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# Smooth varieties with torus actions ☆

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A B S T R A C T

In this paper we provide a characterization of smooth algebraic varieties endowed with a faithful algebraic torus action in terms of a combinatorial description given by Altmann and Hausen. Our main result is that such a variety $X$ is smooth if and only if it is locally isomorphic in the étale topology to the affine space endowed with a linear torus action. Furthermore, this is the case if and only if the combinatorial data describing $X$ is locally isomorphic in the étale topology to the combinatorial data describing affine space endowed with a linear torus action. Finally, we provide an effective method to check the smoothness of a $\mathbb{G}_m$-threefold in terms of the combinatorial data.



## Introduction

Let k be an algebraically closed field of characteristic zero and let $\mathbb{T}$ be the algebraic torus $\mathbb{T} = (\mathbb{G}_m)^k$ of dimension $k$ where $\mathbb{G}_m$ is the multiplicative group of the field $(\mathrm{k}^*, \cdot)$. The variety $\mathbb{T}$ has a natural structure of algebraic group. We denote by $M$ its character lattice and by $N$ its 1-parameter subgroup lattice. In this paper a variety denotes an

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integral separated scheme of finite type. A $\mathbb{T}$-variety is a normal variety $X$ endowed with a faithful action of $\mathbb{T}$ acting on $X$ by regular automorphisms. The assumption that the $\mathbb{T}$-action on $X$ is faithful is not a restriction since given any regular $\mathbb{T}$-action $\alpha$, the kernel $\ker \alpha$ is a normal algebraic subgroup of $\mathbb{T}$ and $\mathbb{T}/(\ker \alpha)$ is again an algebraic torus acting faithfully on $X$.

The complexity of a $\mathbb{T}$-variety is the codimension of the generic orbit. Furthermore, since the action is assumed to be faithful, the complexity of $X$ is given by $\dim X - \dim \mathbb{T}$. The best known example of $\mathbb{T}$-varieties are those of complexity zero, i.e., toric varieties. Toric varieties were first introduced by Demazure in [7] as a tool to study subgroups of the Cremona group. Toric varieties allow a combinatorial description in term of certain collections of strongly convex polyhedral cones in the vector space $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ called fans, see for instance [19,12,6].

For higher complexity there is also a combinatorial description of a $\mathbb{T}$-variety. We will use the language of p-divisors first introduced by Altmann and Hausen in [2] for the affine case and generalized in [3] to arbitrary $\mathbb{T}$-varieties. This description, that we call the A-H combinatorial description, generalizes previously known partial cases such as [14,8,11,24]. See [4] for a detailed survey on the topic.

Since the introduction of the A-H description, a lot of work has been done in generalizing the known results from toric geometry to the more general case of $\mathbb{T}$-varieties. On of the most basic parts of the theory that are still open is the characterization of smooth $\mathbb{T}$-varieties of complexity higher than one. In particular, several classification of singularities of $\mathbb{T}$-varieties are given in [17], but no smoothness criterion is given in complexity higher than one. In this paper we achieve such a characterization in arbitrary complexity.

By Sumihiro's Theorem [23], every (normal) $\mathbb{T}$-variety admits an affine cover by $\mathbb{T}$-invariant affine open sets. Hence, to study smoothness, it is enough to consider affine $\mathbb{T}$-varieties. The A-H description of an affine $\mathbb{T}$-variety $X$ consists in a couple $(Y, \mathcal{D})$, where $Y$ is a normal semiprojective variety that is a kind of quotient of $X$, usually called the Chow quotient and $\mathcal{D}$ is a divisor on $Y$ whose coefficients are not integers as usual, but polyhedra in $N_{\mathbb{Q}}$, see Section 1 for details.

It is well known that an affine toric variety $X$ of dimension $n$ is smooth if and only if it is equivariantly isomorphic to a $\mathbb{T}$-invariant open set in $\mathbb{A}^n$ endowed with the standard $\mathbb{T}$-action of complexity 0 by component-wise multiplication. Our main result states that an affine $\mathbb{T}$-variety $X$ of dimension $n$ and of arbitrary complexity is smooth if and only if it is locally isomorphic in the étale topology to the affine space $\mathbb{A}^n$ endowed with a linear $\mathbb{T}$-action, see Proposition 4. Furthermore, $X$ is smooth if and only if the combinatorial data $(Y, \mathcal{D})$ is locally isomorphic in the étale topology to the combinatorial data $(Y', \mathcal{D}')$ of the affine space $\mathbb{A}^n$ endowed with a linear $\mathbb{T}$-action, see Theorem 7. The main ingredient in our result is Luna's Slice Theorem [18].

In order to effectively apply Theorem 7, it is necessary to know the A-H description of $\mathbb{A}^n$ endowed with all possible linear $\mathbb{T}$-actions. The case of complexity 0 and 1 are well known, see Corollary 9. In Proposition 12, we compute the combinatorial description of all the $\mathbb{G}_m$-action on $\mathbb{A}^3$. We apply this proposition to give several examples illustrating the behavior of the combinatorial data of smooth and singular affine $\mathbb{G}_m$-varieties of complexity 2.

We state our results in the case of an algebraically closed field of characteristic zero since the A-H description is given in that case. Nevertheless, all our arguments are characteristic free. In the introduction of [2] it is stated that they expect their arguments to hold in positive characteristic with the same proofs. Hence, our results are also valid in positive characteristic provided the A-H description is valid too. In particular, see [1] for Luna's Slice Theorem in positive characteristic and [15] for a different proof of the A-H description in complexity one over arbitrary fields.

## 1. Altmann–Hausen presentation of $\mathbb{T}$-varieties

In this section, we present a combinatorial description of affine $\mathbb{T}$-varieties. We use here the description first introduced by Altmann and Hausen in [2]. This description, that we call the A-H description, generalizes that of toric varieties [6] and many particular cases that were treated before, see [14,8,11,24]. Furthermore, this was later generalized to the non-affine case in [3]. See [4] for a detailed survey on the subject.

Let $N \simeq \mathbb{Z}^k$ be a lattice of rank $k$, and let $M = \mathrm{Hom}(N, \mathbb{Z})$ be its dual lattice. We let $\mathbb{T} = \mathrm{Spec}(\mathrm{k}[M])$ be the algebraic torus whose character lattice is $M$ and whose 1-parameter subgroup lattice is $N$. To both these lattices we associate rational vector spaces $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ and $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$, respectively. In all this paper, we let $\sigma \subseteq N_{\mathbb{Q}}$ be a strongly convex polyhedral cone, i.e., the intersection of finitely many closed linear half spaces in $N_{\mathbb{Q}}$ which does not contain any line. Let $\sigma^\vee \subseteq M_{\mathbb{Q}}$ be its dual cone and let $\sigma_M^\vee := \sigma^\vee \cap M$ be the semi-group of lattice point in $M$ contained in $\sigma^\vee$. Let $\Delta \subseteq N_{\mathbb{Q}}$ be a convex polyhedron, i.e., the intersection of finitely many closed affine half spaces in $N_{\mathbb{Q}}$. Then $\Delta$ admits a decomposition using the Minkowski sum as

$$\Delta := \Pi + \sigma = \{v_1 + v_2 \mid v_1 \in \Pi, v_2 \in \sigma\},$$

where $\Pi$ is a polytope, i.e., the convex hull of finitely many points in $N_{\mathbb{Q}}$ and $\sigma$ is a strongly convex polyhedral cone. In this decomposition, the cone $\sigma$ is uniquely determined by $\Delta$. The cone $\sigma$ is called the tail cone of $\Delta$ and the polyhedron $\Delta$ is said to be a $\sigma$-polyhedron. Let $\mathrm{Pol}_\sigma(N_{\mathbb{Q}})$ be the set of all $\sigma$-polyhedra in $N_{\mathbb{Q}}$. The set $\mathrm{Pol}_\sigma(N_{\mathbb{Q}})$ with Minkowski sum as operation forms a commutative semi-group with neutral element $\sigma$.

Let $Y$ be a semiprojective variety, i.e., $\Gamma(Y, \mathcal{O}_Y)$ is finitely generated and $Y$ is projective over $Y_0 = \mathrm{Spec}(\Gamma(\mathrm{Y}, \mathcal{O}_{\mathrm{Y}}))$. A polyhedral divisor $\mathcal{D}$ on $Y$ is a formal sum $\mathcal{D} = \sum \Delta_i \cdot D_i$, where $D_i$ are prime divisors on $Y$ and $\Delta_i$ are $\sigma$-polyhedron with $\Delta_i = \sigma$ except for finitely many of them.

Let $\mathcal{D}$ be a polyhedral divisor on $Y$. We define the evaluation divisor $\mathcal{D}(u)$ for every $u \in \sigma_M^\vee$ as the Weil $\mathbb{Q}$-divisor $Y$ given by

$$\mathcal{D}(u) = \sum \min_{v \in \Delta_i} \langle u, v \rangle D_i \quad \text{for all} \quad u \in \sigma_M^\vee.$$

A polyhedral divisor on $Y$ is called a p-divisor if the evaluation divisor $\mathcal{D}(u)$ is a $\mathbb{Q}$-Cartier semi-ample divisor for each $u \in \sigma_M^\vee$ and big for each $u \in \operatorname{relint}(\sigma^\vee) \cap M$. For every p-divisor $\mathcal{D}$ on $Y$ we can associate a sheaf of $\mathcal{O}_Y$-algebras

$$\mathcal{A}(Y, \mathcal{D}) = \bigoplus_{u \in \sigma_M^\vee} \mathcal{O}_Y(\mathcal{D}(u)) \cdot \chi^u,$$

and its ring of global sections

$$A(Y, \mathcal{D}) = \Gamma(Y, \mathcal{A}(Y, \mathcal{D})) = \bigoplus_{u \in \sigma_M^\vee} A_u, \quad \text{where} \quad A_u = \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(u))) \cdot \chi^u.$$

To the $M$-graded algebra $A(Y, \mathcal{D})$, we associate the scheme $X(Y, \mathcal{D}) = \operatorname{Spec}(A(Y, \mathcal{D}))$. The main result by Altmann and Hausen in [2] is the following.

**Theorem 1** *(Altmann–Hausen). For any p-divisor $\mathcal{D}$ on a normal semi-projective variety $Y$, the scheme $X(Y, \mathcal{D})$ is a normal affine $\mathbb{T}$-variety of dimension* $\dim(Y) + \dim(\mathbb{T})$. *Conversely any normal affine $\mathbb{T}$-variety is isomorphic to an $X(Y, \mathcal{D})$ for some semi-projective variety $Y$ and some p-divisor $\mathcal{D}$ on $Y$.*

The semi-projective variety $Y$ serving as base for the combinatorial data in the above theorem is not unique. Indeed, in the following example we give three different presentations for the same affine $\mathbb{T}$-variety. Computations can be carried out following the method described in [2, Section 11]

**Example 2.** Let $\mathbb{A}^3 = \operatorname{Spec}(\mathrm{k}[x, y, z])$ endowed with the linear $\mathbb{G}_m$-action given by $\lambda \cdot (x, y, z) \to (\lambda x, \lambda^{-1} y, \lambda z)$. We say that the weight matrix is $\begin{pmatrix} 1 & -1 & 1 \end{pmatrix}^t$. In this case, the lattices $M$ and $N$ are isomorphic to $\mathbb{Z}$ and the tail cone must be $\sigma = \{0\}$, then $\mathbb{A}^3$ is equivariantly isomorphic to

(*i*) $X(Y_1, \mathcal{D}_1)$, where $Y_1$ is the is the blow-up of $\mathbb{A}^2$ at the origin and $\mathcal{D}_1 = [-1, 0] \cdot E$, with $E$ the exceptional divisor of the blow up.

(*ii*) $X(Y_2, \mathcal{D}_2)$, where $Y_2$ is the is the blow-up of $\mathbb{A}^2$ at the origin and $\mathcal{D}_2 = [0, 1] \cdot E + D$, with $E$ the exceptional divisor of the blow up and $D$ the strict transform of any curve passing with multiplicity one at the origin.

(*iii*) $X(Y_3, \mathcal{D}_3)$, where $\psi : Y_3 \to Y_1$ is any projective birational morphism with $Y_3$ normal and $\mathcal{D}_3 = \psi^*(\mathcal{D}_1)$ is the total transform of $\mathcal{D}_1$, see [17, lemma 2.1].

Amongst all the possible bases $Y$ for the combinatorial description of a $\mathbb{T}$-variety, there is one of particular interest for us. It corresponds to the definition of minimal p-divisor given in [2, Definition 8.7]. A p-divisor $\mathcal{D}$ on a normal semi-projective variety $Y$ is minimal if given a projective birational morphism $\psi : Y \to Y'$ such that $\mathcal{D} = \psi^*(\mathcal{D}')$, then we have $\psi$ is an isomorphism. Given an affine $\mathbb{T}$-variety $X$, the semi-projective variety $Y$ where a minimal p-divisor describing $X$ lives is called the Chow quotient for the $\mathbb{T}$-action. In the previous example, the first two descriptions are minimal while the last one is only minimal if $\psi$ is an isomorphism. Since every birational projective morphism between normal curves is an isomorphism, we have that in the case of complexity one $\mathbb{T}$-varieties, all p-divisors are minimal.

Let $X$ be an affine $\mathbb{T}$-variety and let $L \to X$ be the trivial line bundle. We need to recall the construction of the Chow quotient of the $\mathbb{T}$-action. By [2, Section 5], every linearization of the trivial line bundle is given by a character $\chi^u$, $u \in M$ and for every $u \in M$ the set of semistable points is given by

$$X^{ss}(u) = \{x \in X \mid \text{there exists } n \in \mathbb{Z}_{\geq 0} \text{ and } f \in A_{nu} \text{ such that } f(x) \neq 0\}\,.$$

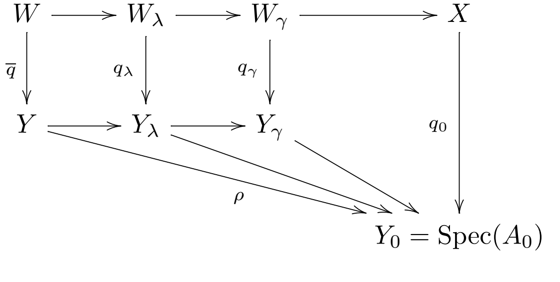
The set $X^{ss}(u)$ is a $\mathbb{T}$-invariant open subset of $X$ admitting a good $\mathbb{T}$-quotient to

$$Y_u = X^{ss}(u)/\!/\mathbb{T} = \operatorname{Proj} \bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_{nu}.$$

There exists a quasifan $\Lambda \in M_{\mathbb{Q}}$ generated by a finite collection of (not necessarily strongly convex) cones $\lambda$ such that for any $u$ and $u'$ in the relative interior of the same cone $\lambda \in \Lambda$, we obtain the same set of semi-stable points, i.e., $X^{ss}(u) = X^{ss}(u')$. Thereby, we can index the open sets of semi-stable points $X^{ss}(u)$ using $\lambda$ so that $X^{ss}(u) = W_\lambda$ for any $u \in \operatorname{relint}(\lambda)$. Furthermore, if $\gamma$ is a face of $\lambda$, $W_\lambda$ is an open subset of $W_\gamma$. Let $W = \bigcap_{\lambda \in \Lambda} W_\lambda$. The quotient maps

$$q_\lambda : W_\lambda \to W_\lambda/\!/\mathbb{T} = \operatorname{Proj} \bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_{nu}$$

for any $u$ in the relative interior of $\lambda$ form an inverse system indexed by the cones in the fan $\Lambda$. We let $q : W \longrightarrow Y' = \varprojlim Y_\lambda$ be the inverse limit of this system. The semi-projective variety $Y$ is then obtained by taking the normalization of the closure of the image of $W$ by $q$ in $Y'$. In the following commutative diagram we summarize all the morphisms involved. We will refer to them by these names in the text. The morphism $\overline{q} : W \to Y$ comes from the universal property of the normalization.



(1.1)

## 2. Smoothness criteria

Our main result in this paper is a characterization of smooth $\mathbb{T}$-varieties in terms of the A-H combinatorial data. This will follow as an application of Luna's Slice Theorem [18]. We will first recall this theorem and the required notation. Throughout this section $G$ will be used to denote a reductive algebraic group. The Slice Theorem describes the local structure in the étale topology of algebraic varieties endowed with a $G$-action. For details and proofs see [18,9]. We will follow the presentation in [13, Appendix to Ch. 1§D], see also [16]. For details on étale morphisms and the étale topology, see [10,22].

Let $X$, $X'$ be algebraic varieties, $f : X \to X'$ be a morphism and $x \in X$. The morphism $f$ is called étale at $x$ if it is flat and unramified at $x$. We say that $f$ is étale if it is étale at every point $x \in X$. For every $x \in X$, an étale neighborhood $\psi$ of $x$ is an étale morphism $\psi : Z \to X$ from some algebraic variety $Z$ such that $x \in \psi(Z)$. We say that two varieties $X$ and $X'$ are locally isomorphic in the étale topology at the points $x \in X$ and $x' \in X'$ if they have a common étale neighborhood, i.e., if there exists a variety $Z$ and a pair of étale morphisms $\psi : Z \to X$ and $\psi' : Z \to X'$ such that $x \in \psi(Z)$ and $x' \in \psi'(Z)$.

If $X$ is a $G$-variety, we say that the neighborhood $\psi$ is $G$-invariant if $Z$ is also a $G$-variety and $\psi$ is equivariant. Furthermore, if both $X$ and $X'$ are $G$-varieties, we say that $X$ and $X'$ are equivariantly locally isomorphic in the étale topology at the points $x$ and $x'$ if they have a common $G$-invariant étale neighborhood.

For the Slice Theorem we need a stronger equivariant notion of étale morphisms. Let $\phi : X \to X'$ be a $G$-equivariant morphism of affine $G$-varieties. The morphism $\phi$ is called strongly étale if

(i) $\phi$ is étale and the induced morphism $\phi_{//G} : X//G \to X'//G$ is étale.
(ii) $\phi$ and the quotient morphism $\pi : X \to X//G$ induce a $G$-isomorphism between $X$ and the fibered product $X' \times_{X'//G} X//G$.

Strongly étale morphism play the role of "local isomorphism" in the Slice Theorem. Let now $H$ be a closed subgroup of $G$ and $V$ an $H$-variety. The twisted product $G \star^H V$ is the algebraic quotient of $G \times V$ with respect to the $H$-action given by

$$H \times (G \times V) \to G \times V, \qquad (h, (g, v)) \mapsto h \cdot (g, v) = (gh^{-1}, hv) .$$

For a smooth point $x \in X$, we denote by $G_x$ the stabilizer of $x$ and by $T_xX$ the fiber at $x$ of the tangent bundle of $X$. Furthermore, for a closed orbit $G \cdot x$, we denote by $N_x$ the fiber at $x$ of the normal bundle to the orbit $G \cdot x$.

**Theorem 3** *(Luna's Slice Theorem). Let $X$ be an smooth affine $G$-variety. Let $x \in X$ be a point such that the orbit $G \cdot x$ is closed and fix a linearization of the trivial bundle on $X$ such that $G \cdot x \subset X^{ss}$. Then, there exists an affine smooth $G_x$-invariant locally* *closed subvariety $V_x$ of $X^{ss}$ (a slice to $G\cdot x$) such that we have the following commutative diagram*

$$\begin{array}{ccccc} G\star^{G_x} T_xV_x & \xleftarrow{\text{st. ét.}} & G\star^{G_x} V_x & \xrightarrow{\text{st. ét.}} & X^{ss} \\ \downarrow & & \downarrow & & \downarrow \\ N_x/\!/G_x & \xleftarrow{\text{ét.}} & (G\star^{G_x} V_x)/\!/G & \xrightarrow{\text{ét.}} & X^{ss}/\!/G \end{array}$$

Our main result will follow from the following proposition that translates Luna's Slice Theorem into the language of the A-H presentation. Let $X = X(Y, \mathcal{D})$. Recall that the morphism $\rho : Y \to Y_0$ in the A-H presentation is the natural morphism from the Chow quotient $Y$ used in the presentation and the algebraic quotient $Y_0 = X/\!/\mathbb{T}$.

**Proposition 4.** *Let $X = X(Y, \mathcal{D})$ be an affine $\mathbb{T}$-variety of dimension $n$. Assume that $Y$ is minimal. Then, $X$ is smooth if and only if for every point $y$ of $Y_0$ there exists a neighborhood $\mathcal{U} \subseteq Y_0$, an algebraic variety $Z$, a smooth toric variety $X'$ of dimension $n$, and strongly étale morphisms $a : Z \to X(\rho^{-1}(\mathcal{U}), \mathcal{D}|_{\rho^{-1}(\mathcal{U})})$ and $b : Z \to X'$ where $\mathbb{T}$ acts on $X'$ via a subtorus of the big torus action. In particular, every smooth point in a $\mathbb{T}$-variety is $\mathbb{T}$-equivariantly locally isomorphic in the étale topology to a point in the affine space endowed with a linear $\mathbb{T}$-action.*

**Proof.** We prove first the "if" of the first assertion. By assumption $Y_0$ is covered by the open sets $\mathcal{U}$ and so the open sets $\rho^{-1}(\mathcal{U})$ cover $Y$. Furthermore, every $X(\rho^{-1}(\mathcal{U}), \mathcal{D}|_{\rho^{-1}(\mathcal{U})})$ is an open set in $X$ by [3, Proposition 3.4] and for every $y \in Y_0$ the fiber $q_0^{-1}(y)$ of the algebraic quotient $q_0 : X \to Y_0$ is contained in $X(\rho^{-1}(\mathcal{U}), \mathcal{D}|_{\rho^{-1}(\mathcal{U})})$. This shows that $X$ is covered by the smooth $\mathbb{T}$-stable open sets $X(\rho^{-1}(\mathcal{U}), \mathcal{D}|_{\rho^{-1}(\mathcal{U})})$. Hence, every point $x \in X$ is locally isomorphic in the étale topology to a open set in a smooth toric variety $X'$. Since smoothness is preserved by étale morphisms, we have that $X$ is smooth.

Assume now that $X$ is smooth and let again $q_0 : X \to Y_0 = X/\!/\mathbb{T}$ be the algebraic quotient. Then for every $y \in Y_0$, $q_0^{-1}(y)$ contains a unique closed orbit that we will denote $C_y$. Choose a linearization of the trivial bundle given by the character $\chi^u$ such that $C_y \subset X^{ss} := X^{ss}(u)$. By the top row in the commutative diagram in Theorem 3, for any point $x \in C_y$ we have a closed smooth $\mathbb{T}_x$-stable subvariety $V_x$ of $X$ and strongly étale morphisms

$$X' := \mathbb{T}\star^{\mathbb{T}_x} T_xV_x \xleftarrow{\ b\ } Z := \mathbb{T}\star^{\mathbb{T}_x} V_x \xrightarrow{\ a\ } X^{ss}.$$

Let $\mathcal{U}$ be the image of $X^{ss}$ in $Y_0$ by $q_0$. Recall that in the commutative diagram (1.1) the variety $X^{ss} = X^{ss}(u) = W_\lambda$ and so by the commutativity of the diagram and [3, Proposition 3.4] we have that $X(\rho^{-1}(\mathcal{U}), \mathcal{D}|_{\rho^{-1}(\mathcal{U})}) = X^{ss}$. Since the $\mathbb{T}_x$-actions on $\mathbb{T}$ and on $T_xV_x$ are linear, we have that $X'$ is a smooth toric variety where $\mathbb{T}$ acts as a subtorus of the big torus. This proves the "only if" part of the first assertion.

For the last statement, it is well known that every smooth affine toric variety of dimension $n$ can be embedded as an open set in the affine space equivariantly with respect to the big torus action. Let $c : X' \hookrightarrow \mathbb{A}^n$ be such an embedding. Since an open embedding is étale, the compositions $c \circ b$ and $a$ are the required étale neighborhoods for every $x \in X^{ss} \subseteq X$. □

**Remark 5.** The statement that every smooth point in a $\mathbb{T}$-variety is $\mathbb{T}$-equivariantly locally isomorphic in the étale topology to a point in the affine space endowed with a linear $\mathbb{T}$-action is generalization of the classical result that every smooth point in a variety is locally isomorphic in the étale topology to a point in the affine space.

For the proof of our main result in Theorem 7, we need the following version of [2, Theorem 8.8]. The statement and proof of this Lemma and the corresponding statement in [2] are very similar. For the proof, we refer the reader to the original paper. Here, we only highlight the main points where the proof in [2] requires to be adapted.

**Lemma 6.** *Let $X = X(Y, \mathcal{D})$ and $X' = X(Y', \mathcal{D}')$ be affine $\mathbb{T}$-varieties, where $\mathcal{D}$ and $\mathcal{D}'$ are minimal p-divisors on normal semi-projective varieties $Y$ and $Y'$, respectively. If $\psi : X' \to X$ is a strongly étale morphism, then, there exists an étale surjective morphism $\varphi : Y' \to Y$ such that $X'$ is equivariantly isomorphic to $X(Y', \varphi^*(\mathcal{D}))$.*

*Conversely, if $\varphi : Y' \to Y$ is an étale surjective morphism and $\mathcal{D}$ is a minimal p-divisor on $Y$, then $\mathcal{D}' = \varphi^*(\mathcal{D})$ is minimal and there is a strongly étale morphism. $\psi : X(Y', \mathcal{D}') \to X(Y, \mathcal{D})$.*

**Proof.** For every $u \in M$, the existence of an étale morphism $(X')^{ss}(u)/\!/\mathbb{T} \to X^{ss}(u)/\!/\mathbb{T}$ is assured by the fact that $\psi$ is strongly étale. Since $\mathcal{D}$ and $\mathcal{D}'$ are minimal, we obtain, as in the proof of [2, Theorem 8.8], an étale morphism $\varphi : Y' \to Y$. In this case, the morphism $\kappa$ therein can be taken as the identity, see [2, p. 600 l.1]. Hence, [2, Theorem 8.8] yields $X' = X(Y', \varphi^*(\mathcal{D}))$ equivariantly.

Conversely, in this case we have a morphism of p-divisors $(\varphi, \mathrm{id}, 1) : \mathcal{D}' \to \mathcal{D}$. Hence, by [2, Proposition 8.6] we obtain the existence of a morphism $\psi : X(Y', \mathcal{D}') \to X(Y, \mathcal{D})$. By the diagram in equation (1.1), the A-H presentation is local relative to the algebraic quotient morphism $q_0 : X \to Y_0 = X/\!/\mathbb{T}$. Hence, we can assume we have a morphism $X = X^{ss}(u) = W_\lambda \to Y = Y_\lambda = W_\lambda/\!/\mathbb{T}$. Let now $W = X \times_Y Y'$ be the fiber product. By definition, the morphism $W \to X$ is strongly étale and the Chow quotient of $W$ is $Y'$. By [2, Theorem 8.8] and its proof, since we have morphism $\varphi : Y' \to Y$ between the Chow quotients, we obtain again that $\kappa$ therein can be taken as the identity and so $W = X(Y', \varphi^*(\mathcal{D}))$. □

Let now $\mathcal{D}$ and $\mathcal{D}'$ be p-divisors on normal semi-projective varieties $Y$ and $Y'$ respectively. We say that $(Y, \mathcal{D})$ is locally isomorphic in the étale topology to $(Y', \mathcal{D}')$ if for every pair of points $y \in Y$ and $y' \in Y'$ we have a variety $V$ and étale neighborhoods $\psi : V \to Y$ and $\psi' : V \to Y'$ such that $\psi^*(\mathcal{D}) = (\psi')^*(\mathcal{D}')$. We can now prove our main theorem.

**Theorem 7.** *Let $X = X(Y, \mathcal{D})$ be an affine $\mathbb{T}$-variety of dimension $n$. Assume that $(Y, \mathcal{D})$ is minimal. Then, $X$ is smooth if and only if for every point $y \in Y_0$ there exists a neighborhood $\mathcal{U} \subseteq Y_0$, a linear $\mathbb{T}$-action on $\mathbb{A}^n$ given by the combinatorial data $(Y', \mathcal{D}')$ with $\mathcal{D}'$ minimal, an algebraic variety $V$, and étale morphisms $\alpha : V \to Y$ and $\beta : V \to Y'$ such that $\rho^{-1}(\mathcal{U}) \subseteq \alpha(V)$ and $\alpha^*(\mathcal{D}) = \beta^*(\mathcal{D}')$. In particular, the combinatorial data $(Y, \mathcal{D})$ is locally isomorphic in the étale topology to the combinatorial data of the affine space endowed with a linear $\mathbb{T}$-action.*

**Proof.** Since $X$ is covered by the smooth $\mathbb{T}$-stable open sets $X(\rho^{-1}(\mathcal{U}), \mathcal{D}|_{\rho^{-1}(\mathcal{U})})$. We obtain that every point $x \in X$ is locally isomorphic in the étale topology to a open set in $\mathbb{A}^n$ endowed with a linear $\mathbb{T}$-action. Hence $X$ is smooth. This shows the "if" of the first assertion.

Assume now that $X$ is smooth. By the diagram in equation (1.1), the A-H presentation is local relative to the algebraic quotient morphism $q_0 : X \to Y_0 = X/\!/\mathbb{T}$. Hence, we can assume we have a morphism $X = X^{ss}(u) = W_\lambda \to Y = Y_\lambda = W_\lambda/\!/\mathbb{T}$. By Proposition 4, there exists an algebraic variety $Z$, a smooth toric variety $X'$ of dimension $n$, and strongly étale morphisms $a : Z \to X(\rho^{-1}(\mathcal{U}), \mathcal{D}|_{\rho^{-1}(\mathcal{U})})$ and $b : Z \to X'$ where and $\mathbb{T}$ acts in $X'$ via a subtorus of the big torus action. By the diagram in Theorem 3 we obtain the following diagram of étale morphisms

$$
\begin{array}{ccccc}
X' := \mathbb{T} \star^{\mathbb{T}_x} T_x V_x & \xleftarrow{\;b\;} & Z := \mathbb{T} \star^{\mathbb{T}_x} V_x & \xrightarrow{\;a\;} & X = X^{ss} \\
\downarrow & & \downarrow & & \downarrow \\
Y' = N_x/\!/\mathbb{T}_x & \xleftarrow{\;\beta\;} & V := (\mathbb{T} \star^{\mathbb{T}_x} V_x)/\!/\mathbb{T} & \xrightarrow{\;\alpha\;} & Y = X^{ss}/\!/\mathbb{T}
\end{array}
$$

The result now follows by applying twice Lemma 6 to the right and left square in the above diagram and by embedding the smooth affine toric variety $X'$ linearly on the affine space $\mathbb{A}^n$ as in the proof of Proposition 4. □

**Remark 8.** By Theorem 7, local models for smooth $\mathbb{T}$-varieties can be obtained via the downgrading procedure described in [2, section 11]. In Section 3, we will use this procedure to compute the local models for $\mathbb{G}_m$-threefolds. Here, we now give as a simple corollary the well known criterion for the smoothness of a complexity one $\mathbb{T}$-variety [17].

**Corollary 9.** *Let $X = X(Y, \mathcal{D})$ be an affine $\mathbb{T}$-variety of complexity one, where $Y$ is a smooth curve and $\mathcal{D} = \sum \Delta_i \cdot z_i$ is a p-divisor on $Y$. Then $X$ is smooth if and only if* (*i*) *$Y$ is affine and the cone spanned in $N_{\mathbb{Q}} \times \mathbb{Q}$ by $(0, \operatorname{tail}(\mathcal{D}))$ and $(1, \Delta_i)$ is smooth for all $i$.*

(*ii*) *$Y = \mathbb{P}^1$ and $X$ is equivariantly isomorphic to an open set in the affine space endowed with a linear $\mathbb{T}$-action.*

*Furthermore, if $X$ is rational then there is a Zariski cover of $X$ by open sets isomorphic to open sets in the affine space endowed with linear $\mathbb{T}$-actions.*

**Proof.** Let $X = X(Y, \mathcal{D})$ be an affine $\mathbb{T}$-variety of complexity one. Recall that all p-divisors in complexity one are minimal. Assume first that $Y$ is projective. In this case we have that $Y_0$ is reduced to a point. Theorem 7 yields that the combinatorial data $(Y, \mathcal{D})$ is locally isomorphic in the étale topology to the combinatorial data $(Y', \mathcal{D}')$ of the affine space endowed with a linear $\mathbb{T}$-action. Let $\alpha : V \to Y$ and $\beta : V \to Y'$ be the étale morphisms in Theorem 7. Since $Y_0$ is a point, we have $\mathcal{U} = Y_0$ and the condition $\rho^{-1}(\mathcal{U}) \subseteq \alpha(V)$ ensures that $\alpha$ is surjective. Hence, $V$ is a projective curve. Furthermore, by the downgrading procedure described in [2, section 11], every linear $\mathbb{T}$-action on the affine space with algebraic quotient reduced to a point is given by a p-divisor on a complete toric variety. Since the complexity is one we have $Y' = \mathbb{P}^1$. Now, by the Riemann–Hurwitz formula, we obtain that $\alpha$ and $\beta$ are isomorphisms. This proves $(ii)$.

In the case where $Y$ is affine, we have that $Y_0 = Y$ and Theorem 7 yields that the combinatorial data $(Y, \mathcal{D})$ is locally isomorphic in the étale topology to the combinatorial data $(Y', \mathcal{D}')$ of the affine space endowed with a linear $\mathbb{T}$-action. By the downgrading procedure in [2, section 11], we obtain that every complexity one linear $\mathbb{T}$-action on the affine space with algebraic quotient of dimension one is given by a p-divisor on $Y' = \mathbb{A}^1$ supported at the origin $\mathcal{D}' = \Delta_0 \cdot [0]$ and such that the cone spanned in $N_{\mathbb{Q}} \times \mathbb{Q}$ by $(0, \operatorname{tail}(\mathcal{D}'))$ and $(1, \Delta_0)$ is smooth. This proves $(i)$.

The last statement of the corollary follows directly from $(i)$ and $(ii)$. □

**Remark 10.** By the results from the second author in [21], even in the case where $X$ is rational, there is not always a Zariski cover of $X$ by open sets isomorphic to open sets in the affine space endowed with linear $\mathbb{T}$-actions in complexity higher than one.

## 3. Smooth threefolds with complexity two torus actions

To be able to effectively apply Theorem 7 as a smoothness criterion, it is useful to compute all possible A-H presentations for the affine space $\mathbb{A}^n$ endowed with a linear torus action. As we show in the previous section, a smoothness criterion in complexity 1 is well known and is easily reproved as a corollary of our criterion. In this section we compute all A-H presentations for the affine space $\mathbb{A}^3$ with a linear torus action of complexity 2. This yields an effective smoothness criterion for threefolds endowed with a $\mathbb{T}$-action of complexity 2.

**Definition 11.** Let $X$ be an affine $\mathbb{T}$-variety, let $q \in \mathbb{C}[X]$ be a semi-invariant polynomial and let $k$ be an positive integer. Then $\varphi_k : X_k = \mathrm{Spec}(\mathbb{C}[X][y]/(q(x) - y^k)) \to X$ is called a cyclic cover of $X$ of order $k$ along the divisor $D = \{q(x) = 0\}$.

To be able to perform computations, we need to put the base variety $Y$ in a special form. This corresponds to taking a suitable $\mathbb{T}$-invariant cyclic cover along a coordinate axis that ensures that the algebraic quotient of the new action is an affine space of dimensions 0 1 or 2, respectively and in the case where it is 0-dimensional, that the corresponding toric variety quotient has a smooth toric chart.

This technical restriction in our Proposition below is not a serious drawback since cyclic covers are well understood for the A-H presentation following the work of the second author, see [20] and any $\mathbb{G}_m$-variety given in terms of the A-H presentation can be easily taken to this special form. Furthermore, recall that a cyclic cover of $\mathbb{A}^n$ along a coordinate axis is isomorphic to $\mathbb{A}^n$. Hence, the cyclic covers under consideration take the affine space into the affine space with a different $\mathbb{T}$-action.

In the following proposition, we denote a section of the matrix $F = (a \quad b \quad \pm c)^t$ by $s = (\alpha \quad \beta \quad \gamma)$. We also denote $\gcd(i,j)$ by $\rho(i,j)$ and we set $\delta = \gcd(\frac{a}{\rho(a,c)}, \frac{b}{\rho(b,c)})$.

**Proposition 12.** *Let $X = \mathbb{A}^3$ endowed with an effective $\mathbb{G}_m$-action. Then, after a suitable $\mathbb{T}$-invariant cyclic cover along a coordinate axis, we have $X \simeq \mathbb{A}^3 \simeq X(Y, \mathcal{D})$ and the weight matrix $F$, $Y$ and $\mathcal{D}$ are given in the following list:*

*(1) $F$ equals $(a \quad b \quad -c)^t$ with $a$, $b$ and $c$ positive integers; $Y$ is isomorphic to a weighted blow-up $\pi : \tilde{\mathbb{A}}^2 \to \mathbb{A}^2$ of $\mathbb{A}^2$ centered at the origin having an irreducible exceptional divisor $E$; and $\mathcal{D}$ is given by*

$$\mathcal{D} = \left\{ \frac{\alpha\rho(a,c)}{c} \right\} \otimes D_1 + \left\{ \frac{\beta\rho(b,c)}{c} \right\} \otimes D_2 + \left[ \frac{\gamma}{\delta}, \frac{\gamma}{\delta} + \frac{1}{\delta c} \right] \otimes E,$$

*with $D_1$, $D_2$ the strict transforms of the coordinate axes in $\mathbb{A}^2$.*

*(2) $F$ equals $(a \quad b \quad c)^t$ with $a$, $b$ and $c$ positive integers; $Y$ is isomorphic to the weighted projective space $\mathbb{P}(a,b,c)$ having a smooth standard chart $U_3 = \{x_3 \neq 0\}$; and $\mathcal{D}$ is given by*

$$\mathcal{D} = \left[ \frac{\alpha\rho(a,c)}{c}; +\infty \right[ \otimes D_1 + \left[ \frac{\beta\rho(b,c)}{c}; +\infty \right[ \otimes D_2 + \left[ \frac{\gamma}{\delta}; +\infty \right[ \otimes D_3,$$

*with $D_1$, $D_2$ and $D_3$ the coordinate axes in $\mathbb{P}(a,b,c)$.*

*(3) $F$ equals $(0 \quad b \quad c)$ with $b$ and $c$ positive integers; $Y$ is isomorphic to $\mathbb{P}^1 \times \mathbb{A}^1$; and $\mathcal{D}$ is given by*

$$\mathcal{D} = \left[ \frac{\beta\rho(b,c)}{c}; +\infty \right[ \otimes D_2 + \left[ -\frac{\beta\rho(b,c)}{c}; +\infty \right[ \otimes D_3,$$

*with $D_2 = \{0\} \times \mathbb{A}^1$ and $D_3 = \{\infty\} \times \mathbb{A}^1$.*

*(4) $F$ equals $(0 \quad b \quad -c)^t$ with $b$ and $c$ positive integers; $Y$ is isomorphic to $\mathbb{A}^2$; and $\mathcal{D}$ is given by*

$$\mathcal{D} = \left[\frac{\gamma\rho(b,c)}{b}, \frac{\beta\rho(b,c)}{c}\right] \otimes D_2,$$

*with $D_2$ a coordinate axis in $\mathbb{A}^2$.*

*(5) $F$ equals $(0 \quad 0 \quad 1)^t$; $Y$ is isomorphic to $\mathbb{A}^2$; and $\mathcal{D} = 0$ with tail cone $[0, +\infty[$.*

**Proof.** Let $\mathbb{A}^3$ be endowed with a faithful linear action of $\mathbb{G}_m$. It is clear that the matrices $F$ in the above list cover all possible cases of faithful linear action of $\mathbb{G}_m$ on $\mathbb{A}^3$. We will give the ingredients to compute the A-H presentation by the downgrading method described in [2, section 11]. Therein, the combinatorial data is obtained by a routine computation from the following exact sequence

$$0 \longrightarrow \mathbb{Z} \underset{F}{\overset{s}{\rightleftarrows}} \mathbb{Z}^3 \xrightarrow[P]{} \mathbb{Z}^2 \longrightarrow 0\ .$$

In general, in cases (1) and (2) it is impossible to compute the cokernel matrix $P$. Nevertheless, the choices made of a suitable cyclic cover in these two cases allows us to compute $P$ and from this data. Indeed, a direct verification show s that the cokernel matrix $P$ can be chosen, respectively, as follows:

$$(1)\quad P = \begin{pmatrix} \frac{c}{\rho(a,c)} & 0 & \frac{a}{\rho(a,c)} \\ 0 & \frac{c}{\rho(b,c)} & \frac{b}{\rho(b,c)} \end{pmatrix}, \qquad (3)\quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{c}{\rho(b,c)} & -\frac{b}{\rho(b,c)} \end{pmatrix},$$

$$(2)\quad P = \begin{pmatrix} \frac{c}{\rho(a,c)} & 0 & -\frac{a}{\rho(a,c)} \\ 0 & \frac{c}{\rho(b,c)} & -\frac{b}{\rho(b,c)} \end{pmatrix}, \qquad (4)\quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{c}{\rho(b,c)} & \frac{b}{\rho(b,c)} \end{pmatrix},$$

$$(5)\quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

To obtain now the combinatorial data from the exact sequence above we follow [2, section 11]. Let $v_i$, for $i \in \{1, 2, 3\}$ be the integral vector determined by the i-th column vector of $P$. The set $\{v_i \mid i = 1, 2, 3\}$ is the set of 1-dimensional cones of the fan of the toric variety $Y$. Moreover each non-zero integral vector $v_i$ corresponds to a prime divisor $D_i$ of $Y$ being a part of the support of the p-divisor $\mathcal{D}$. The coefficients corresponding to each prime divisor $D_i$ is now given by $s\left(\mathbb{R}^3_{\geq 0} \cap P^{-1}(v_i)\right)$. $\square$

In the remaining of this section, we provide several illustrative examples showing the different behaviors of the A-H description in the case of complexity 2 threefolds with respect to our smoothness criterion.

**Example 13** *(A smooth $\mathbb{T}$-variety over an open set in an abelian variety).* Let $E_1 = \{h_1(u_1,v_1) = 0\} \subset \mathbb{A}^2 = \operatorname{Spec}(\mathrm{k}[u_1,v_1])$ and $E_2 = \{h_2(u_2,v_2) = 0\} \subset \mathbb{A}^2 = \operatorname{Spec}(\mathrm{k}[u_2,v_2])$ be two planar affine smooth elliptic curves passing with multiplicity one through the origin. Let $Y$ be the blow up of $E_1 \times E_2$ at the origin of $\mathbb{A}^4 = \mathbb{A}^2 \times \mathbb{A}^2$. Let $X = X(Y,\mathcal{D})$, where

$$\mathcal{D} = \left\{\frac{1}{3}\right\} \otimes D + \left[0, \frac{1}{3}\right] \otimes E,$$

where $D$ is the strict transform of $\{u_1 = 0\} \times E_2 \subseteq E_1 \times E_2$ in $Y$ and $E$ is the exceptional divisor of the blowup. We will show that $X$ is smooth. By Theorem 7, we only need to show that for every $y \in Y_0 = X/\!/\mathbb{T} \simeq E_1 \times E_2$ there is an étale neighborhood $\mathcal{U}$ such that $X\left(\rho^{-1}(\mathcal{U}), \mathcal{D}|_{\rho^{-1}(\mathcal{U})}\right)$ is $\mathbb{T}$-equivariantly isomorphic to an étale neighborhood of $\mathbb{A}^3$ endowed with a linear $\mathbb{G}_m$-action. Hence, we only need to show that, étale locally over $y \in Y_0$, the divisor $\mathcal{D}$ appears in the list in Proposition 12. Indeed, it corresponds to case (1) with $F = (1 \quad 1 \quad -3)^t$ taking $s = (0 \quad 1 \quad 0)$.

On the other hand, with the method in [2, section 11], we can verify that $X = X(Y,\mathcal{D})$ is isomorphic to the closed $\mathbb{G}_m$-stable subvariety of $\mathbb{A}^5 = \operatorname{Spec}(\mathrm{k}[x_1,y_1,x_2,y_2,z])$ with weight matrix $(1 \quad 3 \quad 3 \quad 3 \quad -3)^t$ given by the equations

$$\left\{\tfrac{1}{z} \cdot h_1(x_1^3 z, y_1 z) = 0\,;\ \tfrac{1}{z} \cdot h_2(x_2 z, y_2 z) = 0\right\} \subset \mathbb{A}^5.$$

By the Jacobian criterion, we can check that $X$ is indeed smooth.

**Example 14** *(A smooth $\mathbb{T}$-variety with non-rational support divisor).* Let $E_1 = \{h(u,v) = u^2 - v(v-\alpha)(v-\beta) = 0\} \subset \mathbb{A}^2 = \operatorname{Spec}(\mathrm{k}[u,v])$ be a planar affine smooth elliptic curve. Let $X = X(Y,\mathcal{D})$, where $Y$ is the blowup of $\mathbb{A}^2$ at the origin and

$$\mathcal{D} = \left\{\frac{1}{2}\right\} \otimes D_1 + \left\{-\frac{1}{3}\right\} \otimes D_2 + \left[0, \frac{1}{6}\right] \otimes E,$$

where $D_1$ is the strict transform of the curve $\{h(u,v) = 0\}$, $D_2$ is the strict transform of $\{u = 0\}$ and $E$ is the exceptional divisor of the blowup. Again, by Theorem 7, we only need to show that, étale locally over $y \in Y_0 = X/\!/\mathbb{G}_m \simeq \mathbb{A}^2$, the divisor $\mathcal{D}$ appears in the list in Proposition 12. In an étale neighborhood of the origin in $\mathbb{A}^2$ it corresponds to case (1) with $F = (2 \quad 3 \quad -6)^t$ taking $s = (-1 \quad 1 \quad 0)$. Furthermore, the same local model works in an étale neighborhood of every point different from $(0,\alpha)$ and $(0,\beta)$. In an étale neighborhood $\mathcal{U}$ of these two points, the divisor $\mathcal{D}$ is given by $\mathcal{D}|_{\mathcal{U}} = \left\{\frac{1}{2}\right\} \otimes D_1 + \left\{-\frac{1}{3}\right\} \otimes D_2$. Such a p-divisor corresponds to a $\mathbb{T}$-invariant open set of case (2) with $F = (2 \quad 3 \quad 6)^t$ taking $s = (-1 \quad 1 \quad 0)$, see [3, Proposition 3.4].

On the other hand, we can verify that $X = X(Y,\mathcal{D})$ is isomorphic to the closed $\mathbb{G}_m$-stable subvariety of $\mathbb{A}^4 = \operatorname{Spec}(\mathrm{k}[x,y,z,t])$ with weight matrix $(2 \quad 6 \quad -6 \quad 3)^t$ given by the equation

$$\left\{\tfrac{1}{z}\cdot h(x^3z, yz) = t^2\right\} \subset \mathbb{A}^5.$$

By the Jacobian criterion, we can check that $X$ is indeed smooth.

**Example 15** *(A singular $\mathbb{T}$-variety with smooth combinatorial data).* Let $C = \{h(u,v) = u + v(1-v)^2 = 0\} \simeq \mathbb{A}^1 \subset \mathbb{A}^2 = \operatorname{Spec}(\mathrm{k}[u,v])$. Let $X = X(Y,\mathcal{D})$, where $Y$ is the blowup of $\mathbb{A}^2$ at the origin and

$$\mathcal{D} = \left\{\frac{1}{2}\right\} \otimes D_1 + \left\{-\frac{1}{3}\right\} \otimes D_2 + \left[0, \frac{1}{6}\right] \otimes E,$$

where $D_1$ is the strict transform of the curve $C$, $D_2$ is the strict transform of $\{u = 0\}$ and $E$ is the exceptional divisor of the blowup. By Theorem 7, $X$ is smooth if and only if, étale locally over $y \in Y_0 = X/\!/\mathbb{G}_m \simeq \mathbb{A}^2$, the divisor $\mathcal{D}$ appears in the list in Proposition 12. In an étale neighborhood of every point different from $(0,1)$ it corresponds to case (1) with $F = (2 \quad 3 \quad -6)^t$ taking $s = (-1 \quad 1 \quad 0)$. Nevertheless, $D_1$ and $D_2$ intersect non-normally on the preimage of $(0,1)$. Since all toric divisors intersect normally, such point does not admit an étale neighborhood $\mathcal{U}$ such that $X\left(\rho^{-1}(\mathcal{U}), \mathcal{D}|_{\rho^{-1}(\mathcal{U})}\right)$ is equivariantly isomorphic to an étale open set on a toric variety.

On the other hand, we can verify that $X = X(Y,\mathcal{D})$ is isomorphic to the $\mathbb{G}_m$-stable subvariety of $\mathbb{A}^4 = \operatorname{Spec}(\mathrm{k}[x,y,z,t])$ with weight matrix $(2 \quad 6 \quad -6 \quad 3)^t$ given by the equation

$$\{x^3 + y(1-yz)^2 = t^2\} \subset \mathbb{A}^4.$$

By the Jacobian criterion, we can check that the point $(0,1,1,0)$ is singular.

**Example 16** *(A singular $\mathbb{T}$-variety with irreducible support).* Let $X = X(Y,\mathcal{D})$, where $Y$ is the blowup of $\mathbb{A}^2$ at the origin and

$$\mathcal{D} = [-p, 0] \otimes E,$$

where $E$ is the exceptional divisor of the blowup and $p$ is an integer strictly greater than one. By Theorem 7, $X$ cannot be smooth since an exceptional divisor only appears on case (1) in the list in Proposition 12 and therein, the width of the coefficient polytope is at most 1. On the other hand, we can verify that $X$ is equivariantly isomorphic to the quotient of $\mathbb{A}^3 = \operatorname{Spec}(\mathrm{k}[x,y,z])$ by the finite cyclic group $\mu_p$ of the $p$-th roots of the unit acting via $\epsilon \cdot (x,y,z) = (\epsilon x, \epsilon y, z)$. Since such action is not a pseudo-reflection, the quotient is singular [5]. The $\mathbb{G}_m$-action is given in $\mathbb{A}^3$ by weight matrix $(1 \quad 1 \quad -1)^t$.

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