GEOMETRIC CLASS FIELD THEORY AND AN INTRODUCTION TO THE LANGLANDS PROGRAM

Contents

1. Lecture I : Global fields and class field theory

1.1. **Galois groups.** Let K be a field. We let K^{sep} be a separable closure of K. We let $G_K = \mathrm{Aut}_K(K^{sep})$ be the absolute Galois group of K.

Let $L \subset K^{sep}$ be a finite extension of K. We say that L is Galois if for all $\sigma \in G_K$, $\sigma(L) = L$. The Galois group of L over K is $Gal(L/K) = \mathrm{Aut}_K(L)$.

Proposition 1.1. *Let L/K be a Galois extension.*

(1) *The natural map $G_K \to Gal(L/K)$ is surjective.*
(2) *The group $Gal(L/K)$ has cardinality* $\dim_K L$.

Any finite extension $L \subset K^{sep}$ is contained in a Galois extension. Therefore, $G_K = \lim_{L/K, \text{finite galois}} Gal(L/K)$.

We equip G_K with a topology by declaring that an open basis of neighborhoods of 1 is given by the $G_L = Gal(K^{sep}/L)$ for L/K a finite extension. Then G_K is a profinite group. Moreover the Galois correspondence is :

Theorem 1.1.

$$\begin{array}{rcl} \{\textit{Open subgroups of } G_K\} & \leftrightarrow & \{\textit{Finite separable field extensions of } K\} \\ H & \mapsto & (K^{sep})^H \\ G_L & \leftarrow\!\shortmid & L \end{array}$$

Example 1. Let $q = p^r$ and let $K = \mathbb{F}_q$ be the finite field with q elements. Let $\overline{\mathbb{F}_q}$ be an algebraic closure of $\mathbb{F}_q$. For all $n \geq 0$, there is a unique extension of $\mathbb{F}_q$, $\mathbb{F}_{q^n} \subset \overline{\mathbb{F}_q}$ of degree n. Its Galois group is isomorphic to $\mathbb{Z}/n\mathbb{Z}$, and a generator is given by the Frobenius $Frob_q : x \mapsto x^q$. Therefore $G_{\mathbb{F}_q} \simeq \hat{\mathbb{Z}}$ and $Frob_q$ is a topological generator.

Example 2. Let $K = \mathbb{R}$ be the field of real numbers. We have $\overline{\mathbb{R}} = \mathbb{C}$ and $G_{\mathbb{R}} = \mathbb{Z}/2\mathbb{Z}$, the generator is given by the complex conjugation $c : z \mapsto \overline{z}$.

1.2. **Discrete valuation rings.**

1.2.1. *Valuations.* Let K be a field. A discrete valuation v (of rank 1) on K is a surjective function : $v : K^\times \to \mathbb{Z}$ which satisfies :

(1) $v(xy) = v(x) + v(y)$,
(2) $v(x+y) \geq \inf\{v(x), v(y)\}$.

One extends v to K by setting $v(0) = +\infty$.

Example 3. The trivial valuation on a field K is defined by $v(x) = 1$ for all $x \in K^\times$.

Example 4. Let p be a prime number. For all $x \in \mathbb{Q}^\times$, write $x = x'p^n$ where p does not appear in the prime decomposition of x', and set $v_p(x) = n$. This is the p-adic valuation on $\mathbb{Q}$.

Example 5. Let k be a field. Let $k(T)$ be the field of rational functions over k. Let P be an irreducible polynomial. For any $x \in k(T)^\times$, write $x = x'P^n$ where P does not appear in the decomposition of x in product of prime ideals and let $v_P(x) = n$. This is the P-adic valuation on $k(T)$. Let $deg : k(T) \to \mathbb{Z} \cup \{\infty\}$ be the degree map. Then $-deg$ is a valuation.

Theorem 1.2 (Ostrowski)**.** *The only non-trivial valuations on $\mathbb{Q}$ are (up to equivalence) the p-adic valuations v_p for prime numbers p.*

Proof. [Cas67], section. 3, p. 45. □

Theorem 1.3. *The only non-trivial valuation on $k(T)$ which are trivial on k are the v_P for P an irreducible polynomial and $-deg$.*

1.2.2. *Valuation ring.* We let $A = \{x \in K, v(x) \geq 0\}$. This is the ring of the valuation v. It is easy to check that A is a discrete valuation ring, namely a principal domain which has a unique non-zero prime ideal. Conversely, A determines the valuation v. Indeed, we have a group isomorphism $K^\times/A^\times \simeq \mathbb{Z}$ which sends a generator π of the maximal ideal of A to 1 and we recover v as the composite $K^\times \to K^\times/A^\times \simeq \mathbb{Z}$.

We let $|.|_v = e^{-v(.)}$ be the associated norm. It is called non-archimedean because $|x+y|_v \leq \sup\{|x|_v, |y|_v\}$.

1.2.3. *Completion.* If A is a discrete valuation ring, we can consider its completion $\hat{A}$ with respect to the norm $|.|_v$. Concretely, $\hat{A} = \lim_n A/\mathfrak{p}^n$.

1.3. **Dedekind rings.**

1.3.1. *Definition.*

Definition 1.1. *A Dedekind ring is a noetherian domain which is integrally closed of dimension one.*

Proposition 1.2. *A noetherian domain is a Dedekind ring if and only if, for all maximal ideal $\mathfrak{p}$ of A, the localization $A_{(\mathfrak{p})}$ is a discrete valuation ring.*

Proof. See [Ser68], proposition 4 on p. 22. □

For any maximal ideal $\mathfrak{p}$ of A, we denote by $v_\mathfrak{p}$ the corresponding $\mathfrak{p}$-adic valuation. We will also denote by $A_\mathfrak{p} = \hat{A_{(\mathfrak{p})}}$ the completion of A for the $\mathfrak{p}$-adic topology.

1.3.2. *Fractional ideals.* A fractional ideal of a Dedekind ring A is a non-zero finitely generated submodule of $K = Frac(A)$. The set of fractional ideals is a monoid under multiplication, with neutral element A itself.

Proposition 1.3. *The fractional ideals of a Dedekind ring form a group. Any fractional ideal $\mathfrak{a}$ has a unique expression*

$$\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{n_\mathfrak{p}}$$

where almost all the $n_\mathfrak{p}$ are zero.

Proof. See [Ser68], corollaire and proposition 7 on p. 24. □

1.3.3. *Extension of Dedekind rings.* Let A be a Dedekind ring with fraction field K. Let L be a finite extension of K. Let B be the integral closure of A in K.

Theorem 1.4. *If either A is a finite type algebra over a field, or L is a separable extension of K, B is a finite A-algebra and a Dedekind ring.*

Proof. See [Ser68], part I, chap. 4. □

We assume that the assumptions of the theorem hold. There is a (surjective) map Spec $B \to$ Spec A. We say that a prime ideal $\mathfrak{P}$ in B divides a prime ideal $\mathfrak{p}$ and write $\mathfrak{P} \mid \mathfrak{p}$ if $\mathfrak{P}$ is mapped to $\mathfrak{p}$.

If $\mathfrak{p}$ is a maximal ideal of A, we have $\mathfrak{p} = \prod_{\mathfrak{P}|\mathfrak{p}} \mathfrak{P}^{e_\mathfrak{P}}$. The integer $e_\mathfrak{P}$ is called the ramification index at $\mathfrak{P}$. The residual degree at $\mathfrak{P}$ is the degree of the finite extension $A/\mathfrak{p} \to B/\mathfrak{P}$ and is denoted by $f_\mathfrak{P}$.

Proposition 1.4. *We have the formula $\sum_{\mathfrak{P}|\mathfrak{p}} e_\mathfrak{P} f_\mathfrak{P} = \dim_K L$.*

Proof. $B \otimes A_{(\mathfrak{p})}$ is a finite free $A_{(\mathfrak{p})}$-module of finite rank $\dim_K L$. By reduction modulo $\mathfrak{p}$ we find that $B/\mathfrak{p} \to \prod B/\mathfrak{P}^{e_{\mathfrak{P}}}$ is an isomorphism. The formula is obtained by comparing the dimensions as $A/\mathfrak{p}$-modules on both sides. □

Definition 1.2. *We say that B is unramifed over A at $\mathfrak{P}$ if $e_{\mathfrak{P}} = 1$.*

1.3.4. *Ramification.* Let $K \subset L$ be a finite separable extension of fields. We have a non-degenerate bilinear trace map $\mathrm{Tr} : L \times L \to K$. Let $A \subset K$ be a Dedekind ring with fraction field K. Let B be the integral closure of A in L. We assume that the assumptions of theorem 1.4 hold.

We can define $\mathfrak{D}_{B/A}^{-1} = \{x \in L, \mathrm{Tr}(xB) \subseteq A\}$. This is a fractional ideal of B and its inverse $\mathfrak{D}_{B/A}$ is an ideal called the different of B with respect to A.

Proposition 1.5. *The set of ramified prime of B over A is exactly the set of primes which divide the different $\mathfrak{D}_{B/A}$. In particular this is a finite set.*

Proof. See [Ser68], thm 1 on page 62. □

1.3.5. *Unramified extensions in complete discrete valuation rings.* Let $\mathcal{O}_K$ be a complete discrete valuation ring. Let K be its field of fraction. For any finite separable extension L of K, we let $\mathcal{O}_L$ be the inegral closure of $\mathcal{O}_K$ in L.

Lemma 1.1. *The ring $\mathcal{O}_L$ is a complete discrete valuation ring.*

Proof. We know that $\mathcal{O}_L$ is a Dedekind ring and has finitely many maximal ideals. Each of these ideals induce a topology on L which extends the topology of K. Since K is complete, this topology is unique (this is the product topology on K^n identified with L. Therefore there is a unique maximal prime in $\mathcal{O}_L$. □

Let K^{sep} be a separable closure of K. This is a valued field (in general not complete). Let $\mathfrak{m}_{\mathcal{O}_K^{sep}}$ be the maximal ideal of $\mathcal{O}_{K^{sep}}$. Let $k^{sep} = \mathcal{O}_{K^{sep}}/\mathfrak{m}_{\mathcal{O}_K^{sep}}$.

Theorem 1.5. *k^{sep} is a separable closure of k and there is an equivalence of category :*

$$\begin{array}{rcl} \{\textit{Unramified finite extensions } L \subset K^{sep}\} & \to & \{\textit{finite extensions } \ell \subset k^{sep}\} \\ L & \mapsto & \mathcal{O}_L/\mathfrak{m}_{\mathcal{O}_L} \end{array}$$

Proof. [Fr7], p. 26. □

Assume that L/K is Galois. Let $Gal(L/K)$ be the Galois group. We have a surjective map $Gal(L/K) \to Gal(l/k)$ whose kernel is denoted by $I_{L/K}$ and is called the inertia. Passing to the limit over L we have an exact sequence :

$$1 \to I_K \to G_K \to G_k \to 1.$$

1.4. **Global fields.**

1.4.1. *Definition.* A global field K is either a number field or a function field of one variable over a finite field $\mathbb{F}_q$.

(1) K is a number field. That is K is a field of characteristic 0 and is a finite extension of $\mathbb{Q}$.
(2) K is a function field of one variable over a finite field $\mathbb{F}_q$. That is, K is a field of characteristic p and is a finite type extension of transcendance degree 1 over $\mathbb{F}_p$. Moreover $\mathbb{F}_q$ is the integral closure of $\mathbb{F}_p$ in K. The simplest example of such field is $\mathbb{F}_q(T)$.

If K is a number field, K is finite over $\mathbb{Q}$ and we let $\mathcal{O}_K$ be the ring of integer. If K is a function field, we can choose $T \in K$ which is not algebraic over $\mathbb{F}_p$ and K is finite over $\mathbb{F}_q(T)$. But T is not unique.

1.4.2. *Places.* A place v of K is an equivalence class of non-trivial rank one norm :

$$|.|_v : K \to \mathbb{R}_{\geq 0}.$$

There is the following description of the places of K.

Proposition 1.6. *If K is a number field, the places of K are the non-archimedean norms $|.|_{\mathfrak{P}}$ attached to the maximal ideals $\mathfrak{P} \in$ Spec $\mathcal{O}_K$ and the archimedean norms $|.|_\sigma$ for embeddings $\sigma : K \to \mathbb{C}$.*

Proof. [Cas67], p. 45. □

Remark 1.1. Two conjugate embeddings σ and $\bar{\sigma}$ give the same archimedean norm.

Proposition 1.7. *If K is a function field, there exists a unique non-singular complete curve X with function field K and the places of K are the valuations attached to the closed points of the curve X.*

Proof. See lecture II. □

Remark 1.2. If we consider $\mathbb{F}_p(T)$, the associated curve is $\mathbb{P}^1_{\mathbb{F}_p} = \mathbb{A}^1_{\mathbb{F}_p} \cup \{\infty\}$. The closed points of $\mathbb{A}^1$ are the irreducible monic polynomials $P \in \mathbb{F}_p[T]$ with corresponding norms $|.|_P$, and ∞ corresponds to the valuation $-deg$.

In all cases, we let X (or ${}_KX$ if the context is unclear) be the set of places of K. In the number field case, we have $X = X_{fin} \cup X_\infty$ where $X_{fin} =$ Specmax $\mathcal{O}_K$ is the set of finite places and $X_\infty = \{\sigma : K \to \mathbb{C}\}/\{\text{complex conjugation}\}$ is the set of infinite places.

1.5. **From global to local fields.** If v is a place of K, we let K_v be the completion of K with respect to $|.|_v$. If v is not achimedean, we let $\mathcal{O}_v$ or $\mathcal{O}_{K_v}$ the ring of elements $x \in K_v$ with $v(x) \geq 0$. If v is archimedean, then $K_v = \mathbb{R}$ or $\mathbb{C}$.

Let L/K be a finite field extension of K. Let w be a place of L. Then w restricts to a place v of K and we say $w \mid v$. Therefore, we have a map ${}_LX \to {}_KX$.

We have the following "localization" formula :

Proposition 1.8. *The canonical map $L \otimes_K K_v \to \prod_{w|v} L_w$ is an isomorphism.*

Definition 1.3. *We say that the extension L/K is unramified at a finite place v if all the extensions L_w/K_v are unramified.*

Proposition 1.9. *A finite extension L/K is ramified at only finitely many places of K.*

1.6. **Decomposition group.** Let L/K be a finite Galois extension. Let $f : {}_LX \to {}_KX$. The group $Gal(L/K)$ acts on ${}_LX$, trivially on ${}_KX$.

Proposition 1.10. *For any $v \in {}_KX$, the action of $Gal(L/K)$ is transitive on $f^{-1}(v)$.*

Proof. See [Tat67], prop. 1.2. □

Let $w \in f^{-1}(v)$ and let $D_v = \{\sigma \in Gal(L/K), \sigma w = w\}$.

Proposition 1.11. *The map $D_v \to Gal(L_w/K_v)$ is an isomorphism.*

Proof. See [Tat67], prop. 1.2. □

The group D_v is independant of w and called the decomposition group at v. Its embedding in $Gal(L/K)$ depends on w, but its conjugacy class is independent of w.

1.7. **Frobenius substitution.** If we assume that L/K is unramified at a finite place v, then we have a canonical element $Frob_v \in D_v$, and therefore a conjugacy class $Frob_v \in Gal(L/K)$.

1.8. **The Artin reciprocity map.** We now assume that L/K is abelian. This implies that the conjugacy action of $Gal(L/K)$ on itself is trivial. Let Σ be the set of finite places where L/K is ramified.

Let I^{Σ} be the free abelian group generated by finite places not in Σ.

We define a map :

$$\begin{array}{rcl} \mathrm{rec}_{L/K} : I^{\Sigma} & \to & Gal(L/K) \\ v & \mapsto & Frob_v \end{array}$$

Theorem 1.6 (crude reciprocity law)**.** *The map* $\mathrm{rec}_{L/K}$ *is onto and there exists* $\epsilon > 0$ *such that for all* $a \in K^{\times}$ *which satisfy :*

(1) $|a-1|_v < \epsilon$ *for all* $v \in \Sigma$,
(2) $\sigma(a) > 0$ *for all* $\sigma : K \to \mathbb{R}$ *in the number field case,*

we have $\mathrm{rec}_{L/K}(a) = 1$.

Remark 1.3. By $\mathrm{rec}_{L/K}(a)$ we mean $\mathrm{rec}_{L/K}(\sum_{v \notin \Sigma} v(a).v)$. This is a very hard result, you can consult [Tat67].

In this course, we will be interested in everywhere unramified extensions of K. Let H/K be the maximal abelian everywhere unramified extension of K (also called the Hilbert class field of K).

In the number field case, we have a map $I^{X_\infty} \to Gal(H/K)$. We remark that I^{X_∞} is the group of fractional ideals over Spec $\mathcal{O}_K$. Let

$$Cl^{+}(\mathcal{O}_K) = I^{X_\infty}/\{a \in K^{\times},\ \forall\ \sigma : K \to \mathbb{R},\ \sigma(a) > 0\}$$

be the strict class group.

In the function field case we have a map $I^{\emptyset} \to Gal(H/K)$. We remark that $I^{\emptyset}$ is the group of divisors on the curve X corresponding to K. Let $Pic(X) = Div(X)/\mathrm{div}(K^{\times})$ be the Picard group.

Theorem 1.7. *In the number field case, the map* $Cl^{+}(\mathcal{O}_K) \to Gal(H/K)$ *is an isomorphism. In the function field case, the map* $Pic(X) \to Gal(H/K)$ *is injective with dense image.*

One of the main goal of these lectures is to give Deligne's geometric proof of this theorem in the function field case. We can further geometrize the statement by interpreting $Gal(H/K)$ as $\pi_1(X)^{ab}$. Therefore the theorem reads as an injection with dense image :

$$Pic(X) \to \pi_1(X)^{ab}.$$

One can actually refine the statement. We have a degree map $Pic(X) \to \mathbb{Z}$. We also have a natural map $\pi_1(X) \to \pi_1(\mathrm{Spec}\ \mathbb{F}_q) = \hat{\mathbb{Z}}$. Let us define the Weil group of X, $W(X)$ as the preimage of $\mathbb{Z}$ in $\pi_1(X)$. Then the refined statement is that we have a commutative diagram:

$$\begin{array}{ccc} Pic(X) & \longrightarrow & \pi_1(X)^{ab} \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \hat{\mathbb{Z}} \end{array}$$

which induces an isomorphism between $Pic(X)$ and $W(X)^{ab}$.

1.9. **Adèles and idèles.**

1.9.1. *Adèles.* In this course we will meet at several points the ring $\mathbb{A}_K$ of adèles of a global field K. By definition, $\mathbb{A}_K$ is the subring of $\prod_{v\in\ _K X} K_v$ of elements $(x_v)_v$ such that $x_v \in \mathcal{O}_{K_v}$ for almost all v (all except finitely many ones). We equip $\mathbb{A}_K$ with a ring topology by declaring that a basis of opens of 0 are given by opens $\prod_{v\in_K X} U_v$ where for all v, U_v is an open neighborhood of 0 in K_v, and for almost all v, $U_v = \mathcal{O}_{K_v}$. The diagonal embedding $K \to \prod_v K_v$ factorizes through $\mathbb{A}_K$.

1.9.2. *Idèles.* The group of idèles is $\mathbb{A}_K^\times$ and it carries the subset topology given by the inclusion $\mathbb{A}_K^\times \to \mathbb{A}_K \times \mathbb{A}_K$, $x \mapsto (x, x^{-1})$.

Class field theory is best formulated using idèles (see [Tat67], section 5). Let us simply remark the following :

Proposition 1.12. *In the number field case, there is a natural isomorphism :*

$$K^\times\backslash\mathbb{A}_K^\times/(\prod_{v\in X_{fin}} \mathcal{O}_v^\times \prod_{v\in X_\infty} K_v^{\times,\circ}) \to Cl^+(\mathcal{O}_K).$$

In the function field case there is a natural isomorphism :

$$K^\times\backslash\mathbb{A}_K^\times/\prod_{v\in X} \mathcal{O}_v^\times \to Pic(X).$$

In the above formula $K_v^{\times,\circ}$ is the component of the identity in $K_v^\times$.

2. Lecture II : Curves

2.1. **Algebraic curves.** Let k be a field.

Definition 2.1. *A function field of dimension one over k is a field K of finite type, transcendance degree 1 and such that k is algebraically closed in K.*

We attach a set X (or $_KX$) to K: the set of all non-trivial valuations on K which are trivial on k (up to equivalence). We put a topology on X as follows : the opens are $\emptyset$ and the complements of a finite set of points.

Remark 2.1. We will also add to X a generic point η, which belongs to all non-empty open subsets.

We now equip X with a sheaf of rings $\mathscr{O}_X$. If U is some open, we let $\mathscr{O}_X(U) = \{f \in K, v(f) \geq 0\ \forall v \in U\}$, so that $(X, \mathscr{O}_X)$ becomes a ringed space.

Definition 2.2. *A curve C over* Spec k *is a scheme of pure dimension* 1 *over* Spec k.

It is reasonable to add a few more assumptions.

Definition 2.3. *A Dedekind scheme is a quasi-compact, separated scheme which is covered by affines* Spec A *where A is a Dedekind ring.*

Definition 2.4. *A non-singular curve over* Spec k *is an irreducible, quasi-compact, separated Dedekind scheme over* Spec k.

Let K be the fonction field of an irreducible curve. We say that C is geometrically connected if k is algebraically closed in K.

Definition 2.5. *A scheme X over* Spec k *is projective if it can be embedded as a closed subscheme of a projective scheme $\mathbb{P}_k^N$.*

Theorem 2.1. *Let K be a function field over k. The locally ringed space $(X, \mathscr{O}_X)$ is a geometrically connected, non-singular, projective curve over* Spec k.

A complete proof can be found in [Har77], I, 6. Let us give some elements of proof.

Proposition 2.1. *Let $x \in K \setminus k$. We consider $U = \{v \in X, v(x) \geq 0\}$. Then U is open in X and $\mathscr{O}_X(U)$ is the normalisation of $k[x]$ in K. Moreover, $(U, \mathscr{O}_X|_U) = (\mathrm{Spec} B, \mathscr{O}_{\mathrm{Spec}\ B})$ for $B = \mathscr{O}_X(U)$.*

From this proposition, we deduce that X is a non-singular curve. Indeed, let $V = \{v \in X, v(x^{-1}) \geq 0\}$. Then $X = U \cup V$ is an affine cover of X. Moreover, $U \cap V =$ Spec (Normalization of $k[x, x^{-1}]$ in K) is also affine.

The projectivity is a little bit delicate. Nevertheless one can easily prove the following:

Proposition 2.2. $\mathrm{H}^0(X, \mathscr{O}_X) = k$.

Proof. Let $x \in K \setminus k$. We need to find $v \in X$ such that $v(x) < 0$. Let $V = \{v \in X, v(x^{-1}) \geq 0\}$. Then $\mathscr{O}_X(V) = B$ and $k[x^{-1}] \to B$ is finite flat. We can find a prime ideal above (x^{-1}) in B and it corresponds to a valuation v for which $v(x^{-1}) > 0$. □

2.2. **An equivalence of category.** We now prove that the last construction exhausts all projective non-singular curves.

Lemma 2.1. *Let C be a projective non-singular curve over* Spec k. *Then there is an isomorphism $C \to {}_K X$ where K is the function field of C.*

Proof. We first define a morphism. To any closed point x of C, we have a local ring $\mathcal{O}_{C,x} \hookrightarrow K$ which is a discrete valuation ring because the curve is non-singular. Therefore we have a map $C \to {}_K X$. This map is injective (the curve C is separated). The map extends to a locally ringed space map $(C, \mathscr{O}_C) \to ({}_K X, \mathscr{O}_{{}_K X})$, since for any open U of C, $\mathscr{O}_C(U) = \cap_{x \in U} \mathscr{O}_{C,x}$. The map $C \to {}_K X$ is therefore a map of algebraic curve. Its image is closed since C is projective, it is all of ${}_K X$. □

Let X and Y be two schemes. A morphism $f : X \to Y$ is finite flat if for any affine Spec $A \subset Y$, $f^{-1}(\mathrm{Spec}\ A) = \mathrm{Spec}\ B$ is affine and $A \to B$ is a finite flat map.

Lemma 2.2. *Let $f : X \to Y$ be a non-constant morphism between projective non-singular algebraic curves. Then f maps the generic point η_X of X to the generic point η_Y of Y. The morphism f is finite flat and is determined by the morphism $\mathscr{O}_{Y,\eta_Y} \to \mathscr{O}_{X,\eta_X}$ on generic points.*

Proof. The image of f is a connected closed subset of Y. It is either Y or a closed point of Y. It is therefore Y and the generic point of X maps to the generic point of Y. Therefore we have a map $K \to L$ where K is the function field of Y and L is the function field of X. Let $x \in K$ be an element which is not algebraic over k. The we have finite flat maps $k[x] \to A \to B$ where A is the normalization of $k[x]$ in K and B the normalization of $k[x]$ in B. And $\mathrm{Spec}(B) = D(f^\star(x)) \to D(x)$ is finite flat. □

Theorem 2.2. *The functor "generic point" induces an equivalence of categories between:*

{*Non-singular, geometrically connected projective curves on* Spec k, *non constant morphisms*}

and

{*Function fields of one variable over k*}.

Proof. This is [Har77], corollary 6.12. □

2.3. **Divisors.**

Definition 2.6. *We let $Div(X)$ be the free abelian group generated by the closed points $x \in X$.*

We have a partial order on $Div(X)$. If $D = \sum n_x x$ and $D' = \sum m_x x$, we say that $D \geq D'$ is $n_x \geq m_x$ for all x. We say that a divisor D is effective if $D \geq 0$.

If $f \in K^\times$, we let $div(f) = \sum_{x \in X} v_x(f)x$. These divisors are called principal. We let $deg : Div(X) \to \mathbb{Z}$ which maps $\sum n_x x$ to $\sum n_x [k(x):k]$. We let $Div^0(X)$ be the kernel of deg.

Lemma 2.3. *For all $f \in K^\times$, $deg(div(f)) = 0$.*

Proof. [Ser88], prop. 1, p. 8. □

Definition 2.7. *We let $Pic(X) = Div(X)/div(K^\times)$ be the Picard group of X.*

By lemma 2.3, the map deg passes to the quotient and defines a map $deg : Pic(X) \to \mathbb{Z}$. We let $Pic^r(X) = deg^{-1}(r)$.

2.4. **Geometric interpretation of divisors.** A sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules is called a locally free sheaf of rank n if there is an open covering $X = \cup U_i$ such that $\mathscr{F}|_{U_i} \simeq \mathscr{O}_{U_i}^n$. An invertible sheaf is a locally free sheaf of rank one. Let $D \in Div(X)$. We let $\mathscr{O}_X(D)$ be the invertible sheaf defined by $\mathscr{O}_X(D)(U) = \{x \in K, v(x) + v(D) \geq 0, \ \forall v \in U\}$.

Lemma 2.4. *There is a bijection between :*

$$\{\textit{Locally free sheaves of rank one } \mathcal{L} \ + \ \textit{non-zero rational section } f \in \mathcal{L}_\eta \setminus \{0\}\}/\textit{isom}$$

and $Div(X)$.

Proof. To $D \in Div(X)$ we associate $\mathscr{O}_X(D)$ equipped with the rational section 1. Conversely let $(\mathcal{L}, f)$. Then for all $x \in X$, we consider $\mathcal{L}_x \subset \mathcal{L}_\eta$. This is an $\mathscr{O}_{X,x}$- rank one module inside a K-vector space of dimension 1. The module $f.\mathscr{O}_{X,x}$ is another rank 1 submodule inside $\mathcal{L}_\eta$. Let t_x be a uniformizing parameter at x. Then we have $f.\mathscr{O}_{X,x} = t_x^{-n_x}\mathcal{L}_x$ for a unique integer n_x. We let $D = \sum n_x x$. The map $\mathscr{O}_X(D) \stackrel{\times f}{\to} \mathcal{L}$ is an isomorphism which sends 1 to f. □

Corollary 2.1. *There is a bijection between :*

$$\{\textit{Locally free sheaves of rank one } \mathcal{L}\}/\textit{isom}$$

and $Pic(X)$.

2.5. **Cohomology of line bundles.** Attached to a locally free of rank one $\mathcal{L}$ (in fact any abelian sheaf !), we have the cohomology groups $\mathrm{H}^0(X, \mathcal{L})$ and $\mathrm{H}^1(X, \mathcal{L})$.

Theorem 2.3.
(1) *The k-vector spaces $\mathrm{H}^i(X, \mathcal{L})$ are finite dimensional. Let $g = \dim_k \mathrm{H}^1(X, \mathscr{O}_X)$ be the genus of the curve.*
(2) *We have $\dim_k \mathrm{H}^0(X, \mathcal{L}) - \dim_k \mathrm{H}^1(X, \mathcal{L}) = deg(\mathcal{L}) - g + 1$.*
(3) *Assume that X/k is smooth. There is an invertible line bundle $\Omega^1_{X/k}$ of degree $2g-2$, and a canonical isomorphism $\mathrm{H}^1(X, \Omega^1_{X/k}) \to k$.*
(4) *We have a Serre duality perfect pairing :*

$$\mathrm{H}^0(X, \mathcal{L}) \times \mathrm{H}^1(X, \Omega^1_{X/k} \otimes \mathcal{L}^{-1}) \to k.$$

Proof. See [Ser88], prop. 2 and thm. 1, p. 10 and corollary p. 17. □

Remark 2.2. We notice that if $\deg \mathcal{L} < 0$, then $\mathrm{H}^0(X, \mathcal{L}) = 0$. Using the duality theorem, we deduce that if $\deg \mathcal{L} > 2g-2$, $\mathrm{H}^1(X, \mathcal{L}) = 0$ and $\dim_k \mathrm{H}^0(X, \mathcal{L}) = deg\mathcal{L} - g + 1$.

Remark 2.3. A non-singular curve needs not necessarily be smooth in caracteristic p. For example let $k = \mathbb{F}_p(t)$, and consider the curve of equation $Y^2 = X^p - t$. This curve is regular at $Y = 0$ but not smooth.

2.6. **Explicit definition of the cohomology.** We let $\mathbb{A}_K$ be the ring of adèles of K. For a divisor $D = \sum n_x x$, we let $\hat{\mathcal{O}}(D) = \{(f_x) \in \mathbb{A}_K,\ v_x(f_x) + n_x \geq 0\}$.

Then we have an exact sequence :

$$0 \to \mathrm{H}^0(X, \mathscr{O}_X(D)) \to K \to \mathbb{A}_K/\hat{\mathcal{O}}(D) \to \mathrm{H}^1(X, \mathscr{O}_X(D)) \to 0$$

Indeed, we can consider the following resolution of the sheaf $\mathscr{O}_X(D)$ by skyscraper sheaves (which are acyclic):

$$0 \to \mathscr{O}_X(D) \to (\iota_\eta)_\star K \to \oplus_{x \in X} (\iota_x)_\star K_x / t_x^{-n_x} \mathcal{O}_x \to 0$$

where $\mathcal{O}_x = \hat{\mathscr{O}_{X,x}}$ and t_x is a uniformizing element, $\iota_\eta : \eta \to X$ is the inclusion of the generic point and $\iota_x : x \to X$ is the inclusion of the closed point x.

Remark 2.4. One can therefore interpret $\mathrm{H}^1(X, \mathscr{O}_X)$ as measuring the obstruction to construct a global rational function whose polar part has been given at a finite set of points.

2.7. **Duality.** We follow here [Tat68]. We first construct the dualizing sheaf $J_{X/k}$ as follows. At the generic point, this is the sheaf of continuous linear forms :

$$\ell : K \backslash \mathbb{A}_K \to k$$

On some open U, we let $J_{X/k}(U) = \{\ell : K\backslash\mathbb{A}_K / \prod_{x \in U} \mathcal{O}_x \to k\}$. We see that by definition, $\mathrm{H}^0(X, J_{X/k}(-D)) = \mathrm{H}^1(X, \mathscr{O}_X(D))^\vee$. We now assume that the curve is smooth over k. In such a case, there is an isomorphism given by the residue (see [Tat68] and [Ser88]):

$$\begin{array}{ccc} \Omega^1_{X/k} & \to & J_{X/k} \\ \omega & \mapsto & \sum res_x(f_x \omega) \end{array}$$

2.8. **Weil's formula.** We let $Bun_{GL_n}(X)$ be the set of isomorphism classes of locally free sheaves of rank n. Note that $Bun_{GL_1}(X) = Pic(X)$.

Theorem 2.4. *There is an isomorphism :*

$$Bun_{GL_n}(X) = GL_n(K) \backslash GL_n(\mathbb{A}_K) / \prod_x GL_n(\mathcal{O}_x).$$

Proof. Let $\mathscr{F}$ be a locally free sheaf of rank n. Let $s_1, \cdots, s_n$ be a basis of sections at η. Then for all point $x \in X$, there is a unique element $f_x \in GL_n(K_x)/GL_n(\mathcal{O}_x)$ and an isomorphism $K^n / f_x O_x^n = \mathscr{F}_\eta / \mathscr{F}_x$. Conversely, given a collection $(f_x) \in GL_n(\mathbb{A}_K)$ we can define the subsheaf of $(\iota_\eta)_\star K^n$ by $\mathscr{F}(U) = \{s \in K^n,\ \forall x \in U,\ s \in f_x \mathcal{O}_x^n\}$. □

Here is a similar, but slightly simpler formula for $\mathbb{P}^1$.

Theorem 2.5.

$$Bun_{GL_n}(\mathbb{P}^1) = GL_n(k[x^{-1}]) \backslash GL_n(k[x, x^{-1}]) / GL_n(k[x])$$

Proof. Since $k[x]$ and $k[x^{-1}]$ are principal, any locally free sheaf $\mathscr{F}$ is trivial on Spec $k[x]$ or Spec $k[x^{-1}]$. Elements in $GL_n(k[x, x^{-1}])$ give the gluing data. Namely, we can take a basis $e_1, \cdots, e_n$ of $\mathscr{F}(\text{Spec } k[x])$ and a basis $f_1, \cdots, f_n$ of $\mathscr{F}(\text{Spec } k[x^{-1}])$. Restricting to Spec $k[x, x^{-1}]$, we find a matrix in $GL_n(k[x, x^{-1}])$ which passes from the basis (e_i) to the basis (f_i). □

We deduce from this theorem very easily that $Pic(\mathbb{P}^1) \overset{deg}{\simeq} \mathbb{Z}$. We let $\mathscr{O}(n)$ be a sheaf of degree n. We have the following theorem of Grothendieck :

Theorem 2.6. *Any vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles $\mathscr{O}(n)$.*

Proof. By theorem 2.5, we are reduced to certain matrix computations. See [HM82]. □

2.9. **Finiteness of the $Pic^0(X)$ over finite fields.**

Theorem 2.7. *If k is a finite field $Pic^0(X)$ is finite.*

Proof. It suffices to prove the finiteness of $Pic^n(X)$ for large n. If n is very large, any D is equivalent to an effective divisor because $\dim \mathrm{H}^0(X, \mathscr{O}_X(D)) > 0$. But there are clearly finitely many effective divisors of degree less than n. □

3. Lecture III : Jacobians

3.1. **The Yoneda functor.** Let $\mathcal{C}$ be a category. Let $\mathcal{F}(\mathcal{C}^{op}, SET)$ be the category of contravariant functors from $\mathcal{C}^{op}$ to SET.

Lemma 3.1. *We have a fully faithfull functor :*

$$\begin{array}{rcl} \mathcal{C} & \to & \mathcal{F}(\mathcal{C}^{op}, SET) \\ X & \mapsto & \mathrm{Hom}(-, X) \end{array}$$

A functor $F \in \mathcal{F}(\mathcal{C}^{op}, SET)$ is said to be representable if it is in the essential image of $\mathcal{C}$.

3.2. **The functor of points of a scheme.** If X is a scheme, then we let $X(-) = \mathrm{Hom}(-, X)$ be the corresponding functor of points. Actually this functor of points is a sheaf for the fppf topology.

Definition 3.1. *Let T be a scheme. An $fppf$ covering of T is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that each f_i is flat, locally of finite presentation and such that $T = \cup f_i(T_i)$.*

Remark 3.1. A Zariski or an étale covering is an $fppf$ covering.

3.3. **The relative cohomology over the curve.** Let X be a curve over k as before. For any k-scheme S, we can define a relative curves $X_S = X \times_{\mathrm{Spec}\ k} S$ and we denote by $p_S : X_S \to S$ the projection.

Let $\mathscr{F}$ be a coherent sheaf over X_S. We let $\mathrm{R}^q(p_S)_\star(\mathscr{F})$ be the sheaf associated to the presheaf $U \mapsto \mathrm{H}^q(X_U, \mathscr{F})$.

Theorem 3.1. *The sheaf $\mathrm{R}^q(p_S)_\star(\mathscr{F})$ is a coherent sheaves.*

Assume that $S = \mathrm{Spec}\ A$ is affine and $\mathscr{F}$ is a locally free sheaf of finite rank.

Theorem 3.2. *There exists a perfect complex $M_\bullet : 0 \to M_0 \to M_1 \to 0$ with the property that for any affine scheme $S' = \mathrm{Spec} A' \to S$, $\mathrm{H}^i(X_{S'}, \mathscr{F}) = \mathrm{H}^i(M_\bullet \otimes_A A')$*

Corollary 3.1.
1. *For all $s \in S$, the function $s \mapsto \chi(s) = \dim_{k(s)} \mathrm{H}^1(X_s, \mathscr{F}) - \dim_{k(s)} \mathrm{H}^0(X_s, \mathscr{F})$ is locally constant.*
2. *For all $s \in S$, the function $s \mapsto \dim_{k(s)} \mathrm{H}^i(X_s, \mathscr{F})$ increases under specialization.*
3. *Assume that for all $s \in S$, $\dim_{k(s)} \mathrm{H}^i(X_s, \mathscr{F})$ is constant. Then $\mathrm{R}^i(p_S)_\star(\mathscr{F})$ is a locally free sheaf.*

Corollary 3.2. *Let $\mathcal{L}$ be an invertible sheaf on X_S. For all $s \in S$, the function $s \mapsto deg(\mathcal{L}_s)$ is locally constant. We call it $deg(\mathcal{L})$. If $deg(\mathcal{L}) \geq 2g-1$, then $(p_S)_\star\mathcal{L}$ is a locally free sheaf of rank $deg(\mathcal{L}) - g + 1$ over S.*

3.4. **The relative Picard functor.** In this section we define the Picard functor. A good reference is [Kle05].

Let S be a scheme. We let $Pic(S)$ be the group of isomorphism classes of line bundles over S.

Let X be a curve over k as before. For any k-scheme S, we can define a relative curves $X_S = X \times_{\mathrm{Spec}\ k} S$.

The functor $S \mapsto Pic(X_S)$ cannot be representable because this is not a sheaf. Indeed, let $\mathcal{L} \in Pic(S)$ be a non-trivial sheaf. Let $p : X_S \to S$ be the structural map. Then we see that $p^\star\mathcal{L}$ and $\mathscr{O}_{X_S}$ are not isomorphic. On the other hand, they are locally isomorphic

We can therefore consider the functor $S \mapsto Pic(X_S)/Pic(S)$. There is still an issue. This functor is not a sheaf in general, and cannot be representable.

Example 6. Let $X = V(X^2+Y^2+Z^2) \subset \mathbb{P}^2_{\mathbb{R}}$ be the twisted form of $\mathbb{P}^1_{\mathbb{R}}$. This is a complete curve over $\mathbb{R}$, of genus 0, with no real point. In particular, the degree of any line bundle in X is even. We have $X_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}}$ with a descent automorphism $\sigma : z \mapsto \frac{1}{\bar{z}}$. We consider $\mathscr{O}(1) \in P_X(\mathbb{C})$. Clearly $\sigma^\star\mathscr{O}(1) \simeq \mathscr{O}(1)$. On the other hand there is no degree 1 line bundle on X, so $\mathscr{O}(1)$ does not descend.

We let $P_X(-)$ be the sheafification (in the $fppf$ or étale topology) of $S \mapsto Pic(X_S)/Pic(S)$. This is the relative Picard functor of the curve. Since $P_X(-)$ is a sheafification of a presheaf, it may be hard to describe its value on a given scheme S. We nevertheless have :

Proposition 3.1. *Suppose that X has a k-rational point P. Then*

$$P_X(-) = \{\mathcal{L} \in Pic(X_S),\ \mathcal{L}|_{P\times S} = \mathscr{O}_{\mathcal{S}}\}.$$

Proposition 3.2. *We have an exact sequence :*

$$0 \to Pic(X_k) \to P_X(k) \to \mathrm{Br}(k)$$

In particular, if k is a finite field, $Pic(X_k) = P_X(k)$.

For any $r \in \mathbb{Z}$, we can also define $Pic^r(X_S)$, $P^r_X(-)$....

3.5. **Representability of the relative Picard functor.** We have the classical theorem:

Theorem 3.3. *The relative Picard functor is representable and P^0_X is an abelian scheme, called the Jacobian of the curve.*

We sketch the proof. A good reference is [Mil86].

3.5.1. *Relative Cartier divisors.*

Definition 3.2. *An effective relative Cartier divisor D over X_S is a closed subscheme $D \hookrightarrow X_S$ such that $p_S : D \to S$ is finite flat.*

Attached to D, we have the invertible sheaf $\mathscr{O}_{X_S}(-D) = \mathcal{I}_D$, to which we can attach the pair $(\mathscr{O}_{X_S}(D), 1 : \mathscr{O}_{X_S} \to \mathscr{O}_{X_S}(D))$. The set of effective Cartier divisors over S, $Div_{\geq 0}(S)$, is the set of isomorphism classes of pairs $(\mathcal{L} \in Pic(X_S), f : \mathscr{O}_{X_S} \to \mathcal{L})$ such that $\mathcal{L}/\mathscr{O}_{X_S}$ is finite flat over S. This last condition is also equivalent to asking that f is nowhere identically zero over S.

Let $r \geq 0$. We let $Div^r_{\geq 0}(-)$ be the functor which maps S to the set of isomorphism classes of effective relative Cartier divisors of degree r over X_S.

3.5.2. *Quotienting schemes by finite groups.* Let A be a ring and let G be a finite group acting on A. Let $B = A^G$. Let $p : \mathrm{Spec}\ A = X \to \mathrm{Spec}\ B = Y$ be the corresponding morphism.

Proposition 3.3 ([Gro03], exp V, prop. 1.1)**.** (1) *The ring A is integral over B.*
(2) *The morphism p is surjective and closed and the map $X \to Y$ induces an homeomorphism $Y \simeq X/G$ (where X/G carries the quotient topology).*
(3) *For any scheme Z, we have that* $\mathrm{Hom}(Y, Z) = \mathrm{Hom}(X, Z)^G$.

We now let X be a scheme and we assume that G acts on X and that X admits an affine covering stable by G.

Proposition 3.4 ([Gro03], exp V, prop. 1.8)**.** *There is a scheme Y and a surjective morphism $p : X \to Y$ such that :*
(1) *The morphism p is surjective and closed and the map $X \to Y$ induces an homeomorphism $Y \simeq X/G$ (where X/G carries the quotient topology).*
(2) *For any scheme Z, we have that* $\mathrm{Hom}(Y, Z) = \mathrm{Hom}(X, Z)^G$.
(3) *We have $\mathscr{O}_Y = p_\star \mathscr{O}_X^G$.*

The scheme Y of the proposition (which is unique up to a unique isomorphism) is called a categorial quotient of X by G.

3.5.3. *Representing $Div^r_{\geq 0}(-)$.* We will prove that $Div^r_{\geq 0}(-)$ is representable. Let $X^r = X \times \cdots \times X$ be the r-th fold product of the curve. The symmetric group $\mathcal{S}_r$ acts on X^r by permutation of the factors.

Lemma 3.2. *The categorical quotient $X^r/\mathcal{S}_r = X^{(r)}$ exists and is smooth.*

Proof. See [Mil86], prop. 3.2. □

We have a map $X^r \to Div^r_{\geq 0}$ which sends $(P_1, \cdots, P_r)$ to $(\mathscr{O}_{X_S}(\sum P_i), \mathscr{O}_{X_S} \to \mathscr{O}_{X_S}(\sum P_i))$. This map pass to the quotient to a map $X^{(r)} \to Div^r_{\geq 0}$.

Proposition 3.5. *$X^{(r)} \to Div^r_{\geq 0}$ is an isomorphism.*

Proof. We need to show injectivity and surjectivity. For surjectivity, it suffices to prove the surjectivity of $X^r \to Div^r_{\geq 0}$. We will show that if $(\mathcal{L} \in Pic(X_S), f : \mathscr{O}_{X_S} \to \mathcal{L})$ is a degree r cartier divisor, there is a finite flat map $T \to S$ and sections $P_1, \cdots, P_r \in X(T)$ such that $(\mathcal{L}, f) \simeq (\mathscr{O}_{X_T}(\sum P_i), 1)$. We prove this by induction on r. The case $r = 1$ is trivial. Let us assume $r \geq 2$. Let $T = V(\mathcal{L}^{-1}) \subset X_S$. The map $T \to S$ is finite flat. Over X_T we have the degree 1 divisor $P : T \overset{\Delta}{\hookrightarrow} T \times_S T \hookrightarrow X_T$. We see that $f_T : \mathscr{O}_{X_T} \to \mathcal{L}_T(-P) \to \mathcal{L}_T$ and $\mathcal{L}_T(-P)$ is now of degree $r-1$. We conclude by induction.

We need to show injectivity. We do this when $r = 2$, the general case is left to the reader. Let P_1, P_2 and Q_1, Q_2 by $\mathrm{Spec}\ R$-points of X. We assume that $\mathscr{O}_{X_S}(-P_1 - P_2) = \mathscr{O}_{X_S}(-Q_1 - Q_2)$.

After localizing in $\mathrm{Spec}\ R$, we can find an affine open $\mathrm{Spec}\ A$ of X_S with the property that P_1, P_2, Q_1, Q_2 factor through $\mathrm{Spec}\ A$. Therefore, we have morphisms $Q_i : A \to R$ with kernel I_i and $P_i : A \to R$ with kernel J_i and by assumption $I_1 I_2 = J_1 J_2$. We want to deduce that the maps $P_1 \otimes P_2 : A \otimes A \to R$ and $Q_1 \otimes Q_2 : A \otimes A \to R$ have the same restriction to $(A \otimes A)^{\Sigma_2}$. We claim that for any $a \in A$, $Q_1(a)Q_2(a) = P_1(a)P_2(a)$ and $Q_1(a) + Q_2(a) = P_1(a) + P_2(a)$ because they can be interpreted as the coefficients of the characteristic polyomial of a acting on $A/I_1I_2 = A/J_1J_2$. We deduce that $P_1 \otimes P_2(a \otimes 1 + 1 \otimes a) = Q_1 \otimes Q_2(a \otimes 1 + 1 \otimes a)$ and $P_1 \otimes P_2(a \otimes a) = Q_1 \otimes Q_2(a \otimes a)$. The elements $a \otimes 1 + 1 \otimes a$ and $a \otimes a$ generate $(A \otimes A)^{\Sigma_2}$ as an algebra. □

3.5.4. *The Abel-Jacobi map.* We call the map $AJ_r : X^{(r)} \to P_X^r$ the Abel-Jacobi map. We will use this map to prove the representability of P_X^r.

Let us assume that $r \geq 2g-1$. Let $S \to Spec\ k$ and let $\mathcal{L} \in Pic^r(X_S)$ (corresponding to a point $x : S \to P_X^r$). Then the fiber product $X^{(r)} \times_{AJ_r, P_X^r, x} S$ is the set of nowhere vanishing sections $f \in R^0(p_S)_\star\mathcal{L}$, up to isomorphism.

But since $r \geq 2g-1$, $R^0(p_S)_\star\mathcal{L}$ is a locally free sheaf of rank $r-g+1$, and

$$X^{(r)} \times_{AJ_r, P_X^r, x} S = (R^0(p_S)_\star\mathcal{L} \setminus \{0\})/\mathscr{O}_S^\times$$

is therefore a fibration in projective spaces of dimension $r-g$.

If we had a section $s : P_X^r \to X^{(r)}$, then we would deduce that P_X^r is representable. Indeed, if we let $q : X^{(r)} \to P_X^r \xrightarrow{q} X^{(r)}$ then the morphism p induces an isomorphism between $X^{(r)} \times_{q, X^{(r)}, id} X^{(r)}$ and P_X^r.

We will prove that there are local sections. At this stage, we assume that the field k is separably closed. By Galois descent, we can reduce to this case.

For any $r-g$-uple of points $t = (t_1, \cdots, t_{r-g}) \in X(k)^{r-g}$, we let

$$X_t^{(r)} = \{(P_1, \cdots, P_r), \dim \mathrm{H}^0(\mathscr{O}_X(\sum_{i=1}^{r} P_i - \sum_{i=1}^{r-g} t_j)) = 1\}.$$

This is an open of $X_t^{(r)}$ and moreover, $X^{(r)} = \cup_t X_t^{(r)}$.

We similarly defined $(P_X^r)_t$ has the subfunctor parametrizing $\mathcal{L}$ with the property that $\dim \mathrm{H}^0(\mathcal{L}(-\sum_{i=1}^{r-g} t_j)) = 1$. The map $X_t^{(r)} \to (P_X^r)_t$ is an isomorphism and therefore $(P_X^r)_t$ is representable. And we have a covering $P_X^r = \cup (P_X^r)_t$.

We finally deduce that P_X^r is smooth and geometrically connected because $X^{(r)}$ is.

Finally for any line bunde $\mathcal{L}$ of degree s over X, the map $- \otimes \mathcal{L} : P_X^r \to P_X^{r+s}$ is an isomorphism. We deduce the representability of P_X^r for all r.

4. Lecture IV : Fundamental groups and geometric class field theory

4.1. **The classical fundamental group.** Let S be a connected, locally arcwise connected, locally simply connected topological space. Let $s \in S$ be a point. We can define $\pi_1(S, s)$, the group of homotopy classes of loops $\gamma : \mathrm{S}^1 \to S$ with $\gamma(0) = s$. Let Cov be the category of coverings of S. Recall that $S' \to S$ is a covering if any point $x \in S$ has a neighborhood U_x such that $p^{-1}(U_x) \simeq U_x \times I$ for a discrete set I. We define a functor $F : Cov \to \mathrm{SET}$ by sending $p : S' \to S$ to $p^{-1}(s)$. Let $\pi_1(S, s) - \mathrm{SET}$ be the category of sets equipped with an action of $\pi_1(S, s)$. We have the following classical theorem :

Theorem 4.1. *The functor F can be enriched to an equivalence of categories $Cov \to \pi_1(S, s) - \mathrm{SET}$.*

Moreover, F is representable functor : let $\tilde{p} : \tilde{S} \to S$ be the universal cover of S, and $\xi \in F(\tilde{S}) = \tilde{p}^{-1}(s)$. Then $F(-) = \tilde{S}(-)$.

Remark 4.1. One can recover $\pi_1(S, s)$ abstractly from the functor F, as the group of automorphisms of F.

4.2. **The fundamental group of a field k.** Let k be a field. A connected covering of k is by definition of finite separable field extension of k. A covering of k will be by definition a finite product of finite separable extension of k (we say also a finite étale extension of k). Let Cov be the category of coverings of k. Let $\overline{k}$ be an algebraic closure of k and $k^{sep} \subset k^{alg}$ be the separable closure. We define a functor $F : Cov \to \mathrm{FSET}$ by mapping ℓ/k to $\mathrm{Hom}(\ell, \overline{k})$ where FSET is the category of finite sets. Let $Gal(k^{sep}/k) = G_k$ and $G_k - \mathrm{FSET}$ the category of finite sets equipped with a continuous left action of G_k.

Theorem 4.2. *The functor F can be upgraded to an equivalence of category $Cov \to G_k - \mathrm{FSET}$.*

Proof. This is a reformulation of Galois theory. We exhibit and inverse functor. If I is a G_k-set. We consider the algebra of functions $f : I \to k^{sep}$ which are G_k-equivariant. $\square$

The functor F is pro-representable. We can write $k^{sep} = \cup_i k_i$ has a filtered union of finite extensions, and $F(-) = \mathrm{colim}_i \mathrm{Hom}(-, k_i)$.

4.3. **The étale fundamental group of a scheme.** The original reference is [Gro03]. Another good reference is [Mur67].

4.3.1. *Etale covers.* We let X be a locally noetherian scheme.

Definition 4.1. *A morphism $p : Y \to X$ of schemes is finite étale if*

(1) *For all affine open* $\mathrm{Spec} A \hookrightarrow X$*, the fiber* $\mathrm{Spec}\ A \times_X Y = \mathrm{Spec}\ B$ *is affine and B is a finite projective A-module,*
(2) *For all point $x \in X$, Y_x is the spectrum of a finite étale extension of $k(x)$.*

A finite étale cover $p : Y \to X$ is a finite étale map which is surjective. In general, the image of p is an open and closed subscheme of X. In particular, if X is connected, an finite étale morphism is a cover. We assume that X is connected.

We let Cov be the category whose objects are finite étale schemes $Y \to X$ and morphisms are X-morphisms of schemes.

4.3.2. *The main theorem.* We let $x \in X$ be a point and we pick $\bar{x} \to x$ a geometric point above x. We can define a functor $F : Cov \to \mathrm{FSET}$ by mapping Y to the set $Y \times_X \bar{x}$.

Theorem 4.3 ([Mur67], thm. 4.4.1)**.** (1) *There exists a unique profinite group $\pi_1(X, \bar{x})$ such the functor F can be enriched to an equivalence of categories :*

$$Cov \to \pi_1(X, \bar{x}) - \mathrm{FSET}.$$

(2) *Let $\overline{x'} \to X$ be another point geometric point of X. There exists a topological isomorphism: $\pi_1(X, \bar{x}) \to \pi_1(X, \bar{x'})$, which is unique up to an inner automorphism.*

If $Y \to X$ is a morphism of schemes, the pull-back of étale covers from X to Y induces a morphism $\pi_1(Y, \overline{y}) \to \pi_1(X, \overline{x})$.

Remark 4.2. We can revisit Frobenius substitution. If $X \to \mathrm{Spec}\ \mathbb{Z}$ is a finite type scheme. Then any closed point $s \in X$ has residue field a finite field. Let $\overline{x}$ be a geometric point of X. For any $s \in X$ and any geometric point $\overline{s} \to s$, we get a morphism (well defined up to conjugacy) $\pi_1(s, \overline{s}) \to \pi_1(X, \overline{x})$. If s is a closed point, $\pi_1(s, \overline{s})$ is topologically generated by the Frobenius.

4.4. $\mathbb{P}^1$ **is geometrically simply connected.** In this section we prove :

Theorem 4.4. *Let k be an algebraically closed field. Then $\pi_1(\mathbb{P}^1_k, \overline{x}) = 1$.*

Proof. Let $f : X \to \mathbb{P}^1_k$ be a finite étale cover. Let $f_\star \mathscr{O}_X$. This is a vector bundle over $\mathbb{P}^1_k$. Therefore, $f_\star \mathscr{O}_X = \oplus_{i=1}^r \mathscr{O}(n_i)$ for integers n_i. We will prove that this is the trivial bundle (all n_i are 0). This will prove that $f_\star \mathscr{O}_X = \mathrm{H}^0(X, \mathscr{O}_X) \otimes_k \mathscr{O}_{\mathbb{P}^1_k}$. Since k is algebraically closed, $\mathrm{H}^0(X, \mathscr{O}_X) = k^r$ (as algebra) and X is the disjoint union of r copies of $\mathbb{P}^1_k$. There is a bilinear trace map : $f_\star \mathscr{O}_X \times f_\star \mathscr{O}_X \to \mathscr{O}_{\mathbb{P}^1_k}$ and this is a perfect pairing. Therefore we deduce that it is enough to prove that for all i, $n_i \geq 0$. Let i be the index for which n_i is minimal and assume that $n_i < 0$. The product map $m : f_\star \mathscr{O}_X \otimes f_\star \mathscr{O}_X \to f_\star \mathscr{O}_X$ restricts to a map $\mathscr{O}(n_i) \otimes \mathscr{O}(n_i) \to f_\star \mathscr{O}_X$. But there are no non-zero maps $\mathscr{O}(2n_i) \to f_\star \mathscr{O}_X$. Therefore $m(\mathscr{O}(n_i) \otimes \mathscr{O}(n_i)) = 0$. But X is a smooth curve and therefore it is reduced. $\square$

4.5. **Descent of étale covers.** We consider the following situation : X is a scheme and Γ is a finite group acting on X. We assume that X has an affine covering stable under Γ. We can define the categorical quotient X/Γ (see [Gro03], exposé V, sect. 1). For a point $x \in X$, we let Γ_x be the inertia group at x. This is the subgroup of Γ of elements which stabilize x and act trivially on the residual field at x, $k(x)$.

Let $Y \to X$ be an étale cover. We assume that Y carries an action of Γ compatible with the action on X.

We can therefore consider the quotient Y/Γ and we have a diagram :

$$\begin{array}{ccc} Y & \longrightarrow & Y/\Gamma \\ \downarrow & & \downarrow \\ X & \longrightarrow & X/\Gamma \end{array}$$

The following two propositions are [Gro03], exposé IX, rem. 5.8.

Proposition 4.1. *The map $Y/\Gamma \to X/\Gamma$ is finite étale if and only if, for all $x \in X$, if we let Γ_x the inertia subgroup at x, then Γ_x acts trivially on Y_x.*

Proposition 4.2. *We have an equivalence between the category of finite étale cover of X/Γ and the finite étale cover of X which carry an action of Γ compatible with the action on X and such that for all $x \in X$, Γ_x act trivially on the fiber.*

4.6. $\mathbb{P}^r$ **is geometrically simply connected.**

Theorem 4.5. *Let k be an algebraically closed field. Then $\pi_1(\mathbb{P}^r_k, \overline{x}) = 1$.*

Proof. We first need to prove that $\pi_1((\mathbb{P}^1_k)^r, \overline{x}) = 1$. We prove this by induction on r. The case $r = 1$ is theorem 4.4. We assume $r \geq 2$ and consider the map $p : (\mathbb{P}^1_k)^r \to (\mathbb{P}^1_k)^{r-1}$ given by the projection on the first $r-1$ coordinates. We now let $f : X \to (\mathbb{P}^1_k)^r$ be a finite étale cover of degree d. We claim that $p_\star f_\star \mathscr{O}_X$ is a locally free sheaf of algebras over $(\mathbb{P}^1_k)^{r-1}$. This follows from corollary 3.1. Indeed, for each point $t \in (\mathbb{P}^1_k)^{r-1}$, $p^{-1}(t) = \mathbb{P}^1_{k(t)}$ and $X_t \to \mathbb{P}^1_{k(t)}$ is isomorphic to $\mathbb{P}^1_{k(t)} \times_{\text{Spec } k(t)} \text{Spec } k(t)'$ for a finite étale extension of $k(t)'$ of degree d. We find that $dim_{k(t)}(\mathbb{P}^1_{k(t)}, (f_t)_\star \mathscr{O}_{X_t}) = d$ is constant. Let X' be the spectrum of this sheaf of algebras. We see that $X' \to (\mathbb{P}^1_k)^{r-1}$ is finite flat and moreover, $X' \times_{(\mathbb{P}^1_k)^{r-1}} (\mathbb{P}^1_k)^r \simeq X$. In other words, X' descends X. We see that $X' \to (\mathbb{P}^1_k)^{r-1}$ is smooth, because $X \to (\mathbb{P}^1_k)^r$ is. Therefore we deduce that X' is a finite étale cover. We also deduce that the map $p : \pi_1((\mathbb{P}^1_k)^r, \overline{x}) \to \pi_1((\mathbb{P}^1_k)^{r-1}, p(\overline{x}))$ is an isomorphism. By induction we deduce that $\pi_1((\mathbb{P}^1_k)^r, \overline{x}) = 1$. Then we use proposition 4.2. Indeed, $\mathbb{P}^r_k = (\mathbb{P}^1_k)^r / \mathcal{S}_r$. Let $X \to \mathbb{P}^r_k$ be an étale cover. Its pullback to $(\mathbb{P}^1_k)^r$ is $\tilde{X}$ and it is isomorphic to $(\mathbb{P}^1_k)^r \times I$ where I is a finite set over which Γ acts. Take a point in the diagonal x. Then the inertia group is $(\mathcal{S}_r)_x = \mathcal{S}_r$ and we deduce that $\mathcal{S}_r$ acts trivially on I. Therefore $X = \mathbb{P}^r_k \times I$. □

4.7. **Descending étale covers under projective fibration.**

Theorem 4.6. *Let $f : X \to Y$ be a projective fibration. Then the map $\pi_1(X, \overline{x}) \to \pi_1(Y, f(\overline{x}))$ is an isomorphism.*

Proof. We have seen a proof in theorem 4.5 in the case of a fibration in projective lines. A similar argument applies. □

4.8. **Geometric class field theory.** Let L be a finite abelian group. Let X be a complete non-singular curve over $\mathbb{F}_q$. We will prove the following theorem :

Theorem 4.7. *There is a canonical bijection :*

$$\begin{array}{rcl} \{\chi : \pi_1(X) \to L\} & \to & \{\rho : Pic(X) \to L\} \\ \chi & \mapsto & \rho \end{array}$$

where ρ is defined by the rule that for all $x \in X$, $\rho(\mathscr{O}(x)) = \chi(Frob_x)$.

As a corollary, we deduce :

Theorem 4.8. *We have a commutative diagram:*

$$\begin{array}{ccc} Pic(X) & \longrightarrow & \pi_1(X)^{ab} \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \hat{\mathbb{Z}} \end{array}$$

which induces an isomorphism between $Pic(X)$ and $W(X)^{ab}$.

Proof. The theorem 4.7 implies that the profinite completion of $Pic(X)$ is isomorphic to $\pi_1(X)^{ab}$. Now we have an exact sequence $1 \to Pic^0(X) \to Pic(X) \to \mathbb{Z} \to 1$ (which splits non-canonically) and $Pic^0(X)$ is a finite group. Therefore the profinite completion of $Pic(X)$ is $Pic^0(X) \times \hat{\mathbb{Z}}$. □

4.8.1. *Systems of abelian covers over $\{X^{(r)}\}_{r\geq 0}$, compatible with the monoidal structure.* We recall that we have multiplications $m : X^{(r)} \times X^{(r')} \to X^{(r+r')}$, and projections $p_1 : X^{(r)} \times X^{(r')} \to X^{(r)}$ and $p_2 : X^{(r)} \times X^{(r')} \to X^{(r')}$. It will be convenient to consider also $Div_{\geq 0} = \coprod_{r\geq 0} X^{(r)}$. So that we have three maps, $m, p_1, p_2 : Div_{\geq 0} \times Div_{\geq 0} \to Div_{\geq 0}$. We also let $\pi_1(Div_{\geq 0})^{ab} = \oplus_{r\geq 0}\pi_1(X^{(r)})^{ab}$.

Let L be a finite abelian group. Let $\chi_1 : \pi_1(X) \to L$ be a character.

Proposition 4.3. *There is a unique way to attach to χ_1 a character $\chi = \prod_{r\geq 0}\chi_r$: $\pi_1(Div_{\geq 0})^{ab} \to L$ such that :*

$$m^\star\chi = p_1^\star\chi + p_2^\star\chi$$

as characters of $\pi_1(Div_{\geq 0} \times Div_{\geq 0})^{ab}$.

Remark 4.3. We thus claim that there is a unique system of characters $\{\chi_r : \pi_1(X^{(r)}) \to L\}_{r\geq 1}$ which satisfy that the pull backs of $\chi_r + \chi_{r'}$ and $\chi_{(r+r')}$ to characters of $\pi_1(X^{(r)} \times X^{(r')})$ coincide:

$$\begin{array}{ccccc} \pi_1(X^{(r)} \times X^{(r')}) & \longrightarrow & \pi_1(X^{(r+r')}) & \xrightarrow{\chi_{r+r'}} & L \\ \downarrow & & & & \\ \pi_1(X^{(r)}) \times \pi_1(X^{(r')}) & & & & \\ \downarrow{\scriptstyle \chi_r+\chi_{r'}} & & & & \\ L & & & & \end{array}$$

Proof. Let $Y \to X$ be the abelian cover with group L corresponding to χ_1. We construct an abelian cover over X^r corresponding to $\chi_1^{\oplus r} : \pi_1(X^r) \to L$. This is $Y^r/H \to X^r$ where $H = \mathrm{Ker}(L^r \stackrel{\Sigma}{\to} L)$. Then we check that the action of $\mathcal{S}_r$ on X^r lifts to Y^r and passes to the quotient Y^r/H. Moreover, the action of the inertia group is trivial on the fibers. Therefore the cover descends to $X^{(r)}$. □

Lemma 4.1. *Let $r_0 \geq 0$. Assume that we have a system of characters $\{\chi_r : \pi_1(X^{(r)}) \to L\}_{r \geq r_0}$ which satisfy that the pull backs $p_1^\star \chi_r + p_2^\star \chi_{r'} = m^\star \chi_{(r+r')}$ has characters of $\pi_1(X^{(r)} \times X^{(r')})$.*

Then, there exists a unique character $\chi_1 : \pi_1(X, \overline{x}) \to L$ such that this system arises from χ_1.

Proof. Let x_0 be a rational point on $X^{(r)}$ for $r \geq r_0$. We get a map $X \to X^{(r+1)}$ by sending x to (x, x_0). We let $\chi_1 : \pi_1(X) \to \pi_1(X^{(r+1)}) \to L$. □

4.8.2. *Systems of abelian covers of P_X, compatible with the monoidal structure.* Recall that $P_X(\mathbb{F}_q) = Pic(X)$. We have maps $m : P_X \times P_X \to P_X$ as well as projections $p_i : P_X \times P_X \to P_X$.

A character $\rho : \pi_1(P_X) \to L$ is compatible with the monoidal structure if we have $p_1^\star \rho + p_2 \star \rho = m^\star \rho$ as characters of $\pi_1(P_X \times P_X)$.

To such a character we can associated a group morphism : $\tilde{\rho} : Pic(X) \to L$ by evaluating on $Frob_x$ for each $x \in Pic(X)$.

Proposition 4.4. *The association $\rho \mapsto \tilde{\rho}$ defines an bijection between characters compatible with the monoidal structure on P_X and characters of $Pic(X)$.*

Proof. Let $P_X \overset{Frob_q - 1}{\to} P_X$ be the Lang isogeny which maps $\mathcal{L}$ to $Frob_q^\star \mathcal{L} \otimes \mathcal{L}^{-1}$. Its kernel is precisely $P_X(\mathbb{F}_q) = Pic(X)$. This provides a map $\rho_{Lang} : \pi_1(P_X) \to Pic(X)$. Moreover, for any $\mathcal{L} \in Pic(X)$, $\rho_{Lang}(Frob_{\mathcal{L}}) = \mathcal{L}$. It is an easy exercise to check that $m^\star \rho_{Lang} = p_1^\star \rho_{Lang} + p_2^\star \rho_{Lang}$.

Let $\rho : \pi_1(P_X) \to L$ be a character compatible with the monoidal structure. We need to find a factorization $\rho : \pi_1(P_X) \overset{\rho_{Lang}}{\to} Pic(X) \to L$. We therefore need to prove that $\pi_1(P_X) \overset{Frob_q - 1}{\to} \pi_1(P_X) \overset{\rho}{\to} L$ is the trivial character. But this is nothing else than $Frob_q^\star \rho - \rho$ (because ρ is compatible with the monoidal structure). And we know that $Frob_q^\star \rho = \rho$.

□

4.8.3. *Proof of theorem 4.7.* We see that the following sets are in natural bijection :

(1) Characters $\chi : \pi_1(X) \to L$,
(2) Characters $\{\chi_r : \pi_1(X^{(r)}) \to L\}_{r \geq 0}$, compatible with the monoidal structure,
(3) Characters $\{\chi_r : \pi_1(X^{(r)}) \to L\}_{r \geq r_0}$, compatible with the monoidal structure,
(4) Characters $\{\rho_r : \pi_1(P_X^r) \to L\}_{r \geq r_0}$, compatible with the monoidal structure,
(5) Characters $\{\rho_r : \pi_1(P_X^r) \to L\}_{r \geq 0}$, compatible with the monoidal structure,
(6) Characters $\tilde{\rho} : Pic(X) \to L$.

- (1) ⇔ (2) is proposition 4.3,
- (2) ⇔ (3) is lemma 4.1,
- (3) ⇔ (4) is proposition 4.6,
- (4) ⇔ (5) is similar to lemma 4.1,
- (5) ⇔ (6) is proposition 4.4

Remark 4.4. We can restate our theorem as follows. Given a character $\chi : \pi_1(X) \to L$, there exists a unique character $\rho : \pi_1(P_X) \to L$ such that for $m : X \times P_X \to P_X$ the map which sends $(x, \mathcal{L})$ to $\mathcal{L}(x)$, we have $m^\star \rho = p_1^\star \chi + p_2^\star \rho$.

5. Lecture V : The Langlands correspondence for GL_n over function fields

5.1. **The space of spherical cuspidal automorphic functions.** We consider the $\overline{\mathbb{Q}}$-vector space $\mathcal{F}$ of locally constant functions $\mathrm{GL}_n(K)\backslash\mathrm{GL}_n(\mathbb{A}_K)/\prod_{x\in X}\mathrm{GL}_n(\mathcal{O}_x)\to\overline{\mathbb{Q}}$.

We let Z be the center of GL_n (isomorphic to GL_1). We let $\chi : Z(K)\backslash Z(\mathbb{A}_K)/\prod_x Z(\mathcal{O}_x)\to \overline{\mathbb{Q}}^\times$ be a character.

We let $\mathcal{C}_{cusp}(GL_n,\chi)$ be the subspace of $\mathcal{F}$ of functions f which satisfy :

(1) (central character) $f(zg)=\chi(z)f(g)$,
(2) (cuspidality) For all standard parabolic P of GL_n, with unipotent radical U, for all $x\in\mathrm{GL}_n(\mathbb{A}_K)$, we have
$$\int_{U(K)\backslash U(\mathbb{A}_K)} f(ux)du=0.$$
(3) (growth condition) There is a compact $C\subset\mathrm{GL}_n(\mathbb{A}_K)$ such that f vanishes outside of $Z(\mathbb{A}_f)C$.

Remark 5.1. The locally profinite group $U(\mathbb{A}_f)$ carries a Haar measure du. We normalize the Haar measure by $du(U(\prod\mathcal{O}_x))=1$. Then this Haar measure takes rational values.

Theorem 5.1. *The space $\mathcal{C}_{cusp}(\mathrm{GL}_n,\chi)$ is finite dimensional.*

We will only give the proof of this theorem for the group GL_2.

By the Iwasawa decomposition we have that $\mathrm{GL}_2(\mathbb{A})=\mathrm{B}(\mathbb{A})\prod_x GL_2(\mathcal{O}_x)$ where B is the upper triangular Borel.

We will now define Siegel sets. Let v be a fixed place of K, $C^v\subset(\mathbb{A}_K^v)^\times$ and $C_0\subset\mathbb{A}_K$ be compact open subsets.

Let
$$\mathcal{S}_{C^v,C_0}=\{\begin{pmatrix}1&y\\0&1\end{pmatrix}\begin{pmatrix}x'&0\\0&x\end{pmatrix}k,\ y\in C_0,\ x',x\in K_v^\times\times C_v,\ |x'x^{-1}|\geq 1, k\in\prod_x GL_2(\mathcal{O}_x)\}$$
be a Siegel set.

Lemma 5.1. *For any Siegel set $\mathcal{S}_{C^v,C_0}$ and any $c\in\mathbb{R}_{>0}$, the subset $\mathcal{S}^{\leq c}_{C^v,C_0}$ of elements $\begin{pmatrix}1&y\\0&1\end{pmatrix}\begin{pmatrix}x'&0\\0&x\end{pmatrix}k$ which satisfy the condition that $|(x'x^{-1})_v|_v\leq c$ is compact modulo the center.*

Proof. We have to see that $x'x^{-1}$ belongs to some compact. From the conditions $|x'x^{-1}|\geq 1$ and $x'x^{-1}\in K_v^\times\times C_v$, we deduce that $|(x'x^{-1})_v|_v\geq c_1$ for a constant c_1. Therefore, $(x'x^{-1})_v$ belongs to a compact. □

Lemma 5.2. *For C^v and C_0 big enough, we have that $\mathrm{GL}_2(K).\mathcal{S}_{C^v,C_0}=\mathrm{GL}_2(\mathbb{A}_K)$.*

Proof. We first claim that for any $g\in\mathrm{GL}_2(\mathbb{A})$, there exists $\gamma\in GL_2(K)$ such that
$$g=\begin{pmatrix}x'&y\\0&x\end{pmatrix}k$$
with $k\in K$, and $|x'x^{-1}|\geq 1$.

Indeed, any g has an expression of the form $g=\begin{pmatrix}x'&y\\0&x\end{pmatrix}k$ by the Iwasawa decomposition. Let us assume that $|x'x^{-1}|<1$. We can conjugate g by the element $w=\begin{pmatrix}0&1\\1&0\end{pmatrix}\in GL_2(K)\cap\prod_x\mathrm{GL}_2(\mathcal{O}_x)$ and we find that $wg=\begin{pmatrix}x&0\\y&x'\end{pmatrix}k'$ for $k'\in\prod_x\mathrm{GL}_2(\mathcal{O}_x)$.

We observe that $\begin{pmatrix} x & 0 \\ y & x' \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (x')^{-1}y & 1 \end{pmatrix}$.

Let us put $\alpha = (x')^{-1}y$. For each place v where $|\alpha_v|_v \leq 1$, we have that $\begin{pmatrix} 1 & 0 \\ \alpha_v & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_x)$.

Let S be the finite set of places for which $|\alpha_v|_v > 1$. We find that for $v \in S$, $\begin{pmatrix} 1 & 0 \\ \alpha_v & 1 \end{pmatrix} = \begin{pmatrix} \alpha_v & 1 \\ 0 & \alpha_v^{-1} \end{pmatrix} k''$ with $(k'')^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & \alpha_v^{-1} \end{pmatrix}$.

We deduce that $wg = \begin{pmatrix} x \prod_{v \in S} \alpha_v & y' \\ 0 & x' \prod_{v \in S} \alpha_v^{-1} \end{pmatrix} k'''$.

We claim that there is a compact $C^v \subset (\mathbb{A}_K^v)^\times$ such that $K^\times . K_v C^v = \mathbb{A}_K^\times$. Therefore, for any $g \in GL_2(\mathbb{A}_K)$, there is $\gamma \in GL_2(K)$ such that $\gamma g = \begin{pmatrix} x' & y \\ 0 & x \end{pmatrix} k$ with $k \in K$, and $|x'x^{-1}| \geq 1$ and $x', x^{-1} \in K_v^\times \times C^v$.

We have thus reduced to the case that $g = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' & 0 \\ 0 & x \end{pmatrix} k$ where $|x'x^{-1}| \geq 1$ and $x', x^{-1} \in K_v^\times \times C$. We now we claim that there is a compact C_0 such that $K + C_0 = \mathbb{A}_K$ and we are done. □

Proof.[of theorem] By the cuspidality assumption, we find that $\int_{U(K)\backslash U(\mathbb{A}_K)} f(ux)du = 0$. Observe that $U(\mathbb{A}_K) \simeq \mathbb{A}_K$. There is a compact subgroup $C \subset \mathbb{A}_K$ such that C surjects onto $\mathbb{A}_K/K$. Therefore, $\int_C f(ug)du = 0$.

We apply this to an element $g = \begin{pmatrix} x' & y \\ 0 & x \end{pmatrix}$.

As we have that

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' & y \\ 0 & x \end{pmatrix} = \begin{pmatrix} x' & y \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & x(x')^{-1}u \\ 0 & 1 \end{pmatrix}$$

We therefore deduce that if for all $w \in X$, with $|(x(x')^{-1})_w|_w \leq 1$, then $\int_C f(ug)du = vol(C)f(g) = 0$.

We now take a Siegel set $\mathcal{S}_{C^v,C_0}$ as in lemma 5.2. We deduce that $f|_{\mathcal{S}_{C^v,C_0}}$ is supported on $\mathcal{S}^{\leq 1}_{C^v,C_0}$ which is compact modulo the center.

It follows that any $f \in \mathcal{C}_{cusp}(\mathrm{GL}_2, \chi)$ is dermined by its restriction to a set of representatives of $\mathcal{S}^{\leq 1}_{C^v,C_0} / \prod_x \mathrm{GL}_2(\mathcal{O}_x) Z(\mathbb{A}_K)$ and this set is indeed finite !

□

It is harmless to assume that χ is a finite order because we can always twist the space $\mathcal{C}_{cusp}(GL_n, \chi)$ by a function of the form $\Psi \circ deg \circ \det$ for a character $\Psi : \mathbb{Z} \to \overline{\mathbb{Q}}^\times$. We will make this assumption.

5.2. **The spherical Hecke algebra.**

5.2.1. *The Satake isomorphism.* For all $x \in X$ we let $\mathcal{H}_x$ be the algebra of functions on $f : \mathrm{GL}_n(K_x) \to \overline{\mathbb{Q}}$ with compact support and which are left and right $\mathrm{GL}_n(\mathcal{O}_x)$-invariant, equipped with the convolution product :

$$f \star g(h) = \int_{\mathrm{GL}_n(\mathcal{O}_x)} f(h) g(h^{-1}t) dh.$$

We let T be the maximal diagonal torus of GL_n. We let $X_\star(T)$ be the group of cocharacters. Each such cocharacter is of the form $t \mapsto \mathrm{diag}(t^{k_1}, \cdots, t^{k_n})$ for $(k_1, \cdots, k_n) \in \mathbb{Z}^n$. We say that a cocharacter is dominant if $k_1 \geq \cdots \geq k_n$ and we denote by $X_\star(\mathrm{T})$ the cone of dominant cocharacters.

By the Cartan decomposition, $\mathrm{GL}_n(K_x) = \coprod_{\lambda \in X_\star(T)^+} \mathrm{GL}_n(\mathcal{O}_x)\lambda(t_x)\mathrm{GL}_n(\mathcal{O}_x)$ and therefore the characteristic functions $T_{\lambda,x} = 1_{\mathrm{GL}_n(\mathcal{O}_x)\lambda(t_x)\mathrm{GL}_n(\mathcal{O}_x)}$ form a basis of the $\overline{\mathbb{Q}}$-vector space $\mathcal{H}_x$.

For all $1 \leq i \leq n$, we let λ_i be the cocharacter with coefficient $(1, \cdots, 1, 0, \cdots, 0)$ with i many 1 and we let $T_{i,x} = T_{\lambda_i,x}$.

Theorem 5.2 (Satake isomorphism)**.** *The algebra $\mathcal{H}_x$ is commutative, isomorphic to $\overline{\mathbb{Q}}[T_{1,x}, \cdots, T_{n,x}, T_{n,x}^{-1}]$.*

The proof of this theorem relies on the Satake transform (see [Gro98] for example) :

$$\begin{array}{rcl} \mathcal{H}_x & \to & \overline{\mathbb{Q}}[X_\star(T)]^{\mathcal{S}_n} \\ f & \to & [t \mapsto \delta(t)^{\frac{1}{2}} \int_{U(K_x)} f(tu)du] \end{array}$$

Let $\mathrm{GL}_n(\overline{\mathbb{Q}})^{ss}/conj$ be the set of semi-simple conjugacy classes in $\mathrm{GL}_n(\overline{\mathbb{Q}})$. This set is in bijection with the set of unitary degree n polynomials via the characteristic polynomial function $M \mapsto \det(XId - M)$.

We now define a bijection

$$\mathrm{Spec}(\mathcal{H}_x)(\overline{\mathbb{Q}}) = \mathrm{GL}_n(\overline{\mathbb{Q}})^{ss}/conj$$

by associating to an homorphism $\Theta : \mathcal{H}_x \to \overline{\mathbb{Q}}$ the semi-simple conjugacy class corresponding to the characteristic polynomial $X^n - \Theta(T_{1,x})X^{n-1} + \cdots + (-1)^n\Theta(T_{n,x})$.

5.2.2. *Action of the spherical Hecke algebra.* The spherical Hecke algebra $\mathcal{H}_x$ acts on $\mathcal{C}_{cusp}(GL_n, \chi)$ by convolution. Namely we let

$$h.f(g) = \int_{\mathrm{GL}_n(K_x)} h(u)f(gu)\mathrm{d}u.$$

5.3. **The Langlands correspondence.** The global Hecke algebra $\mathcal{H} = \otimes'_{x\in X}\mathcal{H}_x$ acts on $\mathcal{C}_{cusp}(GL_n, \chi)$ by convolution.

Theorem 5.3 ([Dd80], [Laf02])**.**
(1) *The space $\mathcal{C}_{cusp}(GL_n, \chi)$ has a spectral decomposition into one dimensional eigenspaces for the action of $\mathcal{H}$: there are finitely many distinct homomorphisms $\Theta_1, \cdots, \Theta_r : \mathcal{H} \to \overline{\mathbb{Q}}$ such that $\mathcal{C}_{cusp}(GL_n, \chi) = \oplus_{i=1}^r \mathcal{C}_{cusp}(GL_n, \chi)[\Theta_i]$ and $\dim_{\overline{\mathbb{Q}}} \mathcal{C}_{cusp}(GL_n, \chi)[\Theta_i] = 1$.*
(2) *Let $\ell \neq p$ and fix an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$. To any Θ_i we can attach an irreducible representation :*
$$\rho_i : \pi_1(X, \overline{x}) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$
which satisfies that $det(XId - \rho_i(Frob_x)) = X^n - \Theta_i(T_{1,x})X^{n-1} + \cdots + (-1)^n\Theta_i(T_{n,x})$.
(3) *The map $\Theta_i \mapsto \rho_i$ is a bijection between the set $\{\Theta_1, \cdots, \Theta_r\}$ and the set of isomorphism classes of irreducible representations $\rho : \pi_1(X, \overline{x}) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ such that $\det \rho$ corresponds to χ via class field theory.*

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