A brief introduction to automorphisms of algebraic varieties. Talca. Chile. 2019

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Lecture 1

Biregular automorphisms

For any projective algebraic variety X over a field $\Bbbk$ we set $\mathrm{Aut}_{\Bbbk}(X)$ to be the group of biregular maps of X to itself defined over $\Bbbk$. If $\Bbbk = \mathbb{C}$, it has a structure of a Lie group whose connected component of identity $\mathrm{Aut}(X)^0$ is a complex algebraic group and the group of connected components $\mathrm{Aut}(X)^c$ is a countable discrete group. Over an arbitrary field $\Bbbk$, one defines a group scheme of locally finite type $\underline{\mathrm{Aut}}(X)$ representing the functor from the category of finite algebras over $\Bbbk$ to the category of groups defined by $K \mapsto \mathrm{Aut}_K(X \times K)$. By definition, its set of rational points $\underline{\mathrm{Aut}}(X)(K)$ is the group of automorphisms $\mathrm{Aut}_{\Bbbk}(X)$. We still have the connected component of identity $\underline{\mathrm{Aut}_{\Bbbk}}(X)^0$ which is a group scheme of finite type. If $p = \mathrm{char}(\Bbbk) \neq 0$, this group scheme may be non-reduced, and the reduced one is an algebraic group over $\Bbbk$. The group

$$\mathrm{Aut}(X)^c := \mathrm{Aut}(X)/\,\mathrm{Aut}(X)^0$$

of connected components is a constant group scheme, it is defined by some abstract group Γ together with an action of the Galois group of $\Bbbk$. Until very recently we did not know any example of a smooth algebraic variety X for which the group $\mathrm{Aut}(X)^c$ is not finitely generated. The first example of such a variety (of dimension 6) was given by John Lesieutre in 2018 [**?**]. Since then examples in all dimensions larger than one were given by T. Dinh and K. Oguiso [**?**].

For example, if X is a nonsingular projective algebraic curve of genus g over an algebraically closed field $\Bbbk$, the group $\mathrm{Aut}(X)^0$ is isomorphic to $\mathrm{PGL}(2, \Bbbk)$ if $g = 0$ and to itself, equipped with a group law if $g = 1$. If $g > 1$ $\mathrm{Aut}(X)$ is a finite group (of order $\leq 84(g-1)$ if char($\Bbbk$) $= 0$). If $\Bbbk$ is not algebraically closed, the answer becomes more difficult. For example, if $X(\Bbbk) = \emptyset$, then $\mathrm{Aut}(X)^0 \cong \mathrm{O}(2, \Bbbk) \ncong \mathrm{PGL}(2, \Bbbk)$ if $g = 0$ and if $\Bbbk$ is a number field or a field of rational functions of a curve, then $\mathrm{Aut}(X)$ is finite if $g = 1$.

The study of $\mathrm{Aut}(X)$ is divided into two parts, the study of $\mathrm{Aut}(X)^0$ and the study of the group of connected components $\mathrm{Aut}(X)^c$. The first part is mainly concerned with the geometric invariant theory and we will be more interested in the discrete part $\mathrm{Aut}(X)^c$. The class of varieties with nontrivial group $\mathrm{Aut}(X)^0$ is rather special. Each algebraic group G which stays connected when we replace $\Bbbk$ with its algebraic closure fits in an extension

$$1 \to \mathrm{Lin}(G) \to G \to A \to 1, \tag{1.1}$$

where A is a projective algebraic group (a complex torus if $\Bbbk = \mathbb{C}$) and $\mathrm{Lin}(G)$ is a connected linear algebraic group, i.e. isomorphic to a closed subgroup of $\mathrm{GL}(n, \Bbbk)$. Recall that to any irreducible algebraic variety one can associate a regular map $a : X \to \mathrm{Alb}(X)$ where $\mathrm{Alb}(X)$ is an abelian variety satisfying the following universal property: for any regular map $a' : X \to A$ to an abelian variety there exists a unique map $\phi : A \to \mathrm{Alb}(X)$ such that $a' = \phi \circ a$. For example, when X is a smooth algebraic curve, $\mathrm{Alb}(X)$ coincides with the Jacobian variety $\mathrm{Jac}(X)$. If we take $G = \mathrm{Aut}(X)^0$ in (**??**) and consider a natural homomorphism $\mathrm{Aut}(X)^0 \to \mathrm{Aut}(\mathrm{Alb}(X))$, then we obtain that $\mathrm{Lin}(\mathrm{Aut}(X)^0)$ is mapped to zero and the abelian quotient A is mapped injectively into $\mathrm{Alb}(X) = \mathrm{Aut}(\mathrm{Alb}(X))^0$.

Assuming that $\mathrm{Alb}(X)$ is trivial, we find that $\mathrm{Lin}(\mathrm{Aut}(X)^0)$ acts on X. Each linear algebraic group of positive dimension contains a subgroup isomorphic to $\mathbb{G}_a$ (the additive group of the field) or $\mathbb{G}_m$ (the multiplicative group of the field), so there exists an open subset of X such that through each point $x \in X$ passes a rational curve, the closure of the orbit of $\mathbb{G}_a$ or $\mathbb{G}_m$. Thus X is a *uniruled variety*, i.e. an algebraic variety that contains an open dense U which is the image of an open subset of the product $\mathbb{P}^1 \times B$ for some variety B.

To study the discrete part $\mathrm{Aut}(X)^c$ we seek a representation of this group by automorphisms of some discrete objects assigned to X in a functorial way.

The most natural action is the action of $\mathrm{Aut}(X)^c$ on some cohomology theory. We assume that $\Bbbk = \mathbb{C}$ not to get involved in other arithmetical cohomology theories, e.g. the l-adic cohomology.

Let X be a nonsingular complex projective algebraic variety of dimension n (of real dimension $2n$). Its cohomology $H^k(X, \mathbb{C})$ admits the *Hodge decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $\overline{H^{p,q}}(X) = H^{q,p}(X)$. The dimensions of $H^{p,q}(X)$ are called the *Hodge numbers* and denoted by $h^{p,q}(X)$. Via the de Rham Theorem, each cohomology class in $H^{p,q}(X)$ can be represented by a complex differential form of type (p, q), i.e. locally expressed in terms of the wedge-products of p forms dz_i and q-forms of type $d\bar{z}_j$. We also have

$$H^{p,q}(X) \cong H^q(X, \Omega_X^p), \tag{1.2}$$

where Ω_X^p is the sheaf of holomorphic differential p-forms on X.

We set

$$H^{p,q}(X, \Lambda) = H^{p+q}(X, \Lambda) \cap (H^{p,q}(X) + H^{q,p}(X)),$$

where Λ is a subring of $\mathbb{C}$. Fix a projective embedding of X in some projective space $\mathbb{P}^N$, then cohomology class η of a hyperplane section of X belongs to $H^{1,1}(X, \mathbb{Z})$. Let L be the linear operator on $H^*(X,\Lambda)$ defined by the cup-product with η. We define the *primitive cohomology*

$$H^{2k}(X,\Lambda)_{\mathrm{prim}}: \quad = \quad \mathrm{Ker}(L^{n-k}: H^{2k}(X,\Lambda) \to \Lambda), \tag{1.3}$$

$$H^{2k+1}(X,\Lambda)_{\mathrm{prim}} \quad = \quad H^{2k+1}(X,\Lambda). \tag{1.4}$$

The *Hard Lefschetz Theorem* asserts that

$$L^k: H^{n-k}(X,\mathbb{Q}) \to H^{n+k}(X,\mathbb{Q})$$

is an isomorphism. The cup-product $(\phi,\psi) \mapsto \phi \cup \psi \cup \eta^{d-k}$ defines a bilinear form

$$Q_\eta: H^k(X,\Lambda) \times H^k(X,\Lambda) \to H^{2d}(X,\Lambda) \cong \Lambda,$$

where $\Lambda = \mathbb{R}, \mathbb{C}$ and the last isomorphism is defined by the fundamental class of X.

The *Hodge Index Theorem* asserts that the cup-product Q_η satisfies the following properties

- $Q_\eta(H^{p,q}(X), H^{p',q'}(X)) = 0$, if $(p,q) \neq (q',p')$;
- $\sqrt{-1}^{p-q}(-1)^{(n-k)(n-k-1)/2}Q_\eta(\phi,\bar{\phi}) > 0$, for any $\phi \in H^{p,q}(X)_{\mathrm{prim}}, p+q=k$.

Let us apply this to the case when $n = 2m$. In this case, we have the cup-product on the middle cohomology

$$H^n(X,\Lambda) \times H^d(X,\Lambda) \to H^{2n}(X,\Lambda) \cong \Lambda.$$

By *Poincaré Duality*, this is the perfect symmetric duality modulo torsion (of course, no torsion if Λ is a field). For $\Lambda = \mathbb{R}$, it coincides with Q_η, and the Hodge Index Theorem asserts in this case that Q_η does not depend on η and its restriction to $H^{m,m}(X,\mathbb{R})_{\mathrm{prim}}$ is a definite symmetric bilinear form of sign $(-1)^{n(n-1)/2}$. Assume that

$$h^{m-1,m-1}(X) = \mathbb{C}\eta^{m-1} \tag{1.5}$$

It follows from the Hard Lefschetz Theorem that $L: H^{m-1}(X,\mathbb{R})) \to H^{m+1}(X,\mathbb{R})$ is an isomorphism, hence

$$H^{m,m}(X,\mathbb{R}) = H^{m,m}(X,\mathbb{R})_{\mathrm{prim}} \perp \mathbb{R}\eta^m. \tag{1.6}$$

This implies that the bilinear form Q_η restricted to $H^{m,m}(X,\mathbb{R})$ has Sylvester signature $(1, h^{m,m}(X) - 1)$ if m is odd and $(h^{m.m}, 0)$ otherwise. Note that $H^{m,m}(X,\mathbb{R})_{\mathrm{prim}}$ depends on a choice of an embedding $X \hookrightarrow \mathbb{P}^N$, so the previous orthogonal decomposition depends on it too.

For any $p \geq 0$, let

$$H^{2p}(X,\mathbb{Z})_{\mathrm{alg}} \subset H^{p,p}(X,\mathbb{Z})$$

be the subgroup of cohomology classes generated by the cohomology classes $[Z]$, where Z is an irreducible p-dimensional closed subvariety of X. Its elements are called *algebraic cohomology classes*.

We set

$$N^p(X) = H^{2p}(X,\mathbb{Z})_{\mathrm{alg}}/\text{Torsion}. \tag{1.7}$$

This is a free abelian group of some finite rank $\rho_p(X)$. In the case of algebraic varieties defined over a field $\Bbbk$ different from $\mathbb{C}$ one defines the *Chow group* $\mathrm{CH}^p(X)$ of algebraic cycles of codimension p on X modulo rational equivalence (see [**?**], where they are denoted by $A^k(X)$). Its quotient $\mathrm{CH}_{\mathrm{alg}}(X)$ by the subgroup of p-cycles algebraically equivalent to zero is the closest substitute of $H^{2p}(X,\mathbb{Z})_{\mathrm{alg}}$.

The intersection theory of algebraic cycles defines the symmetric intersection product

$$\mathrm{CH}^k(X)\times \mathrm{CH}^l(X)\to \mathrm{CH}^{k+l}(X),\ (\gamma,\beta)\mapsto \gamma\cdot\beta$$

It descents to the intersection product

$$\mathrm{CH}^k_{\mathrm{alg}}(X)\times \mathrm{CH}^l_{\mathrm{alg}}(X)\to \mathrm{CH}^{k+l}_{\mathrm{alg}}(X).$$

When $\Bbbk=\mathbb{C}$, there is a natural homomorphism $\mathrm{CH}^p_{\mathrm{alg}}(X)\to H^{p,p}(X,\mathbb{Z})$, however it may not be injective if $p>1$.

The group $\mathrm{CH}^1(X)$ coincides with the *Picard group* $\mathrm{Pic}(X)$ of X, the group of divisors modulo linear equivalence. It is naturally identified with the group of isomorphism classes of line bundles (or invertible sheaves) on X . The group $\mathrm{CH}^1_{\mathrm{alg}}(X)$ is denoted by $\mathrm{NS}(X)$ and is called the *Néron-Severi group* of X. It is a finitely generated abelian group.

One defines the group $N^p(X)$ of *numerical equivalence classes* of algebraic cycles as the quotient group of $\mathrm{CH}^p(X)$ modulo the subgroup of elements γ such that $\gamma\cdot\beta=0$ for all $\beta\in\mathrm{CH}^{n-p}(X)$. It is not known whether this definition coincides with the definition (**??**) when $\Bbbk=\mathbb{C}$ and $p>1$.

The statement about the signature of the intersection product on $N^d(X)_{\mathbb{R}}$ over fields of arbitrary characteristic is a conjecture. It follows from *Standard Conjectures* of A. Grothendieck.

The group $N^1(X)$ coincides with the group $\mathrm{Num}(X)$ of numerical classes of divisors on X. It is the quotient of the Néron-Severi group by the torsion subgroup.

Example 1.1. Assume $n=2$, i.e. X is a nonsingular projective algebraic surface. Since $H^0(X,\Lambda)\cong\Lambda$, we have $h^{0,0}=1$, and condition (**??**) is obviously satisfied. In this case

$$N^1(X)=H^2(X,\mathbb{Z})_{\mathrm{alg}}/\text{Torsion}=\mathrm{NS}(X)/\text{Torsion}.$$

The number $\rho_1(X)$ is called the *Picard number* of X and is denoted by $\rho(X)$. Over $\mathbb{C}$, we have the Hodge decomposition

$$H^2(X,\mathbb{C})=H^{2,0}(X)\oplus H^{1,1}(X)\oplus H^{0,2}(X).$$

The Hodge numbers $h^{2,0}=h^{0,2}=\dim H^0(X,\Omega^2_X)=\dim\Omega^2(X)$. By Serre's Duality, $\dim_{\mathbb{C}} H^0(X,\Omega^2_X)=\dim_{\mathbb{C}} H^2(X,\mathcal{O}_X)$, where $\mathcal{O}_X$ is the sheaf of regular (=holomorphic) functions on X. This number is classically denoted by by $p_g(X)$ and is called the *geometric genus* of X. We have

$$\rho(X)\le h^{1,1}(X)=b_2(X)-2p_g(X), \tag{1.8}$$

where $b_k(X)$ denote the Betti numbers of X. In characteristic $p > 0$, this is not true anymore, it could happen that $\rho(X) = b_2(X)$ (defined in terms of the l-adic cohomology) even when $p_g > 0$.

We also have the Hodge decomposition

$$H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X).$$

The Hodge number $h^{1,0} = h^{0,1}$ is denoted by $q(X)$ and is called the *irregularity* of X. Ii $p = 0$, it is equal to the dimension of the Albanese variety $\mathrm{Alb}(X)$.

Note that, in the case of surfaces, the Hodge Index Theorem can be proved without using the Hodge decomposition, and it is true for arbitrary fields. It asserts that the free group $N^1(X)$ equipped with the intersection pairing has the Sylvester signature (afer tensoring with $\mathbb{R}$) equal to $(1, \rho - 1)$.

To every nonsingular n-dimensional variety X we may assign a special class in $\mathrm{Pic}(X)$ or $N^1(X)$, the *canonical class* K_X. Over $\mathbb{C}$ it coincides with $-c_1(X)$, where c_1 is the first Chern class of the corresponding complex manifold. In fact, there is a theory of Chern classes for varieties over any field. There are n Chern classes $c_1, \ldots, c_n$. The top one c_n is an integer that coincides with the topological Euler-Poincaré characteristic $e(X)$ of X. The canonical class K_X is defined as the linear equivalence class of the divisor of zeros and poles of any rational n-form ω on X. The dimension of the linear space $\Omega^n(X)$ of regular differential top forms is denoted by $p_g(X)$ or $h^0(K_X)$ expressing the dimension of the space of sections of the line bundle Ω^n_X, it is called the geometric genus of X. All n-dimensional varieties divided in $n+1$ classes according to how the function $h^0(mK_X)$ grows. Thus we have $h^0(mK_X) \sim m^k$, where k is called the *Kodaira dimension* of X and is denoted by $\mathrm{kod}(X)$.

For example, if X is uniruled, then $\mathrm{kod}(X) = -\infty$ (the converse is true if $n \leq 3$). If $\mathrm{kod}(X) = n$, a variety is said to be of general type. They are higher-dimensional analogs of algebraic curves of genus $g > 1$.

In the case $n = 2$, we have 4 classes with $\mathrm{kod}(X) = -\infty, 0, 1, 2$. The surfaces with $\mathrm{kod}(X) = -\infty$ admit a birational map $X \to V$, where V is a projective bundle over a curve C, or $V \cong \mathbb{P}^2_{\Bbbk}$. The surfaces with $\mathrm{kod}(X) = 0$ come in four types

- *Abelian surfaces*, i.e. abelian varieties of dimension 2;
- *K3 surfaces* characterized by the condition that $K_X = 0$ and $\pi_1(X) = \{1\}$ (or if $\Bbbk \neq \mathbb{C}$ no finite unramified covers of degree larger than 1 exist). An equivalent condition is that $K_X = 0$ and $e(X) = 24$.
- *Enriques surfaces* characterized by the condition that $e(X) = 12$ and $2K_X = 0$. If $p \neq 2$, they are isomorphic to the quotients of a K3 surface by a fixed-point-free involution.
- *Hyperelliptic surfaces*. Certain finite quotients of products $E \times C$, where E is an elliptic curve.

Surfaces with $\mathrm{kod}(X) = 1$ are characterized by the condition that there exists a fibration $f : X \to C$ over a curve with general fiber whose divisor class $[F]$ is proportional to K_X over $\mathbb{Q}$. It is an elliptic curve if $p \neq 2, 3$ or maybe isomorphic to a cuspidal cubic curve if $p = 2, 3$. Algebraic surfaces of Kodaira dimension 0 and 1 are 2-dimensional analogs of elliptic curves.

Example 1.2. Assume that $X : F_d(x_0, \ldots, x_n) = 0$ is a hypersurface in $\mathbb{P}^{n+1}$ with $n > 0$. We have $K_X \sim (d - n - 1)h$, where h is the class of a hyperplane section of X. So, if $d \neq n + 1$, any automorphism preserves K_X and hence preserves the class h. This means that $\mathrm{Aut}(X) \subset \mathrm{PGL}(n+1)$. Also, if $n \geq d + 1$, X is not uniruled and hence $\mathrm{Aut}(X)^0$ is trivial, hence $\mathrm{Aut}(X)$ is finite.

The computation of this group is rather hard even for curves. The classification of projective groups of automorphisms of curves of curves of degree d in characteristic 0 is known for $d \leq 5$. For $d = 4$ this is a classical result that goes back to A. Wiman [**?**] for nonsingular curves (see [**?**, 6.5.2] for a modern exposition) and Ciani [**?**], [**?**] for singular irreducible quartics. For $d = 5$ it is due to V. Snyder [**?**] and E. Ciani [**?**]. No classification in positive characteristic is known yet.

Also there are known some bounds on the aut$|\mathrm{Aut}(X)|$ in terms of degree d (see [**?**]). For example, it is known that $|\mathrm{Aut}(X)| \leq 6d^2$ except when $d = 4$ and X is the Klein quartic in which case $|\mathrm{Aut}(X)| = 168$ and $d = 6$ and X is a Wiman sextic with $|\mathrm{Aut}(X)| = 360$.

The classification of projective groups of automorphisms of hypersurfaces is in a rudimentary state. We know the classification of projective automorphisms of cubic surfaces over algebraically closed fields of arbitrary characteristic. There is no yet a complete classification of projective automorphisms of quartic surfaces although people are working on this (see [**?**]).

We know now the classification of automorphism groups of nonsingular cubic hypersurfaces in $\mathbb{P}^4$ over algebraically closed fields of characteristic 0 [**?**].

Of course computers can help to provide a possible list of groups in each degree but it would be hard to deduce from this the equations of the curves.

If $n = d + 1$, we have $K_X = 0$ and hence $\mathrm{Aut}(X)$ maybe larger than the group of projective automorphisms. In many cases it is an infinite group.

Example 1.3. Since a variety X of general type is not uniruled, its automorphism group coincides with $\mathrm{Aut}(X)^c$. If K_X is ample, it coincides with a finite subgroup of projective automorphisms under a projective embedding given by $|mK_X|$ for some $m > 0$. If K_X is not ample, by a theorem of Matsumura [**?**] (see a modern proof in [**?**]) it is still a finite group. There are known some bounds for the order of the automorphism group of an n-dimensional algebraic variety of general type in terms of the *volume* of X

$$\mathrm{vol}(X, K_X) := \lim.\sup_{m \to \infty} \frac{n! h^0(X, mK_X)}{m^n}$$

[**?**].

In the case of nonsingular minimal varieties $\mathrm{vol}(X, K_X) = K_X^n$. If $n = 1$, $\mathrm{vol}(X, K_X) = 2g-2$. It is conjectured that $|\operatorname{Aut}(X)| \le c\,\mathrm{vol}(X, K_X)$ for some constant c. In the case of surfaces, the best bound is $42^2K_X^2$ in a striking analogy with the Hurwitz bound for curves [**?**].

Let $n = 2m$ and $r : \operatorname{Aut}(X)^c \to \mathrm{O}(N^m(X))$ is the representation of $\operatorname{Aut}(X)$ in the group of isometries of its numerical lattice $N^{2m}(X)$. If $m = 1$, the kernel of r is always finite. In fact, $N^1(X)$ is a finite quotient of $\mathrm{NS}(X)$, it is enough to prove that the group of automorphisms that act trivially on $\mathrm{NS}(X)$ is finite. But its elements preserve the class h of a hyperplane section and hence act linearly on X. So they form an algebraic group G and its connected component of identity G^0 is contained in $\operatorname{Aut}(X)^0$ and G/G^0 is a finite group.

Since $N^m(X)$ is a definite lattice if $n = 2(2m-1)$, we see that $\operatorname{Aut}(X)$ coincides with Ker(r) up to a finite group. However, if $n = 4k+2$, the quadratic form on $N^m(X)$ has a *hyperbolic signature* $(1, r-1)$, where $r = \operatorname{rank} N^{2m}(X)$ and the quotient group $\operatorname{Aut}(X)/\mathrm{Ker}(r)$ could be infinite.

Example 1.4. Let X be a nonsingular quartic surface in $\mathbb{P}^3$. Since $d = n+1$ in this case, $K_X = 0$. A quartic surface is a special case of a K3 surface. Suppose $k = 0$ and $N^1(X) = \mathbb{Z}h$. Then it follows from above that the image of ρ is trivial.

Now let us look at the kernel. It could be non-trivial. To see this we consider an example of a K3 surface which is obtained as the double cover of $\mathbb{P}^2$ branched along a nonsingular plane curve of degree 6. We can write its equation in the form

$$w^2 + f_6(x, y, z) = 0$$

if $\mathrm{char}(\Bbbk) \neq 2$ and

$$w^2 + f_3(x, y, z)w + f_6(x, y, z) = 0,$$

where f_3, f_6 are homogeneous forms of degrees 3 and 6, if $\mathrm{char}(\Bbbk) = 2$. We should consider this equation as a surface of degree 6 in the weighted projective space $\mathbb{P}(1, 1, 1, 3)$. The transformation $(x, y, z, w) \mapsto (x, y, z, -w)$ (resp. $(x, y, z, w) \mapsto (x, y, y, w + f_3(x, y, z))$ is an automorphism of order 2.

Let us prove that this is the only exceptional case for K3 surface X with non-trivial automorphism group and $\rho(X) = 1$. We first assume that $\Bbbk = \mathbb{C}$. We already know that $\operatorname{Aut}(X)$ is finite. Let $g \in \operatorname{Aut}(X)$ be an element of order l which we may assume to be a prime number.

Let $T(X)$ be the orthogonal complement of $\operatorname{Pic}(X) = H^2(X, \mathbb{Z})_{\mathrm{alg}}$ in $H^2(X, \mathbb{Z})$. It is called the *lattice of transcendental cycles*. Its rank is equal to $22 - \rho(X)$. In our case it is equal to 21. In fact, one can define the group $T(Z)_{\mathbb{Q}}$ of (rational) transcendental cycles for any irreducible complex algebraic variety of dimension n admitting a resolution of singularities $\tilde{X}$ and prove that it is a birational invariant [**?**]. We first define

$$H^n(\Bbbk(Z)) = \varinjlim H^n(U, \mathbb{Q}),$$

where U runs through the partially ordered set of open subsets of Z. Then we set

$$T^n(Z)_{\mathbb{Q}} := \mathrm{Image}(H^n(Z) \to H^n(\Bbbk(Z)). \tag{1.9}$$

One can prove that if a finite group G acts on Z with quotient Z/G, then

$$T(Z/G)_{\mathbb{Q}} \cong T(Z)_{\mathbb{Q}}^{G} = \{x \in T(Z)_{\mathbb{Q}} : g^*(x) = x\} \tag{1.10}$$

[**?**, Proposition 5].

Consider the representation of the cyclic group $G = (g)$ on $\Omega^2(X) = \mathbb{C}\omega$. It is given by a homomorphism

$$\chi : G \to \mathbb{C}^*, \quad \sigma^*(\omega) = \chi(g)\omega.$$

An automorphism g is called *symplectic* if χ is trivial and *non-symplectic* otherwise. The reason for the name is that ω has no zeros or poles and defines a holomorphic symplectic structure on X.

Suppose g is non-symplectic. Consider the quotient surface $Y = X/G$. Since $\Omega^2(X)^G = \{0\}$, it follows from the properties of the Kodaira dimension of algebraic surfaces and their classification that a nonsingular model of Y is either an Enriques surface or a rational surface. In the first case the set X^G of fixed points of G is empty and $|G| = 2$. The Picard group of an Enriques surface ius equal to 10 and it embeds into $\mathrm{Pic}(X)$ under the inverse image map $\mathrm{Pic}(X/G) \to \mathrm{Pic}(X)$. We get a contradiction with our assumption that $\rho(X) = 1$. If Y is rational, then $T(Y) = \{0\}$, applying (**??**), we obtain that $T(X)^G = \{0\}$. The group G acts on $T(X)$ and decompose $T(X) \cong \mathbb{Z}^{21}$ into irreducible G-modules. Each such module is either trivial or isomorphic to $\mathbb{Z}/(P(t))$, where $P(t) = \frac{t^l - 1}{t-1}$ is of rank $l - 1$. Since $T(X)^G = \{0\}$ there are no trivial summands, and hence $l - 1$ divides $\mathrm{rank}\, T(X) = 21$. Since only $1, 3, 7, 21$ divide 21, the only possibility is that $l = 2$ and g acts as the minus identity. In this case, g acts locally at a fixed point by $(u, v) \mapsto (-u, v)$ (since ω is locally given by $du \wedge dv$) and the quotient surface is a nonsingular (the local ring of the image of the fixed point is analytically isomorphic to $\mathbb{C}[[u^2, v]]$, i.e. the point is nonsingular). The surface Y is rational and $\rho(Y) = 1$ (otherwise $\rho(X) > 1$). The only rational surface with this property is $\mathbb{P}^2$. This is is our exceptional case.

Assume that g acts symplectically. In this case G acts trivially on $T(X)$. In fact, since $H^2(X, \mathbb{Z})_{\mathrm{alg}} \subset H^{1,1}(X)$ and $\omega \in H^{2,0}(X)$, the multiplication map by ω defines an embedding of $T(X)$ into $\mathbb{C} = H^4(X, \mathbb{C})$. Since $g^*(\omega) = \omega$ and the embedding is g-equivariant, and hence G acts trivially on $T(X)$.

Now we use that $X^g = \emptyset$ (otherwise X/G is a K3 surface that admits an unramified cover of degree l contradicting the definition of a K3 surface). Its local action in a neighborhood of a fixed point is of the form $(u, v) \mapsto (\epsilon_l u, \epsilon_l^{-1} v)$, where ϵ_l is a primitive lth root of 1. The image of a fixed point in $Y = X/(g)$ is *double rational point* of type A_{l-1} (or ordinary node). The formal completion of its local ring is isomorphic to $\mathbb{C}[[u^l, v^l, uv]] \cong \mathbb{C}[[U, V]]/(UV - Z^l)]$. One can resolve this singular point by finding a nonsingular surface $\tilde{Y}$ together with a proper birational map $f : \tilde{Y} \to Y$ such that it is an isomorphism outside the singular point to $\mathbb{P}^1$ and the pre-image of the singular point is a chain of l smooth rational curves. One can show that $\Omega^2_{\tilde{Y}} \cong (\Omega^2_{\tilde{X}})^{(g)} = \Omega^2_{\tilde{X}} \cong \mathbb{C}$, and it follows from the classification of surfaces that $\tilde{Y}$ is a K3 surface with certain (known) number $k(l)$ of curves isomorphic to $\mathbb{P}^1$. They are linearly independent over $\mathbb{Q}$. In particular, $\rho(\tilde{Y}) \geq 1 + k(l)$ and $\dim T(\tilde{Y}) \leq 21 - k(l) \leq 20$ and we get a contradiction with formula (**??**).

If $\mathrm{char}(\Bbbk) = p > 0$, the argument is more involved. First, we can still define $T(X)_{\mathbb{Q}}$ by using the étale l-adic cohomology (l has nothing to do with the notation for the order of g). Formula (**??**) is still true but we have to assume that the order of g is prime to $p = \mathrm{char}(\Bbbk)$. If g acts sympectically, the argument goes through without change and we get a contradiction. If g acts non-symplectically, then its must be coprime to p (since there are no non-trivial p-roots of 1). Then again the argument applies and we get that the only possibility is that X is a double cover of $\mathbb{P}^2$. So, we obtain that no non-trivial group of order prime to p can act on X unless its order is 2 and X is a double cover of the plane. I believe that I can deal with the case when the order is divisible by p but so far I have not found a proof.

Example 1.5. Now suppose that the Picard number ρ of a nonsingular quartic surface X is larger than one. For example, assume that X contains a line ℓ and a smooth curve C_n of degree n intersecting ℓ at $n-1$ points. Consider the pencil of planes through the line ℓ. Each plane H_t from the pencil intersects the surface along the line ℓ and a cubic curve F_t in H. We have $F_3 \cdot \ell = 3$ and $F_t \cdot C_n = 1$. We can choose the intersection point $F_t \cap C_n$ as the origin in the group law on the cubic curve. For any point $x \in X \setminus \ell$ we find a unique F_t containing x and then define $T(x)$ to be the point on F_t such that $x + T(x) = 0$ in the group law. This defines a birational automorphism of X but since it is a minimal surface it extends to a biregular automorphism. Let us see how g acts on the Picard group. We certainly have $g(F_t) = g(h - \ell) = F_t = h - \ell$ and $g(C_n) = C_n$, where h is the class of a plane section. It follows from the definition of the group law on a plane cubic that $\ell + g(\ell) \sim 6C_n + mF_t$ for some integer m (we assume that X is general in the sense that all cubic curves F_t are irreducible). Let

$$S := \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & 6 \\ 0 & 0 & -1 \end{pmatrix}$$

be the matrix of the transformation g^* of $\mathrm{Pic}(X)$ in terms of the basis formed by $e_1 = [h - \ell], e_2 = [C_n], e_3 = [\ell]$. Let

$$A = (e_i \cdot e_j) = \begin{pmatrix} 0 & 1 & 3 \\ 1 & -2 & n-1 \\ 3 & n-1 & -2 \end{pmatrix}$$

be the matrix of the intersection form in the same basis. Here we use that for any smooth rational curve R on a K3 surface we have $R^2 = -2$. This follows from the adjunction formula $K_R = (R + K_X) \cdot R$. Since g^* preserves the intersection form we have

$${}^tS \cdot A \cdot S = A.$$

Computing the product, we find that $n = 2n + 10$. Thus the matrix of the transformation is

$$S := \begin{pmatrix} 1 & 0 & 2n+10 \\ 0 & 1 & 6 \\ 0 & 0 & -1 \end{pmatrix}$$

Let $v = -(n+5)e_1 - 3e_2 + e_3$. We check that $v \cdot (ae_1 + be_2 + ce_3) = -(6n+14)c = \alpha \cdot e_2 = 0$. Thus the vector $\alpha = \frac{1}{6n+14} \in \mathrm{Pic}(X)^\vee = \{x \in \mathrm{Pic}(X)_{\mathbb{Q}} : x \cdot v \in \mathbb{Z} \text{ for any } v \in \mathrm{Pic}(X)\}$ (the *dual Picard lattice*). Our transformation coincides with the reflection

$$r_\alpha : x \mapsto x + 2\frac{x\cdot\alpha}{\alpha^2}\alpha$$

Here for any quadratic lattice L with symmetric bilinear form $(v,w)\mapsto v\cdot w$ and any vector $\alpha\in L^\vee$ with $2\frac{x\cdot\alpha}{\alpha^2}\in\mathbb{Z}$, the reflection transformation is defined by the formula

$$r_\alpha : x \mapsto x + 2\frac{x\cdot\alpha}{\alpha^2}\alpha.$$

Example 1.6. Suppose X acquires an ordinary double point x_0, i.e. locally it is given by equation $uv+w^2=0$. Let $\pi : X'\to X$ be a minimal resolution of this singular point. As I said before, $R=\pi^{-1}(x_0)\cong\mathbb{P}^1$ and π is an isomorphism over $X\setminus\{x_0\}$. Let M be the sublattice of the Picard lattice $\mathrm{Pic}(X')$ generated by the class $r=[R]$ and the class h of the pre-image of a plane section of X. We have $h^2=4, r^2=-2, r\cdot h=1$. Consider the following birational transformation T of X. Take a general point x, join it with the line $\langle x,x_0\rangle$. It intersects X at some other point x' different from x_0. We take for the image $T(x)$ of x. This transformation extends to a biregular automorphism g of X'. Let us compute its action in the basis h,r. Consider the projection p_{x_0} of X to a plane $\Pi\cong\mathbb{P}^2$ with center at x_0. We can identify Π with the exceptional plane of the blow-up of $\mathbb{P}^3$ at x_0. The curve R in this plane is a smooth conic C. Every line through x_0 defines a tangent direction at x_0 in $\mathbb{P}^3$ and hence defines a point in Π. This is our projection point of any point on this line. If we take a plane through x_0, then the transformation T leaves invariant the plane section. Since the proper transform of this plane in X' is the class $h-r$, we see that $g^*(h-r)=h-r$. We also see that the pre-image of the conic in X' is equal to $2h-2r=r+g^*(r)$. This gives $g^*(r)=2h-3r$. Together with the equality $g^*(h-r)=h-r$, we obtain $g^*(h)=3h-4r$. Thus the matrix of the isometry g^* is

$$A=\begin{pmatrix}3 & 2\\ -4 & -3\end{pmatrix}.$$

Again we see that this is a reflection with respect to the vector $\alpha=\frac{1}{2}(h-2r)$ with $\alpha^2=-1$.

Now we assume that we have another node x_1. So we have now two transformations g_1 and g_2 and we can ask whether they commute or not. We write the matrices in the basis $h, r_1=[R_1], r_2=[R_2]$.

$$g_1^*=\begin{pmatrix}4 & 3 & 0\\ -3 & -2 & 0\\ 0 & 0 & 1\end{pmatrix},\quad g_2^*=\begin{pmatrix}4 & 0 & 3\\ 0 & 1 & 0\\ -3 & 0 & -2\end{pmatrix},\quad g_1^*\cdot g_2^*=\begin{pmatrix}16 & 3 & 12\\ -12 & -2 & -9\\ -3 & 0 & -2\end{pmatrix}$$

We find that the characteristic polynomial of the product is equal to $(t-1)^3$. It is of infinite order.

Now suppose we have a third ordinary node (there at most 16 of them on a quartic surface). One can check that the projection transformations g_1,g_2,g_3 is the free product of cyclic groups of order 2 unless they lie on a line which is of course contained in the surface.

Note that we know the classification of all finite groups that can act symplectically on a K3 surface in any characteristic (see [**?**], [**?**], [**?**] if $\Bbbk=\mathbb{C}$ and [**?**] if $\Bbbk$ is of arbitrary characteristic). We know all possible cyclic groups that act on a K3 surface if $\Bbbk = \mathbb{C}$ or $p > 3$ [**?**]. Any finite group G of automorphisms of a K3 surface fits in the extension

$$1 \to G^{\text{symp}} \to G \xrightarrow{\chi} C_k \to 1.$$

where C_k is a cyclic group of some order k and G^{symp} is the subgroup of symplectic automorphisms. We know all possible G^{symp} and all possible C_k but we still do not have a complete list of finite groups of automorphisms even for $\Bbbk = \mathbb{C}$. There is an extensive research on non-symplectic automorphisms of K3 surfaces. We refer an interested reader to Math.Sci. Net.

Lecture 2

Birational automorphisms

Algebraic varieties also form a category with respect to rational maps $X \dashrightarrow Y$. As we know from Claire Voisin's lectures they are defined by equivalence classes of morphisms defined on open dense subset, or in case of irreducible varieties by homomorphisms of the field of rational functions $\Bbbk(Y) \to \Bbbk(X)$. An automorphism in this category is a birational automorphism, i.e. an invertible rational map. We denote the group of birational automorphisms by $\mathrm{Bir}(X)$.

The theory of minimal models implies that, in the case of surfaces, $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ if X is a minimal surface of Kodaira dimension ≥ 0. For higher-dimensional minimal models with Kodaira dimension ≥ 0, $\mathrm{Aut}(X)$ is a subgroup of finite index in $\mathrm{Bir}(X)$. There is something in between, the group of *pseudo-automorphisms*. A birational map $f : X \dashrightarrow Y$ is called a pseudo-automorphism if there exists an open subset U in X of codimension ≥ 2 and an open subset V in Y with the same property such that the restriction of f to U defines an isomorphism $U \cong V$.

Let us start with the case when $X = \mathbb{P}^n$. We have $\mathrm{Aut}(\mathbb{P}^n_{\Bbbk}) \cong \mathrm{PGL}(n+1, \Bbbk)$. But $\mathrm{Bir}_{\Bbbk}(\mathbb{P}^n_{\Bbbk})$ is much much bigger. It was first intensively studied by Luigi Cremona and Ernest de Jonquiéres and is called the Cremona group of degree n. It is denoted by $\mathrm{Cr}_n(\Bbbk)$. Obviously, $\mathrm{Cr}_n(\Bbbk)$ contains the group of projective transformations of $\mathbb{P}^n$ isomorphic to $\mathrm{PGL}(n+1, \Bbbk)$. Its elements correspond to transformations defined by linear homogeneous polynomials. It is clear that $\mathrm{Cr}_1(\Bbbk) = \mathrm{PGL}(2, \Bbbk)$, so we assume $n > 1$.

A Cremona transformation Φ can be defined, algebraically, as an automorphism of the field of rational functions on $\mathbb{P}^n$ isomorphic to $\Bbbk(z_1, \ldots, z_n)$ or, geometrically, as an invertible rational map given by a formula

$$\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n,\ [x_0, \ldots, x_n] \mapsto [F_0(x_0, \ldots, x_n), \ldots, F_n(x_0, \ldots, x_n)], \tag{2.1}$$

where $F_0, \ldots, F_n$ are mutually coprime homogeneous polynomials of some degree d (called the *algebraic degree* of the transformation). The set of indeterminacy points $\mathrm{Bs}(\Phi)$ of Φ is equal to the intersection of the $n+1$ hypersurfaces $V(F_i)$ (in scheme-theoretical sense). It has a structure of a closed subscheme of $\mathbb{P}^n$, called the *base scheme* of Φ. The pre-image of a hyperplane $\sum a_i t_i = 0$

under the transformation given by equation (**??**) is a hypersurface $\sum a_i F_i(t_0,\dots,t_n) = 0$. The pre-image of the intersection of n hyperplanes defining a general point $x \in \mathbb{P}^n$ is the intersection of n corresponding hyperplanes. The transformation is generically one-to-one if this intersection (considered as the scheme-theoretical intersection) is equal to the union of $\mathrm{Bs}(\Phi)$ and a single point, the pre-image of x under Φ. Of course, the assumption that the point x is general is essential. There is a closed subset, even outside of $\mathrm{Bs}(\Phi)$, such that the closure of the fiber of $\Phi : \mathbb{P}^n \setminus \mathrm{Bs}(\Phi)$ over this point is of positive dimension.

When $n = 2$, according to the famous *Noether Theorem*, the group $\mathrm{Cr}_2(\mathbb{C})$ is generated by $\mathrm{PGL}_2(\mathbb{C})$ and the *standard quadratic transformation* T_2 defined, algebraically, by $(z_1, z_2) \mapsto (1/z_1, 1/z_2)$, and, geometrically, by $[t_0, t_1, t_2] \mapsto [t_1t_2, t_0t_2, t_0t_1]$. It is an involution, i.e. T_2^2 is the identity.

A convenient way to partially describe a Cremona transformation Φ uses the definition of the *characteristic matrix*. As any rational map, Φ defines a regular map Φ_U of an open Zariski subset $U = \mathbb{P}^n \setminus \mathrm{Bs}(\Phi)$ to $\mathbb{P}^n$. Let Γ_Φ denote the Zariski closure of the graph of Φ_U in $\mathbb{P}^n \times \mathbb{P}^n$. Let π and σ be the first and the second projection maps, so that we have the following commutative diagram.

$$\begin{tikzcd} & \Gamma_\Phi \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ \mathbb{P}^n \arrow[rr, dashed, "\Phi"] & & \mathbb{P}^n \end{tikzcd} \tag{2.2}$$

Let $\tilde{\Gamma}_\Phi$ be a resolution of singularities of Γ_Φ, if it exists. If $n = 2$, it always exists and, moreover, we can choose it to be minimal, so that it is uniquely defined, up to isomorphism. It is known that any birational map of nonsingular varieties is a composition of the blow-ups with smooth centers. For any such map $f : X \to Y$ one can see what happens with the Picard group; we have

$$\mathrm{Pic}(X) = f^*(\mathrm{Pic}(Y)) \oplus \mathbb{Z}e,$$

where e is the class in $\mathrm{Pic}(X)$ of the exceptional divisor $f^{-1}(Z)$, wherw Z is the center of the blow-up. This allows one to define two bases in $\mathrm{Pic}(\Gamma_\Phi)$, one comes from π and another one comes from σ. We have $\mathrm{Pic}(X) \cong \mathbb{Z}^r$, and the transition matrix of these two bases is the *characteristic matrix* of T.

We write

$$[\Gamma_\Phi] = \sum_{i=0}^{i} d_i h_1^i h_2^{n-i} \in \mathrm{CH}^n(\mathbb{P}^n \times \mathbb{P}^n)$$

as in Voisin's Lecture. Since Φ is birational $d_0 = d_n = 1$. The map Φ is described by the numbers $(d_1, \dots, d_{n-1})$, the *characteristic vector* of Φ. The geometric meaning of them is that d_i is equal to the degree of the pre-image $\Phi^{-1}(L^i)$ of a codimension i linear subspace L^i in $\mathbb{P}^n$. In particular d_1 is equal to the algebraic degree of Φ. The characteristic vector of Φ^{-1} is equal to $(d_{n-1}, \dots, d_1)$. If $n = 3$, then the characteristic vector is (d_1, d_2), where, in general $d_1 \neq d_2$. For example, we have quadro-quadratic or quadro-cubic Cremona transformation of $\mathbb{P}^3$. Many examples known in classical literature are collected in Hilda Hudson book [**?**]. The numbers $(d_1,\dots,d_n)$ satisfy certain inequalities. For example, there are inequalities discovered by Cremona:

$$d_{i+j} \le d_i d_j$$

There are also some inequalities coming from Hodge index inequalities discussed in Lazarsfeld's book [**?**, vol. 1, 1.6]. For example,

$$d_i^2 \ge d_{i-1}d_{i+1}.$$

For example, $(2,3,5)$ satisfies the Cremona inequality but does not satisfy the Hodge inequality. It is a big problem, even for $n=3$, to prove that any vector satisfying the Cremona and the Hodge inequalities is realized as the characteristic vector of a Cremona transformation. It suffices to show that the Chow class defined by this vector represents an irreducible subvariety of $\mathbb{P}^n \times \mathbb{P}^n$. It was shown by June Huh, that the latter is true if one multiplies by sufficiently large positive integer [**?**, Theorem 21]. Then it will represent a *correspondence* not a birational isomorphism.

Example 2.1. Let Φ be given by the automorphism of $\Bbbk(t_1,\dots,t_n)$ defined by inverting t_i. The corresponding Cremona transformation is defined by

$$\Phi : [x_0,\dots,x_n] \mapsto [x_1\cdots x_n, x_0x_2\cdots x_n,\dots,x_0\cdots x_{n-1}].$$

It is called the standard Cremona transformation. We denote it by T_n. The characteristic vector of T_n is equal to $(d_1,\dots,d_{n-1})$, where $d_i = \binom{n}{i}$. The transformation preserves the n-dimensional torus equal to the complement of the hyperplanes $x_i = 0$ and restricts to this torus as the automorphism $x \mapsto x^{-1}$.

Example 2.2. Let f_{d-2}, f_{d-1}, f_d be homogeneous forms of the indicated degrees in variables x_1, x_2, x_3. Consider the transformation given by the formula:

$$[x_0,x_1,x_2,x_3] \mapsto [-x_0f_{d-2} - f_{d-1}, f_{d-2}x_1, f_{d-2}x_2, f_{d-2}x_3]. \tag{2.3}$$

Let

$$V_d : x_0^2 f_d(x_1,\dots,x_n) + x_0 f_{d-1}(x_1,\dots,x_n) + f_d(x_1,\dots,x_n) = 0. \tag{2.4}$$

In affine open subset $x_0 \ne 0$, the point $[0,\dots,0]$ is a singular point of V_d of multiplicity $d-2$. We observe that the surface V_d given by equation (**??**) is left invariant under Φ. A line joining any point $Q = [y_0,\dots,y_n] \in \mathbb{P}^3$ with $P = [1,0,\dots,0]$ is preserved under the transformation. Indeed, we write it in a parametric form $(s, ty_0,\dots,ty_n]$ and plugging in equation (**??**), we see that the line goes to $[-s - t\frac{f_{d-1}(y)}{f_{d-2}(y)}, ty_1,\dots,ty_n]$, it is the same line. This is also an involution, because if we do it again we get the identity transformation of each line. Restricting the transformation to V_d, we obtain a birational transformation defined by the projection from the point P. In the case when $d=4$, it was considered in Example **??**.

Assume now that $n = 2$. Then any birational morphism is a composition of the blow-ups at points. Let X be a basic rational surface, i.e. a rational surface that admits a birational morphism to $\mathbb{P}^2$

$$\pi : X = X_N \xrightarrow{\pi_N} X_{N-1} \xrightarrow{\pi_{N-1}} \ldots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbb{P}^2, \tag{2.5}$$

where $\pi_i : X_i \to X_{i-1}$ is the blow-up of a point $x_i \in X_{i-1}$. Note that the images of the points x_i in $\mathbb{P}^2$ may coincide (we say that these points are *infinitely near* to their images). Let

$$E_i = \pi_i^{-1}(x_i), \quad \mathcal{E}_i = (\pi_{i+1} \circ \ldots \pi_N)^{-1}(E_i). \tag{2.6}$$

Let e_i denote the cohomology class $[\mathcal{E}_i]$ of the (possibly reducible) curve $\mathcal{E}_i$. It satisfies $e_i^2 = e_i \cdot K_X = -1$. One easily checks that $e_i \cdot e_j = 0$ if $i \neq j$. Let $e_0 = \pi^*([\ell])$, where ℓ is a line in $\mathbb{P}^2$. We have $e_0 \cdot e_i = 0$ for all i. The classes $e_0, e_1, \ldots, e_N$ form a basis in $N^1(X)$ which we call a *geometric basis*. The Gram matrix J_N of a geometric basis is the diagonal matrix $\mathrm{diag}(1, -1, \ldots, -1)$. Thus the factorization (**??**) defines an isomorphism of quadratic lattices

$$\phi_\pi : I^{1,N} \to N^1(X),\ \mathbf{e}_i \mapsto e_i,$$

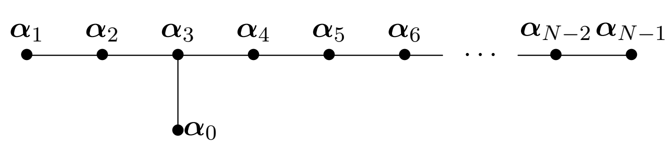
where $\mathbf{e}_0, \ldots, \mathbf{e}_N$ is the standard basis of the standard odd unimodular quadratic lattice $I^{1,N}$ of signature $(1, N)$. It follows from the formula for the behavior of the canonical class under a blow-up that K_X is equal to the image of the vector

$$\mathbf{k}_N = -3\mathbf{e}_0 + \mathbf{e}_1 + \ldots + \mathbf{e}_N.$$

This implies that the quadratic lattice $K_X^\perp$ is isomorphic to the orthogonal complement of the vector $\mathbf{k}_N$ in $I^{1,N}$. It has a basis formed by the vectors

$$\alpha_0 = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3, \alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, \ldots, \mathbf{e}_N = \mathbf{e}_{N-1} - \mathbf{e}_N.$$

Each basis vector satisfies $\alpha_i^2 = -2$ and $\alpha_i \cdot \alpha_j \in \{0, 1\}$. A lattice isomorphic to this lattice is denoted by E_N. One can describe it by a graph whose vertices are the vectors α_i and two vertices are connected by an edge if $\alpha_i \cdot \alpha_j = 1$. We get the following graph:



If $4 \leq N \leq 8$, they are the *Dynkin diagrams* of types A_4, D_5, E_6, E_7 and E_8 familiar from the theory of Lie algebras and Lie groups. Any basis vector α_i defines a reflection isometry

$$r_i : v \mapsto v + (v \cdot \alpha_i)\alpha_i.$$

The subgroup of the orthogonal group $\mathrm{O}(\mathsf{E}_N)$ generated by these reflections is denoted by $W(\mathsf{E}_N)$. It is called the *Weyl group* of E_N. If $N = 8, 9, 10$ it coincides with this group. If $N \leq 13$, it is a subgroup of finite index of $\mathrm{O}(\mathsf{E}_N)$.

One can define similar diagrams by extending the arms in all directions to get a graph $T_{p,q,r}$, where the numbers mean the lengths of the arms including the 3-valent point. This defines a lattice $\mathsf{E}_{p,q,r}$ and the Weyl group $W_{p,q,r} := W(\mathsf{E}_{p,q,r})$. Our diagrams correspond to $(p,q,r) = (2,3,N-3)$.

The following theorem is due to S. Kantor and goes back to the end of the 19th century. It has been reproved in modern terms by M. Nagata and others.

Theorem 2.3. *Let $(e_0,\dots,e_N)$ be a geometric basis defined by the birational morphism π. Then the geometric basis $(e_0', e_1', \dots, e_N')$ is expressed in terms of $(e_0,e_1,\dots,e_N)$ by a matrix A which defines an orthogonal transformation of $I^{1,N}$ equal to the composition of reflections with respect to the root basis $(\alpha_0,\dots,\alpha_{N-1})$ of E_N.*

The matrix A is the characteristic matrix of Φ. For any plane $H(a) : \sum a_i y_i = 0$ in $\mathbb{P}^s$, its pre-image under the rational map Φ is equal to a curve $V(a) := \sum a_i P_i(t) = 0$ of degree d, where P_i are the polynomials defining Φ. One can show that the class $\pi^{-1}(e_0') = de_0 - m_1e_1 - \cdots - m_Ne_N$, where m_i are the multiplicities of a general hypersurface $V(a)$ at the points x_i (this has to be carefully defined if the points are infinitely near).

Also, the class $\pi(\sigma^*(e_i'))$ is the class of a curve C_i in $\mathbb{P}^2$ whose image under T is the point y_i defined by e_i'. We have $\sigma^*(e_i') = d_ie_0 - m_{1i}e_1 - \cdots - m_{Ni}e_N$, where d_i is the degree of the curve C_i and m_{ki} are the multiplicities of C_i at the points x_i. Thus the characteristic matrix A has the following form

$$A = \begin{pmatrix} d & d_1 & \dots & d_N \\ -m_1 & -m_{11} & \dots & m_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ -m_N & -m_{N1} & \dots & -m_{NN} \end{pmatrix}.$$

The matrix A is orthogonal with respect to the inner product given by the matrix J_N.

Example 2.4. Let us find the characteristic matrix of the standard Cremona transformation T_2. The base locus consists of three points $p_1 = [1,0,0], p_2 = [0,1,0], p_3 = [0,0,1]$. The lines $\langle p_i, p_j\rangle$ are blown down to the point p_k, where i,j,k are distinct. This shows that the characteristic matrix looks like

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}.$$

We notice that it acts as the reflection with respect to the vector

$$\alpha = e_0 - e_1 - e_2 - e_3$$

with $\alpha^2 = -2$.

One must be warned that the assigning to a Cremona transformation in the plane its characteristic matrix is not a homomorphism. A subgroup G of $\mathrm{Cr}_n(\Bbbk)$ is called *regularizable* if there exists a rational surface and a birational map $\phi : X \to \mathbb{P}^2$ such that $\phi^{-1} \circ G \circ \phi \subset \mathrm{Aut}(X)$. Here we may assume that X is a basic rational surface. Any finite group is regularizable but not every infinite group is. Let $(e_0, e_1, \ldots, e_n)$ be a geometric basis in $N^1(X)$. Let G be a regularizable subgroup of $\mathrm{Cr}_2(\Bbbk)$ which is realized as a group of biregular automorphisms of X. Then we can realize the characteristic matrix of any $g \in G$ as a matrix of the automorphism $\pi^{-1} \circ g \circ \pi$ in the basis $(e_0, e_1, \ldots, e_n)$. This will give a matrix realization of the natural action homomorphism $G' \to \mathrm{O}(N^1(X))$ and also a homomorphism $G' \to \mathrm{O}(K_X^{\perp}) \cong \mathrm{O}(\mathsf{E}_n)$. The image of this homomorphism is contained in the reflection group $W_{2,3,n-3}$.

Example 2.5. Let $X = X(p_1, \ldots, p_5)$ be the blow-up of N points. If $N \le 8$, the groups $W(\mathsf{E}_N)$ are finite. This means that the group $\mathrm{Aut}(X)^c$ is finite. However, the group $\mathrm{Aut}(X)^0$ could be non-trivial. This always happens if $N \le 3$. If we assume additionally that $-K_X$ is ample (in this case the surface X is called a *del Pezzo surface* of degree $9 - N$), then $\mathrm{Aut}(X)^0$ is trivial. The condition $-K_X$ is ample is satisfied if and only if no root (i.e. a vector $\alpha \in E_N$ with $\alpha^2 = -2$) is represented by an effective divisor. If $p = 0$ all automorphism groups of del Pezzo surfaces are known. If $p > 0$ they are known if $N \le 6$.

It follows from the theory of minimal models of *G-surfaces* (i.e, surfaces equipped with an action of a finite group G and morphism equal to G-equivariant morphisms) that any finite subgroup of $\mathrm{Cr}_2(\Bbbk)$ is isomorphic to a group of automorphisms of a del Pezzo surface or a *conic bundle* (a conic bundle is a subfibration of a $\mathbb{P}^2$-bundle over a curve whose fibers are conics). fibration over a curve with general fiber isomor. When $\Bbbk$ is algebraically closed of zero characteristic all such groups can be listed [**?**].

Example 2.6. Assume $N = 9$. The lattice E_9 is degenerate. Its radical is generated by the vector $K_X \in K_X^{\perp}$ (because $K_X^2 = 0$ in this case). The lattice $K_X^{\perp}/\mathbb{Z}K_X$ is isomorphic (as a lattice) to E_8. The Weyl group $W(\mathsf{E}_9)$ fits in the extension

$$1 \to \mathbb{Z}^8 \to W(\mathsf{E}_9) \to W(\mathsf{E}_8) \to 1. \tag{2.7}$$

Take now the points $p_1, \ldots, p_9$ equal to the intersection points of two nonsingular cubics F_1, F_2. The rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ given by the pencil of cubics generated by F_1, F_2 can be lifted to a biregular map $f : X \to \mathbb{P}^1$. Its general fiber is isomorphic to a general member of the pencil. We obtain an example of a rational elliptic surface. We can take the exceptional curves E_9 over p_9 as a section. It defined a group law on each nonsingular points. The Mordell-Weil group of a general fiber (considered as an elliptic curve over $\Bbbk(\mathbb{P}^1)$) acts birationally on X (by translations). When all fibers of the fibration are irreducible, the Mordell-Weil group is isomorphic to $\mathbb{Z}^8$ and its action on $\mathrm{Pic}(X)$ defined an isomorphism with the kernel of $W(\mathsf{E}_9) \to W(\mathsf{E}_8)$ from exact sequence (**??**).

Lecture 3

Kummer surfaces and dynamics

Let us use what have learnt to explain the structure of automorphism group of a *Kummer surface*. In general, a Kummer variety $\mathrm{Kum}(A)$ associated with an abelian variety A is the quotient of A by the involution $\tau : a \mapsto -a$. For example, when $n = \dim A = 1$, it is isomorphic to $\mathbb{P}^1$. If $g \geq 2$ it is alway singular and its locus of singular points consists of 2^{2g} points locally isomorphic to the affine cone over the Veronese curve of degree g. If A is principally polarized, i.e. contains an ample divisor Θ with the property $h^0(\Theta) = 1$, then the linear system $|2\Theta|$ maps A two-to-one to $\mathbb{P}^{2^g-1}$ with the image isomorphic to $\mathrm{Kum}(A)$. The images of the divisors $\tau_\epsilon^*(\Theta)$, where τ_ϵ denote the translation automorphism of A with respect to a 2-torsion point ϵ, are cut out by hyperplanes with multiplicity 2 (the hyperplanes are called *tropes*). They contain $2^{g-1}(2^g - 1)$ singular points of $\mathrm{Kum}(A)$. Each singular point lies in $2^{g-1}(2^g - 1)$ tropes. This defines the famous *Kummer configuration* (N_k) of points and hyperplanes, where $N = 2^{2g}$ and $k = 2^{g-1}(2^g - 1)$.

Each translation automorphism τ_ϵ descends to a projective automorphism of $\mathrm{Kum}(A) \subset \mathbb{P}^{2^g-1}$. Its set of fixed points in the projective space consists of two projective subspaces of dimension $2^{g-1}-1$. Each intersects $\mathrm{Kum}(A)$ in a subvariety isomorphic to the Kummer variety of a principally polarized abelian variety of dimension $g-1$. If $A = \mathrm{Jac}(C)$, then this variety if the *Prym variety* of the pair (C, ϵ).

We will be concerned with the case $g = 2$. An example of a principally polarized abelian surface is the Jacobian variety of a genus 2 curve C. In this case $X = \mathrm{Kum}(\mathrm{Jac}(C))$ is a quartic surface in $\mathbb{P}^3$ with 16 nodes. It also has 16 tropes, planes that cut out in X a conic taken with multiplicity 2. Each conic contains 6 nodes, they can be identified with 6 Weierstrass points of C. In this case the set of fixed points of the translation automorphism in $\mathbb{P}^3$ consists of two skew lines. Each intersect $\mathrm{Kum}(A)$ at 4 points, the double cover of $\mathbb{P}^1$ branched along these four points is the Prym variety of the pair (C, ϵ).

The minimal resolution Y of X is a K3 surface. It contains 17 linearly independent classes: the class of a hyperplane section h and the classes e_i of 16 exceptional curves of the resolution. It also contains 16 classes T_i of smooth rational curves equal to the proper transforms of trope-conics.

They can be expressed (over $\mathbb{Q}$) in terms of the previous 17 classes. The Picard number ρ satisfies $17 \leq \rho \leq 20$. The maximal number is achieved at the Fermat surface $x^4 + y^4 + z^4 + w^4 = 0$ which is birationally isomorphic to the Kummer surface isogenous to the product of the elliptic curve with complex multiplication by $\sqrt{-1}$ with itself.

The Kummer surface is self-dual, i.e. its *projective dual variety* (the closure of all tangent planes to nonsingular points in the dual projective space) is isomorphic to itself. This defines a birational automorphism of X, called a *switch*. It exchanges singular points with tropes. Two switches differ by a translation automorphism.

We will assume that $\rho = 17$ and explain the structure of $\mathrm{Bir}(\mathrm{Kum}(\mathrm{Jac}(C))) \cong \mathrm{Aut}(Y)$.

It contains several obvious isomorphisms

- 16 automorphisms defined by projections from 16 modes;
- 15 projective transformations induced by translations t_ϵ of $\mathrm{Jac}(C)$;
- a switch;

It contains also much less obvious involutions

- 60 Hutchinson-Göpel involutions;
- 192 Hutchinson-Weber involutions.

Let us define them. A *Göpel tetrad* is a set of four nodes such that any plane containing three of them is not a trope plane. A *Weber hexad* is a set of 6 nodes such that no four are contained in a trope and no four is a Göpel tetrad. There are 60 Göpel tetrads and 192 Weber hexads (see Hudson's book [**?**], he was a brother of Hilda Hudson).

In 1902 John Hutchinson showed that, for each Göpel tetrad, one can write the equation of the Kummer surface in the form

$$q(xy + zw, xz + yw, xw + yz) + xyzw = 0,$$

where q is a quadratic form in 3 variables [**?**]. One immediately checks that the substitution $(x, y, z, w) \mapsto (1/x, 1/y, 1/z, 1/w)$ transforms this equation to itself. This means that the standard Cremona involution defines an automorphism of order 2 of the Kummer surface. This is a Hutchinson-Göpel involution. One can realize this birational automorphism on a birational model of X called the *Weddle surface*. It is a quartic surface in $\mathbb{P}^3$ with 6 nodes. It contains the 15 lines joining the pairs of these points and the rational twisted cubic passing through the six nodes. The linear system of quadric surfaces passing through the six nodes defines a birational map from the

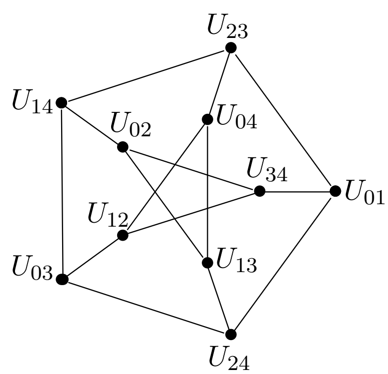
Weddle surface to the Kummer surface that blows down the 15 lines and the twisted cubic to the 16 nodes of the Kummer surface. Hutchinson showed that the Weddle surface can be given by equation

$$\det\begin{pmatrix} xyz & w & a & a' \\ yzw & x & b & b' \\ xzw & y & c & c' \\ xyw & z & d & d' \end{pmatrix} = 0,$$

where the coordinates of the six nodes are $[1,0,0,0]$, $[0,1,0,0]$, $[0,0,1,0]$, $[0,0,0,1]$, $[a,b,c,d]$, $[a',b',c',d']$. The standard Cremona transformation T_3 leaves the equation invariant and defined an birational involution of the Weddle surface and hence also of the Kummer surface.

Hutchinson had made another remarkable discovery [**?**].

Let $V(F_3)$ be a nonsingular cubic surface in $\mathbb{P}^3$. The Hessian matrix of second partial derivatives of $F_3(x_0,x_1,x_2,x_3)$ has linear homogeneous polynomials as its entries. The determinant of this matrix defines a quartic surface H, called the *Hessian surface* of $V(F_3)$. It has 10 nodes and 10 lines, each node lies on 3 lines and each line contains 3 nodes. For any Weber hexad on X the map given by quadrics through the hexad defines a birational map $\phi : X \dashrightarrow H$ from X to the Hessian surface H of some cubic surface. Now H has a natural birational involution. The surface H can be defined as the locus of points $x = [a_0,a_1,a_2,a_3] \in \mathbb{P}^3$ such that the polar quadric $V(\sum a_i \frac{\partial F_3}{\partial x_i})$ is singular. Then ϕ assigns to $x \in H$ the singular point of the polar quadric is a birational involution τ of H [**?**, 1.1.4]. When lifted to a nonsingular model it becomes a biregular involution that has no fixed points. The quotient is an Enriques surface. It is very special one since it depends only on 4 parameters (the moduli of cubic surfaces). It contains a set of 10 smooth rational curves whose mutual intersections are described by the famous Petersen graph with $\mathfrak{S}_5$-symmetry.



Now the composition $\phi^{-1} \circ \tau \circ \phi$ is a birational involution of our Kummer surface. This is the Hutchinson-Weber involution.

Theorem 3.1 (S. Kondo, 1998)**.** *The group of birational automorphisms of a general Kummer surface is generated by the itemized involutions.*

Let me give you an idea how one proves such a theorem. It is based on Borcherd's method. There are 24 even unimodular negative-definite lattices of rank 24. One of them is distinguished itself by the property that it does no contain vectors v with $v \cdot v = -2$. It is called the *Leech lattice* and it is denoted by Λ. The orthogonal sum of Λ and the rank 2 lattice $\mathbb{Z}f + \mathbb{Z}g$ defined by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is denoted by $II_{1,25}$. It is a unique unimodular even lattice of signature $(1, 25)$. The lattice $II_{1,25}$ has many vectors of square norm -2. For example, vectors of the form $(\lambda, f + \frac{1}{2}(-2 - \lambda^2)g)$, where $\lambda \in \Lambda$. They are called *Leech roots*. John Conway has shown that the group of isometries of $II_{1,25}$ generated by all roots (i.e. vectors of square norm -2) has a fundamental domain Ω which is a convex polyhedron bounded by hyperplanes orthogonal to the Leech roots. The symmetry group of Ω is the semi-direct product $\Lambda \rtimes \mathrm{O}(\Lambda)$. Now suppose we can find a primitive embedding of the Picard lattice $i : \mathrm{Pic}(\tilde{X}) \hookrightarrow II_{1,25}$. Then the pre-image of Ω in $\mathrm{Pic}(\tilde{X})$ intersects the *nef cone* of $\tilde{X}$ (this a convex cone in $\mathrm{Pic}(X) \otimes \mathbb{R}$ of vectors that intersect any class of a curve on $\tilde{X}$ non-negatively) along some convex polytope Ω' bounded by some exterior hyperplanes that bound the nef cone and some interior hyperplanes. One could be lucky (as in the case of Kummer surfaces) to be able to show that the reflection into an interior wall corresponds to an action of some involution of the surface. Then we fix the class of an ample divisor η, apply g to η and then find a composition of the reflections $r = r_1 \circ \cdots \circ_N$ to obtain that $(r \circ g)(\eta) \in \Omega'$. If no symmetry of Ω' are realized by an automorphism of $\tilde{X}$, we conclude that $r \circ g$ is the identity transformation, hence g belongs to the subgroup of $\mathrm{Aut}(\tilde{X})$ generated by the involutions σ_i such that σ^* is a reflection into an inner bounding hyperplane.

The same method applies to some other cases. For example one can find the group of birational automorphisms of the Hessian quartic (Dolgachev-Keum), of the Kummer surface of the product of two non-isogenous general elliptic curves (Keum-Kondo), a general 15-nodal quartic surface (Dolgachev-Shimada), and also some families of special Enriques surfaces (Shimada, Allcock-Dolgachev, Mukai-Ohashi).

Now let us switch the topic and discuss some relationship with complex dynamics. It studies the behavior of iterates of a holomorphic automorphism of a complex manifold.

We start with a planar Cremona transformation $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Take a line ℓ and apply the nth iterate f^n to it to obtain a curve $f^n(\ell)$ of certain degree $d_f(n)$. A fundamental theorem of Marat Gizatullin says that there are four different types of behavior of the function $d_f(n)$. Note that this function depends only on the conjugacy class of f in the Cremona group.

- $d_f(n)$ is bounded. Then f is conjugate to a projective automorphism or f is of finite order.
- $d_f(n)$ is bounded by a linear function. Then f preserves a pencil of rational curves.
- $d_f(n)$ is bounded by a quadratic function. Then f preserves a pencil of elliptic curves.
- $d_f(n)$ is an exponential function of n.

Define the dynamical degree of f to be the number

$$\lambda(f) := \overline{\lim}_{n\to\infty} d_f(n)^{1/n}.$$

We have $\lambda(f) = 1$ if f is not of the last type in which case $\lambda)F) > 1$ The number $\lambda(f)$ is always an algebraic integer. In fact, it is very special one.

An algebraic integer α is called a *Pisot number* if it has one real root > 1 and all other roots have absolute value < 1. An algebraic integer α is called a *Salem number* if it has one real root $\lambda > 1$, one real root $1/\lambda$ and all other roots have absolute value equal to 1. It is known that the set of Pisot numbers is a closed subset of $\mathbb{R}$. It is contained in the closure of the set of Salem numbers and its minimum is equal to the real root of $x^3 - x - 1$ which is approximately equal to 1.324717 (it is called the *plastic* or *padovian* number). The smallest accumulation point is the golden ratio $\frac{1}{2}(1 + \sqrt{5})$. All Pisot numbers between these two numbers are known.

We do not know the smallest Salem number, it is conjectured to be the Lehmer number λ_{Lehmer} equal to the real root of the polynomial

$$P(x) = x^{10} + x^9 - (x^7 + \cdots + x^3) + x + 1.$$

It is equal approximately to 1.7628.. The computer search proves that its is indeed the smallest Salem number of degree less than or equal to 42.

We have the following theorem

Theorem 3.2.
- *$\lambda(f) > 1$ and $d_f(n)$ is a Pisot number and f is not conjugate to an automorphism of a rational surface.*
- *$\lambda(f) > 1$ and $\lambda(f)$ is a Salem number and f is conjugate to an automorphism of a rational surface*

The fact that $\lambda(f)$ is an algebraic integer is not obvious at all. To prove one uses the notion of a *stable rational map*. This is a transformation such that the characteristic matrix Char(f^n) is equal to Char$(f)^n$. This happens if and only if $\mathrm{Bs}(f) \cap \mathrm{Bs}(f^{-1}) = \emptyset$. By a result of J. Diller and C. Favre, one can replace f by a conjugate transformation such that it becomes algebraically stable. In this case we can compute $\lambda(f)$ as the spectral radius of the characteristic matrix. It is equal to largest absolute value of its eigenvalues. They showed that the characteristic polynomial of the matrix is a Pisot or Salem polynomial, hence the spectral radius is equal to $\lambda(f)$.

Using the notion of the spectral radius, one can define the dynamical degrees $\lambda_k(g)$. of a biregular automorphism g of any smooth algebraic variety by considering its action on the cohomology $H^{2k}(X, \mathbb{C})$ (in fact, it is enough to look at $H^{2k}(X, \mathbb{C})_{\mathrm{alg}}$). For algebraic surfaces, there is only one dynamical degree $\lambda(g)$, and it is always a Salem number.

It was show by C. McMullen that the Lehmer number is realized as the dynamical degree of an automorphism of a K3-surface [**?**] and an automorphism of a rational surface obtained from the projective plane by blowing up a special set of 10 points on an irreducible cuspidal cubic curve [**?**]. The polynomial $P(x)$ is the characteristic polynomial of an element of the Weyl group $W(\mathsf{E}_{10})$ equal to the product of the reflections r_{α_i} (it does not depend on their order). It is called the *Coxeter element* of the Weyl group.

It was proven by K. Oguiso [**?**] that the Lehmer number cannot be realized as the dynamical degree of an automorphism of an Enriques surface. There is still going a search for the smallest Salem number realized as an automorphism of an Enriques surface.

It is known that the automorphism group of an Enriques surface is very large. In fact, Enriques surfaces depend on 10 moduli and, for a moduli general surface S, the group $\mathrm{Aut}(S)$ is isomorphic (in its representation on $\mathrm{O}(N^1(S))$) to a subgroup of finite index, namely the 2-level congruence subgroup $\{g \in W(\mathsf{E}_{10}) : \frac{1}{2}(g(x) - x) \in N^1(X)\}$. When the surface specializes, the group becomes smaller, in fact even could become finite group. All such surfaces were classified over $\mathbb{C}$ by S. Kondo and V. Nikulin and over fields of arbitrary characteristic by T. Katsura, S. Kondo and G. Martin. Since the dynamical degree of an automorphism of finite order is equal to 1 one looks for a family of Enriques surfaces that closest in some sense to surfaces with finite automorphism group. An example of such a family is the family of Enriques surfaces that are obtained from the Hessian surfaces of cubic surfaces with automorphism group $\mathfrak{S}_4$ as the quotient by the fixed-point-free involution.

Bibliography

[1] M. Brion, *Notes on automorphism groups of projective varieties*, Notes de lécole dété "Varieties and Group Actions" (Varsovie, 2018), 29 pages, https://www-fourier.ujfpgrenoble.fr/ mbrion.

[2] E. Ciani, *I gruppi finiti di collineazioni piane dotati di una quartica invariante irreducibile*, Rendiconti del R. Istituto Lombardo, serie II, **33** (1900), 1I70–1176.

[3] E. Ciani, *Le quartiche piane proiettive a ss̀tesse*. Rendiconti Circolo Mat. di Palermo, **28** (1909), 217–232.

[4] E. Ciani, *Le quintiche piane autoprojettive*. Rendiconti Circolo Mat. di Palermo, **36** (1913), 58–78.

[5] T. Dinh, K. Oguiso, *A surface with discrete and nonfinitely generated automorphism group*. Duke Math. J. 168 (2019), no. 6, 941-966

[6] I. Dolgachev, V. Iskovskikh, *Finite subgroups of the plane Cremona group*. Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, 443-548, Progr. Math., 269, BirkhŁuser Boston, Inc., Boston, MA, 2009.

[7] I. Dolgachev, J. Keum, *Birational automorphisms of quartic Hessian surfaces*. Trans. Amer. Math. Soc. **354** (2002), no. 8, 3031-3057.

[8] I. Dolgachev, J. Keum, *Finite groups of symplectic automorphisms of K3 surfaces in positive characteristic*. Ann. of Math. (2) **169** (2009), no. 1, 269-313.

[9] I. Dolgachev, *Classical algebraic geometry:a modern view*, Cambridge Univ. Press, 2012.

[10] D. Eisenbud, J. Harris, *3264 and all thata second course in algebraic geometry*. Cambridge University Press, Cambridge, 2016.

[11] W. Fulton, *Intersection theory*. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 2. Springer-Verlag, Berlin, 1998.

[12] C. Hacon, J. McKernan, C. Xu, *On the birational automorphisms of varieties of general type*. Ann. of Math. (2) **177** (2013), no. 3, 1077-1111.

[13] T. Harui,*Automorphism groups of smooth plane curves*. Kodai Math. J. **42** (2019), no. 2, 308-331.

[14] R. Hudson, *Kummer's quartic surface*, Cambridge University Press, 1905 [reprinted in 1990 with a forword by W. Barth].

[15] H. Hudson, *Cremona transformations in plane and space*, Cambridge Univ. Press. 1927.

[16] J. Huh, *Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs*. J. Amer. Math. Soc. **25** (2012), no. 3, 907-927.

[17] J. I. Hutchinson, *The Hessian of the cubic surface*, Bull. Amer. Math. Soc. **5**(6) (1899), 282–292.

[18] J. I. Hutchinson, *On some birational transformations of the Kummer surface into itself*, Bull. Amer. Math. Soc., **7**(5) (1901), 211–217.

[19] J. Keum, *Orders of automorphisms of K3 surfaces*. Adv. Math. **303** (2016), 39-87.

[20] S. Kondo, *Niemeier lattices, Mathieu groups, and finite groups of symplectic automorphisms of K3 surfaces. With an appendix by Shigeru Mukai*. Duke Math. J. **92** (1998), no. 3, 593-603.

[21] R. Lazarsfeld, *Positivity in algebraic geometry*, vol. I and vol. II, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, **49**. Springer-Verlag, Berlin, 2004.

[22] J. Lesieutre, *A projective variety with discrete, non-finitely generated automorphism group*. Invent. Math. 212 (2018), no. 1, 189-211.

[23] C. McMullen, *Dynamics on blowups of the projective plane*. Publ. Math. Inst. Hautes tudes Sci. No. **105** (2007), 49-89.

[24] C. McMullen, *K3 surfaces, entropy and glue*. J. Reine Angew. Math. **658** (2011), 1-25.

[25] H. Matsumura, *On algebraic groups of birational transformations*. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **34** (1963), 151-155.

[26] S. Mukai, *Finite groups of automorphisms of K3 surfaces and the Mathieu group*. Invent. Math. **94** (1988), no. 1, 183-221.

[27] S. Marcugini, F. Pambianco, H. Kaneta, *Projective Automorphism Groups of Nonsingular Quartic Surfaces*, math.AG. arXiv:1611.10101

[28] K. Oguiso, *The third smallest Salem number in automorphisms of K3 surfaces*. Algebraic geometry in East Asia–Seoul 2008, 331–360, Adv. Stud. Pure Math., 60, Math. Soc. Japan, Tokyo, 2010.

[29] T. Shioda, *An explicit algorithm for computing the Picard number of certain algebraic surfaces*. Amer. J. Math. **108** (1986), no. 2, 415-432.

[30] V. Snyder, *Plane Quintic Curves Which Possess a Group of Linear Transformations*. Amer. J. Math. **30** (1908), no. 1, 1-9.

[31] L. Wei, X. Yu,*Automorphism groups of smooth cubic threefolds*. math.AG.arXiv:1907.00392.

[32] A. Wiman, *Zur Theorie endlichen Gruppen von birationalen Transformationen in der Ebene*, Math. Ann. **48** (1896), 195–240.

[33] G. Xiao, *Bound of automorphisms of surfaces of general type*. II. J. Algebraic Geom. **4** (1995), no. 4, 701-793.

[34] G. Xiao, *Galois covers between K3 surfaces*. Ann. Inst. Fourier (Grenoble) **46** (1996), no. 1, 73-88.