PRYM VARIETIES AND PRYM MAP MINI-COURSE

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1. Basics on abelian varieties

Through this notes we work on $\mathbb{C}$.

Definition 1.1. A complex torus A is a quotient V/Λ, with $V \simeq \mathbb{C}^g$ a $\mathbb{C}$- vector space and $\Lambda \simeq \mathbb{Z}^{2g}$ a full rang lattice inside V. A polarization on A is an ample line bundle[1] L on A. An abelian variety is a complex torus admitting a polarization, so (A, L) is polaraized abelian variety.

Remark 1.2. In particular, with the addition operation inherited from V, an abelian variety is an abelian group.

By definition of ampleness, given a line bundle L on A we have that the map

$$\begin{aligned} \varphi_{L^{\otimes k}}: \qquad & A \hookrightarrow \mathbb{P}H^0(A, L^{\otimes k})^* \\ & x \mapsto [s_0(x) : s_1(x) : \cdots : s_N(x)], \end{aligned}$$

defined by the sections of $L^{\otimes k}$ is an embedding for some $k > 1$. In fact, in the case of polarized abelian varieties it suffices to take $k = 3$. Then an abelian variety is also a projective variety.

Different encarnations of a polarization on A. The following data are equivalent:

- A first Chern class $c_1(L) \in H^2(A, ZZ)$ of an ample line bundle L on A.
- A non degenerated alternating form $E : V \times V \to \mathbb{R}$ such that $E(\Lambda, \Lambda) \subset \mathbb{Z}$ and $E(iv, iw) = E(v, w)$.
- A non degenerated Hermitian form $H : V \times V \to \mathbb{C}$ with $H(\Lambda, \Lambda) \subset \mathbb{Z}$.
- An isogeny $\phi_L : A \to \widehat{A} := \mathrm{Pic}^0(A)$
- A Weil divisor $\Theta \subset A$ such that the subgroup $\{x \in A \mid t_x^*\Theta \sim \Theta\}$ is finite.

Let E be an alternating form representing a polarization on $A = V/\Lambda$. There exists a basis $\lambda_1, \dots, \lambda_g, \mu_1, \dots \mu_g$ of Λ with respect to which E is given by the matrix $\left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right)$, where D is a diagonal matrix with positive integer entries $d_1, \dots, d_g$ satisfiying $d_i | d_{i+1}$ for $i = 1, \dots, g-1$.

Definition 1.3. The vector $(d_1, \dots, d_g)$ is called the *type of the polarization* of L and when it is of the form $(1, \dots, 1)$ the polarization is *principal*.

Given an abelian variety (A, L), the sections of $L^{\otimes k}$, for some $k > 0$, provide an embedding of A into a projective space:

$$\begin{aligned} \varphi_{L^{\otimes k}}: \qquad & A \hookrightarrow \mathbb{P}H^0(A, L^{\otimes k})^* \simeq \mathbb{P}^N \\ & x \mapsto [s_0(x) : s_1(x) : \cdots : s_N(x)], \end{aligned}$$

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[1] In fact, the polarization depends only on the first Chern class $c_1(L)$

where $s_0, s_1, \dots, s_N$ form a basis of the space of sections $H^0(A, L^{\otimes k})$.

Let $H_1(C,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$ be the group of closed paths in C (which does not depend on the starting point) modulo homology. This group can be seen as a full rank lattice inside of $H^0(C,\omega_C)^*$, via the injective map

$$\gamma \mapsto \left\{\omega \mapsto \int_\gamma \omega\right\}$$

assigning to a path γ the functional which integrates the holomorphic differentials along γ.

Definition 1.4. The Jacobian of an algebraic curve C (or compact Riemann surface) is the complex torus

$$JC = H^0(C,\omega_C)^*/H_1(C,\mathbb{Z}).$$

The intersection product on $H_1(C,\mathbb{Z})$ induces an alternating form E on $V := H^0(C,\omega_C)^*$. More precisely, if we choose a basis over $\mathbb{Z}$, $\gamma_1, ..., \gamma_{2g}$ of $H_1(C,\mathbb{Z})$ as in the Figure 1, the intersection product has as matrix $\begin{pmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{pmatrix}$. As $H_1(C,\mathbb{Z})$ is a full rank lattice in V, the $\{\gamma_i\}$ form also a basis of V as $\mathbb{R}$-vector space. One verifies then, that with respect to this basis, the intersection matrix gives an alternating form E on V defining a principal polarization Θ.

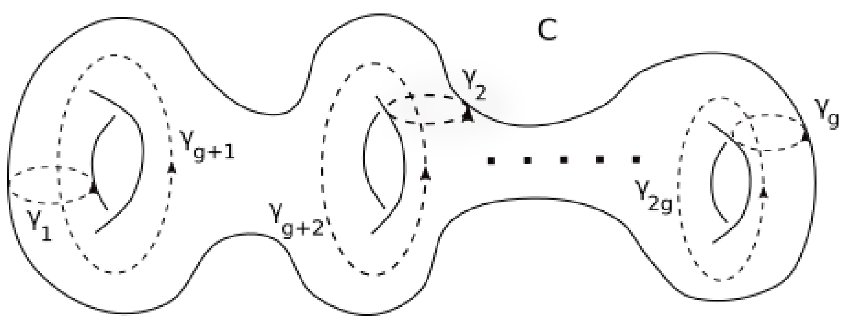


FIGURE 1. Curve of genus g

A one-dimensional abelian variety is also an algebraic curve of genus one, that is, an elliptic curve. The Jacobian of a genus one curve is then isomorphic to the curve itself.

Algebraic geometers typically gather their objects of study in families to investigate a *general* property or single out interesting elements. Ideally, the set of all the objects forms itself an algebraic variety where one can apply known tools. This leads to the notion of moduli space, which is the variety parametrising the objects. Fortunately, there exists a nice parameter space for all principally polarized abelian varieties (ppav) of fixed dimension g (up to isomorphism classes). Let $\mathfrak{h}_g$ be the Siegel upper half plane

$$\mathfrak{h}_g := \{\tau \in M_{g\times g}(\mathbb{C}) \ \mid \ \tau^t = \tau,\ \operatorname{Im}\tau > 0\}$$

and

$$Sp_{2g}(\mathbb{Q}) = \left\{ M \in GL_{2g}(\mathbb{Q}) \ : \ M \begin{pmatrix} 0 & \mathbf{1_g} \\ -\mathbf{1_g} & 0 \end{pmatrix} {}^tM = \begin{pmatrix} 0 & \mathbf{1_g} \\ -\mathbf{1_g} & 0 \end{pmatrix} \right\}$$

the symplectic group, which acts on $\mathfrak{h}_g$ by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_{2g}(\mathbb{Q}), \quad M \cdot \tau = (a + b\tau)(c + d\tau)^{-1}$$

Thus, every point in the quotient $\mathfrak{h}_g/Sp_g$ represents an isomorphism class of a principally polarized abelian variety of dimension g: for each $\tau \in \mathfrak{h}_g$ set $A_\tau = \mathbb{C}^g/\tau\mathbb{Z}^g \oplus \mathbb{Z}^g$, then

$$A_\tau \simeq A_{\tau'} \text{ as ppav} \quad \Leftrightarrow \ \exists \ M \in Sp_{2g}(\mathbb{Z}) \quad \text{ s.t. } \quad \tau' = M \cdot \tau.$$

In the sequel, we denote by $\mathcal{A}_g$ the moduli space of principally polarized abelian varieties of dimension g. Observe that the dimension of this space is the same as the dimension of the space of symmetric matrices of size g, thus $\dim A_g = \frac{g(g+1)}{2}$.

Let $\mathcal{M}_g$ be the moduli space of smooth projective curves of genus g, it is an irreducible algebraic variety of dimension $3g-3$. By associating to each smooth curve $[C] \in \mathcal{M}_g$ its Jacobian we get the *Torelli map:*

$$\mathfrak{t} : \mathcal{M}_g \to \mathcal{A}_g, \qquad [C] \mapsto (JC, \Theta).$$

Theorem 1.5. *The Torelli map* $\mathfrak{t}$ *is injective.*

Comparing the dimensions of both spaces, one deduces that the general principally polarized abelian variety of dimension < 3 is the Jacobian of some curve.

1.1. **Prym varieties.** Consider a finite covering $f : \widetilde{C} \to C$ of degree d between two smooth projective curves and let g and $\tilde{g}$ denote the genera of C and $\widetilde{C}$ respectively. By the Hurwitz formula these genera are related by

$$\tilde{g} = d(g-1) + \frac{\deg R}{2} + 1 \tag{1.1}$$

where R denotes the ramification divisor of f, that is the set of points in $\widetilde{C}$ (counted with multiplicities) where the map is not locally a homeomorphism. The map f induces a map between the Jacobians of the curves, the *norm map.* As a group, the Jacobian JC is generated by the points of the curve C, and in fact JC parametrizes classes of linear equivalence of divisors of degree zero. With this in mind one can simply define the norm map as the push forward of divisors from $\widetilde{C}$ to C:

$$\mathrm{Nm}_f : J\widetilde{C} \to JC, \qquad \left[\sum_i n_i p_i\right] \mapsto \left[\sum_i n_i f(p_i)\right]$$

where the sum is finite, $\sum n_i = 0$ with $n_i \in \mathbb{Z}$ and the bracket denotes the class of linear equivalence. The kernel of Nm_f is not necessarily connected but since Nm_f is a homomorphism of groups the connected component containing the zero is naturally a subgroup of $J\widetilde{C}$. This subgroup is the *Prym variety of f* denoted by

$$P(f) := (\mathrm{Ker}\,\mathrm{Nm}_f)^0 \subset J\widetilde{C}. \tag{1.2}$$

Moreover, the restriction Ξ of the principal polarization Θ on $J\widetilde{C}$ to $P(f)$, defines a polarization so $(P(f), \Xi)$ is an abelian subvariety of the Jacobian $J\widetilde{C}$ of dimension

$$\dim P(f) = \dim J\widetilde{C} - \dim JC = \tilde{g} - g.$$

The Prym variety can be regarded as the complementary variety of the image of $f^* : JC \to J\widetilde{C}$ inside of $J\widetilde{C}$.

Theorem 1.6. *(Wirtinger, Mumford) Let $f : \widetilde{C} \to C$ of deg $d \geq 2$ $g \geq 1$. Then Ξ defines a principal polarization if and only if it is one of the following cases:*

(a) *f is étale of degree 2, in this case $\Theta_{|P} \equiv 2\Xi$, with Ξ principal.*
(b) *f is a double covering ramified in exactly 2 points, so $\Theta_{|P} \equiv 2\Xi$.*
(c) *$g(\widetilde{C}) = 2, g = 1$ (any degree).*
(d) *$g = 2$, $d = 3$, f is non-cyclic.*

Proof. Uses that $(f^*)^*\tilde{\Theta} \equiv n\Theta$ and that P and f^*JC are complementary subvarieties of a ppav. □

Assume now that f is an étale double covering, according to (2.1) the dimension of the corresponding Prym variety is $\dim P(f) = 2(g-1) - g = g - 1$. Thus, this construction provides us a way to associate to each étale double covering $f : \widetilde{C} \to C$ over a smooth curve C of genus g a principally polarized abelian variety, this is the Prym map. In order to make the definition precise we need to introduce the moduli space

$$\mathcal{R}_g := \{[C, \eta] \mid [C] \in \mathcal{M}_g,\ \eta \in \mathrm{Pic}^0(C),\ \eta^{\otimes 2} \simeq \mathcal{O}_C\}$$

parametrizing all the étale double coverings over curves of genus g up to isomorphism. Given a pair $[C, \eta] \in \mathcal{R}_g$ the isomorphism $\eta^{\otimes 2} \simeq \mathcal{O}_C$ endows $\mathcal{O}_C \oplus \eta$ with a ring structure (actually with a structure of $\mathcal{O}_C$-algebra). Thus, the corresponding double covering is given by taking the spectrum $\widetilde{C} := \mathrm{Spec}(\mathcal{O}_C \oplus \eta)$ and the map f is just the natural projection $\mathrm{Spec}(\mathcal{O}_C \oplus \eta) \to C = \mathrm{Spec}\,\mathcal{O}_C$, induced by the inclusion $\mathcal{O} \hookrightarrow \mathcal{O}_C \oplus \eta$. There are finitely many "square roots" of $\mathcal{O}_C$, that is, line bundles η with $\eta^{\otimes 2} \simeq \mathcal{O}_C$. In other words, the forgetful map

$$\mathcal{R}_g \to \mathcal{M}_g, \qquad [C, \eta] \mapsto \eta$$

is finite of degree $2^{2g} - 1$ and hence $\dim \mathcal{R}_g = \dim M_g = 3g - 3$. The *Prym map* is then defined as

$$Pr_g : \mathcal{R}_g \to \mathcal{A}_{g-1} \qquad [C, \eta] \mapsto (P(f), \Xi).$$

By comparing the dimensions on both sides, one sees that $\dim \mathcal{R}_g \geq \dim \mathcal{A}_{g-1} = \frac{g(g-1)}{2}$ for $g \leq 6$, so it makes sense to ask if for low values of g the Prym map is dominant, i.e. if we can realize a (general) principally polarized abelian varieties of dimension ≤ 6 as the Prym variety of some covering. The following theorem gives a positive answer to this question and summarizes the situation for the classical Prym map, to which proof several mathematicians have contributed.

Theorem 1.3. *(a) The Prym map is dominant if $g \leq 6$.*
(b) The Prym map is generically injective if $g \geq 7$.
(c) The Prym map is never injective.

Let $\mathcal{P}_{g-1}$ denote the image of Pr_g. Wirtinger showed ([19]) that the closure $\overline{\mathcal{P}}_{g-1}$ is an irreducible subvariety in $\mathcal{A}_{g-1}$ of dimension $3g-3$, so $\overline{\mathcal{P}}_{g-1} = \mathcal{A}_{g-1}$ for $g \leq 6$, which implies part (a). Moreover, he also proved that the Jacobian locus in

$\mathcal{A}_{g-1}$ (i.e. the image of the Torelli map $\mathfrak{t}$) is contained in $\overline{\mathcal{P}}_{g-1}$. In this sense, Pryms are a generalization of Jacobians. Part (b) was first proved by R. Friedman and R. Smith ([8]) and for $g \geq 8$, by V. Kanev ([9]) by using degeneration methods. More geometric proofs were given by G. Welters ([18]) and later by O. Debarre ([4]), on the spirit of the proof of Torelli's theorem. The fact that the Prym map is non-injective was first observed by Beauville ([2]) who produced, using *Recillas' trigonal construction*, examples of non-isomorphic coverings $(\widetilde{C}_1, C), (\widetilde{C}_2, C_2)$ in $\mathcal{R}_g$ for $g \leq 10$, whose Prym varieties are isomorphic as principally polarized abelian varieties. Later, Donagi's tetragonal construction ([5]) (of which Recillas' construction is a degeneration) provides examples for the non-injectivity of the Prym map in any genus. When $g < 6$ the fibers of the Prym map are positive dimensional and they are geometrically well-understood.

2. Non-principally polarized abelian varieties

One can of course consider more general coverings to construct abelian varieties, for instance coverings of degree higher than 2 or ramified ones. In these cases one obtains abelian varieties which are no longer principally polarized. In this section we will discuss cyclic coverings ramified at $2r$ points with $r \geq 0$.

By definition, an étale cyclic covering $\widetilde{C} \to C$ of degree d admits an automorphism σ of order d on $\widetilde{C}$ such that $C = \widetilde{C}/\langle\sigma\rangle$. For simplicity, we can asume that d is a prime number (otherwise one can factorize the map through cyclic coverings of smaller degree). If the cyclic covering is branched over a divisor $B \subset C$, the ramification index at each ramified point is d, i.e. all the branches come together in that point. In particular, if $\deg B = dm$, for some integer $m > 0$, the ramification divisor is of degree $2r = (d-1)dm$. Similarly to the double étale case, giving a ramified cyclic covering is equivalent to give a triple (C, B, η) with η a line bundle on C satisfying $\eta^{\otimes d} \simeq \mathcal{O}_C(B)$. More precisely, take a section s of $\mathcal{O}_C(B)$ vanishing exactly along B (If $B = \emptyset$, take s the constant section 1). Denote by $|\eta|$ the total space of η and let $p : |\eta| \to C$ the bundle projection. If t is the tautological section of the line bundle $p^*\eta \to |\eta|$, then the locus where the section $p^*s - t$ vanishes defines the curve $\widetilde{C}$ inside $|\eta|$.

Recall that points in the Jacobian represent line bundles of degree zero on the corresponding curve. Hence, every automorphism of a curve induces an automorphism of its Jacobian by taking the pullback of the bundle under the automorphism. If we denote by the same letter the automorphism on $J\widetilde{C}$ induced by σ, it is more convenient to define the Prym variety of the covering f as

$$P(f) := \mathrm{Im}(1-\sigma) \subset J\widetilde{C}$$

which is an abelian subvariety of $J\widetilde{C}$ of dimension

$$\dim P(f) = \tilde{g} - g = (d-1)(g-1) + r =: p$$

with polarization Ξ given by the restriction of the principal polarization $\tilde{\Theta}$ on $J\widetilde{C}$; it is a polarization of type $D = (1, \dots, 1, d, \dots, d)$, where 1 occurs $p-(g-1)$ times and d occurs $g-1$ times if $r = 0$, and 1 occurs $p-g$ times and d occurs g times if $r > 0$. This definition coincides with (2.2) since in that case $\mathrm{Im}(1-\sigma) = \ker(1+\sigma)^0$. Let

$$\mathcal{R}_g(d, r) := \{(C, B, \eta) \ \mid \ \eta \in \mathrm{Pic}^m(C), \ B \text{ reduced divisor in } |\eta^{\otimes d}|\}$$

denote the moduli space parametrizing all the étale cyclic coverings of degree d over a curve of genus g and let $\mathcal{A}_p^D$ be the moduli space of abelian varieties of dimension p and polarization type D. In this case the Prym map is given by

$$Pr_g(d,r) : \mathcal{R}_g(d,r) \to \mathcal{A}_p^D, \qquad [\widetilde{C} \xrightarrow{f} C] \mapsto (P(f), \Xi).$$

In order to compute the dimension of its image one needs to know when this map is generically finite. This is the case when the differential map of $Pr_g(d,r)$ is injective at a generic point of $\mathcal{R}_g(d,r)$ or equivalently, when the codifferential map $d^*Pr_g(d,r)$ is surjective. One of the advantages of considering cyclic covering is that the tangent space at $0 \in P(f)$ to the Prym variety can be identified with the direct sum of space of sections

$$T_0P \simeq \bigoplus_{i=1}^{d-1} H^0(C, \omega_C \otimes \eta^i)^*,$$

where each summand is an eigenspace for the action of σ on $H^0(\widetilde{C}, \omega_{\widetilde{C}})^*$. Notice that the forgetful map $[C, B, \eta] \mapsto [C, B]$ is finite over the moduli space $\mathcal{M}_{g,dm}$ of dm-pointed smooth curves of genus g. Therefore the cotangent space to a generic point $[C, \eta, B] \in \mathcal{R}_g(d,r)$ can be identified to the cotangent space to $\mathcal{M}_{g,dm}$ at $[C, B]$. By identifying the cotangent spaces

$$T^*_{(P,\Xi)}\mathcal{A}_p^D \simeq \mathrm{Sym}^2(T_0P)^*, \qquad T^*_{[C,\eta,B]}\mathcal{R}_g(d,r) \simeq H^0(C, \omega_C^2(B)),$$

we obtain that the codifferential of $P_g(d,r)$ at a generic point $[C, B, \eta]$ is given by the multiplication of sections

$$d^*Pr_g(d,r) : \mathrm{Sym}^2(T_0P)^* \to H^0(C, \omega_C^2 \otimes \mathcal{O}(B)).$$

In the cases when this map is surjective a the generic point $[(C, B, \eta)]$ we get that the Prym map $Pr_g(d,r)$ is generically finite onto its image.

Theorem 2.1. *([11]) If*

- $g \geq 2$ *and* $r \geq 6$ *for* d *even or* $r \geq 7$ *for* d *odd;*
- $g \geq 3$ *and* $d = r = 4$ *or* 5 *or* $(d,r) = (2,4)$ *or* $(3,6)$*;*
- $g \geq 5$ *and* $d = r = 2$ *or* 3*;*

the Prym map $Pr_g(d,r)$ *is generically finite.*

In the case of ramified double coverings we know that the Prym map is generically injective as soon as the dimensions of the moduli spaces in the source and target allow it (see [12], [13], [17]), except when $g = 3, r = 4$, where $\deg P_3(2,4) = 3$; there are actually at least to two different ways to interpret this degree ([1], [16]). As we have seen in the previous section, it is particularly appealing to study the geometry of the fibres of the Prym map. Let $\mathcal{B}_D$ be the component of the moduli space $\mathcal{A}_p^D$ of elements (P, Ξ) such that the polarization Ξ is compatible with σ, i.e. $\sigma^*\Xi \equiv \Xi$. In particular, $\mathcal{B}_D = \mathcal{A}_p^D$ when $d = 2$. One checks that $\mathrm{Im}\, Pr_g(d,r) \subset \mathcal{B}_D$. By computing the dimension of $\mathcal{B}_D$ ([11]) one can prove that only in the following cases $Pr_g(d,r)$ is generically finite and dominant over $\mathcal{B}_D$ ([11]):

(g,n,r)	(6,2,0)	(3,2,2)	(1,2,3)	(4,3,0)	(2,7,0)	(2,3,3)
$\deg Pr_g(n,d)$	27	3	1	16	10	?
$p = \dim P$	5	4	3	6	6	5

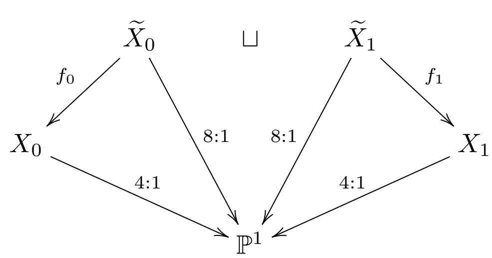
The degree of the Prym map $Pr_2(3,3)$ seems to be unknown. We would like to emphasize that the fibre of the different Prym maps carries a peculiar structure, so the way of computing the degree has been *ad hoc* for each case.

3. The Prym map $Pr_6 : \mathcal{R}_6 \to \mathcal{A}_5$.

The case of étale double coverings over a genus 6 curve deserves special attention. We have $\dim \mathcal{R}_6 = \dim \mathcal{A}_5 = 15$ and the map Pr_6 being dominant implies that it is also generically finite. Can one determine the degree or even describe its generic fibre? There is not only a positive answer to this question but also a beautiful one. The degree of $Pr_6 : \mathcal{R}_6 \to \mathcal{A}_5$ is 27, which is also the number of lines on every smooth cubic surface. This is not a coincidence because the monodromy group of the Prym map equals the Weyl group $W(E_6)$, which governs the incidence structure of the lines in the smooth cubic surface ([6]). On the other hand, Pr_6 fails to be finite over the locus of all the Jacobians of curves of genus 5, as well as over the locus of intermediate Jacobians of cubic threefolds.[2] The fibres and the blow up of the Prym map along these loci are explicitly described in [6]. There is even a procedure to pass from an element to another on a general fibre of Pr_6: the tetragonal construction. First, notice that a general curve $[C] \in \mathcal{M}_6$ carries exactly 5 g^1_4's, that is line bundles of degree 4 on C whose space of sections is two-dimensional. Thus, a g^1_4 on C is equivalent to having a 4:1 map $C \to \mathbb{P}^1$. Now, given a double covering $[f : \widetilde{C} \to C] \in \mathcal{R}_6$ and a g^1_4 on C one can construct two other coverings in $\mathcal{R}_6$ as follows. Let $C^{(4)}$ (respectively $\widetilde{C}^{(4)}$) be the 4th. symmetric product of C (respectively $\widetilde{C}$), so they parametrize divisors of degree 4 on the corresponding curves. Define the curve $\widetilde{X}$ by the cartesian diagram

$$\begin{array}{ccc} \widetilde{X} & \hookrightarrow & \widetilde{C}^{(4)} \\ {\scriptstyle 16:1}\downarrow {\scriptstyle f^{(4)}_{|\widetilde{X}}} & & {\scriptstyle 2^4:1}\downarrow {\scriptstyle f^{(4)}} \\ \mathbb{P}^1 = g^1_4 & \hookrightarrow & C^{(4)} \end{array}$$

where $f^{(4)}$ is the natural map $p_1 + \cdots + p_4 \mapsto f(p_1) + \cdots + f(p_4)$. The involution σ on $\widetilde{C}$ induces an involution $\tilde{\sigma} : p_1 + \cdots + p_4 \mapsto \sigma(p_1) + \cdots + \sigma(p_4)$ on $\widetilde{X}$. It turns out that $\widetilde{X}$ consists of two disjoint non-singular connected components $\widetilde{X}_0$, $\widetilde{X}_1$ and the involution $\tilde{\sigma}$ acts without fixed points on each of these components. Moreover, the restriction of the map $f^{(4)}$ to $\widetilde{X}_0$ and $\widetilde{X}_1$ defines 8:1 maps fitting in the following diagram



[2] The intermediate of a cubic threefold Y is the complex torus $H^{1,2}(Y)/H_3(Y, \mathbb{Z})$.

where $X_i := \widetilde{X}_i/\tilde{\sigma}$ for $i = 0, 1$. Therefore, from an element $[\widetilde{C} \stackrel{f}{\to} C] \in \mathcal{R}_6$ together with a pencil g^1_4 on C we have constructed two other elements in $\mathcal{R}_6$, namely $[\widetilde{X}_0 \stackrel{f_0}{\to} X_0]$ and $[\widetilde{X}_1 \stackrel{f_1}{\to} X_1]$, together with g^1_4's on X_0 and X_1. These 3 pairs are in general non-isomorphic to each other, nevertheless the associated Pryms are isomorphic ([5]). If one applies the construction to $[\widetilde{X}_0 \stackrel{f_0}{\to} X_0]$ using the obtained g^1_4 one gets back the original two coverings, but if one uses another pencil on X_0 one gets two other non-isomorphic coverings. By repeating this procedure on the coverings using different pencils on their base curves, we obtain eventually all the elements on the fibre. This is also the structure on the lines of a smooth cubic surface. Two coverings in the fibre are related by the tetragonal construction as two lines on a cubic surface are incident. The triad $\{(\widetilde{C}, C), (\widetilde{X}_0, X_0), (\widetilde{X}_1, X_1)\}$ corresponds to a triangle in the surface. The fact that each line in the cubic surface is a "side" of five triangles is reflected in the construction by the five g^1_4's that a curve C possesses. This particular case shows that the fibers of the Pym map displays rich and beautiful geometry.

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