remark 1.6.** In [13], a $k$-derivation $\partial : K \to K$ of a field extension $K/k$ is called locally nilpotent if $K$ is equal to the field of fractions of its subring $\operatorname{Nil}(\partial) = \bigcup_{n \geq 0} \operatorname{Ker} \partial^n$. In the case where $K = K_X$ is the field of rational functions on a $k$-variety $X$, this property turns out to be equivalent to the rational integrability of the associated derivation $\partial : \mathcal{K}_X \to \mathcal{K}_X$. Indeed, by virtue of [13, Lemma 2, p. 13] and Proposition 1.2 the two notions are both equivalent to the property that

$K_X$ is a purely transcendental extension of its subfield $\mathrm{Ker}\,\partial$. The formulation in terms of rational integrability has the advantage to be easier to check in practice: by definition, if $K_X = k(f_1, \ldots, f_n)$ then a $k$-derivation $\partial : K_X \to K_X$ is rationally integrable if and only if $\exp(t\partial)(f_i) \in K_X(t)$ for every $i = 1, \ldots, n$.

**Example 1.7.** The derivation $\partial = x\frac{\partial}{\partial x} : k[x] \to k[x]$ is not locally nilpotent, and since $\exp(t\partial)(x) = x\exp(t) \in k(x)[[t]]\backslash k(x)(t)$, the induced $k$-derivation of $k(x)$ is not rationally integrable, with field of invariants $\mathrm{Ker}\,\partial = k$.

**Example 1.8.** Let $\tilde{\partial} : \mathcal{O}_{\mathbb{A}^1} \to \mathcal{O}_{\mathbb{A}^1}$ be the $k$-derivation associated with the regular action of $\mathbb{G}_a$ on $\mathbb{A}^1 = \mathrm{Spec}(k[x])$ by translations. Then $\Gamma(\mathbb{A}^1, \tilde{\partial}) = \frac{\partial}{\partial x}$ is a locally nilpotent derivation of $k[x]$. On the other hand, for every nonconstant polynomial $p \in k[x]$, the $k$-derivation of $k[x]_{p(x)}$ induced by $\tilde{\partial}$ is rationally integrable but not locally nilpotent, defining a rational $\mathbb{G}_a$-action of the principal open subset $U_p = \mathrm{Spec}(k[x]_{p(x)})$ of $\mathbb{A}^1$.

**Example 1.9.** The derivation $\partial = -x^2\frac{\partial}{\partial x} : k[x] \to k[x]$ is not locally nilpotent. However, the equality

$$\exp(t\partial)(x) = \sum_{n=0}^{\infty} \frac{\partial^n x}{n!} t^n = \sum_{n=0}^{\infty} (-1)^n x^{n+1} t^n = \frac{x}{1+tx}$$

in $k(t)[[x]]$ implies that the induced derivation of $k(x)$ is rationally integrable with $s = x^{-1}$ as a slice, and hence defines a rational $\mathbb{G}_a$-action $\alpha : \mathbb{G}_a \times \mathbb{A}^1 \dashrightarrow \mathbb{A}^1$ on $\mathbb{A}^1 = \mathrm{Spec}(k[x])$. In fact, $\alpha$ coincides simply with the restriction to the open subset $\mathbb{P}^1\backslash\{[1:0]\}$ of $\mathbb{P}^1 = \mathrm{Proj}(k[u,v])$ of the regular $\mathbb{G}_a$-action $t \cdot [u:v] = [u : v + tu]$.

In the examples above, the derivation $\tilde{\partial} : \mathcal{O}_X \to \mathcal{K}_X$ factors through $\mathcal{O}_X$, in other words, the *a priori* rational vector field is in fact regular. The following examples illustrate the situation where the $\mathbb{G}_a$-action is induced by genuinely rational vector fields.

**Example 1.10.** The $k$-derivation $\partial = x^{-1}\frac{\partial}{\partial y} : k(x,y) \to k(x,y)$ is rationally integrable and its associated rational $\mathbb{G}_a$-action $\alpha : \mathbb{G}_a \times X \dashrightarrow X$, $(x,y) \mapsto (x, y + \frac{t}{x})$ on $X = \mathrm{Spec}(k[x,y])$ restricts to a regular one on the open subset $U = X_x = \mathrm{Spec}(k[x^{\pm 1}, y])$ where $\partial$ is actually locally nilpotent. But $\mathrm{dom}(\alpha) \cap (\{0\} \times X) = \{0\} \times U$ and in fact, $(t,p) \notin \mathrm{dom}(\alpha)$ for all $p \in X\backslash U$ and $t \in \mathbb{G}_a$.

**Example 1.11.** By virtue of Proposition 1.2 and Theorem 1.5, a nonzero $k$-derivation $\partial : k(x_1, \ldots, x_n) \to k(x_1, \ldots, x_n)$ is rationally integrable if and only if there exists an element $s \in k(x_1, \ldots, x_n)$ purely transcendental over $K_0 = \mathrm{Ker}\,\partial$ such that $\partial(s) = 1$ and an isomorphism $k(x_1, \ldots, x_n) \simeq K_0(s)$.

If $n = 2$, then by Lüroth theorem $K_0$ is itself purely transcendental over $k$, say $K_0 = k(c_1)$ for some $c_1 \in k(x_1, x_2)$ transcendental of $k$ and we obtain that every nontrivial rational $\mathbb{G}_a$-action on $\mathbb{A}^2 = \mathrm{Spec}(k[x_1, x_2])$ is birationally conjugated to a translation defined by a locally nilpotent derivation $\partial = \frac{\partial}{\partial s}$ for some isomorphism $k(x_1, x_2) \simeq k(c_1, s)$. This provides a rational counterpart of a classical result of Rentschler [17] which asserts that every nontrivial regular $\mathbb{G}_{a,k}$-action on $\mathbb{A}^2$ is biregularly conjugated to a "twisted translation" associated with a locally nilpotent derivation of the form $r(x)\frac{\partial}{\partial y}$ for some nonzero polynomial $r(x) \in k[x]$.

If $n = 3$ and $k$ is algebraically closed then $K_0$ is the function field of a unirational hence rational surface, so $K_0 \simeq k(c_1, c_2)$ for some algebraically independent elements $c_1, c_2 \in k(x_1, x_2, x_3)$, and we conclude again that every nontrivial rational $\mathbb{G}_a$-action on $\mathbb{A}^3 = \mathrm{Spec}(k[x_1, x_2, x_3])$ is birationally conjugated to a translation $\partial = \frac{\partial}{\partial s}$ for some isomorphism $k(x_1, x_2) \simeq k(c_1, c_2)(s)$. Note that in contrast, there exist locally nilpotent derivations of $k[x_1, x_2, x_3]$ which are not biregularly triangularizable [1].

The same conclusion holds for $n = 4$ when $k$ is algebraically closed. Indeed, by virtue of [5], $K_0$ is $k$-ruled, i.e. isomorphic over $k$ to a purely transcendental extension $K_1(c_3)$ of a subfield $K_1 \subset K_0$ of transcendence degree 2 over $k$. The latter being in turn the function field of a unirational hence rational surface, we obtain isomorphisms $K_1 \simeq k(c_1, c_2)$, $K_0 \simeq k(c_1, c_2)(c_3)$ and finally $k(x_1, x_2, x_3, x_4) \simeq k(c_1, c_2, c_3)(s)$ for which $\partial = \frac{\partial}{\partial s}$.

## 2. Regular Actions of the Additive Group on Semi-Affine Varieties

Recall that a $k$-variety $X$ is called semi-affine if the canonical morphism $p : X \to X_0 = \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ is proper. In this case $\Gamma(X, \mathcal{O}_X)$ is finitely generated and so $X_0$ is an affine variety [8, Corollary 3.6]. For instance, complete or affine $k$-varieties are semi-affine. By the previous subsection, every regular $\mathbb{G}_a$-action $\alpha : \mathbb{G}_a \times X \to X$ on a $k$-variety $X$ gives rise to a rationally integrable $k$-derivation $\tilde{\partial} : \mathcal{O}_X \to \mathcal{O}_X$. Conversely, the following theorem shows that in the case where $X$ is semi-affine, a rationally integrable derivation $\tilde{\partial} : \mathcal{O}_X \to \mathcal{O}_X$ corresponds to a regular $\mathbb{G}_a$-action if and only if the associated global $k$-derivation $\Gamma(X, \tilde{\partial}) : \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X)$ is locally nilpotent.

**Theorem 2.1.** *Regular $\mathbb{G}_a$-actions on a semi-affine variety $X$ are in one-to-one correspondence with rationally integrable $k$-derivations $\tilde{\partial} : \mathcal{O}_X \to \mathcal{O}_X$ such that the derivation $\Gamma(X, \tilde{\partial}) : \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X)$ on the ring of global regular functions is locally nilpotent.*

**Proof.** By Rosenlicht theorem [18], for any regular $\mathbb{G}_a$-action on $X$ there exists a nonempty $\mathbb{G}_a$-invariant affine open subset $U$. Hence, $\Gamma(U, \tilde{\partial})$ is locally nilpotent and since $\Gamma(X, \mathcal{O}_X) \subset \Gamma(U, \mathcal{O}_X)$ it follows that $\Gamma(X, \tilde{\partial})$ is a locally nilpotent derivation of $\Gamma(X, \mathcal{O}_X)$. Conversely, let $\tilde{\partial} : \mathcal{O}_X \to \mathcal{O}_X$ be a derivation such that $\partial_0 = \Gamma(X, \tilde{\partial}) : \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X)$ is locally nilpotent. Then $\partial_0$ induces a possibly trivial regular $\mathbb{G}_a$-action $\alpha_0 : \mathbb{G}_a \times X_0 \to X_0$ on $X_0 = \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ for which the canonical morphism $p : X \to X_0$ is $\mathbb{G}_a$-equivariant. In particular, for every point $x \in X$, letting $\xi = \alpha|_{\mathbb{G}_a \times \{x\}} : \mathbb{G}_a \dashrightarrow X$, $t \mapsto \alpha(t,x)$ and $\xi_0 = \alpha_0|_{\mathbb{G}_a \times p(x)} : \mathbb{G}_a \to X_0, t \mapsto \alpha_0(t, p(x))$, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{G}_a & \overset{\xi}{\dashrightarrow} & X \\ & \underset{\xi_0}{\searrow} & \downarrow p \\ & & X_0. \end{array}$$

Since $p$ is proper, we deduce from the valuative criterion for properness applied to the local ring of every closed point $t \in \mathbb{G}_a$ that $\alpha$ is defined at every point $(x,t) \in \mathbb{G}_a \times X$ whence is a regular $\mathbb{G}_a$-action on $X$. □

As a consequence of the proof of the above theorem, we obtain the following criterion to decide whether a derivation gives rise to a regular $\mathbb{G}_a$-action on a semi-affine variety.

**Corollary 2.2.** *Let $X$ be a semi-affine variety and let $\tilde{\partial} : \mathcal{O}_X \to \mathcal{O}_X$ be a $k$-derivation. Then $\tilde{\partial}$ defines a regular $\mathbb{G}_a$-action on $X$ if and only if there exists a nonempty affine open subset $U \subset X$ such that $\Gamma(U, \tilde{\partial}) : \Gamma(U, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ is locally nilpotent.*

**Example 2.3.** The semi-affineness hypothesis cannot be weakened. For instance, letting $X = \mathbb{A}^2_* = \mathrm{Spec}(k[x,y]) \setminus \{(0,0)\}$, the derivation $\tilde{\partial} = \frac{\partial}{\partial x} : \mathcal{O}_X \to \mathcal{O}_X$ only defines a rational $\mathbb{G}_a$-action $\alpha : \mathbb{G}_a \times X \dashrightarrow X$ since for a point of the form $p = (x_0, 0) \in X$ the orbit map $\xi : \mathbb{G}_a \dashrightarrow X$, $t \mapsto \alpha(t,p) = (x_0 + t, 0)$ is not defined at $t_0 = -x_0$. On the other hand, the restriction of $\frac{\partial}{\partial x}$ to the principal affine open subset $\{y \neq 0\}$ of $X$ is locally nilpotent.

The previous example illustrates the typical situation where a rationally integrable $k$-derivation $\tilde{\partial} : \mathcal{O}_X \to \mathcal{K}_X$ factoring through $\mathcal{O}_X$ does not give rise to a regular $\mathbb{G}_a$-action $\alpha : \mathbb{G}_a \times X \to X$. Namely, even though $\{0\} \times X$ is contained in the domain of definition $\mathrm{dom}(\alpha)$ of $\alpha$, the $\mathbb{G}_a$-orbit of a point $x$ might not be defined for every time $t \in \mathbb{G}_a$. Nevertheless, in such situations, the following result, which is consequence of a general construction due to Zaitsev [19, Theorem 4.12] (see also [3]), shows that it is always possible to find a minimal equivariant partial completion of $X$ on which the $\mathbb{G}_a$-action extends to a regular one.

**Proposition 2.4.** *Let $X$ be an algebraic variety equipped with a rational $\mathbb{G}_a$-action $\alpha : \mathbb{G}_a \times X \dashrightarrow X$ associated to a rationally integrable $k$-derivation $\tilde{\partial} : \mathcal{O}_X \to \mathcal{O}_X$. Then there exists an algebraic variety $\overline{X}$ equipped with a regular $\mathbb{G}_a$-action $\overline{\alpha} : \mathbb{G}_a \times \overline{X} \to \overline{X}$ and a $\mathbb{G}_a$-equivariant open immersion $j : X \hookrightarrow \overline{X}$. Furthermore, such a triple $(\overline{X}, \overline{\alpha}, j)$ with the additional property that $\overline{X} \setminus X$ contains no $\mathbb{G}_a$-orbits is unique up to equivalence.*

## 3. Applications

### 3.1. *The rational Makar–Limanov invariant*

By analogy with the usual Makar–Limanov invariant [13] of an affine $k$-variety $X = \mathrm{Spec}(A)$, which is defined as the subalgebra $\mathrm{ML}(A)$ of $A$ consisting of regular functions on $X$ which are invariant under all regular $\mathbb{G}_a$-action on $X$, it is natural to define the *rational Makar–Limanov invariant* of a $k$-variety $X$ as the subfield $\mathrm{RML}(X)$ of $K_X$ consisting of rational functions on $X$ which are invariant under all rational $\mathbb{G}_a$-actions on $X$. Equivalently, $\mathrm{RML}(X)$ is equal to the intersection in $K_X$ of the kernels of all rationally integrable $k$-derivations of $K_X$. The RML invariant of a $k$-rational variety is clearly equal to $k$ while Proposition 1.2 shows in particular that $\mathrm{RML}(X) = K_X$ if and only if $X$ is not birationally ruled. The following proposition provides the rational counterpart of a result due to Makar–Limanov [13, Lemma 21] which asserts that if $A$ is a $k$-algebra such that $\mathrm{ML}(A) = A$ then $\mathrm{ML}(A[x]) = A$.

**Proposition 3.1.** *If $X$ is not birationally ruled then the projection $\mathrm{pr}_X : X \times \mathbb{P}^1 \to X$ is invariant under all rational $\mathbb{G}_a$-actions on $X \times \mathbb{P}^1$.*

**Proof.** Let $K_{X\times\mathbb{P}^1} = K_X(u)$ where $u$ is transcendental over $K_X$. By virtue of Proposition 1.2, a rational $\mathbb{G}_a$-action $\alpha : \mathbb{G}_a \times (X \times \mathbb{P}^1) \dashrightarrow X \times \mathbb{P}^1$ on $X \times \mathbb{P}^1$ gives rise to a decomposition $K_{X\times\mathbb{P}^1} = K^{\mathbb{G}_a}_{X\times\mathbb{P}^1}(s)$ for a suitable rational slice $s$. Letting $\nu_0$ be the restriction of the $u^{-1}$-adic valuation on $K_{X\times\mathbb{P}^1}$ to the subfield $K^{\mathbb{G}_a}_{X\times\mathbb{P}^1}$, it is enough to show that $\nu_0(x) = 0$ for every $x \in K^{\mathbb{G}_a}_{X\times\mathbb{P}^1}$. Indeed, noting that the residue field of the $u^{-1}$-adic valuation on $K_{X\times\mathbb{P}^1}$ is equal to $K_X$, this will imply that $K^{\mathbb{G}_a}_{X\times\mathbb{P}^1}$ is contained in $K_X$ whence is equal to it since these two fields have the same transcendence degree over $k$ and are both algebraically closed in $K_{X\times\mathbb{P}^1}$. So suppose on the contrary that there exists $x \in K^{\mathbb{G}_a}_{X\times\mathbb{P}^1}$ transcendental over $k$ with $\nu_0(x) \neq 0$. Up to changing $x$ for its inverse we may assume that $\nu_0(x) < 0$. It follows that the transcendence degree of the residue field $\kappa_0$ of $\nu_0$ over $k$ is strictly smaller than that of $K^{\mathbb{G}_a}_{X\times\mathbb{P}^1}$. The Ruled Residue Theorem [16] then implies that $K_X$ is a simple transcendental extension of the algebraic closure of $\kappa_0$ in $K_X$, in contradiction with the hypothesis that $X$ is not birationally ruled. □

**Corollary 3.2.** *A $k$-variety admits two rational $\mathbb{G}_a$-actions $\alpha_i : \mathbb{G}_a \times X \dashrightarrow X$, $i = 1, 2$, such that for a general $k$-rational point $x \in X$ the rational orbit maps $\alpha_i|_{\mathbb{G}_a\times\{x\}} : \mathbb{G}_a \dashrightarrow X$, $t \mapsto \alpha_i(t, x)$ do not coincide if and only if it is birationally isomorphic over $k$ to $Y \times \mathbb{P}^2$ for some $k$-variety $Y$.*

One could have expected more generally that if $X$ is not birationally ruled then for every $n \geq 1$ the projection $\mathrm{pr}_X : X \times \mathbb{P}^n \to X$ is invariant under all rational $\mathbb{G}_a$-actions on $X \times \mathbb{P}^n$. But this is wrong, as shown by the following example derived from a famous counterexample to the birational version of the Zariski Cancellation Problem [2].

**Example 3.3.** The affine threefold $X \subset \mathbb{A}^4_{\mathbb{C}} = \operatorname{Spec}(\mathbb{C}[x,y,z,t])$ defined by the equation

$$y^2 + (t^4+1)(t^6+t^4+1)z^2 = 2x^3 + 3t^2x^2 + +t^4 + 1$$

has no nontrivial rational $\mathbb{G}_a$-actions but $\operatorname{RML}(X \times \mathbb{A}^3) = \mathbb{C}$.

**Proof.** By virtue of [15, Example 2.9], $X$ is a unirational, non-rational affine variety with the property that $X \times \mathbb{A}^3$ is rational. So $\operatorname{RML}(X \times \mathbb{A}^3) = \mathbb{C}$ and it remains to check that $\operatorname{RML}(X) = K_X$. By virtue of Proposition 1.2, the existence of a nontrivial rational $\mathbb{G}_a$-action on $X$ would imply that $X$ is birationally isomorphic to $S \times \mathbb{A}^1$ for a smooth affine surface $S$. But since $X$ is unirational, $S$ would be unirational whence rational and so would be $X$, a contradiction. □

**Remark 3.4.** In the regular case, an example of a smooth rational affine surface $S = \operatorname{Spec}(A)$ such that $\operatorname{ML}(S) = A$ but $\operatorname{ML}(S \times \mathbb{A}^2) = \mathbb{C}$ was given in [6].

### 3.2. *Homogeneous rational* $\mathbb{G}_a$*-actions on toric varieties*

Recall that a *toric variety* $X$ is a normal $k$-variety equipped with an effective regular action $\mu : \mathbb{T} \times X \to X$ of a split torus $\mathbb{T} = \mathbb{G}^n_{m,k}$ having an open orbit. A rational $\mathbb{G}_a$-action $\alpha : \mathbb{G}_a \times X \dashrightarrow X$ on $X$ is said to be $\mathbb{T}$-homogeneous if it semi-commutes with the action of $\mathbb{T}$. This means equivalently that the subgroup of $\operatorname{Bir}_k(X)$ generated by the regular action $\mu$ of $\mathbb{T}$ and the rational action $\alpha$ of $\mathbb{G}_a$ is isomorphic to an algebraic group of the form $\mathbb{T} \ltimes \mathbb{G}_a$. In this subsection, we give a combinatorial characterization of homogeneous rational $\mathbb{G}_a$-actions on a toric variety $X$ in terms of their corresponding rationally integrable derivations.

Let us briefly recall from [7] some basic facts about the combinatorial description of toric varieties. Let $M = \operatorname{Hom}(\mathbb{T}, \mathbb{G}_{m,k})$ be the character lattice and let $N = \operatorname{Hom}(\mathbb{G}_{m,k}, \mathbb{T})$ be the 1-parameter subgroup lattice of $\mathbb{T}$. Following the usual convention, we consider $M$ and $N$ as additive lattices and we let $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ and $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$. A fan $\Sigma \in N_{\mathbb{Q}}$ is a finite collection of strongly convex polyhedral cones such that every face of $\sigma \in \Sigma$ is contained in $\Sigma$ and for all $\sigma, \sigma' \in \Sigma$ the intersection $\sigma \cap \sigma'$ is a face in both cones $\sigma$ and $\sigma'$. A toric variety $X_\Sigma$ is built from $\Sigma$ in the following way. For every $\sigma \in \Sigma$, we define an affine toric variety $X_\sigma = \operatorname{Spec}(k[\sigma^\vee \cap M])$, where $\sigma^\vee \subseteq M_{\mathbb{Q}}$ is the dual cone of $\sigma$ and $k[\sigma^\vee \cap M]$ is the semigroup algebra of $\sigma^\vee \cap M$, i.e.

$$k[\sigma^\vee \cap M] = \bigoplus_{m \in \sigma^\vee \cap M} k \cdot \chi^m, \quad \text{with } \chi^0 = 1, \quad \text{and}$$

$$\chi^m \cdot \chi^{m'} = \chi^{m+m'}, \quad \forall\, m, m' \in \sigma^\vee \cap M.$$

Furthermore, if $\tau \subseteq \sigma$ is a face of $\sigma$, then the inclusion of algebras $k[\sigma^\vee \cap M] \hookrightarrow k[\tau^\vee \cap M]$ induces a $\mathbb{T}$-equivariant open embedding $X_\tau \hookrightarrow X_\sigma$. The toric variety $X_\Sigma$

associated to the fan $\Sigma$ is then defined as the variety obtained by gluing the family $\{X_\sigma \,|\, \sigma \in \Sigma\}$ along the open embeddings $X_\sigma \hookleftarrow X_{\sigma\cap\sigma'} \hookrightarrow X_{\sigma'}$ for all $\sigma, \sigma' \in \Sigma$.

Let $X_\Sigma$ be a toric variety. Since the torus $\mathbb{T}$ acts on $X_\Sigma$ with an open orbit, the field of fractions $K_X$ of $X$ is equal to $K_{\mathbb{T}} = \mathrm{Frac}(k[M])$ which is a purely transcendental extension of $k$ of degree $n = \dim \mathbb{T}$. Let $\alpha : \mathbb{G}_a \times X \dashrightarrow X$ be a rational $\mathbb{T}$-homogeneous $\mathbb{G}_a$-action on $X$, let $\tilde{\partial} : \mathcal{K}_{\mathbb{T}} \to \mathcal{K}_{\mathbb{T}}$ be the corresponding rational $k$-derivation and let $\partial = \Gamma(\mathbb{T}, \tilde{\partial}) : K_{\mathbb{T}} \to K_{\mathbb{T}}$ be the induced $k$-derivation of $K_{\mathbb{T}}$. In the case where $\alpha$ is regular, it is well known that $\alpha$ is $\mathbb{T}$-homogeneous if and only if $\partial$ is homogeneous, i.e. homogeneous as a linear map with respect to the $M$-grading on $k[M]$. In the rational case, the field $K_{\mathbb{T}}$ is not graded but it is the fraction field of the $M$-graded ring $k[M]$, so we say that $f \in K_{\mathbb{T}}$ is homogeneous if $f$ is a quotient of homogeneous elements. We say that a derivation $\partial : K_{\mathbb{T}} \to K_{\mathbb{T}}$ is homogeneous if it sends homogeneous elements to homogeneous elements.

**Lemma 3.5.** *A rational $\mathbb{G}_a$-action $\alpha : \mathbb{G}_a \times \mathbb{T} \dashrightarrow \mathbb{T}$ is $\mathbb{T}$-homogeneous if and only if the corresponding $k$-derivation $\partial : k[M] \to K_{\mathbb{T}}$ is homogeneous. Furthermore, every homogeneous rational $k$-derivation $\partial : k[M] \to K_{\mathbb{T}}$ is regular, i.e. factors through $k[M]$.*

**Proof.** The first assertion follows from the same argument as in the regular case, see e.g. [12, Lemma 2]. Since every homogeneous element in $k[M]$ is invertible, it follows that the only homogeneous elements in $K_{\mathbb{T}}$ are the characters $\chi^m$, $m \in M$, which are regular functions on $\mathbb{T}$. □

Regular homogeneous $k$-derivations on $\mathbb{T}$ were already described in [4], see also [11, Proposition 3.1]. Let $p \in N$ and let $e \in M$. The linear map $\partial_{p,e} : k[M] \to k[M]$, $\chi^m \mapsto p(m)\chi^{m+e}$ is a homogeneous $k$-derivation on $\mathbb{T}$ and every homogeneous $k$-derivation on $\mathbb{T}$ is a multiple $\partial_{p,e}$ for some $e \in M$ and some $p \in N$. Without loss of generality we may assume that $p$ is primitive.

**Lemma 3.6.** *Let $p \in N$ be a primitive vector and let $e \in M$. The $k$-derivation $\partial_{p,e}$ is rationally integrable if and only if $p(e) = \pm 1$.*

**Proof.** Since $\partial_{-p,e} = -\partial_{p,e}$, we may assume without loss of generality that $p(e) \geq 0$. Choosing mutually dual basis for $M$ and $N$, we may assume $p = (1, 0, \ldots, 0)$ and $e = (e_1, \ldots, e_n)$ with $e_1 \geq 0$. Letting $x_i = \chi^{\beta_i}$, where $\{\beta_1, \ldots, \beta_n\}$ is the basis for $M$, the $k$-derivation $\partial_{p,e}$ becomes

$$\partial_{p,e} = x_1^{e_1+1} x_2^{e_2} \cdots x_n^{e_n} \frac{\partial}{\partial x_1}.$$

A direct computation now shows that $\partial_{p,e}$ is rationally integrable if and only if $e_1 = p(e) = 1$. □

The following lemma gives conditions for a derivation $\partial_{p,e}$ to extend to a regular $k$-derivation of an affine toric variety $X_\sigma$. It was first proven in [4] in a slightly different form (see also [11, Proposition 3.1] for a modern proof). For a fan $\Sigma$ or a cone $\sigma$ the notation $\Sigma(1)$ and $\sigma(1)$ refers to the set of primitive vectors of the rays in $\Sigma$ and $\sigma$, respectively.

**Lemma 3.7.** *Let $X_\sigma$ be an affine toric variety. Then the homogeneous $k$-derivation $\partial_{p,e}$ on $\mathbb{T}$ extends to a $k$-derivation on $X_\sigma$ if and only if*

(1) *$e \in \sigma_M^\vee$, or*
(2) *there exists $\rho_e \in \sigma(1)$ such that $p = \pm\rho_e$, $\rho_e(e) = -1$, and $\rho(e) \geq 0$ for all $\rho \in \sigma(1)\backslash\{\rho_e\}$.*

*Furthermore, $\partial_{e,p}$ is locally nilpotent if and only if it is as in* (3.7).

By the valuative criterion for properness, a toric variety $X_\Sigma$ is semi-affine if and only if $\mathrm{Supp}(\Sigma) = \bigcup_{\sigma\in\Sigma}\sigma$ is convex. We can now apply Corollary 2.2 to recover a description of regular $\mathbb{G}_a$-actions on semi-affine toric varieties which was obtained by Demazure [4] using lengthy explicit computations.

**Proposition 3.8.** *Let $X_\Sigma$ be a semi-affine toric variety. Then $\partial_{p,e}$ is the derivation of a $\mathbb{T}$-homogeneous regular $\mathbb{G}_a$-actions $\alpha_{p,e} : \mathbb{G}_a \times X_\Sigma \to X_\Sigma$ on $X_\Sigma$ if and only if there exists $\rho_e \in \Sigma(1)$ such that $p = \pm\rho_e$, $\rho_e(e) = -1$, and $\rho(e) \geq 0$ for all $\rho \in \Sigma(1)\backslash\{\rho_e\}$.*

**Proof.** By Corollary 2.2, the $k$-derivation $\partial_{p,e}$ is the derivation of a $\mathbb{T}$-homogeneous regular $\mathbb{G}_a$-action if and only if there exists an affine open $\mathbb{G}_a$-invariant subset $U \subseteq X_\Sigma$ such that $\Gamma(U, \tilde{\partial})$ is locally nilpotent. Since the action is $\mathbb{T}$-homogeneous, we can assume that $U$ is also $\mathbb{T}$-invariant. Now the proposition follows from Lemma 3.7. □

### 3.3. *Rational $\mathbb{G}_a$-actions associated with affine-linear bundles of rank one*

Here we consider a class of rational $\mathbb{G}_a$-actions coming from regular actions of certain nonconstant group schemes, locally isomorphic to $\mathbb{G}_a$. We characterize the simplest possible ones in terms of their corresponding rationally integrable $k$-derivations.

Let us first note that every line bundle $p : L \to Z$ over a $k$-variety $Z$ carries a canonical rationally integrable $\mathcal{O}_Z$-derivation $d_{L/Z} : \mathcal{O}_L \to \Omega^1_{L/Z} \hookrightarrow \mathcal{K}_L$ with the property that over every affine open subset $Z_i$ on which $L$ becomes trivial, the $\Gamma(Z_i, \mathcal{O}_{Z_i})$-derivation

$$\Gamma(p^{-1}(Z_i), d_{L/Z}) : \Gamma(p^{-1}(Z_i), \mathcal{O}_L) \to \Gamma(p^{-1}(Z_i), \Omega^1_{L/X}) \simeq \Gamma(p^{-1}(Z_i), \mathcal{O}_L)$$

is locally nilpotent. Indeed, writing $p : L = \mathrm{Spec}_Z(\mathrm{Sym}_Z\mathcal{L}^\vee) \to Z$ for a certain invertible sheaf $\mathcal{L}$, we have $\Omega^1_{L/Z} \simeq p^*\mathcal{L}^\vee$ and for every affine open subset $Z_i$ of $Z$ over which $L$-becomes trivial, say $L|_{Z_i} \simeq \mathrm{Spec}(\mathcal{O}_{Z_i}[s_i])$, $\Gamma(p^{-1}(Z_i), d_{L/Z})$ coincides with the derivation $\frac{\partial}{\partial s_i}$.

A line bundle is in fact a group scheme over $Z$, locally isomorphic to $\mathbb{G}_{a,Z} = \mathbb{G}_a \times_{\mathrm{Spec}(k)} Z$, whose group law $m : L \times_Z L \to L$ is induced by the diagonal homomorphism $\mathcal{L} \to \mathcal{L} \oplus \mathcal{L}$ of the invertible sheaf $\mathcal{L}$ of germs of sections of $p : L \to Z$, and whose neutral section $e : Z \to L$ corresponds to the zero section of $\mathcal{L}$. In this context, the correspondence between regular $\mathbb{G}_a$-actions of an affine variety $X = \mathrm{Spec}(A)$ and locally nilpotent $k$-derivation of $A$ extends to a correspondence between regular actions $\mu : L \times_Z X \to X$ of $L$ on a variety $q : X \to Z$ affine over $Z$ and "locally nilpotent" $\mathcal{O}_Z$-derivations $\tilde{\partial} : \mathcal{O}_X \to q^*\mathcal{L}^\vee$. Namely, the derivation $\tilde{\partial}$ is the composition of the canonical $\mathcal{O}_Z$-derivation $d_{X/Z} : \mathcal{O}_X \to \Omega^1_{X/Z}$ with the homomorphism of $\mathcal{O}_X$-module $\Omega^1_{X/Z} \to q^*\mathcal{L}^\vee$ obtained similarly as in Sec. 1.2 by pulling back the homomorphism $\eta : \mu^*\Omega^1_{X/Z} \to \Omega^1_{L \times_Z X/X} \simeq \mathrm{pr}_X^* q^*\mathcal{L}^\vee$ of $\mathcal{O}_{L \times_Z X}$-module by the zero section morphism $e \times \mathrm{id}_X : X \to L \times_Z X$. This derivation is locally nilpotent in the sense that $q_*\mathcal{O}_X$ is the union of the kernels of the $\mathcal{O}_Z$-linear homomorphisms $\partial^n_{L,X} : q_*\mathcal{O}_X \to q_*\mathcal{O}_L \otimes_{\mathcal{O}_Z} (\mathcal{L}^\vee)^{\otimes n}$, $n \geq 1$, defined inductively by $\partial^1_{L,X} = q_*\tilde{\partial} : q_*\mathcal{O}_X \to q_*q^*\mathcal{L}^\vee \simeq q_*\mathcal{O}_X \otimes_{\mathcal{O}_Z} \mathcal{L}^\vee$ and, for every $n \geq 2$, as the composition $\partial^n_{L,Z} = (\partial^1_{L,Z} \otimes \mathrm{id}) \circ \partial^{n-1}_{L,X}$ where

$$\begin{aligned}(\partial^1_{L,Z} \otimes \mathrm{id}) : q_*\mathcal{O}_X \otimes_{\mathcal{O}_Z} (\mathcal{L}^\vee)^{\otimes n-1} &\to (q_*\mathcal{O}_X \otimes_{\mathcal{O}_Z} \mathcal{L}^\vee) \otimes_{\mathcal{O}_Z} (\mathcal{L}^\vee)^{\otimes n-1} \\ &\simeq q_*\mathcal{O}_X \otimes_{\mathcal{O}_Z} (\mathcal{L}^\vee)^{\otimes n}.\end{aligned}$$

The action $\mu : L \times_Z X \to X$ is then recovered as the morphism induced by the formal exponential homomorphism

$$\begin{aligned}\exp(t\partial_{L,X}) = \sum_{n \geq 0} \frac{\partial^n_{L,X}}{n!} t^n : q_*\mathcal{O}_X &\to q_*\mathcal{O}_X \otimes_{\mathcal{O}_Z} \left( \bigoplus_{n \geq 0} (\mathcal{L}^\vee)^{\otimes n} t^n \right) \\ &\simeq q_*\mathcal{O}_X \otimes_{\mathcal{O}_Z} \mathrm{Sym}_Z \mathcal{L}^\vee.\end{aligned}$$

The simplest examples of varieties admitting an action of a line bundle $p : L \to Z$ are principal homogeneous $L$-bundles, that is, varieties $q : X \to Z$ equipped with an action of $L$ which are locally equivariantly isomorphic over $Z$ to $L$ acting on itself by translations. For such varieties, the corresponding rationally integrable $\mathcal{O}_Z$-derivations $\tilde{\partial} : \mathcal{O}_X \to q^*\mathcal{L}^\vee$ have the additional property that there exists a covering of $Z$ by affine open subsets $Z_i \subset Z$ on which $\mathcal{L}$ becomes trivial and such that the induced derivation

$$\Gamma(q^{-1}(Z_i), \tilde{\partial}) : \Gamma(q^{-1}(Z_i), \mathcal{O}_X) \to \Gamma(q^{-1}(Z_i), q^*\mathcal{L}^\vee) \simeq \Gamma(q^{-1}(Z_i), \mathcal{O}_X)$$

is locally nilpotent, with a regular slice $s_i \in \Gamma(q^{-1}(Z_i), \mathcal{O}_X)$. The following proposition shows conversely that the existence on a variety $X$ of a structure of principal homogeneous bundle under a suitable line bundle $p : L \to Z$ can be decided, without prior knowledge of $L$ and $Z$, from the consideration of certain rationally integrable $k$-derivations $\tilde{\partial} : \mathcal{O}_X \to \mathcal{K}_X$.

**Proposition 3.9.** *Let $X$ be a $k$-variety and let $\tilde{\partial} : \mathcal{O}_X \to \mathcal{N}$ be a rationally integrable $k$-derivation with value in an invertible subsheaf $\mathcal{N}$ of $\mathcal{K}_X$. Suppose that* *there exists a covering of* $X$ *by affine open subsets* $X_i$, $i \in I$, *and trivializations* $\psi_i : \mathcal{N}|_{X_i} \xrightarrow{\sim} \mathcal{O}_{X_i}$ *such that the following hold*:

(a) *For every* $i \in I$, *the* $k$*-derivation* $\Gamma(X_i, \psi_i \circ \tilde{\partial}) : \Gamma(X_i, \mathcal{O}_X) \to \Gamma(X_i, \mathcal{O}_X)$ *is locally nilpotent with a regular slice* $s_i \in \Gamma(X_i, \mathcal{O}_X)$.

(b) *For every* $i, j \in I$, *the invertible function* $\psi_i \circ \psi_j^{-1}|_{X_i \cap X_j} \in \Gamma(X_i \cap X_j, \mathcal{O}_X^*)$ *is contained in* $\mathrm{Ker}(\Gamma(X_i \cap X_j, \tilde{\partial}))$.

*Then there exists a geometrically integral scheme* $Z$ *of finite type over* $k$, *a morphism* $q : X \to Z$ *and an invertible sheaf* $\mathcal{L}$ *on* $Z$ *such that* $\mathcal{N} \simeq q^*\mathcal{L}^\vee$ *and* $q : X \to Z$ *is a principal homogeneous bundle under the line bundle* $p : \mathrm{Spec}_Z(\mathrm{Sym}_Z \mathcal{L}^\vee) \to Z$.

**Proof.** Letting $\alpha_i : \mathbb{G}_a \times X_i \to X_i$ be the $\mathbb{G}_a$-action generated by the $k$-derivation $\partial_i = \Gamma(X_i, \psi_i \circ \tilde{\partial})$ and $Z_i = \mathrm{Spec}(\Gamma(X_i, \mathcal{O}_X)/(s_i)) \subset X_i$, the first hypothesis implies that $\Phi_i : \mathbb{G}_a \times Z_i \to X_i$, $(t, z_i) \mapsto \alpha_i(t, z_i)$ is a $\mathbb{G}_a$-equivariant isomorphism between $\mathbb{G}_a \times Z_i$ equipped with the action by translations on the first factor and $X_i$ equipped with the action $\alpha_i$. By definition, $\partial_i|_{X_i \cap X_j} = a_{ij}\partial_j|_{X_i \cap X_j}$ where $a_{ij} = \psi_i \circ \psi_j^{-1}|_{X_i \cap X_j} \in \Gamma(X_i \cap X_j, \mathcal{O}_X^*)$ and condition (b) says in particular that $a_{ij} \in \mathrm{Ker}\, \partial_i|_{X_i \cap X_j} = \mathrm{Ker}\, \partial_j|_{X_i \cap X_j}$. This implies in turn that every element of $\Gamma(X_i \cap X_j, \mathcal{O}_X)$ which is in the canonical image of $\Gamma(X_i, \mathcal{O}_X)$ or $\Gamma(X_j, \mathcal{O}_X)$ is annihilated by a certain power of $\partial_i$. Since $X$ is separated, $\Gamma(X_i \cap X_j, \mathcal{O}_X)$ is generated by these canonical images [9, I.5.5.6] and so $\partial_i|_{X_i \cap X_j}$ and $\partial_j|_{X_i \cap X_j}$ are locally nilpotent derivations of $\Gamma(X_i \cap X_j, \mathcal{O}_X)$. This shows that $X_i \cap X_j$ is stable under the $\mathbb{G}_a$-actions $\alpha_i$ on $X_i$ and $\alpha_j$ on $X_j$. Therefore there exist open subsets $Z_{ij} \simeq \mathrm{Spec}(\mathrm{Ker}\, \partial_i|_{X_i \cap X_j})$ and $Z_{ji} \simeq \mathrm{Spec}(\mathrm{Ker}\, \partial_j|_{X_i \cap X_j})$ of $Z_i$ and $Z_j$ respectively such that $X_i \cap X_j$ is simultaneously $\mathbb{G}_a$-equivariantly isomorphic to $\mathrm{Spec}_{Z_{ij}}(\mathcal{O}_{Z_{ij}}[s_i])$ and $\mathrm{Spec}_{Z_{ji}}(\mathcal{O}_{Z_{ji}}[s_j])$ with respect to the action $\alpha_i$ and $\alpha_j$. Furthermore, since $a_{ij} \in \mathrm{Ker}\, \partial_i|_{X_i \cap X_j}$ we have

$$\partial_i|_{X_i \cap X_j}(a_{ij}s_i) = a_{ij}\partial_i|_{X_i \cap X_j}(s_i) = a_{ij} = \partial_i|_{X_i \cap X_j}(s_j)$$

and so, there exists $b_{ij} \in \mathrm{Ker}\, \partial_i|_{X_i \cap X_j} = \mathrm{Ker}\, \partial_j|_{X_i \cap X_j}$ such that $s_j|_{X_i \cap X_j} = a_{ij}s_i|_{X_i \cap X_j} + b_{ij}$. The same argument applied to a triple intersection $X_i \cap X_j \cap X_k$ shows that the natural isomorphisms $\varphi_{ij} : Z_{ji} \xrightarrow{\sim} Z_{ij}$ induced by the equality $\mathrm{Ker}\, \partial_i|_{X_i \cap X_j} = \mathrm{Ker}\, \partial_j|_{X_i \cap X_j}$ satisfy $\varphi_{jk}(Z_{ki} \cap Z_{kj}) \subset Z_{jk} \cap Z_{ji}$ and $\varphi_{ik}|_{Z_{ki} \cap Z_{kj}} = \varphi_{ij}|_{Z_{jk} \cap Z_{ji}} \circ \varphi_{jk}|_{Z_{ki} \cap Z_{kj}}$. This implies the existence of a unique $k$-scheme $Z$ together with open immersions $\zeta_i : Z_i \hookrightarrow Z$ such that $\xi_i \circ \varphi_{ij} = \xi_j$. Furthermore, the local projections $\mathrm{pr}_{Z_i} : X_i \simeq Z_i \times \mathbb{A}^1 \to Z_i$ glue to a locally trivial $\mathbb{A}^1$-bundle $q : X \to Z$ with trivializations $\rho^{-1}(Z_i) \simeq \mathrm{Spec}_{Z_i}(\mathcal{O}_{Z_i}[s_i])$, $i \in I$, where we identified $Z_i$ with its image in $Z$. The invertible functions $a_{ij} \in \Gamma(X_i \cap X_j, \mathcal{O}_X^*) \cap \mathrm{Ker}\, \partial_i|_{X_i \cap X_j} \simeq \Gamma(Z_i \cap Z_j, \mathcal{O}_Z^*)$ form a Čech 1-cocycle with value in $\mathcal{O}_Z^*$ defining a unique invertible sheaf $\mathcal{L}^\vee$ such that $\mathcal{N} \simeq q^*\mathcal{L}^\vee$, and the identity $s_j|_{X_i \cap X_j} = a_{ij}s_i|_{X_i \cap X_j} + b_{ij}$ says precisely that $q : X \to Z$ is in fact a principal homogeneous bundle under the line bundle $p : \mathrm{Spec}_Z(\mathrm{Sym}_Z \mathcal{L}^\vee) \to Z$. ◻

**Example 3.10.** Let $S$ be the smooth affine surface in $\mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x,y,z,u])$ defined by the equations

$$\begin{cases} xz = y(y-1), \\ yu = z(z+1), \\ xu = (y-1)(z+1) \end{cases}$$

and let $\partial, \partial' : A = \Gamma(S, \mathcal{O}_S) \to K_S$ be the $k$-derivations defined respectively by

$$\begin{cases} \partial x = 0, \\ \partial y = x^2, \\ \partial z = (2y-1)x, \\ \partial u = x(z+1) + (2y-1)(y-1) \end{cases} \quad \text{and} \quad \begin{cases} \partial' x = \omega^3, \\ \partial' y = \omega^2, \\ \partial' z = \omega, \\ \partial' u = 1, \end{cases}$$

where $\omega = x/(y-1) \in K_S$.

It is straightforward to check that $\partial$ is a locally nilpotent $\mathbb{C}[x]$-derivation of $A$, thus defining a regular $\mathbb{G}_a$-action $\alpha : \mathbb{G}_a \times S \to S$. The surface $S$ is covered by the two $\mathbb{G}_a$-invariant affine open subsets

$$S_0 = S \backslash \{x = y - 1 = 0\} \simeq \mathrm{Spec}(\mathbb{C}[x, v_0]) \quad \text{and}$$

$$S_1 = S \backslash \{x = y = z + 1 = 0\} \simeq \mathrm{Spec}(\mathbb{C}[x, v_1]),$$

where $v_0$ and $v_1$ denote the restriction to $S_0$ of the rational functions $(y-x)/x^2$ and $\omega^{-1}$. The restrictions of $\partial$ to $S_0$ and $S_1$ coincide respectively the locally nilpotent derivations $\frac{\partial}{\partial v_0}$ and $x\frac{\partial}{\partial v_1}$. So letting $C_1 \subset S$ be the curve $\{x = y - 1 = 0\}$, we see that the derivation of $\mathcal{O}_S$ into itself associated to $\partial$ factors through a derivation $\tilde{\partial} : \mathcal{O}_S \to \mathcal{N} = \mathcal{O}_S(-C_1)$. By definition, $\mathcal{N}|_{S_0} = \mathcal{O}_{S_0}$, $\mathcal{N}|_{S_1} = x\mathcal{O}_{S_1}$ and using the isomorphisms $\psi_0 = \mathrm{id}_{\mathcal{O}_{S_0}}$ and $\psi_1 : x\mathcal{O}_{S_1} \to \mathcal{O}_{S_1}$, $x \mapsto 1$, we obtain that the two derivations $\partial_0 = \Gamma(S_0, \psi_0 \circ \tilde{\partial}) = \frac{\partial}{\partial v_0}$ and $\partial_1 = \Gamma(S_1, \psi_1 \circ \tilde{\partial}) = \frac{\partial}{\partial v_1}$ are locally nilpotent with respective slices $s_0 = v_0$ and $s_1 = v_1$ and respective geometric quotients $S_0/\mathbb{G}_a = S_1/\mathbb{G}_a = \mathrm{Spec}(\mathbb{C}[x])$. Since $x^{-1} \in \Gamma(S_0 \cap S_1, \mathcal{O}_S^*) = \mathbb{C}[x^{\pm 1}]$ belongs to $\mathrm{Ker}(\Gamma(S_0 \cap S_1, \tilde{\partial}))$, the hypothesis of Proposition 3.9 is satisfied. In this example, the corresponding scheme $Z$ is isomorphic to the affine line with a double origin, obtained by gluing $S_0/\mathbb{G}_a$ and $S_1/\mathbb{G}_a$ by the identity outside their respective origins $o_0$ and $o_1$, and $\mathcal{L}^\vee = \mathcal{O}_Z(-o_1)$. The initial $\mathbb{G}_a$-action defined by $\partial$ is recovered from the action $\mu : L \times_Z S \to S$ of $L = \mathrm{Spec}_Z(\mathrm{Sym}_Z \mathcal{L}) \to Z$ as the composition

$$\alpha = \mu \circ (\sigma \times \mathrm{id}_S) : \mathbb{G}_a \times S \simeq \mathbb{G}_{a,Z} \times_Z S \to L \times_Z S \to S,$$

where $\sigma : \mathbb{G}_{a,Z} = \mathbb{G}_a \times_{\mathrm{Spec}(\mathbb{C})} Z = \mathrm{Spec}_Z(\mathcal{O}_Z[t]) \to L = \mathrm{Spec}_Z(\mathrm{Sym}_Z \mathcal{L}^\vee)$ is the group scheme homomorphism induced by the canonical global section $\sigma$ of $\mathcal{O}_Z(o_1) = \mathcal{H}om_Z(\mathcal{L}^\vee, \mathcal{O}_Z)$ with divisor equal to $o_1$.

The second derivation $\partial'$ is not locally nilpotent. However, noting that $\partial'\omega = 0$ and that the restriction of $\partial'$ to the open subset $S_1 \simeq \mathrm{Spec}(\mathbb{C}[x, v_1])$ coincides with the derivation $v_1^{-3}\frac{\partial}{\partial x} = \omega^3\frac{\partial}{\partial x}$, we conclude that the associated derivation $\tilde{\partial} : \mathcal{O}_S \to \mathcal{K}_S$ is rationally integrable. Furthermore $\partial'$ restricts on the open subset $S_0' = S\setminus\{y-1=z=u=0\} \simeq \mathrm{Spec}(\mathbb{C}[u,v_0'])$, where $v_0' = \omega|_{S_0'}$ to the locally nilpotent derivation $\frac{\partial}{\partial u}$. The open subsets $S_0'$ and $S_1$ cover $S$ and letting $C_0' \subset S$ be the curve $\{y-1=z=u=0\}$, we see that $\tilde{\partial}$ factors through the invertible subsheaf $\mathcal{N}' = \mathcal{O}_S(3C_0')$ of $\mathcal{K}_S$. By definition, $\mathcal{N}'|_{S_0'} = \mathcal{O}_{S_0'}$, $\mathcal{N}'|_{S_1} = \omega^{-3}\mathcal{O}_{S_1}$ and using the isomorphisms $\psi_0' = \mathrm{id}_{\mathcal{O}_{S_0'}}$ and $\psi_1' : \omega^{-3}\mathcal{O}_{S_1} \to \mathcal{O}_{S_1}$, $\omega^{-3} \mapsto 1$, we obtain that the two derivations $\partial_0' = \Gamma(S_0', \psi_0' \circ \tilde{\partial}') = \frac{\partial}{\partial u}$ and $\partial_1' = \Gamma(S_1, \psi_1' \circ \tilde{\partial}') = \frac{\partial}{\partial x}$ are locally nilpotent with respective slices $s_0' = u$ and $s_1' = x$, and respective geometric quotients $S_0'/\mathbb{G}_a = \mathrm{Spec}(\mathbb{C}[v_0'])$ and $S_1/\mathbb{G}_a = \mathrm{Spec}(\mathbb{C}[v_1])$. Since $\omega^{-3} \in \Gamma(S_0' \cap S_1, \mathcal{O}_S^*) = \mathbb{C}[\omega^{\pm 1}]$ belongs to $\mathrm{Ker}(\Gamma(S_0' \cap S_1, \tilde{\partial}'))$, the hypothesis of Proposition 3.9 is again satisfied. Here the corresponding scheme $Z$ is isomorphic to $\mathbb{P}^1$ obtained by gluing $S_0'/\mathbb{G}_a$ and $S_1/\mathbb{G}_a$ outside their respective origins $o_0'$ and $o_1$ by the isomorphism $v_0' \mapsto v_1^{-1}$, and $\mathcal{L}^\vee \simeq \mathcal{O}_Z(3o_0')$. The resulting morphism $q : S \to Z \simeq \mathbb{P}^1$, which coincides with the one $(x,y,z,u) \mapsto [x : y-1]$, is thus a principal homogeneous bundle under the geometric line bundle $L = \mathcal{O}_{\mathbb{P}^1}(-3)$ on $\mathbb{P}^1$.

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## References

[1] H. Bass, A non-triangular action of $\mathbb{G}_a$ on $\mathbb{A}^3$, *J. Pure Appl. Algebra* **33** (1984) 1–5.
[2] A. Beauville, J.-L. Colliot-Thélène, J.-J. Sansuc and P. Swinnerton-Dyer, Variétés stablement rationnelles non rationnelles, *Ann. Math.* **121**(2) (1985) 283–318.
[3] A. Cohen and J. Draisma, From Lie algebras of vector fields to algebraic group actions, *Transform. Groups* **8**(1) (2003) 51–68.
[4] M. Demazure, Sous-groupes algébriques de rang maximum du groupe de Cremona, *Ann. Sci. École Norm. Sup. (4)* **3** (1970) 507–588.
[5] J. K. Deveney and D. R. Finston, Fields of $\mathbb{G}_a$-invariants are ruled, *Canad. Math. Bull.* **37**(1) (1994) 37–41.
[6] A. Dubouloz, Flexible bundles over rigid affine surfaces, *Comment. Math. Helv.* **90** (2015) 121–137.
[7] W. Fulton, Introduction to Toric Varieties, *Annals of Mathematics Studies*, Vol. 131 (Princeton University Press, Princeton, NJ, 1993).
[8] J. Goodman and A. Landman, Varieties proper over affine schemes, *Invent. Math.* **20** (1973) 267–312.
[9] A. Grothendieck and J. Dieudonné, Éléments de géométrie algébrique. I: Le langage des schémas, *Publ. Math. Inst. Hautes Études Sci.* **4** (1960).
[10] E. G. Koshevoi, Birational representations of multiplicative and additive groups, *Sib. Math. J.* **8**(6) (1967) 1016–1021.
[11] F. Kutzschebauch, M. Leuenberger and A Liendo, The algebraic density property for affine toric varieties, preprint (2014), arXiv:1402.2227.
[12] A. Liendo, Roots of the affine Cremona group, *Transform. Groups* **16**(4) (2011) 1137–1142.

[13] L. Makar-Limanov, Locally nilpotent derivations, a new ring invariant and applications, Lecture notes, Bar-Ilan University (1998), http://www.math.wayne.edu/ lml/.

[14] M. Miyanishi, A remark on an iterative infinite higher derivation, *J. Math. Kyoto Univ.* **8** (1968) 411–415.

[15] L. Moret-Bailly, Variétés stablement rationnelles non rationnelles, *Séminaire N. Bourbaki*, 1984–1985, Exp. No. 643, pp. 223–236.

[16] J. Ohm, The ruled residue theorem for simple transcendental extensions of valued fields, *Proc. Amer. Math. Soc.* **89**(1) (1983) 16–18.

[17] R. Rentschler, Opérations de groupe additif sur le plan affine, *C. R. Acad. Sci. Paris* **267** (1968) 384–387.

[18] M. Rosenlicht, Some basic theorems on algebraic groups, *Amer. J. Math.* **78** (1956) 401–443.

[19] D. Zaitsev, Regularization of birational group operations in the sense of Weil, *J. Lie Theory* **5**(2) (1995) 207–224.