BIFURCATION ANALYSIS OF A SINGULAR NON-LINEAR STURM-LIOUVILLE EQUATION

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ABSTRACT. In this paper we study existence of positive solutions to the following singular non-linear Sturm-Liouville equation

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + u^p \text{ in } (0,1), \\ u(1) = 0, \end{cases}$$

where $\alpha > 0$, $p > 1$ and λ are real constants.

We prove that when $0 < \alpha \leq \frac{1}{2}$ and $p > 1$ or when $\frac{1}{2} < \alpha < 1$ and $1 < p \leq \frac{3-2\alpha}{2\alpha-1}$, there exists a branch of continuous positive solutions bifurcating to the left of the first eigenvalue of the operator $\mathcal{L}_\alpha u = -(x^{2\alpha}u')'$ under the boundary condition $\lim_{x\to 0} x^{2\alpha}u'(x) = 0$. The projection of this branch onto its λ component is unbounded in two cases: when $0 < \alpha \leq \frac{1}{2}$ and $p > 1$, and when $\frac{1}{2} < \alpha < 1$ and $p < \frac{3-2\alpha}{2\alpha-1}$. On the other hand, when $\frac{1}{2} < \alpha < 1$ and $p \geq \frac{3-2\alpha}{2\alpha-1}$, the projection of the branch has a positive lower bound below which no positive solution exists.

When $0 < \alpha < \frac{1}{2}$ and $p > 1$, we show that a second branch of continuous positive solution can be found to the left of the first eigenvalue of the operator $\mathcal{L}_\alpha$ under the boundary condition $\lim_{x\to 0} u(x) = 0$.

Finally, when $\alpha \geq 1$, the operator $\mathcal{L}_\alpha$ has no eigenvalues under its canonical boundary condition at the origin, and we prove that in fact there are no positive solutions to the equation, regardless of $\lambda \in \mathbb{R}$ and $p > 1$.

1. Introduction

We are interested in the problem of existence of a function u satisfying the non-linear singular Sturm-Liouville equation

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + u^p & \text{in } (0,1), \\ u > 0 & \text{in } (0,1), \\ u(1) = 0, \end{cases} \tag{1}$$

where $\alpha > 0$, $p > 1$ and $\lambda \in \mathbb{R}$ are parameters. A motivation for studying equation (1) is presented in detail in section 1.4. Also, we would like to mention that a more general general class of singular Sturm-Liouville equations have been studied using different techniques in the past, and we refer the interested reader to the following papers [17, 27, 28, 30–32] for further reading.

In our work, it is important to remark that by a solution to equation (1) we will mean a function u belonging to $C^2(0,1]$ which solves equation (1). This will become

Date: November 11, 2013.

2010 *Mathematics Subject Classification.* 34B16, 34B18, 34C23.

Key words and phrases. singular Sturm-Liouville equation, bifurcation analysis, positive solutions.

specially relevant when proving non-existence results, as no *a priori* assumption about the behavior of u near the origin is being made.

As it will be seen later, it is convenient to divide the exposition of our results into the following five cases:

(A) $0 < \alpha < \frac{1}{2}$ and $p > 1$,
(B) $\frac{1}{2} \leq \alpha < 1$ and[1] $1 < p < \frac{3-2\alpha}{2\alpha-1}$,
(C) $\frac{1}{2} < \alpha < 1$ and $p = \frac{3-2\alpha}{2\alpha-1}$,
(D) $\frac{1}{2} < \alpha < 1$ and $p > \frac{3-2\alpha}{2\alpha-1}$, and
(E) $\alpha \geq 1$ and $p > 1$.

The exponent

$$2_\alpha := \frac{3-2\alpha}{2\alpha-1} + 1 = \frac{2}{2\alpha-1} \tag{2}$$

plays an important role, as it is critical in the sense that the weighted Sobolev space (introduced in [22])

$$X_0^\alpha := X_0^{\alpha,2}(0,1) = \left\{ u \in H^1_{loc}(0,1) \ : \ u, \ x^\alpha u' \in L^2(0,1), \ u(1) = 0 \right\}$$

is embedded into $L^q(0,1)$ if and only if $q \leq 2_\alpha$ (this follows from the Caffarelli-Kohn-Nirenberg (CKN) inequality [15]; see also [22, Appendix] for the treatment of this particular case).

When dealing with cases (A), (B) and (C) our approach to prove existence results for equation (1) will be to minimize the energy functional

$$I_{\lambda,\alpha}(u) := \int_0^1 \left| x^\alpha u'(x) \right|^2 dx - \lambda \int_0^1 \left| u(x) \right|^2 dx \tag{3}$$

over the manifold

$$\mathcal{M} := \mathcal{M}_{\alpha,p} = X_0^\alpha \cap \left\{ u \in L^{p+1}(0,1) : \|u\|_{p+1} = 1 \right\}.$$

The solutions obtained by this method turn out to be bounded solutions and they bifurcate to the left of the first eigenvalue of the linear problem

$$\begin{cases} -(x^{2\alpha}\varphi')' = \lambda\varphi \quad \text{in } (0,1), \\ \varphi(1) = 0, \\ \lim_{x\to 0^+} x^{2\alpha}\varphi'(x) = 0. \end{cases} \tag{4}$$

We refer the reader to [22, Theorem 1.17] for a complete analysis of the spectrum of the linear operator

$$\mathcal{L}_\alpha \varphi := -(x^{2\alpha}\varphi')',$$

but in particular, the first eigenvalue of equation (4), hereafter denoted by λ_1, can be characterized by

$$\lambda_1 := \inf_{\varphi \in X_0^\alpha} \frac{\int_0^1 \left| x^\alpha \varphi'(x) \right|^2 dx}{\int_0^1 \left| \varphi(x) \right|^2 dx} = \frac{\int_0^1 \left| x^\alpha \varphi_1'(x) \right|^2 dx}{\int_0^1 \left| \varphi_1(x) \right|^2 dx}. \tag{5}$$

Further details about λ_1 and φ_1 will be given later in section 2.

The above is in sharp contrast with the case $\alpha \geq 1$, as the operator $\mathcal{L}_\alpha$ has only essential spectrum (no eigenvalues) and bifurcation becomes a delicate issue, in fact, we prove that no positive solutions exist in this case.

[1]When $\alpha = \frac{1}{2}$ we are abusing the notation and consider that $\frac{3-2\alpha}{2\alpha-1} = +\infty$.

1.1. **The case $0 < \alpha < \frac{1}{2}$ and $p > 1$.** In this case the embedding $X_0^\alpha \hookrightarrow L^{p+1}(0,1)$ is compact for all $p > 1$, hence a standard variational method allows us to prove the existence of a minimizer for $I_{\lambda,\alpha}$ in $\mathcal{M}$ and as a consequence the following

Theorem 1 (Existence for the Neumann problem)**.** *Suppose $0 < \alpha < \frac{1}{2}$ and $p > 1$, then for every $\lambda < \lambda_1$ there exists a solution u_N to equation* (1) *satisfying the following properties:*

(i) $u_N \in C[0,1]$*, with* $u_N(0) > 0$*,*
(ii) $x^{2\alpha-1}u_N' \in C[0,1]$*, in particular* $u_N \in C^1[0,1]$ *and* $u_N'(0) = 0$*,*
(iii) $x^{2\alpha}u_N'' \in C[0,1]$*.*

Remark 1.1. The solution u_N to equation (1) obtained in Theorem 1 turns out to be the *unique* solution of equation 1 satisfying $x^{2\alpha-1}u' \in C[0,1]$. This, together with other uniqueness results related to equation (1), will be the content of the forthcoming paper [21].

As we mentioned earlier, bifurcation only occurs to the left of λ_1, and this is the content of the following

Theorem 2 (Non-existence for the Neumann problem)**.** *Suppose $0 < \alpha < \frac{1}{2}$, $p > 1$ and that $\lambda \geq \lambda_1$. Then equation* (1) *has no solution satisfying* $\lim\limits_{x\to 0^+} x^{2\alpha}u'(x) \leq 0$*.*

Observe that the above non-existence theorem requires the additional assumption $\lim\limits_{x\to 0^+} x^{2\alpha}u'(x) \leq 0$. The reason behind this extra assumption comes from the fact that equation (1) does have (bounded) solutions satisfying $\lim\limits_{x\to 0^+} x^{2\alpha}u'(x) > 0$ for some $\lambda \geq \lambda_1$. This phenomenon occurs because, when $0 < \alpha < \frac{1}{2}$, one can minimize the energy functional $I_{\alpha,\lambda}$ over $\mathcal{M}_0$, the sub-manifold of $\mathcal{M}$ defined by

$$\mathcal{M}_0 := \mathcal{M}_{\alpha,p,0} = X_{00}^\alpha \cap \left\{ u \in L^{p+1}(0,1) : \|u\|_{p+1} = 1 \right\},$$

where $X_{00}^\alpha := \{u \in X_0^\alpha : u(0) = 0\}$ is a well defined (closed) subspace of X_0^α for each $0 < \alpha < \frac{1}{2}$ (see [22, Appendix] for further details about this space). This allows us to prove a second existence theorem: For $0 < \alpha < \frac{1}{2}$, let $\lambda_{1,0}$ be the first eigenvalue of

$$\begin{cases} -(x^{2\alpha}\varphi')' = \lambda\varphi & \text{in } (0,1), \\ \varphi(1) = 0, \\ \lim\limits_{x\to 0^+} \varphi(x) = 0, \end{cases} \tag{6}$$

which can be characterized by

$$\lambda_{1,0} := \inf_{\varphi \in X_{00}^\alpha} \frac{\int_0^1 |x^\alpha \varphi'(x)|^2\, dx}{\int_0^1 |\varphi(x)|^2\, dx} = \frac{\int_0^1 \left|x^\alpha \varphi_{1,0}'(x)\right|^2 dx}{\int_0^1 |\varphi_{1,0}(x)|^2\, dx}. \tag{7}$$

We have the following

Theorem 3 (Existence for the Dirichlet problem)**.** *Suppose $0 < \alpha < \frac{1}{2}$ and $p > 1$, then for every $\lambda < \lambda_{1,0}$ there exists a solution u_D to equation* (1) *satisfying the following properties:*

(i) $u_D \in C[0,1]$*, with* $u_D(0) = 0$*,*
(ii) $x^{2\alpha-1}u_D \in C[0,1]$*, and*
(iii) $x^{2\alpha}u_D' \in C^1[0,1]$*.*

Remark 1.2. The solution u_D from Theorem (3) is the unique solution of equation (1) satisfying $u(0) = 0$ (see [21]).

Remark 1.3. Observe that property (iii) in Theorem 3 above only says that $x^{2\alpha}u'$ belongs to $C^1[0,1]$. This *does not* mean that each term in

$$(x^{2\alpha}u'(x))' = x^{2\alpha}u''(x) + 2\alpha x^{2\alpha-1}u'(x)$$

is continuous in $[0,1]$. This can be seen even for the linear equation (6), as for the eigenfunction $\varphi_{1,0}$ one has that $x^{2\alpha-1}\varphi_{1,0}'(x) \sim x^{-1}$ and $x^{2\alpha}\varphi_{1,0}''(x) \sim x^{-1}$ near the origin, but due to some cancellation of the non-integrable term, one can obtain that $x^{2\alpha}\varphi_{1,0}' \in C^1[0,1]$.

Remark 1.4. It turns out that $\lambda_{1,0} > \lambda_1$ for all $0 < \alpha < \frac{1}{2}$. This implies that when $\lambda < \lambda_1$ we have the existence of *at least* two distinct continuous solutions to equation (1): one satisfying $u(0) > 0$ - the solution given by Theorem 1 - and another solution satisfying $u(0) = 0$ - the solution given by Theorem 3 (see Figure 2 below). However, these two solutions can be considered as part of a *continuum* of bounded solutions to equation (1). See section 1.5 for further comments.

As a counterpart we have the following non-existence result, which does not require any assumptions on the behavior of the solution near the origin.

Theorem 4. *Suppose* $0 < \alpha < \frac{1}{2}$, $p > 1$ *and that* $\lambda \geq \lambda_{1,0}$. *Then equation* (1) *has no positive solution.*

1.2. **The case** $\frac{1}{2} \leq \alpha < 1$**.** As explained earlier, in this range of α's the embedding $X_0^\alpha \hookrightarrow L^{p+1}(0,1)$ is compact if and only if $1 < p < \frac{3-2\alpha}{2\alpha-1}$, so it is convenient to divide the results into three cases:

- ⋄ $1 < p < \frac{3-2\alpha}{2\alpha-1}$,
- ⋄ $p = \frac{3-2\alpha}{2\alpha-1}$ and
- ⋄ $p > \frac{3-2\alpha}{2\alpha-1}$.

1.2.1. *The sub-critical case* $1 < p < \frac{3-2\alpha}{2\alpha-1}$. The embedding $X_0^\alpha \hookrightarrow L^{p+1}(0,1)$ is compact, so we can use a standard variational method to prove

Theorem 5 (Existence for the sub-critical "Canonical" problem)**.** *Suppose* $\frac{1}{2} \leq \alpha < 1$ *and* $1 < p < \frac{3-2\alpha}{2\alpha-1}$, *then for all* $\lambda < \lambda_1$ *there exists a solution* u_C *to equation* (1) *satisfying the following properties:*

(i) $u_C \in C[0,1]$, *with* $u_C(0) > 0$,

(ii) $x^{2\alpha-1}u_C' \in C[0,1]$, *in particular* $\lim_{x\to 0^+} x^{2\alpha}u_C'(x) = 0$, *and*

(iii) $x^{2\alpha}u_C'' \in C[0,1]$.

Remark 1.5. This solution is the unique solution to equation (1) satisfying the regularity condition $x^{2\alpha-1}u_C' \in C[0,1]$ (see [21]).

Bifurcation also occurs to the left of λ_1 in this case, and this is proved in the following

Theorem 6. *Suppose* $\frac{1}{2} \leq \alpha < 1$, $p > 1$ *and that* $\lambda \geq \lambda_1$. *Then equation* (1) *has no solution.*

Remark 1.6. Unlike Theorem 2, no *a priori* behavior of u near the origin is required in the above result. The reason behind this is that when $\alpha \geq \frac{1}{2}$ one can show that all $C^2(0,1]$-solutions of equation (1) satisfy $\lim_{x\to 0^+} x^{2\alpha}u'(x) \leq 0$ (see Corollary 2.8).

1.2.2. *The critical case* $p = \frac{3-2\alpha}{2\alpha-1}$. In order to prove existence in this case, we still look for minimizers of $I_{\lambda,\alpha}$ over the manifold $\mathcal{M}$. The difficulty in doing so comes from the fact that $X_0^\alpha \hookrightarrow L^{2_\alpha}(0,1)$ is *not* compact and as a consequence the standard variational approach does not work. To overcome this issue, we will follow the approach taken by Brezis and Nirenberg in [10] and we will show that it is enough to prove that for suitable λ's

$$\inf_{\mathcal{M}} I_{\lambda,\alpha} < \inf_{\mathcal{M}} I_{0,\alpha}. \tag{8}$$

To do so, notice that

$$S_\alpha := \inf_{\mathcal{M}} I_{0,\alpha} \tag{9}$$

corresponds to the best constant in the CKN inequality

$$S_\alpha \|u\|^2_{L^{2_\alpha}(0,1)} \le \|x^\alpha u'\|^2_{L^2(0,1)}.$$

The key ingredient in proving (8) is to evaluate $I_{\lambda,\alpha}$ at functions of the form

$$u_\varepsilon(x) = \phi(x)U_\varepsilon(x),$$

where ϕ is a suitable chosen cut-off function and

$$U_\varepsilon(x) = \left(\varepsilon + x^{2-2\alpha}\right)^{\frac{1-2\alpha}{2-2\alpha}}$$

corresponds to the basic extremal profile for

$$S_\alpha \|U\|^2_{L^{2_\alpha}(0,\infty)} \le \|x^\alpha U'\|^2_{L^2(0,\infty)}.$$

More details about S_α and its extremal functions will be given in section 2 below.

Theorem 7 (Existence for the critical "Canonical" problem)**.** *Suppose* $\frac{1}{2} < \alpha < 1$ *and that* $p = \frac{3-2\alpha}{2\alpha-1}$. *Then there exists* $\Lambda^*_\alpha \in [0,\lambda_1)$, *such that if* $\lambda \in (\Lambda^*_\alpha, \lambda_1)$, *then equation* (1) *has a solution* u_C *satisfying:*

(i) $u_C \in C[0,1]$, *with* $u_C(0) > 0$,
(ii) $x^{2\alpha-1}u'_C \in C[0,1]$, *in particular* $\lim_{x\to 0^+} x^{2\alpha}u'_C(x) = 0$, *and*
(iii) $x^{2\alpha}u''_C \in C[0,1]$.

Remark 1.7. The solution given by Theorem 7 is the unique solution to equation (1) satisfying $x^{2\alpha-1}u' \in C[0,1]$ (see [21]).

Remark 1.8. The number Λ^*_α can be defined by

$$\Lambda^*_\alpha := \begin{cases} \lambda^*_\alpha & \text{if } \frac{1}{2} < \alpha < \frac{3}{4}, \\ 0 & \text{if } \frac{3}{4} \le \alpha < 1, \end{cases}$$

where $\lambda^*_\alpha > 0$ is a continuous function of α for all $\frac{1}{2} < \alpha < \frac{3}{4}$, and λ^*_α can be explicitly computed by

$$\lambda^*_\alpha := \inf_{\psi \in X_0^{1-\alpha}} \frac{\int_0^1 \left|x^{1-\alpha}\psi'(x)\right|^2 dx}{\int_0^1 \left|x^{1-2\alpha}\psi(x)\right|^2 dx} = \frac{\int_0^1 \left|x^{1-\alpha}\psi'_\alpha(x)\right|^2 dx}{\int_0^1 \left|x^{1-2\alpha}\psi_\alpha(x)\right|^2 dx}. \tag{10}$$

We will show that $\lambda^*_\alpha \underset{\alpha\to\frac{3}{4}^-}{\longrightarrow} 0$ thus making Λ^*_α a continuous function of α, and that $|\Lambda^*_\alpha - \lambda_1| \underset{\alpha\to\frac{1}{2}^+}{\longrightarrow} 0$ (see Figure 1 below). Further properties of λ^*_α and ψ_α will be given later in section 2.

On the other hand, we have the following non existence result

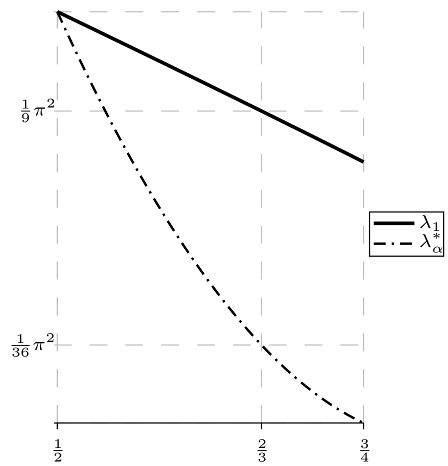


FIGURE 1. λ_1 and λ^*_α when $\frac{1}{2} < \alpha < \frac{3}{4}$.

Theorem 8. *Suppose* $\frac{1}{2} < \alpha < 1$, $p = \frac{3-2\alpha}{2\alpha-1}$ *and that* $\lambda \le \Lambda^*_\alpha$. *Then equation* (1) *has no solution.*

1.2.3. *The super-critical case* $p > \frac{3-2\alpha}{2\alpha-1}$. In this case, we can no longer use the previous approach to prove existence of positive solutions. The reason is that the space X^α_0 is not even embedded into $L^{p+1}(0,1)$. Instead, we have available the global bifurcation result of Rabinowitz [45, Theorem 1.3] which tells us that there exists a branch of bounded positive solutions (λ, u) emanating from $(\lambda_1, 0)$ and going to infinity in $\mathbb{R} \times C[0,1]$, but no further information is obtained from this abstract result of Rabinowitz.

One thing that can be easily seen is that the branch emanating from λ_1 must be bounded below in its λ component, and this is the content of the following

Theorem 9. *Suppose* $\frac{1}{2} < \alpha < 1$ *and that* $p > \frac{3-2\alpha}{2\alpha-1}$. *Suppose* $\lambda \le \bar{\lambda}_{\alpha,p}$, *where*

$$\bar{\lambda}_{\alpha,p} := \lambda_1 \left(\frac{\alpha - \frac{1}{2} - \frac{1}{p+1}}{\frac{1}{2} - \frac{1}{p+1}} \right),$$

then equation (1) *has no solution.*

Remark 1.9. If one defines a *regular solution* as a function u solution to equation (1) so that both u and $x^{2\alpha-1}u'$ belong to $C[0,1]$, and let

$$\hat{\lambda}_{\alpha,p} := \inf \{\lambda > 0 : \text{Equation (1) has a regular solution}\},$$

then Theorem 9 shows that $\bar{\lambda}_{\alpha,p} \le \hat{\lambda}_{\alpha,p}$. However, numerical computations indicate two things: that the inequality is strict, i.e., $\bar{\lambda}_{\alpha,p} < \hat{\lambda}_{\alpha,p}$ and that for every $\hat{\lambda}_{\alpha,p} \le \lambda < \lambda_1$ at least one solution to equation (1) exists (see figures 5 and 6 below). This leads us to raise

Question 1. Is it true that for $\hat{\lambda}_{\alpha,p}$ one has that for each $\hat{\lambda}_{\alpha,p} \le \lambda < \lambda_1$ there exists a regular solution u_λ to equation (1)? More precisely, we believe that for $\lambda = \hat{\lambda}_{\alpha,p}$ a *unique* regular solution exists, and that there exists $\varepsilon > 0$ small enough such that for $\hat{\lambda}_{\alpha,p} < \lambda \le \hat{\lambda}_{\alpha,p} + \varepsilon$ *exactly two* regular solutions exist.

Question 2. If one defines

$$\hat{\hat{\lambda}}_{\alpha,p} := \inf\left\{\lambda > 0 : \text{Equation (1) has a solution}\right\},$$

that is, we drop the regularity assumption, we have that by definition $\hat{\hat{\lambda}}_{\alpha,p} \leq \hat{\lambda}_{\alpha,p}$. A natural question is to determine whether the inequality is strict or not.

1.3. **The case $\alpha \geq 1$.** Before presenting the main result for this case, it is important to emphasize the distinction between $\alpha < 1$ and $\alpha \geq 1$. As seen in [22], the main difference that can be observed between these two cases is that the spectrum of the linear operator $\mathcal{L}_\alpha$ under the homogeneous boundary conditions given in equation (4) consists only of isolated eigenvalues when $\alpha < 1$, but, because of the lack of compactness of the operator $T_\alpha := (\mathcal{L}_\alpha)^{-1}$, the spectrum becomes a continuum when $\alpha \geq 1$, in fact, the spectrum has no eigenvalues in this situation.

As we have established, the solutions obtained when $0 < \alpha < 1$ are solutions that bifurcate from the first eigenvalue of the operator $\mathcal{L}_\alpha$. This phenomenon is in concordance with results about global bifurcation from isolated points in the spectrum (see for instance [26, 45]). However, when $\alpha \geq 1$, the spectrum of $\mathcal{L}_\alpha$ is purely essential and has no isolated points:

- ⋄ $\sigma(\mathcal{L}_1) = \sigma_e(\mathcal{L}_1) = \left[\frac{1}{4}, \infty\right)$, and
- ⋄ $\sigma(\mathcal{L}_\alpha) = \sigma_e(\mathcal{L}_\alpha) = [0, \infty)$ for $\alpha > 1$,

hence, the results mentioned above do not apply.

Besides the lack of compactness and the lack of isolated eigenvalues of the operator $\mathcal{L}_\alpha$, one has that for every $p > 1$ we are dealing with what can be considered a super-critical equation. All these conditions seem to be very restrictive and as a result we obtain that there are no positive solutions, as the following theorem shows.

Theorem 10 (Non-existence when $\alpha \geq 1$)**.** *Let $\alpha \geq 1$, $p > 1$ and λ be real constants, then equation* (1) *has no solution.*

Remark 1.10. In fact one can prove a stronger result. Indeed, our proof of Theorem 10 allows us to show that the equation

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + |u|^{p-1}u & \text{in } (0,1), \\ u(1) = 0, \\ u > 0 \text{ near } 0, \end{cases}$$

has no solution for any $\alpha \geq 1$, $\lambda \in \mathbb{R}$ and $p > 1$. See section 1.5 for further comments about this.

It is worth mentioning that when $\alpha = 1$, Theorem 10 is in sharp contrast with the work done by Berestycki and Esteban in [8]. In that article, the authors study the model equation

$$\begin{cases} -x^2u''(x) = \lambda u + u^p & \text{in } (0,1), \\ u > 0 & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

which can be regarded as a simplified version of the Wheeler-DeWitt equation. In [8], the authors prove, among other things, that the above equation has uncountably many solutions when $0 < \lambda < \frac{1}{4}$. Their result put alongside Theorem 10 shows that the first order term $-2xu'(x)$ plays a crucial role in the existence question.

Even though we did not use general tools from bifurcation theory, it is important to remark that bifurcation from the essential spectrum is a topic that has been greatly studied in the past. One of the founders of the research in this area is C. Stuart who started studying such phenomenon in the '70s. The interested reader might want to check the nice papers written by Stuart himself [47, 48] and the references therein. We also refer to the series of papers published by Stuart and Vuillaume [49–51] where bifurcation from the essential spectrum of a non-linear Sturm-Liouville equation is studied.

1.4. **Connection with an elliptic equation in the ball.** The results from Theorems 5 and 7 suggest that equation (1) is closely related to the elliptic equation

$$\begin{cases} -\Delta v = \lambda v + v^p & \text{in } B(0,R) \subset \mathbb{R}^N, \\ v > 0 & \text{in } B(0,R), \\ v = 0 & \text{on } \partial B(0,R), \end{cases} \tag{11}$$

where $\lambda \in \mathbb{R}$, $p > 1$, $R > 0$ and $B(0,R)$ denotes the ball centered at the origin with radius R. In their celebrated work [10], Brezis and Nirenberg proved, among other things, that for the critical exponent $p = \frac{N+2}{N-2}$, the dimension plays an important role in the existence/non-existence question. They showed that when $N \geq 4$ a solution to equation (11) is guaranteed to exist if and only if[2] $0 < \lambda < \lambda_1(-\Delta)$; but when $N = 3$, they proved that existence only occurs if $\lambda^* < \lambda < \lambda_1(-\Delta)$, where $\lambda^* = \frac{1}{4}\lambda_1(-\Delta) > 0$.

The phenomenon described above is exactly the same as the one occurring for equation (1) when $p = \frac{3-2\alpha}{2\alpha-1}$, as if $\frac{3}{4} < \alpha < 1$, existence only occurs when $0 < \lambda < \lambda_1$, and if $\frac{1}{2} < \alpha < \frac{3}{4}$, solutions only exist when $\lambda^*_\alpha < \lambda < \lambda_1$, with $\lambda^*_\alpha > 0$. An explanation for this connection can be seen by means of a change of variables. Recall that by the result of Gidas, Ni and Nirenberg [35], all solutions to equation (11) are radially symmetric, hence $v(r) = v(|x|)$ satisfies the ODE

$$\begin{cases} -v'' - \dfrac{N-1}{r} v' = \lambda v + v^p & \text{in } (0,R), \\ v > 0 & \text{in } (0,R), \\ v(R) = 0. \end{cases} \tag{12}$$

Now, for $0 < \alpha < 1$, let u be a solution to equation (1) and consider $r = (1-\alpha)^{-1} x^{1-\alpha}$. Define $w(r) = u(x)$, then a direct computation shows that w is a solution to

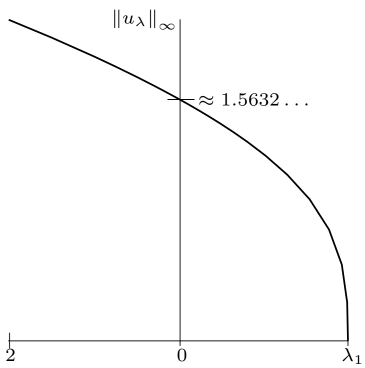
$$\begin{cases} -w'' - \dfrac{N_\alpha - 1}{r} w' = \lambda w + w^p & \text{in } (0,R_\alpha), \\ w > 0 & \text{in } (0,R_\alpha), \\ w(R_\alpha) = 0, \end{cases} \tag{13}$$

where $N_\alpha = (1-\alpha)^{-1}$ and $R_\alpha = (1-\alpha)^{-1}$. Hence, when N_α is an integer (that is when $\alpha = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$) the ODE satisfied by w is exactly equation (12).

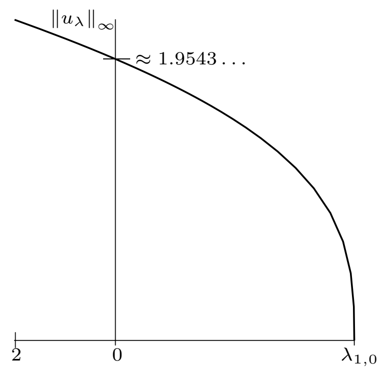
The literature about equation (12) is extensive. For instance, regarding the existence of solutions to equation (11) in the sub-critical case ($p > 1$ when $N = 1, 2$ and $1 < p < \frac{N+2}{N-2}$ when $N \geq 3$), we can mention the works of Berestycki [7],

[2]The number $\lambda_1(-\Delta)$ denotes the first eigenvalue of $-\Delta$ in $B(0,R)$ under Dirichlet boundary condition.

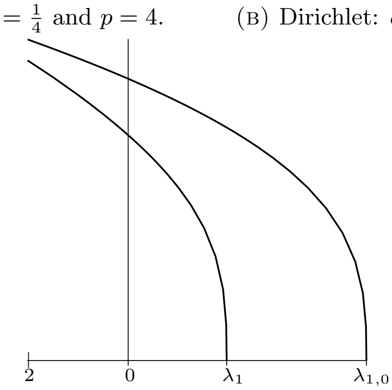
Castro and Lazer [20], de Figueiredo, Lions and Nussbaum [29], Esteban [34] and Lions [36] among others. Most of these results are quite general as they apply to general bounded domains and a large class of non-linearities with sub-critical growth. However, it is apparent to us that the case of non-integer dimension for equation (12) has not been covered in the literature, and the results from Theorems 1, 3 and 5 seem to close that gap in this case. In particular, when $1 < N < 2$ we have the existence of *at least* two bounded solutions satisfying equation (12), one of them satisfies $v(0) > 0$ and $v'(r) \sim r$ for $r \sim 0$ and the other satisfies $v(0) = 0$ and $v'(r) \sim r^{1-N}$ for $r \sim 0$: notice that since $1 < N < 2$, this second solution has a singular derivative at 0 (see figures 2 and 3).



(A) Neumann: $\alpha = \frac{1}{4}$ and $p = 4$.



(B) Dirichlet: $\alpha = \frac{1}{4}$ and $p = 4$.



(C) Figures 2a and 2b together.

FIGURE 2. Bifurcation diagrams when $0 < \alpha < \frac{1}{2}$ and $p < \frac{3-2\alpha}{2\alpha-1}$.

For the critical case, $N \geq 3$ and $p = \frac{N+2}{N-2}$, the behavior of the branch of solutions emanating from $\lambda_1(-\Delta)$ has been fully understood in the case of the ball. We have already mentioned the result of Brezis and Nirenberg [10], and the interested reader might want to check the works of Atkinson and Peletier [1, 2], Bandle and Benguria [3], Bandle and Peletier [4], Benguria, Frank and Loss [6], Brezis and Peletier [11, 12], Cao and Li [18], Capozzi, Fortunato and Palmieri [19], Cerami, Fortunato and Struwe [24] and Cerami, Solimini and Struwe [25], Mancini and Sandeep [37] for further reference on related problems. However, to our knowledge, the fact that the bifurcation picture when $N = 3$ is different from the case $N \geq 4$ has not been fully generalized to cover the case of non integer dimension N in

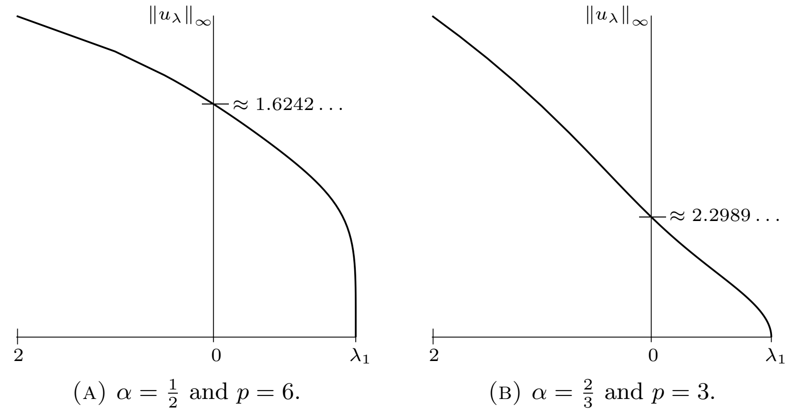


FIGURE 3. Bifurcation diagrams when $\frac{1}{2} \leq \alpha < 1$ and $p < \frac{3-2\alpha}{2\alpha-1}$.

equation (11). In [44], Pucci and Serrin suggest that the non-existence part of their result should hold for any dimension, but an improved version of the identity shown in [43] was required to support their claim; nonetheless, if one formally extends the identity from [43] to cover non-integer dimensions, the result that one obtains is not sharp. Theorem 7 provides a sharp answer to both the existence and non-existence questions in any dimension $N > 2$. In fact, our result implies that the sharp lower bound for which solutions to equation (11) exist is given by a continuous function $\lambda^* = \lambda^*(N)$ which is identically 0 for all $N \geq 4$, positive when $2 < N < 4$ and $|\lambda^*(N) - \lambda_1(-\Delta)| \to 0$ as $N \to 2^+$ (see figures 1 and 4).

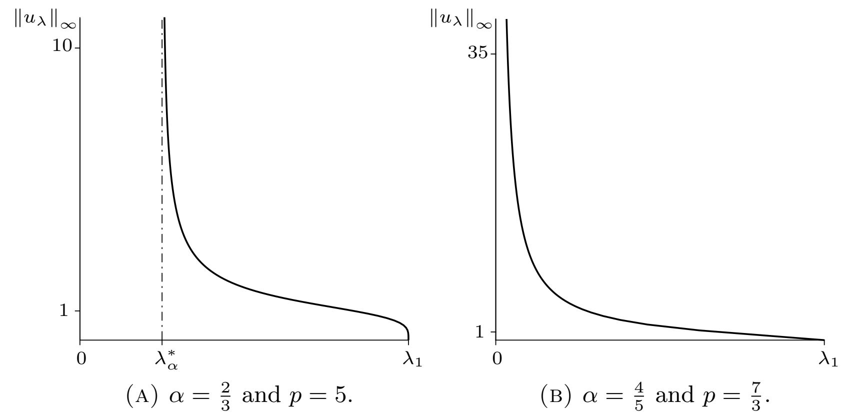
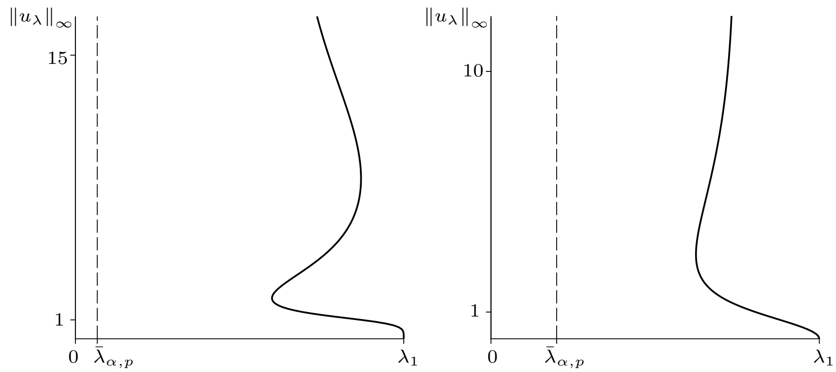


FIGURE 4. Bifurcation diagrams when $\frac{1}{2} < \alpha < 1$ and $p = \frac{3-2\alpha}{2\alpha-1}$.

For the super-critical case, $N \geq 3$ and $p > \frac{N+2}{N-2}$, Rabinowitz [46], Brezis and Nirenberg [10] and Pucci and Serrin [43] proved that there exists a constant $\bar{\lambda}_{N,p} > 0$ such that equation (12) has no solution when $\lambda \leq \bar{\lambda}_{N,p}$. Their proofs are general enough to work on any bounded domain Ω, but the case of a ball was not considered separately and as a consequence non-integer dimensions were not studied. To our knowledge this gap has not been closed, and Theorem 9 provides a proof of that, in fact, $\bar{\lambda}_{N,p} > 0$ is defined for all $N > 2$ and all p super-critical. However, as mentioned earlier, we strongly believe that his lower bound is not sharp (recall Question 1; see figure 5 below).



(A) $\alpha = \frac{2}{3}$ and $p = 6$. (B) $\alpha = \frac{4}{5}$ and $p = 3$.

FIGURE 5. Bifurcation diagrams when $\frac{1}{2} < \alpha < 1$ and $p > \frac{3-2\alpha}{2\alpha-1}$.

On the other hand in terms of the existence question, a complete understanding of the branch of solutions emanating from $\lambda_1(-\Delta)$ has not been fully developed in the super-critical case. Among the interesting results that can be found in the literature, it is worth mentioning the work of Budd and Norbury [13], who, for $N = 3$ and $p > 5$, describe the behavior of the branch for large values of $\|v\|_\infty$ and show that the branch oscillates about a unique value $\lambda^* > 0$, which is also the asymptotic value of the branch. They also characterize λ^* as the unique λ for which a singular H_0^1 solution to equation (12) exists ([13, Lemma 4.1]). Later, Merle and Peletier [38] showed that such $\lambda^* > 0$ can be found for every (not necessarily integer) dimension $N > 2$, and Peihao and Chengkui [41] fully generalized the result of Budd and Norbury for any dimension $2 < N \leq 6$ and only partially in the case $N > 6$. Other interesting results about the super-critical case can be found in the works of Budd and Peletier [14] and of Merle, Peletier and Serrin [39].

We would like to emphasize that our proofs do not rely in the change of variables introduced before, instead we work directly with equation (1). This approach allows us to study the cases $0 < \alpha < 1$ (or $N > 1$ if one thinks of equation (12)) all at once, and most importantly, it allows us to go beyond the $\alpha = 1$ barrier (notice that the change of variables does not work for $\alpha = 1$). When $\alpha > 1$ one could still use the change of variables, but the nature of equation (13) would change, as the coefficient $N_\alpha - 1$ becomes negative and the domain becomes the unbounded interval $(-\infty, R_\alpha)$. By avoiding the use of the change of variables we were able to prove that equation (1) has no solutions when $\alpha \geq 1$, regardless of λ and $p > 1$ with no major effort (Theorem 10). Also, by treating equation (1) directly, we shed some light into what might happen for more general degenerate elliptic operators in higher dimensions.

1.5. **Shooting for solutions and some questions.** For $\alpha > 0$, $p > 1$, $\lambda \in \mathbb{R}$ and $\theta < 0$, consider the "final" value problem

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + |u|^{p-1}u & \text{in } (0,1), \\ u(1) = 0, \\ u'(1) = \theta, \end{cases} \tag{14}$$

and denote by $u(x;\theta) = u(x;\alpha,\lambda,p,\theta)$ its unique solution, which is guaranteed to exists in a left neighborhood of 1. Moreover, it is not difficult to prove that such solution satisfies

$$|u(x;\theta)| \leq C(\theta,\alpha,p,\lambda)x^{-2\alpha} \quad \text{for all } 0 < x < 1, \tag{15}$$

and

$$|u'(x;\theta)| \leq C(\theta,\alpha,p,\lambda)x^{-1-2\alpha} \quad \text{for all } 0 < x < 1, \tag{16}$$

from which we deduce that blow up, if any, can only occur at the origin.

Given that, we would like to have a better understanding of the properties of $u(x;\theta)$, in particular, since we are interested in equation (1), we can consider the sets

$$\mathcal{S}_+ := \{\theta < 0 : u(x;\theta) > 0 \text{ for all } 0 < x < 1\},$$

and

$$\mathcal{S}_- := \{\theta < 0 : u(\bar{x};\theta) = 0 \text{ for some } 0 < \bar{x} < 1\}.$$

We have the following questions:

Question 3. Can we describe the sets $\mathcal{S}_+$ and $\mathcal{S}_-$? How do the parameters $\alpha > 0$, $p > 1$ and $\lambda \in \mathbb{R}$ affect their description?

Question 4. Suppose $\theta \in \mathcal{S}_+$. How does $u(x;\theta)$ behave near the origin?

Question 5. Suppose $\theta \in \mathcal{S}_-$. How do the parameters affect the number of zeros and the behavior near the origin of $u(x;\theta)$?

This questions have been partially answer in a paper of Benguria, Dolbeaut and Esteban [5], as one can consider the change of variables that transforms equation (1) into equation (13), then, after a suitable scaling, one could apply [5, Theorems 4.1, 4.2, 4.3] for the sub-critical case $1 < p < \frac{N+2}{N-2}$; [5, Theorem 3.4] for the critical case $p = \frac{N+2}{N-2}$; and [5, Theorem 5.3] for the super-critical case $p > \frac{N+2}{N-2}$. However, in [5] the results seem more qualitative than quantitative in the description of the sets $\mathcal{S}_+$ and $\mathcal{S}_-$, as the authors define such sets in an almost implicit fashion.

To fix ideas, as a consequence of the result of Budd and Norbury [13] about equation (12), we have that for $\alpha = \frac{2}{3}$ and $p > 5$, there exists $0 < \lambda^* < \lambda_1$, such that for $\lambda = \lambda^*$, equation (1) has a an infinite sequence $\{u_{C,k}(x)\}_{k=1}^{\infty}$ of positive solutions satisfying $u,\ x^{2\alpha-1}u' \in C[0,1]$, and numerical computations suggest that in a small neighborhood of λ^* one can find several such solutions as well (See figure 6 below). The work of Budd and Norbury [13] (Peihao and Chengkui [41] in more generality) seems to describe part of situation, however, it seems to us that in order to have a better understanding of the set $\mathcal{S}_+$, it would be desirable to know the precise values of λ for which the branch turns to one side or another, because it would allow us to precisely count the number of regular positive solutions that exist for each λ . In particular, as we raised in Question 1, it would be nice to know the value of $\hat{\lambda}_{\alpha,p}$.

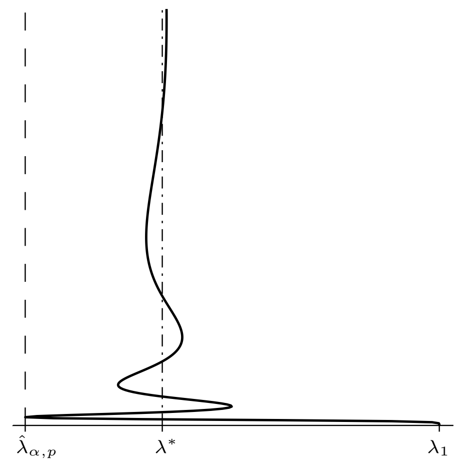


FIGURE 6. Oscillation of the branch of regular solutions near λ^*

One comment that is worth making is that for $\lambda = \lambda^*$ one has the inclusion

$$\left\{u'_{C,k}(1)\right\}_{k=1}^{\infty} \subseteq \mathcal{S}_+,$$

but a complete characterization of the set $\mathcal{S}_+$ would be desirable in this situation. Some numerical computations seem to indicate that in fact

$$\mathcal{S}_+ = \left\{u'_{C,k}(1)\right\}_{k=1}^{\infty}.$$

Concerning the case $\alpha \geq 1$, to our knowledge there is no literature on Questions 3, 4 and 5. Theorem 10 shows that there are no positive solutions of any kind, regardless of $\lambda \in R$ and $p > 1$, in other words we have shown that

$$\mathcal{S}_+ = \emptyset.$$

Moreover, our proof says more, namely that there are no solutions to equation (14) that are positive *near* the origin (*a fortiori* we can also rule out solutions that are negative near the origin). This means that for $\theta \in (-\infty, 0)$, the only possibility is that $u(x;\theta)$ is defined in the whole interval $(0,1)$ (recall estimates 15 and 16), and that it has an infinite sequence of zeros

$$\{z_k\}^{\infty} \subset (0,1),$$

such that $z_k \to 0$ as $k \to \infty$. However, estimates (15) and (16) are very rough (are they sharp?), so it remains to obtain a better understanding of the behavior of $u(x;\theta)$ as $x \to 0$. In particular we would like the answer to two questions:

Question 6. Do bounded (regular) solutions exist to the equation? If so, how many?

Question 7. How fast does the sequence of zeros z_k accumulate as $k \to \infty$?

The rest of this paper is organized as follows: in section 2 we introduce some preliminary results needed to prove our theorems. Section 3 deals with the proof of Theorems 1, 2, 5 and 6. Next in section 4 we prove Theorems 7 and 8. In section 5 we handle the super critical case and prove Theorem 9. In section 6 we prove the non-existence result for $\alpha \geq 1$ given in Theorem 10, and finally in section 7 we prove Theorems 3 and 4.

2. Preliminaries

2.1. **Eigenvalues and Eigenfunctions.** We begin this section by giving some properties of λ_1 and φ_1 defined at (5). Notice that $\mu_1 := (\lambda_1)^{-1}$ corresponds to the first eigenvalue of the operator $\tilde{T}_\alpha : L^2(0,1) \to L^2(0,1)$ defined by $\tilde{T}_\alpha f = u$, where u is the unique solution of

$$\int_0^1 x^{2\alpha} u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx, \text{ for all } v \in X_0^\alpha.$$

The operator $T_\alpha := \tilde{T}_\alpha + I$ was studied in [22], where it was shown that T_α is compact if and only if $\alpha < 1$, and in that case the eigenvalues and eigenfunctions of T_α are completely determined (see [22, Theorem 1.17]). From that result it is easily deduced that when $0 < \alpha < 1$,

$$\lambda_1 = (1-\alpha)^2 j_{\nu 1}^2, \tag{17}$$

where $j_{\nu 1}$ is the first positive zero of $J_\nu : (0, +\infty) \to \mathbb{R}$, the Bessel function of the first kind of order ν (see [52] for a complete treatment of Bessel functions and its properties), and ν is defined in terms of α by

$$\nu := \frac{2\alpha - 1}{2 - 2\alpha}. \tag{18}$$

The corresponding eigenspace is generated by $\varphi_1(x) := x^{\frac{1}{2}-\alpha} J_\nu(j_{\nu 1} x^{1-\alpha})$, and about this function we have

Lemma 2.1. *For $0 < \alpha < 1$, λ_1 and φ_1 as above. Then φ_1 satisfies*

$$\begin{cases} -(x^{2\alpha}\varphi')' = \lambda_1 \varphi & \text{in } (0,1), \\ \varphi(1) = 0, \\ \lim\limits_{x \to 0^+} x^{2\alpha}\varphi'(x) = 0, \end{cases} \tag{19}$$

together with the following properties:

(i) $\varphi_1 \in C^{0,2-2\alpha}[0,1]$,
(ii) $x^{2\alpha-1}\varphi_1' \in C[0,1]$,
(iii) $x^{2\alpha}\varphi_1'' \in C[0,1]$, *and*
(iv) $\varphi_1 > 0$ *in* $[0,1)$.

Proof. The fact that $\varphi_1(x) = x^{\frac{1}{2}-\alpha} J_\nu(j_{\nu 1} x^{1-\alpha})$ solves equation (19) follows from [22, Theorem 1.17]. We have the following series expansion of $J_\nu(y)$ near the origin

$$J_\nu(y) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}, \tag{20}$$

which can be found for instance in [52, p. 40], from here we deduce that

$$\varphi_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+1+\nu)} \left(\frac{j_{\nu 1}}{2}\right)^{2m+\nu} x^{2m(1-\alpha)}.$$

The regularity properties are readily deduced from this series expansion. Finally, the positivity of φ_1 can be obtained from the explicit formula and the fact that λ_1 is given by (17). We omit the details. □

On the other hand, when $0 < \alpha < \frac{1}{2}$, one can also define $\lambda_{1,0}$ and $\varphi_{1,0}$ as in (7). In this case $\mu_{1,0} := (\lambda_{1,0})^{-1}$ corresponds to the first eigenvalue of the operator $\tilde{T}_{\alpha,0} : L^2(0,1) \to L^2(0,1)$ defined by $\tilde{T}_{\alpha,0} f = u$, where u is the unique solution of

$$\int_0^1 x^{2\alpha} u'(x) v'(x) dx = \int_0^1 f(x) v(x) dx, \text{ for all } v \in X_{00}^{\alpha}.$$

The operator $T_{\alpha,0} := \tilde{T}_{\alpha,0} + I$ was also studied in [22], and it was shown that $T_{\alpha,0}$ is compact for all $0 < \alpha < \frac{1}{2}$, and that the eigenvalues and eigenfunctions of $T_{\alpha,0}$ are fully determined (see [22, Theorem 1.16]). From that result we obtain that for $0 < \alpha < \frac{1}{2}$,

$$\lambda_{1,0} = (1-\alpha)^2 j_{\nu_0 1}^2, \tag{21}$$

where, as before, $j_{\nu_0 1}$ denotes the first positive zero of J_{ν_0}, the Bessel function of the first kind of order ν_0, and ν_0 is defined in terms of α by

$$\nu_0 := \frac{1-2\alpha}{2-2\alpha}. \tag{22}$$

Notice that $-\frac{1}{2} < \nu < 0 < \nu_0 < \frac{1}{2}$, where ν is the value used to define λ_1. From this observation one can see that $\lambda_1 < \lambda_{1,0}$ for all $0 < \alpha < \frac{1}{2}$. Now the corresponding eigenspace is generated by $\varphi_{1,0}(x) := x^{\frac{1}{2}-\alpha} J_{\nu_0}(j_{\nu_0 1} x^{1-\alpha})$, and about this function we have

Lemma 2.2. *For $0 < \alpha < \frac{1}{2}$, $\lambda_{1,0}$ and $\varphi_{1,0}$ as above. Then $\varphi_{1,0}$ satisfies*

$$\begin{cases} -(x^{2\alpha}\varphi')' = \lambda_{1,0}\varphi & \text{in } (0,1), \\ \varphi(1) = 0, \\ \lim_{x\to 0^+} \varphi(x) = 0, \end{cases} \tag{23}$$

together with the following properties:

(i) $\varphi_{1,0} \in C^{0,1-2\alpha}[0,1]$,
(ii) $x^{2\alpha-1}\varphi_{1,0} \in C^1[0,1]$,
(iii) $x^{2\alpha}\varphi_{1,0}' \in C^1[0,1]$, *and*
(iv) $\varphi_{1,0} > 0$ *in* $(0,1)$.

Proof. The fact that $\varphi_{1,0}(x) = x^{\frac{1}{2}-\alpha} J_{\nu_0}(j_{\nu_0 1} x^{1-\alpha})$ solves equation (23) follows from [22, Theorem 1.16]. Using the series expansion for $J_{\nu_0}(y)$ given in (20) we deduce that

$$\varphi_{1,0}(x) = x^{1-2\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+1+\nu_0)} \left(\frac{j_{\nu_0 1}}{2}\right)^{2m+\nu_0} x^{2m(1-\alpha)}.$$

The regularity properties and the positivity of $\varphi_{1,0}$ can be obtained from the explicit formula and the definition of $\lambda_{1,0}$. We omit the details. □

As announced in the introduction, we need to study λ_α^* and ψ_α defined by (10). We have the following

Lemma 2.3. *Let* $\frac{1}{2} < \alpha < \frac{3}{4}$ *and define* λ_α^* *as in* (10)*, then the infimum is achieved by a function* $\psi_\alpha \in X_0^{1-\alpha}$ *which satisfies the following equation*

$$\begin{cases} -(x^{2-2\alpha}\psi')' = \lambda_\alpha^* x^{2-4\alpha}\psi \quad \text{in } (0,1), \\ \psi(1) = 0, \\ \lim_{x\to 0^+} x^{2-2\alpha}\psi'(x) = 0. \end{cases} \tag{24}$$

Moreover, $\lambda_\alpha^* = j_{-\nu 1}^2(1-\alpha)^2$*, and* $\psi_\alpha(x) = x^{\alpha-\frac{1}{2}} J_{-\nu}(j_{-\nu 1}x^{1-\alpha})$*, where* $j_{-\nu 1}$ *denotes the first positive zero of* $J_{-\nu}$*, and* ν *is defined by* (18)*. About* ψ_α*, we have the following properties*

(i) $\psi_\alpha \in C^{0,2-2\alpha}[0,1]$*,*
(ii) $x^\alpha \psi_\alpha' \in C[0,1]$*, and*
(iii) $\psi_\alpha > 0$ *in* $[0,1)$*.*

Proof. Notice that the embedding

$$X_0^{1-\alpha} \hookrightarrow \left\{\psi \in L^1_{loc}(0,1) : \left\|x^{1-2\alpha}\psi\right\|_{L^2} < \infty\right\}$$

is compact (this follows from [22, Theorem A.2], because $X_0^{1-\alpha} \hookrightarrow C^{0,\alpha-\frac{1}{2}}[0,1] \subset\subset C^0[0,1]$). With that in mind, it is easy to see that the infimum defining λ_α^* is achieved by a function ψ_α, which must satisfy equation (24). Now, a direct computation shows that if f solves Bessel's equation

$$y^2 f'' + yf' + (y^2 - \nu^2)f = 0,$$

with parameter $\nu = \frac{2\alpha-1}{2-2\alpha}$, then $x^{\alpha-\frac{1}{2}} f\left(\frac{\sqrt{\lambda_\alpha^*}}{1-\alpha}x^{1-\alpha}\right)$ solves

$$-(x^{2-2\alpha}\psi')' = \lambda_\alpha^* x^{2-4\alpha}\psi.$$

Since $\frac{1}{2} < \alpha < \frac{3}{4}$, we have that $0 < \nu < 1$, hence the general solution to Bessel's equation is given by

$$f(y) = AJ_\nu(y) + BJ_{-\nu}(y),$$

where $J_\nu(y)$ is defined in (20). The above implies that ψ_α is given by

$$\psi_\alpha(x) = x^{\alpha-\frac{1}{2}}\left[AJ_\nu\left(\frac{\sqrt{\lambda_\alpha^*}}{1-\alpha}x^{1-\alpha}\right) + BJ_{-\nu}\left(\frac{\sqrt{\lambda_\alpha^*}}{1-\alpha}x^{1-\alpha}\right)\right]$$

for some constants A and B. The series expansion (20) tells us that in order to meet the boundary condition $x^{2-2\alpha}\psi_\alpha'(x) \underset{x\to 0^+}{\longrightarrow} 0$ one has to set $A = 0$. The condition $\psi_\alpha(1) = 0$ implies that

$$\lambda_\alpha^* = (1-\alpha)^2 j_{-\nu 1}^2,$$

where $j_{-\nu 1}$ is the first positive zero of $J_{-\nu}$. Without loss of generality, we fix the solution to be the one with $B = 1$. The regularity properties are obtained from the series expansion (deduced from (20))

$$\psi_\alpha(x) = \sum_{m=0}^\infty \frac{(-1)^m}{m!\Gamma(m+1-\nu)}\left(\frac{j_{-\nu 1}}{2}\right)^{2m-\nu} x^{2m(1-\alpha)},$$

we omit the details. The positivity is readily obtained from the definition of λ_α^* and ψ_α.

About λ_α^*, notice that $j_{\nu 1}$ depends continuously on ν (in fact the dependence is analytic as one can see in [33] or in [52, p. 507]), then λ_α^* depends continuously on α; also, from [42] we deduce that

$$\lambda_\alpha^* = 2(1-\alpha)(3-4\alpha) + O\left((3-4\alpha)^2\right),$$

therefore $\lambda_\alpha^* \to 0$ as $\alpha \to \frac{3}{4}^-$. Also, since $j_{-\nu 1} < j_{\nu 1}$ for all $0 < \nu < 1$ we deduce that $\lambda_\alpha^* < \lambda_1$. Finally, notice that when $\alpha \to \frac{1}{2}^+$ one has $\nu \to 0^+$, hence it is easily seen that $|\lambda_1 - \lambda_\alpha^*| \underset{\alpha\to\frac{1}{2}^+}{\longrightarrow} 0$. This proves the conclusion of Remark 1.8 □

2.2. **Best Constants and extremals.** Another topic that needs to be addressed before proving our results concerns the best constant and extremals for (9), or in general for inequalities of the form

$$C\,\|u\|_{L^{2\alpha}(0,a)} \le \|x^\alpha u'\|_{L^2(0,a)},$$

where $a > 0$. Let $X_0^\alpha(0,a)$ be the set of functions $u \in H^1_{loc}(0,a]$ such that $u, x^\alpha u' \in L^2(0,a)$ and $u(a) = 0$ (when $\frac{1}{2} < \alpha < 1$, one could also define this space as the closure of $C_0^\infty(0,a)$ under the norm $\|x^\alpha u'\|_2$, this follows from [22, Theorem A.4]). Define

$$S_\alpha(a) := \inf_{u\in X_0^\alpha(0,a)} \frac{\int_0^a |x^\alpha u'(x)|^2\,dx}{\left(\int_0^a |u(x)|^{2_\alpha}\,dx\right)^{\frac{2}{2_\alpha}}}.$$

Concerning $S_\alpha(a)$ we have the following

Lemma 2.4. *Let $\frac{1}{2} < \alpha < 1$, $a > 0$ and $S_\alpha(a)$ as above. Then $S_\alpha(a) = S_\alpha(1)$ for all $a > 0$; the infimum in the definition of $S_\alpha(a)$ is not achieved unless $a = +\infty$, in which case the basic extremal profile is given by*

$$U(x) = C\left(1 + x^{2-2\alpha}\right)^{\frac{1-2\alpha}{2-2\alpha}},$$

or after scaling, for every $\varepsilon > 0$ by

$$U_\varepsilon(x) = C_\varepsilon\left(\varepsilon + x^{2-2\alpha}\right)^{\frac{1-2\alpha}{2-2\alpha}}, \tag{25}$$

where C and C_ε are normalization constants. Moreover, we have that

$$S_\alpha = (2\alpha-1)\left[\frac{1}{2-2\alpha}\cdot\frac{\Gamma^2\left(\frac{1}{2-2\alpha}\right)}{\Gamma\left(\frac{1}{1-\alpha}\right)}\right]^{2-2\alpha}, \tag{26}$$

where Γ denotes the Gamma function.

Proof. To see that $S(a) = S(1)$, notice that the quotient $\|x^\alpha u'\|_2^2 / \|u\|_{2_\alpha}^2$ is invariant under the scaling $u_a(x) = u(ax)$. To prove that the infimum is not achieved when $0 < a < +\infty$, notice that it is enough to prove it for $a = 1$, and in that case the proof will be done later when proving Theorem 8 (also check [16, Section 4] where a different approach is taken).

To prove that the infimum is achieved when $a = +\infty$, we use a result from [23, Section 7.1], where the authors study best constants and extremals for the Caffarelli-Kohn-Nirenberg inequalities

$$\left(\int_{\mathbb{R}} \left|x^{-b}u(x)\right|^p dx\right)^{\frac{2}{p}} \le C(a,b)\int_{\mathbb{R}} \left|x^{-a}u'(x)\right|^2 dx,$$

for $a < -\frac{1}{2}$, $a + \frac{1}{2} < b \le a + 1$ and $p = \frac{2}{2(b-a)-1}$. Using their result it is easily deduced that the extremals are of the form (25). Finally, (26) is just a direct evaluation of $\|x^\alpha U'\|_2^2 / \|U\|_{2_\alpha}^2$ using the definition of the Gamma function. We omit the details. □

2.3. **A Pohozaev type identity.** The purpose of this section is to establish a family of Pohozaev type identities satisfied by all solutions of

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + |u|^{p-1} u & \text{in } (0,1), \\ u(1) = 0. \end{cases} \tag{27}$$

To do this, for each $\beta \in \mathbb{R}$, let us define the "energy" functional

$$\begin{aligned} E_{\lambda,\beta}(u)(x) := {} & \frac{1}{2}x^{2\alpha+1+\beta}u'(x)^2 + \frac{1}{p+1}x^{\beta+1}|u(x)|^{p+1} + \frac{\lambda}{2}x^{\beta+1}u(x)^2 \\ & + \frac{1}{2}(2\alpha-1-\beta)x^{2\alpha+\beta}u'(x)u(x) - \frac{\beta}{4}(2\alpha-1-\beta)x^{2\alpha-1+\beta}u(x)^2 \end{aligned} \tag{28}$$

and prove the following

Lemma 2.5. *Let $\alpha > 0$, $p > 1$ and $\beta, \lambda \in \mathbb{R}$. Let u be a solution of equation (27), then, for every $x \in (0,1)$ one has*

$$\begin{aligned} \frac{1}{2}u'(1)^2 = E_{\lambda,\beta}(u)(x) + \lambda(1-\alpha+\beta)\int_x^1 s^\beta u^2 + \frac{\beta}{4}\left(\beta^2 - (2\alpha-1)^2\right)\int_x^1 s^{2\alpha-2+\beta}u^2 \\ + \left((\beta+1)\left(\frac{p+3}{2(p+1)}\right) - \alpha\right)\int_x^1 s^\beta |u|^{p+1}. \end{aligned}$$

Proof. Multiply equation (27) by $s^\beta u(s)$ and integrate over $(x,1)$ to obtain

$$\begin{aligned} \lambda\int_x^1 s^\beta u^2 + \int_x^1 s^\beta |u|^{p+1} &= \int_x^1 s^{2\alpha}u'(s^\beta u)' + x^{2\alpha+\beta}u'(x)u(x) \\ &= \beta\int_x^1 s^{2\alpha+\beta+1}u'u + \int_x^1 s^{2\alpha+\beta}u'^2 + x^{2\alpha+\beta}u'(x)u(x) \\ &= -\frac{\beta}{2}(2\alpha+\beta-1)\int_x^1 s^{2\alpha-2+\beta}u^2 + \int_x^1 s^{2\alpha+\beta}u'^2 + x^{2\alpha+\beta}u'(x)u(x) \\ &\quad - \frac{\beta}{2}x^{2\alpha-1+\beta}u(x)^2, \end{aligned}$$

hence

$$\begin{aligned} \int_x^1 s^{2\alpha+\beta}u'^2 = {} & \lambda\int_x^1 u^2 + \int_x^1 |u|^{p+1} + \frac{\beta}{2}x^{2\alpha-1+\beta}u(x)^2 \\ & + \frac{\beta}{2}(2\alpha+\beta-1)\int_x^1 s^{2\alpha-2+\beta}u^2 - x^{2\alpha+\beta}u'(x)u(x). \end{aligned} \tag{29}$$

Now multiplying equation (27) by $s^{\beta+1}u'(s)$ and integrating over $(x,1)$ gives

$$\begin{gathered} \lambda\int_x^1 s^{\beta+1}uu' + \int_x^1 s^{\beta+1}|u|^{p-1}uu' = \int_x^1 s^{2\alpha}u'(s^{\beta+1}u')' - s^{2\alpha+1+\beta}u'(s)^2\Big|_x^1 \\ I_1 = I_2. \end{gathered}$$

After integrating by parts, we obtain that

$$I_1 = -\frac{\lambda}{2}(\beta+1)\int_x^1 s^\beta u^2 - \frac{\beta+1}{p+1}\int_x^1 s^\beta |u|^{p+1} - \frac{\lambda}{2}x^{\beta+1}u(x)^2 - \frac{1}{p+1}x^{\beta+1}|u(x)|^{p+1}.$$

and that

$$\begin{aligned} I_2 &= (\beta+1)\int_x^1 s^{2\alpha+\beta}u'^2 + \int_x^1 s^{2\alpha+\beta+1}u'u'' - s^{2\alpha+1+\beta}u'(s)^2\Big|_x^1 \\ &= (\beta+1)\int_x^1 s^{2\alpha+\beta}u'^2 - \frac{2\alpha+1+\beta}{2}\int_x^1 s^{2\alpha+\beta}u'^2 - \frac{1}{2}s^{2\alpha+1+\beta}u'(s)^2\Big|_x^1 \\ &= \frac{1}{2}(\beta+1-2\alpha)\int_x^1 s^{2\alpha+\beta}u'^2 - \frac{1}{2}u'(1)^2 + \frac{1}{2}x^{2\alpha+1+\beta}u'(x)^2. \end{aligned}$$

Combining the results of I_1 and I_2 yields

$$\begin{aligned} \frac{1}{2}(\beta+1-2\alpha)\int_x^1 s^{2\alpha+\beta}u'^2 &= -\frac{\lambda}{2}(\beta+1)\int_x^1 s^\beta u^2 - \frac{\beta+1}{p+1}\int_x^1 s^\beta |u|^{p+1} \\ &\quad - \frac{\lambda}{2}x^{\beta+1}u(x)^2 - \frac{1}{p+1}x^{\beta+1}|u(x)|^{p+1} + \frac{1}{2}u'(1)^2 - \frac{1}{2}x^{2\alpha+1+\beta}u'(x)^2. \end{aligned} \tag{30}$$

The result is then obtained from (29) and (30). □

Remark 2.1. For simplicity we have stated and proved the result if the equation is satisfied in the interval $(0,1)$, however, the result remains valid if we replace the interval $(0,1)$ by any interval of the form $(0,a)$, $a>0$, that is: Suppose u solves

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + |u|^{p-1}u & \text{in } (0,a), \\ u(a) = 0, \end{cases}$$

then for all $0<x<a$

$$\begin{aligned} \frac{1}{2}u'(a)^2 = E_{\lambda,\beta}(u)(x) + \lambda(1-\alpha+\beta)\int_x^a s^\beta u^2 + \frac{\beta}{4}\left(\beta^2-(2\alpha-1)^2\right)\int_x^a s^{2\alpha-2+\beta}u^2 \\ + \left((\beta+1)\left(\frac{p+3}{2(p+1)}\right)-\alpha\right)\int_x^a s^\beta |u|^{p+1}. \end{aligned}$$

2.4. **Some regularity results.** We continue with some regularity results for $u \in C^2(0,1]$ solving

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + u^p & \text{in } (0,1), \\ u \geq 0 & \text{in } (0,1), \\ u(1) = 0. \end{cases} \tag{31}$$

Lemma 2.6. *Let $\alpha \geq \frac{1}{2}$, and suppose $u \in C^2(0,1]$, $u(x) \geq 0$ for all $0<x<1$. Then there exists a sequence $0 < x_n < \frac{1}{n}$ such that*

$$x_n^{2\alpha}u'(x_n) \leq \frac{1}{n}.$$

Proof. By contradiction, assume there exists $r>0$ such that $x^{2\alpha}u'(x) \geq r$ for all $0<x<r$, then after integrating, we obtain that for all $x<r$

$$u(r) \geq u(x) + \frac{r}{(2\alpha-1)}\left(x^{1-2\alpha} - r^{1-2\alpha}\right) \geq C_r x^{1-2\alpha}.$$

when $\alpha > \frac{1}{2}$, and that

$$u(r) \geq u(x) + r\ln r - r\ln x \geq -C_r \ln x,$$

when $\alpha = \frac{1}{2}$, for some constant $C_r > 0$. By letting $x \to 0^+$, we obtain that that $u(r) = +\infty$, contradicting the fact that $u \in C^2(0,1]$. □

Lemma 2.7. *Let $\alpha \geq \frac{1}{2}$, $p > 1$ and $\lambda \in \mathbb{R}$. Suppose u solves equation* (31)*, then $u \in L^p(0,1)$.*

Proof. Integrate equation (31) over $[x_n, 1]$, where x_n is taken from lemma 2.6 to obtain

$$\lambda \int_{x_n}^1 u + \int_{x_n}^1 u^p = -u'(1) + x_n^{2\alpha} u'(x_n) \leq -u'(1) + \frac{1}{n}.$$

If $\lambda \geq 0$, by taking the limit as $n \to \infty$ we obtain

$$\lambda \int_0^1 u + \int_0^1 u^p \leq -u'(1),$$

hence $u \in L^p(0,1)$. If $\lambda < 0$, notice that for all $0 < x < 1$ we have $\int_x^1 u \leq \left(\int_x^1 u^p\right)^{\frac{1}{p}}$, therefore

$$\lambda \left(\int_{x_n}^1 u^p\right)^{\frac{1}{p}} + \int_{x_n}^1 u^p \leq \lambda \int_{x_n}^1 u + \int_{x_n}^1 u^p \leq -u'(1) + \frac{1}{n},$$

thus

$$\left(\int_0^1 u^p\right)^{\frac{1}{p}} \left(\lambda + \left(\int_0^1 u^p\right)^{\frac{p-1}{p}}\right) \leq -u'(1),$$

and since $p > 1$, we deduce from here that $\int_0^1 u^p$ must be bounded. □

Corollary 2.8. *Let α, p, λ and u be as in lemma 2.7. Then $L = \lim_{x\to 0^+} x^{2\alpha} u'(x)$ exists and $L \leq 0$.*

Proof. Notice that by integrating equation (31) one obtains

$$x^{2\alpha} u'(x) = u'(1) + \lambda \int_x^1 u(s)ds + \int_x^1 u(s)^p ds,$$

but since $u \in L^p(0,1)$, the right hand side converges as $x \to 0$, so $L = \lim_{x\to 0^+} x^{2\alpha} u'(x)$ exists. Finally, using x_n from lemma 2.6 one gets $L \leq 0$. □

Corollary 2.9. *Let $\alpha > \frac{1}{2}$, $\lambda \in \mathbb{R}$, $p \geq \frac{1}{2\alpha - 1}$ and suppose u solves equation* (31)*. Then $L = \lim_{x\to 0^+} x^{2\alpha} u'(x) = 0$.*

Proof. Suppose there exists $\delta > 0$ such that $x^{2\alpha} u'(x) \leq -\delta$ for all $0 < x < \delta$. Integrating this inequality yields

$$u(x) \geq \frac{\delta}{2\alpha - 1}\left(x^{1-2\alpha} - \delta^{1-2\alpha}\right) \geq C_\delta x^{1-2\alpha},$$

thus $u(x)^p \geq C_\delta x^{(1-2\alpha)p}$, but since $p \geq \frac{1}{2\alpha-1}$ we obtain that $(1-2\alpha)p \leq -1$, a contradiction with the fact that $u \in L^p(0,1)$. Hence there is a sequence such that $x_n^{2\alpha} u'(x_n) \geq -\frac{1}{n}$, so $L \geq 0$; but we already knew that $L \leq 0$. □

Corollary 2.10. *Let α, p and λ as in lemma 2.7. Suppose u solves equation* (31). *Then $x^{2\alpha-1}u = O(\log x)$ if $\alpha = \frac{1}{2}$ and $x^{2\alpha-1}u = O(1)$ if $\alpha > \frac{1}{2}$.*

Proof. Since $x^{2\alpha}u'(x) = O(1)$, the result follows from integration. We omit the details. □

The next lemma shows that positive solutions are monotone near the origin when p is large enough.

Lemma 2.11. *Let $\alpha > \frac{1}{2}, \lambda \in \mathbb{R}$, $p \geq 2_\alpha - 1$ and u be a solution to equation* (31). *Then there exists $0 < \hat{x} \leq 1$ such that $u'(x) \neq 0$ for all $0 < x < \hat{x}$.*

Proof. If $u \equiv 0$ there is nothing to prove, so we assume that $u \not\equiv 0$. We start by proving that there exists $0 < x_0 \leq 1$ such that for all $x < x_0$, either $u'(x) \neq 0$ or $u''(x) < 0$. The proof of this is by contradiction, so we assume that there exists a sequence $x_n \to 0$ such that $u'(x_n) = 0$ and that $u''(x_n) \geq 0$. From the equation we then obtain that

$$\lambda u(x_n) + u(x_n)^p = -x_n^{2\alpha}u''(x_n) - 2\alpha x_n^{2\alpha-1}u'(x_n) \leq 0.$$

Thus, if $\lambda \geq 0$ we obtain that $u(x_1) = u'(x_1) = 0$, this and the existence and uniqueness theorem for ODEs imply that $u \equiv 0$, a contradiction. On the other hand if $\lambda < 0$, the above inequality implies that $u(x_n) \leq (-\lambda)^{-\frac{1}{p-1}}$ for all $n \geq 1$. The Pohozaev identity from lemma 2.5 with $\beta = 0$ and $\varepsilon = x_n$ gives that

$$\frac{1}{2}u'(1)^2 - E_{\lambda,0}(u)(x_n) = \lambda(1-\alpha)\int_{x_n}^1 u^2 + \left(\frac{1}{2} - \alpha + \frac{1}{p+1}\right)\int_{x_n}^1 u^{p+1},$$

but, since $\lambda < 0$ and $p \geq 2_\alpha - 1$ we obtain that the right hand side is non-positive, hence

$$\frac{1}{2}u'(1)^2 \leq E_{\lambda,0}(u)(x_n).$$

But

$$\begin{aligned} E_{\lambda,0}(u)(x_n) &= \frac{\lambda}{2}x_n u(x_n)^2 + \frac{1}{p+1}x_n u(x_n)^{p+1} + \frac{1}{2}x_n^{2\alpha+1}u'(x_n)^2 \\ &\quad + \left(\alpha - \frac{1}{2}\right)x_n^{2\alpha}u'(x_n)u(x_n) \\ &= o(1) \end{aligned}$$

as x_n goes to 0, since $u'(x_n) = 0$ and $u(x_n) = O(1)$, thus proving that $u'(1) = 0$ (and as a consequence, $u \equiv 0$), also a contradiction. So we have the existence of such x_0.

The above proves that all critical points less than x_0 are local maxima, so the only possibility is that there is at most one of them (if there were two local maxima, there must be a local minima in between). This shows that $u'(x) \neq 0$ for all x near the origin. □

Lemma 2.12. *Let $\alpha > \frac{1}{2}$, $p \geq 2_\alpha - 1$ and $\lambda \in \mathbb{R}$. Suppose u solves equation* (31). *Assume in addition that there exists $\varepsilon \geq 0$ such that $x^{-\varepsilon}u^p \in L^1(0,1)$. Then for any $\gamma < \min\left\{2\alpha - 1 - \frac{1-\varepsilon}{p}, 1 - \frac{1-\varepsilon}{p}\right\}$ one has*

(i) $x^{-\gamma}u^p \in L^1(0,1)$,
(ii) $x^{2\alpha-2-\gamma}u \in L^1(0,1)$ *and* $\lim_{x\to 0^+} x^{2\alpha-1-\gamma}u(x) = 0$,

(iii) $x^{2\alpha-1-\gamma}u' \in L^1(0,1)$ *and* $\lim_{x\to 0^+} x^{2\alpha-\gamma}u'(x) = 0$.

Proof. We begin the proof with a claim: there exists a sequence $0 < \delta_n \le \frac{1}{n}$ such that

$$\delta_n^{2\alpha-1-\gamma}u(\delta_n) \le \frac{1}{n}.$$

We prove this by contradiction: if we assume the claim is false, then there would exist $r > 0$ such that $x^{2\alpha-1-\gamma}u(x) \ge r$ for all $x < r$, which implies that

$$x^{-\varepsilon}u(x)^p \ge r^p x^{(1+\gamma-2\alpha)p-\varepsilon},$$

but since $\gamma < 2\alpha - 1 - \frac{1-\varepsilon}{p}$ then $x^{(1+\gamma-2\alpha)p-\varepsilon} \ge x^{-1}$, this contradicts the assumption $x^{-\varepsilon}u^p \in L^1$.

Now, for δ_n as above, define

$$\eta_n(x) = \begin{cases} x^{-\gamma} & \text{if } x > \delta_n, \\ \delta_n^{-\gamma} & \text{if } x \le \delta_n. \end{cases}$$

Notice that $\eta_n \in H^1(0,1)$ for all n. Let $x > 0$ and multiply equation (1) by η_n and integrate by parts over $[x,1]$ to obtain

$$\int_x^1 \eta_n(s)u(s)^p ds = -u'(1) + x^{2\alpha}u'(x)\eta_n(x) + \int_x^1 s^{2\alpha}u'(s)\eta_n'(s)ds - \lambda\int_x^1 \eta_n(s)u(s)ds. \tag{32}$$

First, from corollary 2.9 we know that $\lim_{x\to 0^+} x^{2\alpha}u'(x)\eta_n(x) = \delta_n^{-\gamma}\lim_{x\to 0^+} x^{2\alpha}u'(x) = 0$, also

$$\int_x^1 \eta_n(s)u(s)ds \le \int_0^1 s^{-\gamma}u(s)ds,$$

but

$$\begin{aligned}\int_0^1 s^{-\gamma}u(x)dx &= \int_0^1 s^{-\frac{\varepsilon}{p}}u(s)s^{-\gamma+\frac{\varepsilon}{p}}ds \\ &\le \left(\int_0^1 s^{-\varepsilon}u(s)^p\right)^{\frac{1}{p}}\left(\int_0^1 s^{\left(-\gamma+\frac{\varepsilon}{p}\right)\left(\frac{p}{p-1}\right)}ds\right)^{\frac{p-1}{p}},\end{aligned}$$

and since $\gamma < 1 - \frac{1-\varepsilon}{p}$ one can write

$$1 + \left(-\gamma + \frac{\varepsilon}{p}\right)\left(\frac{p}{p-1}\right) > 0,$$

so the second integral is finite, and as a consequence, $x^{-\gamma}u \in L^1(0,1)$. Therefore

$$\lim_{x\to 0^+}\int_x^1 \eta_n(s)u(s)^p ds \le -u'(1) + |\lambda|\int_0^1 s^{-\gamma}u(s)ds + \lim_{x\to 0^+}\int_x^1 s^{2\alpha}u'(s)\eta_n'(s)ds.$$

Let us study that last term of the right hand side. Suppose $x < \delta_n$

$$\begin{aligned}\int_x^1 s^{2\alpha} u'(s)\eta_n'(s)ds &= -\gamma \int_{\delta_n}^1 s^{2\alpha-1-\gamma} u'(s)ds \\ &= \gamma(2\alpha-1-\gamma)\int_{\delta_n}^1 s^{2\alpha-2-\gamma}u(s)ds + \gamma\delta_n^{2\alpha-1-\gamma}u(\delta_n) \\ &\le \gamma(2\alpha-1-\gamma)\int_0^1 s^{2\alpha-2-\gamma}u(s)ds + \frac{\gamma}{n}.\end{aligned}$$

Notice that,

$$\int_0^1 s^{2\alpha-2-\gamma}u(s)ds \le \left(\int_0^1 s^{-\varepsilon}u(s)^p ds\right)^{\frac{1}{p}} \left(\int_0^1 s^{\left(2\alpha-2-\gamma+\frac{\varepsilon}{p}\right)\left(\frac{p}{p-1}\right)}\right)^{\frac{p-1}{p}},$$

but since $\gamma < 2\alpha - 1 - \frac{1-\varepsilon}{p}$, we obtain that $1 + \left(2\alpha - 2 - \gamma + \frac{\varepsilon}{p}\right)\left(\frac{p}{p-1}\right) > 0$, so the second integral is finite and one concludes that

$$\int_x^1 s^{2\alpha}u'(s)\eta_n'(s)ds \le C\left(\int_0^1 s^{-\varepsilon}u(s)^p ds\right)^{\frac{1}{p}} + O\left(\frac{1}{n}\right).$$

Putting the above estimates together yield

$$\int_0^1 \eta_n(s)u(s)^p ds \le -u'(1) + C\left(\int_0^1 s^{-\varepsilon}u(s)^p ds\right)^{\frac{1}{p}} + O\left(\frac{1}{n}\right).$$

so by letting $n \to \infty$, we conclude that

$$\int_0^1 s^{-\gamma}u(s)^p ds \le -u'(1) + C\left(\int_0^1 s^{-\varepsilon}u(s)^p ds\right)^{\frac{1}{p}}.$$

This proves (i).

Now we prove (iii). Using (32) one obtains

$$\int_x^1 s^{2\alpha}u'(s)\eta_n'(s)ds = u'(1) + \int_x^1 \eta_n(s)u(s)^p ds + \lambda\int_x^1 \eta_n(s)u(s)ds - \delta_n^{-\gamma}x^{2\alpha}u'(x),$$

but, for fixed n, the right hand side converges as $x \to 0$ to

$$u'(1) + \int_0^1 \eta_n(s)u(s)^p ds + \lambda\int_0^1 \eta_n(s)u(s)ds,$$

which converges as $n \to \infty$ to $u'(1) + \int_0^1 s^{-\gamma}u(s)^p ds + \lambda\int_0^1 s^{-\gamma}u(s)ds$, this shows that the left hand side also converges, thus

$$\begin{aligned}-\gamma\int_0^1 s^{2\alpha-1-\gamma}u'(s)ds &= \lim_{n\to\infty}\lim_{x\to 0^+}\int_x^1 s^{2\alpha}u'(s)\eta_n'(s)ds \\ &= u'(1) + \int_0^1 s^{-\gamma}u(s)^p ds + \lambda\int_0^1 s^{-\gamma}u(s)ds,\end{aligned}$$

where we have used lemma 2.11 to say that $d\mu(s) := s^{2\alpha-1-\gamma}u'(s)ds$ defines a signed measure, and hence monotone convergence applies.

To prove that $\lim_{x\to 0^+} x^{2\alpha-\gamma}u'(x) = 0$, multiply equation (1) by $s^{-\gamma}$ and integrate by parts over $[x, 1]$ to obtain

$$x^{2\alpha-\gamma}u'(x) = u'(1) + \int_x^1 s^{-\gamma}u(s)^p ds + \lambda\int_x^1 s^{-\gamma}u(s)ds + \gamma\int_x^1 s^{2\alpha-1-\gamma}u'(s)ds,$$

but we proved that the right hand side converges, and it converges to 0.

To prove (ii), notice that we already proved $x^{2\alpha-2-\gamma}u \in L^1(0,1)$ and that by (iii) the right hand side of

$$x^{2\alpha-1-\gamma}u(x) = \int_x^1 s^{2\alpha-1-\gamma}u'(s)ds - (2\alpha-1-\gamma)\int_x^1 s^{2\alpha-2-\gamma}u(s)ds$$

converges; also, since $\lim_{n\to\infty} \delta_n^{2\alpha-1-\gamma}u(\delta_n) = 0$, then $\lim_{x\to 0^+} x^{2\alpha-1-\gamma}u(x) = 0$. □

We conclude this section by improving lemma 2.7 and Corollaries 2.10, 2.8. Recall that those results deal with the fact that $u \in L^p$ and the behavior of $x^{2\alpha}u'$ and $x^{2\alpha-1}u$ near the origin. We claim that when $p > 2_\alpha - 1$, we have more, namely

Lemma 2.13. *Let $\frac{1}{2} < \alpha < 1$, $p > \max\{2_\alpha - 1, 1\}$ and $\lambda \in \mathbb{R}$. Let u be a solution of equation* (1)*, then $u \in X_0^\alpha(0,1) \cap L^{p+1}(0,1)$, and*

(i) $\lim_{x\to 0^+} x^{\frac{1}{p+1}}u(x) = 0$,

(ii) $\lim_{x\to 0^+} x^{\alpha+\frac{1}{2}}u'(x) = 0$.

Proof of Lemma 2.13. Lemma 2.7 gives that $u \in L^p(0,1)$, so we can apply lemma 2.12 for $\varepsilon_0 = 0$ and obtain that for $\gamma < \gamma_0 = \min\left\{2\alpha - 1 - \frac{1}{p}, 1 - \frac{1}{p}\right\}$, (i), (ii) and (iii) in lemma 2.12 hold. By choosing $\varepsilon_1 < 2\alpha - 1 - \frac{1}{p}$ but arbitrarily close to it, we can repeat the argument one more time, and obtain that (i), (ii) and (iii) in lemma 2.12 hold for all

$$\gamma < \gamma_1 = \min\left\{\left(2\alpha - 1 - \frac{1}{p}\right)\left(1 + \frac{1}{p}\right), \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p}\right)\right\}.$$

Continuing in this fashion we obtain that (i), (ii) and (iii) in lemma 2.12 hold for all γ such that

$$\gamma < \gamma_n = \min\left\{\left(2\alpha - 1 - \frac{1}{p}\right)\sum_{j=0}^{n}\frac{1}{p^j}, \left(1 - \frac{1}{p}\right)\sum_{j=0}^{n}\frac{1}{p^j}\right\}$$

for any $n \in \mathbb{N}$. Hence, if we define

$$\gamma_\infty := \lim_{n\to\infty}\gamma_n = \min\left\{\left(2\alpha - 1 - \frac{1}{p}\right)\frac{p}{p-1}, 1\right\},$$

then (i), (ii) and (iii) from lemma 2.12 hold for all $\gamma < \gamma_\infty$.

First we deal with the case $\frac{1}{2} < \alpha < 1$ and $p + 1 > 2_\alpha = \frac{2}{2\alpha-1}$, we obtain that

$$2\gamma_\infty - (2\alpha - 1) = \frac{1}{p-1}\left((2\alpha-1)(p+1) - 2\right) > 0,$$

so, we can find $\gamma < \gamma_\infty$ such that $2\gamma - (2\alpha - 1) = 0$. Using this γ in (ii) gives that $\lim_{x\to 0^+} x^\gamma u(x) = \lim_{x\to 0^+} x^{2\alpha-1-\gamma}u(x) = 0$. In particular, since $u \in C^2(0,1]$, this shows that $x^\gamma u \in C^0[0,1]$, and we can write

$$\int_0^1 u(s)^{p+1}ds = \int_0^1 s^{-\gamma}u(s)^p s^\gamma u(s)ds \le \|s^\gamma u\|_\infty \int_0^1 s^{-\gamma}u(s)^p ds < +\infty,$$

so $u \in L^{p+1}(0,1)$.

To prove that $u \in X_0^\alpha$, fix $N > 1$ and define $u_N(x) = \max\{u(x), N\}$. Multiply equation (31) by u_N and integrate by parts to obtain

$$\int_{u\le N} x^{2\alpha}u'(x)^2dx = \lambda\int_0^1 u(x)u_N(x) + \int_0^1 u(x)^p u_N(x)dx,$$

where we have used corollary 2.9 to say that $\lim_{x\to 0^+} x^{2\alpha}u'(x)u_N(x) = 0$ and that $u_N(1) = 0$. Since $u \in L^{p+1}(0,1)$, the right hand side converges to $\lambda\int_0^1 u^2 + \int_0^1 u^{p+1} < +\infty$ as $N \to +\infty$, this shows that $u \in X_0^\alpha$.

Now, notice that by our initial choice of γ, we have that

$$x^{\alpha+\frac{1}{2}}u'(x) = x^{2\alpha-\gamma}u'(x) \to 0 \text{ as } x \to 0^+.$$

Similarly $x^{\alpha-\frac{1}{2}}u(x) = x^{2\alpha-\gamma-1}u(x) \to 0$ as $x \to 0^+$. To prove that $x^{\frac{1}{p+1}}u(x) \to 0$, multiply equation (1) by $xu'(x)$ and integrate by parts over $[x,1]$ to obtain

$$\frac{1}{p+1}xu(x)^{p+1} = \frac{1}{2}u'(1)^2 + \left(\alpha - \frac{1}{2}\right)\int_x^1 s^{2\alpha}u'(s)^2ds - \frac{1}{2}x^{2\alpha+1}u'(x)^2$$
$$- \frac{\lambda}{2}\int_x^1 u(s)^2ds - \frac{1}{p+1}\int_x^1 u(s)^{p+1}ds - \frac{\lambda}{2}xu(x)^2,$$

notice that every term in the right hand side converges when $x \to 0^+$, then so must $xu(x)^{p+1}$. Also, the limit $\lim_{x\to 0^+} xu(x)^{p+1} = 0$, because otherwise, $u(x)^{p+1} \sim x^{-1}$ near the origin, contradicting the fact that $u \in L^{p+1}(0,1)$.

We now consider the case $\alpha \ge 1$ and $p > 1$. Notice that as in the previous case, it is enough to prove $u \in L^{p+1}(0,1)$, and to do so, it is again enough to prove that $x^{-\gamma}u^p \in L^1(0,1)$ and that $x^\gamma u \in C[0,1]$ for some γ. Observe that by lemma 2.12, for $\gamma < 1$, $x^{-\gamma}u^p \in L^1(0,1)$; by Hölder inequality

$$x^{\gamma-1}u(x) = x^{-\frac{\gamma}{p}}u(x)x^{\gamma(1+\frac{1}{p})-1} \in L^1(0,1)$$

for all $\frac{1}{p+1} < \frac{1}{2} < \gamma < 1$. Now notice that for $\varepsilon > 0$

$$\int_\varepsilon^1 x^\gamma u'(x)dx = -\gamma\int_\varepsilon^1 x^{\gamma-1}u(x) - \varepsilon^\gamma u(\varepsilon).$$

On one hand, by monotone convergence, we have that $\int_\varepsilon^1 x^\gamma u'(x)dx \to \int_0^1 x^\gamma u'(x)dx$ as $\varepsilon \to 0^+$, and on the other hand, for $\gamma > \frac{1}{p+1}$ there exists a sequence $\varepsilon_n \to 0^+$ such that $\varepsilon_n^\gamma u(\varepsilon_n) \to 0$ (otherwise we would contradict the fact that $x^{-\gamma}u^p \in L^1(0,1)$). Therefore, along ε_n we have that $-\gamma\int_\varepsilon^1 x^{\gamma-1}u(x) - \varepsilon^\gamma u(\varepsilon) \to -\gamma\int_0^1 x^{\gamma-1}u(x)dx$, so by the uniqueness of the limit

$$\int_0^1 x^\gamma u'(x)dx = -\gamma\int_0^1 x^{\gamma-1}u(x)dx,$$

and as a consequence, $x^\gamma u(x) \to 0$ as $x \to 0^+$, in particular $x^\gamma u \in C[0,1]$ for all such γ. Now proceeding as in the previous case, we conclude that $u \in L^{p+1}(0,1)$, $u \in X_0^\alpha$, $xu(x)^{p+1} = x^{\alpha+\frac{1}{2}}u'(x) = o(1)$ as $x \to 0^+$, we omit the details. □

Remark 2.2. Although the case $\frac{1}{2} < \alpha < 1$ and $p = 2_\alpha - 1$ is not considered in lemma 2.13, we can repeat the idea of the proof above and obtain a slightly weaker result: if u solves equation (1) for $p = 2_\alpha - 1$, then for all $\delta > 0$ we have

(i) $x^\delta u^{p+1} \in L^1(0,1)$,

(ii) $u \in X_0^{\alpha+\frac{\delta}{2}}$, and
(iii) $x^{1+\delta}u(x)^{p+1} = x^{\alpha+\frac{1}{2}+\frac{\delta}{2}}u'(x) = o(1)$ as $x \to 0^+$.

Notice that the above properties imply that $u \in L^2(0,1)$. This allows us to write for $p = 2_\alpha - 1$ that

$$\frac{dE_{\lambda,0}(u)(x)}{dx} = \lambda(1-\alpha)u(x)^2 \in L^1(0,1),$$

from where it follows that $E_{\lambda,0}(u)(x) \in C[0,1]$ and that $x^{\alpha-\frac{1}{2}}u(x) = x^{\alpha+\frac{1}{2}}u'(x) = O(1)$ as $x \to 0^+$.

Remark 2.3. With obvious modifications, all the results in this section remain valid for solutions of

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + u^p & \text{in } (0,a), \\ u \geq 0 & \text{in } (0,a), \\ u(a) = 0, \end{cases}$$

where $a > 0$.

3. The sub-critical case

3.1. Proof of Theorems 1 and 5. Let

$$S_{\lambda,\alpha} := \inf_{v \in \mathcal{M}} I_{\lambda,\alpha}(v). \tag{33}$$

First, notice that since

$$\lambda < \lambda_1 \leq \frac{\int_0^1 |x^\alpha v'(x)|^2\,dx}{\int_0^1 |v(x)|^2\,dx}, \text{ for all } v \in X_0^\alpha,$$

we have that $0 < S_{\lambda,\alpha} < \infty$. With this in mind, we claim that $S_{\lambda,\alpha}$ is achieved by some $v \in X_0^\alpha \setminus \{0\}$. Indeed, let $v_n \in X_0^\alpha$ be a minimizing sequence such that $\int_0^1 |v_n(x)|^{p+1}\,dx = 1$, that is

$$S_{\lambda,\alpha} = \lim_{n\to\infty} \int_0^1 |x^\alpha v_n'(x)|^2\,dx - \lambda \int_0^1 |v_n(x)|^2\,dx.$$

The above implies there is a constant $C > 0$, such that

$$\int_0^1 |x^\alpha v_n'(x)|^2\,dx \leq C.$$

Indeed, for $\lambda \geq 0$ and all n large we can write

$$\begin{aligned} \int_0^1 |x^\alpha v_n'(x)|^2\,dx &\leq (S_{\lambda,\alpha}+1) + \lambda \int_0^1 |v_n(x)|^2\,dx \\ &\leq (S_{\lambda,\alpha}+1) + \frac{\lambda}{\lambda_1}\int_0^1 |x^\alpha v_n'(x)|^2\,dx, \end{aligned}$$

therefore

$$\int_0^1 |x^\alpha v_n'(x)|^2\,dx \leq (S_{\lambda,\alpha}+1)\left(1 - \frac{\lambda}{\lambda_1}\right)^{-1}.$$

And for $\lambda < 0$ we immediately obtain that

$$\int_0^1 |x^\alpha v_n'(x)|^2\,dx \leq S_{\lambda,\alpha} + 1.$$

Hence, the sequence v_n is uniformly bounded in X_0^α. Now, since the embedding $X_0^\alpha \hookrightarrow L^{p+1}(0,1)$ is compact (the proof of [22, Theorem A.3] can be copied line by line to obtain this compactness, or one could use [40, Theorem 7.13]), we can assume, after extracting a sub-sequence, that there exists $v \in X_0^\alpha$ such that

- $v_n \to v$ strongly in L^{p+1},
- $v_n \to v$ strongly in L^2, and
- $v_n \rightharpoonup v$ weakly in X_0^α,

thus implying that

$$\begin{aligned}\int_0^1 |x^\alpha v'(x)|^2\,dx - \lambda\int_0^1 |v(x)|^2\,dx &\le \liminf_{n\to\infty}\int_0^1 |x^\alpha v_n'(x)|^2\,dx - \lambda\int_0^1 |v_n(x)|^2\,dx\\ &= S_{\lambda,\alpha}.\end{aligned}$$

Hence $S_{\lambda,\alpha}$ is achieved by $v \not\equiv 0$, which one can assume to be non-negative as one can replace v by $|v|$. Now it is easy to see that v is a solution of

$$\begin{cases} -(x^{2\alpha}v')' = \lambda v + \mu v^p \quad \text{in } (0,1),\\ v(1) = 0,\\ \lim_{x\to 0^+} x^{2\alpha}v'(x) = 0,\end{cases}$$

where $\mu = \mu_{\alpha,\lambda} > 0$ is a suitable Lagrange multiplier. If one lets $u(x) = \mu^{\frac{1}{p-1}}v(x)$ then u is a non trivial non-negative solution of

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + u^p \quad \text{in } (0,1),\\ u(1) = 0,\\ \lim_{x\to 0^+} x^{2\alpha}u'(x) = 0.\end{cases}$$

To prove the regularity properties, notice that from the equation and the fact that $u \in X_0^\alpha \hookrightarrow L^{2_\alpha}$, we have $\left(x^{2\alpha}u'\right)' \in L^{\frac{2_\alpha}{p}}$, and since $\lim_{x\to 0^+} x^{2\alpha}u'(x) = 0$, we can write, using Hardy's Inequality,

$$x^{2\alpha-1}u' = \frac{1}{x}\int_0^x \left(s^{2\alpha}u'(s)\right)'\,ds \in L^{\frac{2_\alpha}{p}},$$

that is, $u \in X_0^{2\alpha-1,\frac{2_\alpha}{p}}(0,1)$. With the aid of [22, Theorem A.2] and a bootstrap argument, we obtain the regularity properties claimed. We omit the details.

To prove that $u > 0$ in $(0,1)$, let $Z := \{x \in [0,1) : u(s) > 0,\ \forall s > x\}$. Since $u \not\equiv 0$ we have that $x_0 := \sup Z < 1$. If $x_0 = 0$ we are done, otherwise $0 < x_0 < 1$ and $u'(x_0) = 0$ (it is an interior minimum), but by the definition of x_0, $u(s) > 0$ for all $s \in (x_0, 1)$. Since the equation is elliptic in $(x_0,1)$, Hopf's lemma applies and we obtain $u'(x_0) > 0$, a contradiction.

□

3.2. **Proof of Theorems 2 and 6.** Suppose we have a solution and multiply equation (1) by φ_1 and integrate by parts over $[\varepsilon, 1]$ to obtain

$$(\lambda - \lambda_1)\int_\varepsilon^1 u(x)\varphi_1(x)dx + \int_\varepsilon^1 u(x)^p\varphi_1(x)dx = \varepsilon^{2\alpha}u'(\varepsilon)\varphi_1(\varepsilon) - \varepsilon^{2\alpha}\varphi_1'(\varepsilon)u(\varepsilon).$$

If $\alpha < \frac{1}{2}$, then we are assuming that $\varepsilon^{2\alpha}u'(\varepsilon) \le o(1)$ and as a consequence we obtain that $\varepsilon u(\varepsilon) = o(1)$ as $\varepsilon \to 0^+$.

If $\alpha \geq \frac{1}{2}$, we do not have the assumption near the origin but we have Corollaries 2.8 and 2.10, which allows us to write $\varepsilon^{2\alpha}u'(\varepsilon) \leq o(1)$ and $\varepsilon u(\varepsilon) = o(1)$.

Therefore in all cases we can write, with the aid of lemma 2.1

$$(\lambda - \lambda_1)\int_\varepsilon^1 u(x)\varphi_1(x)dx + \int_\varepsilon^1 u(x)^p\varphi_1(x)dx \leq o(1), \text{ for all } \varepsilon > 0$$

but since $\lambda \geq \lambda_1$, $\varphi_1 > 0$ and $u > 0$, we reach a contradiction when we send ε to 0^+.

□

4. The critical case: $p = 2_\alpha - 1$

We begin this section with the key ingredient in proving Theorem 7. As announced in the introduction, we will follow the approach taken by Brezis and Nirenberg in [10] and we will prove that $S_{\lambda,\alpha}$ defined at (33) is achieved by some function $v \in \mathcal{M}$. In order to do so, we will prove that it is enough to show that

$$S_{\lambda,\alpha} < S_\alpha,$$

where S_α is defined in (9).

Lemma 4.1. *Suppose $\lambda > 0$. If $S_{\lambda,\alpha} < S_\alpha$, then $S_{\lambda,\alpha}$ is achieved.*

Proof. Let $v_n \in X_0^\alpha$ be a minimizing sequence for $S_{\lambda,\alpha}$, i.e.,

$$\|x^\alpha v_n'\|_2^2 - \lambda \|v_n\|_2^2 = S_{\lambda,\alpha} + o(1), \ \|v_n\|_{p+1} = 1.$$

As we did in the proof Theorem 5, we deduce that v_n is uniformly bounded in X_0^α, so without loss of generality, one can assume that there exists $v \in X_0^\alpha$ such that

$$\begin{aligned} &v_n \rightharpoonup v \text{ in } X_0^\alpha,\\ &v_n \to v \text{ in } L^2,\\ &v_n \to v \text{ a.e. in } (0,1). \end{aligned}$$

Also we have that $\|v\|_{p+1} \leq 1$. Following [10], let $w_n = v_n - v$. It is not difficult to see that $w_n \rightharpoonup 0$ in X_0^α, and certainly we have $w_n \to 0$ a.e. in $(0,1)$. Now, notice that

$$S_\alpha = \inf_{v\in\mathcal{M}} \int_0^1 |x^\alpha v'(x)|^2\, dx \leq \int_0^1 |x^\alpha v_n'(x)|^2\, dx,$$

hence, $S_{\lambda,\alpha} \geq S_\alpha - \lambda\|v\|_2^2$, and since $S_{\lambda,\alpha} < S_\alpha$ and $\lambda > 0$ one deduces that

$$\|v\|_2^2 \geq \frac{S_\alpha - S_{\lambda,\alpha}}{\lambda} > 0.$$

Using that $w_n \rightharpoonup 0$ one obtains

$$\|x^\alpha v_n\|_2^2 = \|x^\alpha v'\|_2^2 + \|x^\alpha w_n'\|_2^2 + o(1),$$

which implies

$$S_{\lambda,\alpha} = \|x^\alpha v'\|_2^2 + \|x^\alpha w_n'\|_2^2 - \lambda\|v\|_2^2 + o(1). \tag{34}$$

Also, Theorem 1 from Brezis and Lieb [9] gives

$$\|v + w_n\|_{p+1}^{p+1} = \|v\|_{p+1}^{p+1} + \|w_n\|_{p+1}^{p+1} + o(1),$$

so $1 \leq \|v\|_{p+1}^2 + \|w_n\|_{p+1}^2 + o(1)$ and as a consequence

$$1 \leq \|v\|_{p+1}^2 + \frac{1}{S_\alpha} \|x^\alpha w_n'\|_2^2 + o(1). \tag{35}$$

To conclude the proof, we identify two cases:

- If $S_{\lambda,\alpha} \leq 0$: from (34) we deduce

$$\begin{aligned} \|x^\alpha v'\|_2^2 - \lambda \|v\|_2^2 &\leq \|x^\alpha v'\|_2^2 + \|x^\alpha w_n'\|_2^2 - \lambda \|v\|_2^2 \\ &= S_{\lambda,\alpha} + o(1) \\ &\leq S_{\lambda,\alpha} \|u\|_{p+1}^2 + o(1). \end{aligned}$$

- If $S_{\lambda,\alpha} > 0$: multiply (35) by $S_{\lambda,\alpha}$ to obtain

$$S_{\lambda,\alpha} \leq S_{\lambda,\alpha} \|v\|_{p+1}^2 + \frac{S_{\lambda,\alpha}}{S_\alpha} \|x^\alpha w_n'\|_2^2 + o(1),$$

hence

$$\begin{aligned} \|x^\alpha v'\|_2^2 - \lambda \|v\|_2^2 &\leq S_{\lambda,\alpha} \|v\|_{p+1}^2 + \left(\frac{S_{\lambda,\alpha}}{S_\alpha} - 1 \right) \|x^\alpha w_n'\|_2^2 + o(1) \\ &\leq S_{\lambda,\alpha} \|v\|_{p+1}^2 + o(1). \end{aligned}$$

Either way, one obtains

$$\|x^\alpha v'\|_2^2 - \lambda \|v\|_2^2 \leq S_{\lambda,\alpha} \|v\|_{p+1}^2 ,$$

thus completing the proof. □

4.1. **Proof of Theorem 7.** To prove this theorem we will evaluate $I_{\lambda,\alpha}$ at $u_\varepsilon(x) = \phi(x) \left(\varepsilon + x^{2-2\alpha}\right)^{\frac{1-2\alpha}{2-2\alpha}}$, where ϕ is to be chosen, and prove that $I_{\lambda,\alpha}(v_\varepsilon) < S_\alpha$ when ε is small enough, which, with the aid of lemma 4.1, allows us to conclude that $S_{\lambda,\alpha}$ is achieved by some function $v \in X_0^\alpha$.

The case $\frac{3}{4} \leq \alpha < 1$

Let $\phi : [0,1] \to [0,1]$ be a smooth function such that $\phi(x) \equiv 1$ for $x \in \left[0, \frac{1}{3}\right]$ and $\phi(x) \equiv 0$ for $x \in \left[\frac{2}{3}, 1\right]$, and consider $v_\varepsilon(x) = \phi(x) \left(\varepsilon + x^{2-2\alpha}\right)^{\frac{1-2\alpha}{2-2\alpha}}$. In order to evaluate $I_{\lambda,\alpha}(v_\varepsilon)$ one has to estimate $\|x^\alpha v_\varepsilon'\|_2^2$, $\|v_\varepsilon\|_2^2$ and $\|v_\varepsilon\|_{p+1}^2$. Firstly, notice that

$$\begin{aligned} \int_0^1 |x^\alpha v_\varepsilon'(x)|^2 \, dx = (2\alpha - 1)^2 &\int_0^{\frac{2}{3}} x^{2-2\alpha} \left(\varepsilon + x^{2-2\alpha}\right)^{\frac{-2}{2-2\alpha}} \phi^2(x) dx \\ &+ \int_{\frac{1}{3}}^{\frac{2}{3}} x^{2\alpha} \left(\varepsilon + x^{2-2\alpha}\right)^{\frac{1-2\alpha}{1-\alpha}} |\phi'(x)|^2 \, dx \\ &+ (1-2\alpha) \int_{\frac{1}{3}}^{\frac{2}{3}} x \left(\varepsilon + x^{2-2\alpha}\right)^{\frac{-2\alpha}{2-2\alpha}} \phi(x)\phi'(x) dx \\ &= I_1 + I_2 + I_3. \end{aligned}$$

To estimate $I_1,\ I_2,\ I_3$, notice that for $\beta > 0, \gamma > 0, 0 < \alpha < 1$ and $\varepsilon > 0$ we have

$$\int_{\frac{1}{3}}^{\frac{2}{3}} x^\beta \left(\varepsilon + x^{2-2\alpha}\right)^{-\gamma} dx \leq \int_{\frac{1}{3}}^{\frac{2}{3}} x^{\beta - 2\gamma(1-\alpha)} dx = O(1). \tag{36}$$

To estimate I_1, let $\beta = 2-2\alpha$, $\gamma = \frac{2}{2-2\alpha}$ and use (36) to obtain

$$\begin{aligned}
I_1 &= (2\alpha-1)^2 \int_0^{\frac{2}{3}} x^{2-2\alpha}\left(\varepsilon + x^{2-2\alpha}\right)^{\frac{-2}{2-2\alpha}} \phi^2(x)dx \\
&= (2\alpha-1)^2 \int_0^{\frac{1}{3}} x^{2-2\alpha}\left(\varepsilon + x^{2-2\alpha}\right)^{\frac{-2}{2-2\alpha}} dx + O\left(\int_{\frac{1}{3}}^{\frac{2}{3}} x^{2-2\alpha}\left(\varepsilon + x^{2-2\alpha}\right)^{\frac{-2}{2-2\alpha}} dx\right) \\
&= (2\alpha-1)^2 \int_0^{\frac{1}{3}} x^{2-2\alpha}\left(\varepsilon + x^{2-2\alpha}\right)^{\frac{-2}{2-2\alpha}} dx + O(1).
\end{aligned}$$

Using the change of variables $x = \varepsilon^{\frac{1}{2-2\alpha}} y$ in the above integral gives

$$I_1 = (2\alpha-1)^2 \varepsilon^{\frac{1-2\alpha}{2-2\alpha}} \int_0^{\infty} y^{2-2\alpha}\left(1 + y^{2-2\alpha}\right)^{\frac{-2}{2-2\alpha}} dy + O(1).$$

For I_2 and I_3, since $\|\phi\|_\infty, \|\phi'\|_\infty < \infty$, one can apply (36) once again to obtain

$$I_2 + I_3 = O(1).$$

Hence

$$\int_0^1 \left|x^\alpha v_\varepsilon'(x)\right|^2 dx = (2\alpha-1)^2 \varepsilon^{\frac{1-2\alpha}{2-2\alpha}} \int_0^{\infty} y^{2-2\alpha}\left(1 + y^{2-2\alpha}\right)^{\frac{-2}{2-2\alpha}} dy + O(1). \tag{37}$$

On the other hand we compute

$$\begin{aligned}
\int_0^1 \left|v_\varepsilon(x)\right|^2 dx &= \int_0^{\frac{1}{3}} \left(\varepsilon + x^{2-2\alpha}\right)^{\frac{1-2\alpha}{1-\alpha}} dx + \int_{\frac{1}{3}}^{\frac{2}{3}} \left(\varepsilon + x^{2-2\alpha}\right)^{\frac{1-2\alpha}{1-\alpha}} \phi^2(x)dx \\
&= J_1 + J_2.
\end{aligned}$$

To estimate J_2, notice that

$$\int_{\frac{1}{3}}^{\frac{2}{3}} \left(\varepsilon + x^{2-2\alpha}\right)^{\frac{1-2\alpha}{1-\alpha}} dx \leq \int_{\frac{1}{3}}^{\frac{2}{3}} x^{2-4\alpha} dx = O(1).$$

To estimate J_1 we need to study two cases: $\frac{3}{4} < \alpha < 1$ and $\alpha = \frac{3}{4}$. If $\frac{3}{4} < \alpha < 1$ we use the change of variables $x = \varepsilon^{\frac{1}{2-2\alpha}} y$ and obtain

$$\begin{aligned}
J_1 &= \int_0^{\frac{1}{3}} \left(\varepsilon + x^{2-2\alpha}\right)^{\frac{1-2\alpha}{1-\alpha}} dx = \varepsilon^{\frac{3-4\alpha}{2-2\alpha}} \int_0^{\frac{1}{3}\varepsilon^{-\frac{1}{2-2\alpha}}} \left(1 + y^{2-2\alpha}\right)^{\frac{1-2\alpha}{1-\alpha}} dy \\
&= \varepsilon^{\frac{3-4\alpha}{2-2\alpha}} \int_0^{\infty} \left(1 + y^{2-2\alpha}\right)^{\frac{1-2\alpha}{1-\alpha}} dy + O(1).
\end{aligned}$$

If $\alpha = \frac{3}{4}$, the change of variables $x = \varepsilon^2 y$ gives

$$\begin{aligned}
J_1 &= \int_0^{\frac{1}{3}} \left(\varepsilon + x^{\frac{1}{2}}\right)^{-2} dx = \int_0^{\frac{1}{3}\varepsilon^{-2}} \left(1 + y^{\frac{1}{2}}\right)^{-2} dy \\
&= 2\left[\ln\left(1 + x^{\frac{1}{2}}\right) + \left(1 + x^{\frac{1}{2}}\right)^{-1}\right]\Bigg|_0^{\frac{1}{3}\varepsilon^{-2}} \\
&= 2\left|\ln\varepsilon\right| + O(1).
\end{aligned}$$

Therefore

$$\int_0^1 \left|v_\varepsilon(x)\right|^2 dx = \begin{cases} 2\left|\ln\varepsilon\right| + O(1) & \text{if } \alpha = \frac{3}{4}, \\ \varepsilon^{\frac{3-4\alpha}{2-2\alpha}} \int_0^\infty \left(1 + y^{2-2\alpha}\right)^{\frac{1-2\alpha}{1-\alpha}} dy + O(1) & \text{if } \frac{3}{4} < \alpha < 1. \end{cases} \tag{38}$$

Finally, we need to estimate $\|v_\varepsilon\|_{p+1}^2$.

$$\begin{aligned}\int_0^1 |v_\varepsilon(x)|^{\frac{2}{2\alpha-1}}\,dx &= \int_0^{\frac13} \left(\varepsilon + x^{2-2\alpha}\right)^{-\frac{2}{2-2\alpha}}\,dx \\ &\quad + \int_{\frac13}^{\frac23} \left(\varepsilon + x^{2-2\alpha}\right)^{-\frac{2}{2-2\alpha}} |\phi(x)|^{\frac{2}{2\alpha-1}}\,dx \\ &= M_1 + M_2.\end{aligned}$$

For M_2, notice that

$$\int_{\frac13}^{\frac23} \left(\varepsilon + x^{2-2\alpha}\right)^{-\frac{2}{2-2\alpha}}\,dx \le \int_{\frac13}^{\frac23} x^{-2}dx = O(1),$$

and for M_1, the change of variables $x = \varepsilon^{\frac{1}{2-2\alpha}} y$ gives

$$\int_0^{\frac13} \left(\varepsilon + x^{2-2\alpha}\right)^{-\frac{2}{2-2\alpha}}\,dx = \varepsilon^{-\frac{1}{2-2\alpha}} \int_0^\infty \left(1 + y^{2-2\alpha}\right)^{-\frac{2}{2-2\alpha}}\,dy + O(1).$$

Thereafter

$$\int_0^1 |v_\varepsilon(x)|^{\frac{2}{2\alpha-1}}\,dx = \varepsilon^{-\frac{1}{2-2\alpha}} \int_0^\infty \left(1 + y^{2-2\alpha}\right)^{-\frac{2}{2-2\alpha}}\,dy + O(1). \tag{39}$$

Now, putting together estimates (37), (38) and (39) gives

$$\begin{aligned}I_{\lambda,\alpha}(v_\varepsilon) &= \frac{\|x^\alpha v_\varepsilon'\|_2^2 - \lambda\|v_\varepsilon\|_2^2}{\|v_\varepsilon\|_{p+1}^2} \\ &= \begin{cases} (2\alpha-1)^2 K_1 - \varepsilon\lambda K_2 + O\left(\varepsilon^{\frac{2\alpha-1}{2-2\alpha}}\right) & \text{if } \alpha > \frac34, \\ (2\alpha-1)^2 K_1 - \varepsilon\,|\ln\varepsilon|\,\lambda \tilde{K}_2 + O(\varepsilon) & \text{if } \alpha = \frac34,\end{cases}\end{aligned}$$

where

$$\begin{aligned}K_1 &= \frac{\int_0^\infty y^{2-2\alpha}\left(1+y^{2-2\alpha}\right)^{-\frac{1}{1-\alpha}}\,dx}{\left[\int_0^\infty (1+y^{2-2\alpha})^{-\frac{1}{1-\alpha}}\,dx\right]^{2\alpha-1}} \\ &= \frac{1}{(2\alpha-1)^2}\frac{\int_0^\infty |y^\alpha U'(y)|^2\,dy}{\left(\int_0^\infty |U(y)|^{p+1}\,dy\right)^{\frac{2}{p+1}}} \\ &= \frac{1}{(2\alpha-1)^2} S_\alpha\end{aligned}$$

and

$$\begin{aligned}K_2 &= \frac{\int_0^\infty \left(1+y^{2-2\alpha}\right)^{\frac{1-2\alpha}{1-\alpha}}\,dx}{\left[\int_0^\infty (1+y^{2-2\alpha})^{-\frac{2}{2-2\alpha}}\,dx\right]^{2\alpha-1}} < +\infty, \\ \tilde{K}_2 &= \frac{2}{\left[\int_0^\infty (1+y^{2-2\alpha})^{-\frac{2}{2-2\alpha}}\,dx\right]^{2\alpha-1}} < +\infty,\end{aligned}$$

Finally, since $\alpha > \frac34$ ($\alpha = \frac34$ resp.), for every $\lambda > 0$ there exists $\varepsilon > 0$ sufficiently small such that $-\varepsilon\lambda K_2 + O\left(\varepsilon^{\frac{2\alpha-1}{2-2\alpha}}\right) < 0$ ($-\varepsilon\,|\ln\varepsilon|\,\lambda\tilde{K}_2 + O(\varepsilon) < 0$ resp.), hence

$$S_{\lambda,\alpha} \le I_{\lambda,\alpha}(v_\varepsilon) < S_\alpha,$$

as claimed.

□

The case $\frac{1}{2} < \alpha < \frac{3}{4}$

In this case, we choose $\phi = \psi_\alpha$, the minimizer for λ_α^* given by lemma 2.3. As before we need to evaluate $I_{\lambda,\alpha}(v_\varepsilon)$, where $v_\varepsilon(x) = \left(\varepsilon + x^{2-2\alpha}\right)^{\frac{1-2\alpha}{2-2\alpha}} \psi_\alpha(x)$. Notice that

$$\begin{aligned}\int_0^1 \left|x^\alpha v_\varepsilon'(x)\right|^2 dx = (2\alpha-1)^2 \int_0^1 x^{2-2\alpha} \left(\varepsilon + x^{2-2\alpha}\right)^{\frac{-2}{2-2\alpha}} \psi_\alpha^2(x) dx \\ + \int_0^1 x^{2\alpha} \left(\varepsilon + x^{2-2\alpha}\right)^{\frac{1-2\alpha}{1-\alpha}} \left|\psi_\alpha'(x)\right|^2 dx \\ + (1-2\alpha) \int_0^1 x \left(\varepsilon + x^{2-2\alpha}\right)^{\frac{-2\alpha}{2-2\alpha}} \psi_\alpha(x)\psi_\alpha'(x) dx \\ = I_1 + I_2 + I_3.\end{aligned}$$

We begin by estimating I_3: We integrate by parts and use the fact that $x^{\frac{1}{2}}\psi_\alpha \to 0$ as $x \to 0^+$ (see lemma 2.3), to obtain

$$\begin{aligned}I_3 &= (1-2\alpha) \int_0^1 x \left(\varepsilon + x^{2-2\alpha}\right)^{\frac{-2\alpha}{2-2\alpha}} \psi_\alpha(x)\psi_\alpha'(x) dx \\ &= \varepsilon(2\alpha-1) \int_0^1 \left(\varepsilon + x^{2-2\alpha}\right)^{-\frac{1}{1-\alpha}} \psi_\alpha^2(x) dx \\ &\quad - (2\alpha-1)^2 \int_0^1 x^{2-2\alpha} \left(\varepsilon + x^{2-2\alpha}\right)^{\frac{-2}{2-2\alpha}} \psi_\alpha^2(x) dx \\ &= \varepsilon(2\alpha-1) \int_0^1 \left(\varepsilon + x^{2-2\alpha}\right)^{-\frac{1}{1-\alpha}} \psi_\alpha^2(x) dx - I_1.\end{aligned}$$

To conclude the estimate of I_3 we need to rewrite $\int_0^1 \left(\varepsilon + x^{2-2\alpha}\right)^{-\frac{1}{1-\alpha}} \psi_\alpha^2(x) dx$. Observe that

$$\begin{aligned}\int_0^1 \left(\varepsilon + x^{2-2\alpha}\right)^{-\frac{1}{1-\alpha}} \psi_\alpha^2(x) dx = \psi_\alpha^2(0) \int_0^1 \left(\varepsilon + x^{2-2\alpha}\right)^{-\frac{1}{1-\alpha}} dx \\ + \int_0^1 \left(\varepsilon + x^{2-2\alpha}\right)^{-\frac{1}{1-\alpha}} \left(\psi_\alpha^2(x) - \psi_\alpha^2(0)\right) dx,\end{aligned}$$

and then we notice that by lemma 2.3 we know that $\left|\psi_\alpha^2(x) - \psi_\alpha^2(0)\right| = O\left(x^{2-2\alpha}\right)$, so we can write

$$\begin{aligned}\left|\int_0^1 \left(\varepsilon + x^{2-2\alpha}\right)^{-\frac{1}{1-\alpha}} \left(\psi_\alpha^2(x) - \psi_\alpha^2(0)\right) dx\right| &\le C \int_0^1 \left(\varepsilon + x^{2-2\alpha}\right)^{-\frac{1}{1-\alpha}} x^{2-2\alpha} dx \\ &= \varepsilon^{\frac{1-2\alpha}{2-2\alpha}} \int_0^{\varepsilon^{-\frac{1}{2-2\alpha}}} (1 + y^{2-2\alpha})^{-\frac{1}{1-\alpha}} y^{2-2\alpha} dy \\ &= \varepsilon^{\frac{1-2\alpha}{2-2\alpha}} \int_0^\infty (1 + y^{2-2\alpha})^{-\frac{1}{1-\alpha}} y^{2-2\alpha} dy \\ &\quad + O(1).\end{aligned}$$

The above means that

$$
\begin{aligned}
I_3 &= \varepsilon\psi_\alpha^2(0)(2\alpha-1)\int_0^1 \left(\varepsilon+x^{2-2\alpha}\right)^{-\frac{1}{1-\alpha}}dx - I_1 + O\left(\varepsilon^{\frac{3-4\alpha}{2-2\alpha}}\right)\\
&= \varepsilon^{\frac{1-2\alpha}{2-2\alpha}}\psi_\alpha^2(0)(2\alpha-1)\int_0^\infty \left(1+y^{2-2\alpha}\right)^{-\frac{1}{1-\alpha}}dy - I_1 + O\left(\varepsilon^{\frac{3-4\alpha}{2-2\alpha}}\right).
\end{aligned} \tag{40}
$$

Now we estimate I_2:

$$
\begin{aligned}
I_2 &= \int_0^1 x^{2\alpha}\left(\varepsilon+x^{2-2\alpha}\right)^{\frac{1-2\alpha}{1-\alpha}}\left|\psi_\alpha'(x)\right|^2 dx\\
&= \int_0^1 x^{2-2\alpha}\left|\psi_\alpha'(x)\right|^2 dx + \int_0^1 \left[\left(\varepsilon+x^{2-2\alpha}\right)^{\frac{1-2\alpha}{1-\alpha}} - x^{2-4\alpha}\right]\left|x^\alpha\psi_\alpha'(x)\right|^2 dx\\
&= \int_0^1 x^{2-2\alpha}\left|\psi_\alpha'(x)\right|^2 dx + I_4.
\end{aligned}
$$

To estimate I_4, we notice that by lemma 2.3, we have that $x^\alpha\psi_\alpha' \in C^{0,1-\alpha}[0,1]$, hence it is enough to estimate

$$
\tilde{I}_4 := \int_0^1 \left[\left(\varepsilon+x^{2-2\alpha}\right)^{\frac{1-2\alpha}{1-\alpha}} - x^{2-4\alpha}\right]dx
$$

Define $f(t) := \left(t\varepsilon+x^{2-2\alpha}\right)^{\frac{2\alpha-1}{1-\alpha}}$, and notice that

$$
\left|\varepsilon+x^{2-2\alpha}\right|^{\frac{2\alpha-1}{1-\alpha}} - x^{4\alpha-2} = |f(1)-f(0)| \le \sup_{t\in[0,1]}|f'(t)|.
$$

A direct computation shows that $f'(t) = \frac{2\alpha-1}{1-\alpha}\varepsilon\left(t\varepsilon+x^{2-2\alpha}\right)^{\frac{3\alpha-2}{1-\alpha}}$. Now, using the monotonicity of $f'(t)$, it is easy to see that for all $t\in[0,1]$ we have

$$
|f'(t)| \le C\varepsilon\begin{cases} x^{6\alpha-4} & \text{if } \frac12<\alpha<\frac23\\ 1 & \text{if } \alpha=\frac23\\ \left(\varepsilon+x^{2-2\alpha}\right)^{\frac{3\alpha-2}{1-\alpha}} & \text{if } \frac23<\alpha<\frac34\end{cases}. \tag{41}
$$

From (41) we deduce that

$$
\begin{aligned}
\left|\tilde{I}_4\right| &= \left|\int_0^1 \left[\left(\varepsilon+x^{2-2\alpha}\right)^{\frac{1-2\alpha}{1-\alpha}} - x^{2-4\alpha}\right]dx\right|\\
&\le \int_0^1 \left|\frac{\left(\varepsilon+x^{2-2\alpha}\right)^{\frac{2\alpha-1}{1-\alpha}} - x^{4\alpha-2}}{x^{4\alpha-2}\left(\varepsilon+x^{2-2\alpha}\right)^{\frac{2\alpha-1}{1-\alpha}}}\right|dx\\
&\le C\varepsilon\begin{cases} \int_0^1 x^{2\alpha-2}\left(\varepsilon+x^{2-2\alpha}\right)^{\frac{1-2\alpha}{1-\alpha}} & \text{if } \frac12<\alpha<\frac23,\\ \int_0^1 x^{-\frac23}\left(\varepsilon+x^{\frac23}\right)^{-1}dx & \text{if } \alpha=\frac23,\\ \int_0^1 x^{2-4\alpha}\left(\varepsilon+x^{2-2\alpha}\right)^{-1} & \text{if } \frac23<\alpha<\frac34,\end{cases}\\
&= C\varepsilon\begin{cases} \varepsilon^{\frac{1-2\alpha}{2-2\alpha}}\int_0^\infty y^{2\alpha-2}\left(1+y^{2-2\alpha}\right)^{\frac{1-2\alpha}{1-\alpha}} + O(1) & \text{if } \frac12<\alpha<\frac23,\\ \varepsilon^{-\frac12}\int_0^\infty y^{-\frac23}\left(1+y^{\frac23}\right)^{-1}dx + O(1) & \text{if } \alpha=\frac23,\\ \varepsilon^{\frac{1-2\alpha}{2-2\alpha}}\int_0^\infty y^{2-4\alpha}\left(1+y^{2-2\alpha}\right)^{-1} + O(1) & \text{if } \frac23<\alpha<\frac34,\end{cases}\\
&= O\left(\varepsilon^{\frac{3-4\alpha}{2-2\alpha}}\right).
\end{aligned}
$$

So we can conclude that

$$I_2 = \int_0^1 x^{2-2\alpha} \left|\psi_\alpha'(x)\right|^2 dx + O\left(\varepsilon^{\frac{3-4\alpha}{2-2\alpha}}\right). \tag{42}$$

Putting together (40) and (42) we deduce that

$$\begin{aligned}\int_0^1 \left|x^\alpha v_\varepsilon'(x)\right|^2 dx = \varepsilon^{\frac{1-2\alpha}{2-2\alpha}} \psi_\alpha^2(0)(2\alpha-1) \int_0^\infty \left(1+y^{2-2\alpha}\right)^{-\frac{1}{1-\alpha}} dy \\ + \int_0^1 x^{2-2\alpha} \left|\psi_\alpha'(x)\right|^2 dx + O\left(\varepsilon^{\frac{3-4\alpha}{2-2\alpha}}\right).\end{aligned}$$

Now, we estimate $\|v_\varepsilon\|_2^2$: Since $\psi_\alpha \in L^\infty$, we use the same estimate obtained for $\tilde{I}_4$, to write

$$\begin{aligned}\int_0^1 v_\varepsilon^2(x)dx &= \int_0^1 \left(\varepsilon + x^{2-2\alpha}\right)^{\frac{1-2\alpha}{1-\alpha}} \psi_\alpha^2(x)dx \\ &= \int_0^1 x^{2-4\alpha}\psi_\alpha^2(x)dx + \int_0^1 \left[\left(\varepsilon + x^{2-2\alpha}\right)^{\frac{1-2\alpha}{1-\alpha}} - x^{2-4\alpha}\right]\psi_\alpha^2(x)dx \\ &= \int_0^1 x^{2-4\alpha}\psi_\alpha^2(x)dx + O\left(\varepsilon^{\frac{3-4\alpha}{2-2\alpha}}\right).\end{aligned}$$

Finally, we estimate $\|v_\varepsilon\|_{p+1}^2$: the same idea used to estimate I_3 gives

$$\begin{aligned}\int_0^1 \left|v_\varepsilon(x)\right|^{p+1} dx &= \int_0^1 \left(\varepsilon + x^{2-2\alpha}\right)^{-\frac{1}{1-\alpha}} \left|\psi_\alpha(x)\right|^{p+1} dx \\ &= \left|\psi_\alpha(0)\right|^{p+1} \int_0^1 \left(\varepsilon + x^{2-2\alpha}\right)^{-\frac{1}{1-\alpha}} dx \\ &\quad + \int_0^1 \left(\varepsilon + x^{2-2\alpha}\right)^{-\frac{1}{1-\alpha}} \left[\left|\psi_\alpha(x)\right|^{p+1} - \left|\psi_\alpha(0)\right|^{p+1}\right] dx \\ &= \left|\psi_\alpha(0)\right|^{p+1} \int_0^1 \left(\varepsilon + x^{2-2\alpha}\right)^{-\frac{1}{1-\alpha}} dx + O\left(\varepsilon^{\frac{1-2\alpha}{2-2\alpha}}\right) \\ &= \varepsilon^{-\frac{1}{2-2\alpha}} \left|\psi_\alpha(0)\right|^{p+1} \int_0^\infty \left(1+y^{2-2\alpha}\right)^{-\frac{1}{1-\alpha}} dy \cdot (1+O(\varepsilon)).\end{aligned}$$

Using the definition of λ_α^* and ψ_α and the above estimates give

$$\begin{aligned}I_{\lambda,\alpha}(v_\varepsilon) &= \frac{\|x^\alpha v_\varepsilon'\|_2^2 - \lambda \|v_\varepsilon\|_2^2}{\|v_\varepsilon\|_p^2} \\ &= (2\alpha-1)K_3 + \varepsilon^{\frac{2\alpha-1}{2-2\alpha}} \left(\lambda_\alpha^* - \lambda\right) K_4 + O(\varepsilon)\end{aligned}$$

where

$$K_3 = \left[\int_0^\infty \left(1+y^{2-2\alpha}\right)^{-\frac{1}{1-\alpha}} dy\right]^{2-2\alpha} = \left[\frac{1}{2-2\alpha} \frac{\Gamma^2\left(\frac{1}{2-2\alpha}\right)}{\Gamma\left(\frac{1}{1-\alpha}\right)}\right]^{2-2\alpha},$$

and

$$K_4 = \left|\psi_\alpha(0)\right|^{-2} \left[\int_0^\infty \left(1+y^{2-2\alpha}\right)^{-\frac{1}{1-\alpha}} dy\right]^{1-2\alpha} \cdot \int_0^1 \left|x^{1-2\alpha}\psi_\alpha(x)\right|^2 dx < +\infty$$

Using lemma 26, one obtains that $K_3 = \frac{S_\alpha}{2\alpha-1}$. Now, since $\frac{1}{2} < \alpha < \frac{3}{4}$, for given $\lambda > \lambda_\alpha^*$ there exists $\varepsilon > 0$ such that $\varepsilon^{\frac{2\alpha-1}{2-2\alpha}} (\lambda_\alpha^* - \lambda) K_4 + O(\varepsilon) < 0$

$$S_{\lambda,\alpha} \leq I_{\lambda,\alpha}(v_\varepsilon) < S_\alpha,$$

thus concluding the proof.

□

The next results show that the solution obtained in Theorem 7 is in fact continuous up to the origin.

Lemma 4.2. *Let $\frac{1}{2} < \alpha < 1$ and $a(x) \in L^{q_\alpha}(0,1)$, where $q_\alpha = \frac{2\alpha}{2\alpha-2}$, and suppose $u \in L^2(0,1)$ solves*

$$\begin{cases} -(x^{2\alpha}u'(x))' = a(x)u(x) \quad \text{in } (0,1), \\ u(1) = 0, \\ \lim_{x\to 0^+} x^{2\alpha}u'(x)u(x) = 0, \end{cases} \tag{43}$$

then $u \in L^t(0,1)$ for all $t \geq 2$.

Corollary 4.3. *Let u be the solution given by Theorem 7, then $u \in C^0[0,1]$. Moreover $x^{2\alpha-1}u'$ and $x^{2\alpha}u''$ are also continuous up to the origin.*

Proof of Lemma 4.2. For a given positive integer n, define

$$u_n(x) := \begin{cases} 0 & \text{if } u(x) < 0, \\ u(x) & \text{if } 0 \leq u(x) \leq n, \\ n & \text{if } u(x) > n. \end{cases}$$

For fixed $\beta \geq 0$, let $\phi(x) = u^+(x)u_n^{2\beta}(x)$. Multiply equation (43) by ϕ and integrate by parts to obtain

$$\int_{u\geq 0} a(x)(u^+(x))^2 u_n^{2\beta} dx = \int_{u\geq 0} x^{2\alpha}u'(x)^2 u_n^{2\beta}(x)dx \\ + 2\beta \int_{0\leq u\leq n} x^{2\alpha}u'(x)^2(u^+(x))^{2\beta}dx.$$

On the other hand, we can write

$$\int_0^1 x^{2\alpha}\left|(u^+(x)u_n^\beta(x))'\right|^2 dx = \int_{u\geq 0} x^{2\alpha}u'(x)^2 u_n^{2\beta}(x)dx \\ + (\beta^2 + 2\beta)\int_{0\leq u\leq n} x^{2\alpha}u'(x)^2(u^+(x))^{2\beta}dx,$$

hence, with the aid of [22, Theorem A.2] one obtains for $M > 1$

$$\begin{aligned}\left(\int_0^1 \left|u^+(x)u_n^\beta(x)\right|^{2_\alpha}\right)^{\frac{2}{2_\alpha}} &\le C_{\alpha,\beta}\int_0^1 a(x)(u^+(x))^2u_n^{2\beta}(x)dx\\ &= C_{\alpha,\beta}\left(\int_{|a|\le M} a(x)(u^+(x))^2u_n^{2\beta}(x)dx\right.\\ &\quad\left.+\int_{|a|>M} a(x)(u^+(x))^2u_n^{2\beta}(x)dx\right)\\ &\le C_{\alpha,\beta}M\int_0^1 (u^+(x))^2u_n^{2\beta}(x)dx\\ &\quad + C_{\alpha,\beta}\left(\int_{|a|>M}|a(x)|^{q_\alpha}\right)^{\frac{1}{q_\alpha}}\left(\int_0^1\left|u^+(x)u_n^\beta(x)\right|^{2_\alpha}\right)^{\frac{2}{2_\alpha}}.\end{aligned}$$

Now, fixing $M = M_\beta$ sufficiently large so that $C_{\alpha,\beta}\left(\int_{|a|>M}|a(x)|^{q_\alpha}\right)^{\frac{1}{q_\alpha}} \le \frac{1}{2}$, gives

$$\left(\int_0^1 \left|u^+(x)u_n^\beta(x)\right|^{2_\alpha}\right)^{\frac{2}{2_\alpha}} \le 2MC_{\alpha,\beta}\int_0^1 u^+(x)^2u_n^{2\beta}(x)dx.$$

By passing to the limit $n\to\infty$ in the above inequality (notice that the constants do not depend on n), we obtain

$$\left(\int_0^1 (u^+(x))^{2_\alpha(1+\beta)}\right)^{\frac{2}{2_\alpha}} \le 2MC_{\alpha,\beta}\int_0^1 (u^+(x))^{2+2\beta}dx.$$

Similarly, one can prove the same inequality for u^-, thus obtaining

$$\left(\int_0^1 |u(x)|^{2_\alpha(1+\beta)}\right)^{\frac{2}{2_\alpha}} \le 2MC_{\alpha,\beta}\int_0^1 |u(x)|^{2+2\beta}\,dx.$$

The above inequality shows that if $u\in L^{2+2\beta}$, then $u\in L^{2_\alpha(1+\beta)}$. Since $u\in L^2$, we can start with $\beta_0 = 0$ and obtain $u\in L^{2_\alpha}$. So by letting $\beta_0 = 0$ and $\beta_{i+1} = \frac{2_\alpha}{2}(1+\beta_i)-1$, we obtain that

$$u\in L^{2_\alpha(1+\beta_i)},\ \text{for all } i\ge 0.$$

Notice that $\beta_i = \left(\frac{2_\alpha}{2}-1\right)\sum_{j=0}^i\left(\frac{2_\alpha}{2}\right)^j$, and since $2_\alpha > 2$ when $0<\alpha<1$, we obtain that $\beta_i\to\infty$, hence $u\in L^t$ for all $t\ge 1$, as claimed. □

Proof of Corollary 4.3. Notice first that by construction, the solution given by Theorem 7 satisfies equation (43), so lemma 4.2 applies, and $u\in L^t(0,1)$ for any $t\ge 1$. Now, we also now that $\lim\limits_{x\to 0^+} x^{2\alpha}u'(x) = 0$, so we can write

$$x^{2\alpha-1}u'(x) = \frac{1}{x}\int_0^x g(s)ds,$$

where $g(s) = -\lambda u(s) - u(s)^p$. Since $u\in L^t$ for all t, we obtain that $g\in L^t$ for all t, hence by Hardy's inequality, we obtain that $x^{2\alpha-1}u'(x)\in L^t$ for all t. This means that $u\in X_0^{2\alpha-1,t}$, so [22, Theorem A.2] applies and we deduce that if t is sufficiently large, $u\in C^0[0,1]$ (in fact one gets $u\in C^{0,\gamma}[0,1]$ for all $\gamma < 2-2\alpha$). So g above is also continuous, which in turn implies that $\lim\limits_{x\to 0^+}\frac{1}{x}\int_0^x g(s)ds$ exists, so

$x^{2\alpha-1}u'(x)$ must also be continuous. Finally the equation implies that $x^{2\alpha}u''(x) = -\lambda u(x) - u(x)^p - 2\alpha x^{2\alpha-1}u'(x) \in C^0[0,1]$. □

4.2. **An equation in the half-line.** In this section we will study the equation

$$-\left(x^{2\alpha}w'\right)' = |w|^{p-1}\, w \text{ in } (0,\infty), \tag{44}$$

where $p = 2_\alpha - 1$ and $\frac{1}{2} < \alpha < 1$. The motivation behind studying this equation comes from the fact that if u solves

$$-\left(x^{2\alpha}u'\right)' = \lambda u + |u|^{p-1}\, u \text{ in } (0,1), \tag{45}$$

then, $u_\delta(x) := \delta^{\alpha-\frac{1}{2}}u(\delta x)$ solves

$$-(x^{2\alpha}u_\delta')' = \lambda\delta^{2-2\alpha}u_\delta + |u_\delta|^{p-1}\, u_\delta \text{ in } (0,\delta^{-1}).$$

So, equation (44) is the limiting equation as $\delta \to 0$ (in a sense that will be made clear later) for u_δ, and for δ small enough u_δ should be close to a solution w of equation (44). If we are able to classify the solutions of equation (44), then we could understand how u is.

Equation (44) is the equation satisfied by the critical points of

$$J_\alpha(w) := \frac{\int_0^\infty |x^\alpha w'(x)|^2\, dx}{\left(\int_0^\infty |w(x)|^{2_\alpha}\, dx\right)^{\frac{2}{2_\alpha}}},$$

in particular $U_\varepsilon(x) = C_\varepsilon\left(\varepsilon + x^{2-2\alpha}\right)^{\frac{1-2\alpha}{2-2\alpha}}$, the extremal family for the Caffarelli-Kohn-Nirenberg inequality introduced in lemma 2.4 are solutions to equation (44). As we will see, these are the only solutions that are bounded at the origin, and this is the content of the following

Lemma 4.4. *Let $w \in C^2(0,\infty)$ be a solution of equation (44), then there are four possibilities*

(i) $w = U_\varepsilon$ for some $\varepsilon > 0$,
(ii) $w = Cx^{\frac{1}{2}-\alpha}$, where C is a normalization constant,
(iii) $w = x^{\frac{1}{2}-\alpha}f(-\ln x)$, where $f : [0,\infty) \to (0,\infty)$ is a periodic smooth function, which is bounded away from zero, or
(iv) $w = x^{\frac{1}{2}-\alpha}g(-\ln x)$, where $g : [0,\infty) \to (-\infty,\infty)$ is a sign changing periodic smooth function.

Proof. To prove this lemma, notice that if w solves equation (44), then $v(y) = e^{\left(\frac{1}{2}-\alpha\right)y}w(e^{-y})$ solves

$$v'' = \left(\alpha - \frac{1}{2}\right)^2 v - |v|^{p-1}\, v \quad \text{ in } \mathbb{R}. \tag{46}$$

The solutions of equation (46) can be easily classified by means of the energy functional

$$E(v)(y) := \frac{1}{2}v'(y)^2 - \frac{1}{2}\left(\alpha - \frac{1}{2}\right)^2 v(y)^2 + \frac{1}{p+1}\,|v(y)|^{p+1},$$

which is constant for every solution, as one can see by multiplying equation (46) by v'. By looking at the phase plane, one obtains that for

$$A := \min\left\{\frac{1}{2}a^2 - \left(\alpha - \frac{1}{2}\right)^2\frac{b^2}{2} + \frac{|b|^{p+1}}{p+1}; a, b \in \mathbb{R}\right\} < 0,$$

then

- ⋄ If $E(v) > 0$, then v must be a sign changing periodic function,
- ⋄ if $E(v) = 0$, then v is a homoclinic orbit for the unstable point $(0,0)$,
- ⋄ if $A < E(v) < 0$, then v is a periodic function that is bounded away from zero, and
- ⋄ if $E(v) = A$, then $v \equiv \pm \left[\frac{2\alpha-1}{4}\right]^{\frac{2\alpha-1}{4-4\alpha}}$.

The homoclinic orbit is given (up to translation) by

$$V(y) = \left(\frac{2\alpha-1}{4}\right)^{\frac{1}{p-1}} \left[\cosh\left(\frac{(p-1)(2\alpha-1)}{4}y\right)\right]^{-\frac{2}{p-1}}$$

and a direct computation shows that $U(x) = x^{\frac{1}{2}-\alpha}V(-\ln x)$. This finishes the proof. □

Remark 4.1. As seen in the proof, the energy functional

$$E(v) := \frac{1}{2}v'^2 - \frac{1}{2}\left(\alpha - \frac{1}{2}\right)^2 v^2 + \frac{1}{p+1}|v|^{p+1}$$

classifies the solutions of equation (46). Since it will be used later, let us introduce the corresponding energy functional for w solution of equation (44) by

(47)
$$E_0(w)(x) := E(v)(y) = \frac{1}{2}x^{2\alpha+1}w'(x)^2 + \frac{1}{p+1}|w(x)|^{p+1} + \left(\alpha - \frac{1}{2}\right)x^{2\alpha}w'(x)w(x),$$

where $v(y) := e^{\left(\frac{1}{2}-\alpha\right)y}w(e^{-y})$ and $y = -\ln x$. Notice that $E_0(w) = E_{0,0}(w)$, where $E_{\lambda,\beta}(u)$ is defined in (28). Now we can say that if $E_0(w) > 0$, then w is unbounded, with infinitely many sign changes near the origin. If $E_0(w) = 0$, then w is a bounded function which is positive (or negative) near the origin, and if $E_0(w) < 0$, then w is an unbounded function positive (or negative) near the origin.

Now, let us establish that if u solves equation (45), then $u_\delta(x) = \delta^{\alpha-\frac{1}{2}}u(\delta x)$ converges to a solution of equation (44), and this is the content of the following

Lemma 4.5. *Suppose $u \in C^2(0,1)$ solves equation (45). Suppose also that there exists a constant $C > 0$ such that*

(48)
$$|u(x)| \le Cx^{\frac{1}{2}-\alpha} \text{ and } |u'(x)| \le Cx^{-\frac{1}{2}-\alpha},$$

then there exists $w \in C^2(0,\infty)$ solution of equation (44) and a sequence $\delta_n \to 0$, such that for all $x > 0$

$$\lim_{\delta_n \to 0} |u_{\delta_n}(x) - w(x)| + \left|u'_{\delta_n}(x) - w'(x)\right| = 0.$$

Moreover, if

$$E_\lambda(u)(x) := E_{\lambda,0}(u)(x) = \frac{1}{2}x^{2\alpha+1}u'(x)^2 + \frac{\lambda}{2}xu(x)^2 + \frac{1}{p+1}x|u(x)|^{p+1} + \left(\alpha - \frac{1}{2}\right)x^{2\alpha}u'(x)u(x),$$

one has that $E := \lim_{x\to 0^+} E_\lambda(u)(x)$ exists and w is characterized by $E_0(w) = E$.

Remark 4.2. This type of lemma has already been proven by Benguria, Dolbeault and Esteban in [5], where they classify, among other things, the solutions of

$$\begin{cases} -\Delta u = \lambda u + |u|^{p-1} u & \text{in } B(0,1), \\ \quad u = 0 & \text{on } \partial B(0,1), \end{cases}$$

where $p = \frac{N+2}{N-2}$ is the critical Sobolev exponent.

Proof. Notice that by our assumption on the growth of u and u' and the definition of u_δ we have that

$$\left| x^{\alpha-\frac{1}{2}} u_\delta(x) \right| \le C \text{ and } \left| x^{\alpha+\frac{1}{2}} u'_\delta(x) \right| \le C$$

uniformly on δ. Also from the equation, one has that

$$\left| x^{\alpha+\frac{3}{2}} u''_\delta(x) \right| \le C.$$

By means of Arzela-Ascoli theorem, one can find a function $w \in C^1(0,\infty)$ and a sequence $\delta \to 0^+$ such that $u_\delta \to w$ and $u'_\delta \to w'$ uniformly in compacts subsets of $(0,\infty)$. Also, it is clear that w must solve equation (44), and as a consequence $w \in C^2(0,\infty)$.

What is left to prove is that $E = \lim\limits_{x\to 0} E_\lambda(u)(x)$ exists, is finite and that $E = E_0(w)$. To see this, notice that by lemma 2.5 we have

$$\frac{dE_\lambda(u)(x)}{dx} = \lambda(1-\alpha)u(x)^2,$$

where we have used $\beta = 0$ and $p = 2_\alpha - 1$. The above shows that $E_\lambda(u)(x)$ is monotone or constant (depending only on λ), so the limit exists in the extended sense. To see that $|E| < \infty$, notice that by the growth condition $u \in L^2(0,1)$, hence

$$|E| = \left| E_\lambda(u)(1) - \lambda(1-\alpha)\int_0^1 u(x)^2 dx \right| \le \frac{1}{2}u'(1)^2 + |\lambda|\,(1-\alpha)\int_0^1 u(x)^2 dx < \infty.$$

Finally, notice that for $x > 0$ and $\delta \to 0^+$ as before, $E_\lambda(u_\delta)(x) \to E_0(w)(x)$ and that $E_\lambda(u_\delta)(x) = E_\lambda(u)(\delta x) \to E$, so $E_0(w) = E$ as claimed. □

The way we will use the above results is in the form of the following direct corollary of lemmas 4.4 and 4.5

Corollary 4.6. *Let $u \in C^2(0,1)$ be as in lemma 4.5, and let $E = \lim\limits_{x\to 0^+} E_\lambda(u)(x)$. Then*

(i) If $E > 0$, then u is unbounded and has infinitely many sign changes near the origin.
(ii) If $E = 0$, then u is bounded and has a finite number of zeros in $(0,1)$.
(iii) If $E < 0$, then u is unbounded and has a finite number of zeros in $(0,1)$.

4.3. **Proof of Theorem 8.** We want to prove that if $\lambda \le \Lambda^*_\alpha$ then no solution exists. To do this, recall the definition of Λ^*_α

$$\Lambda^*_\alpha := \begin{cases} \lambda^*_\alpha & \text{if } \frac{1}{2} < \alpha < \frac{3}{4}, \\ 0 & \text{if } \frac{3}{4} \le \alpha < 1. \end{cases}$$

So we will first prove that no solution exists for all $\lambda \le 0$ and all $\frac{1}{2} < \alpha < 1$, and then we will prove that no solution exists when $0 < \lambda \le \lambda^*_\alpha$ and $\frac{1}{2} < \alpha < \frac{3}{4}$.

The case $\frac{1}{2}<\alpha<1$ ***and*** $\lambda\leq 0$***:***

In this case, we will use lemma 2.5 with $\beta=0$ and corollary 4.6 to show that if u is a solution equation (1), then $E_\lambda(u)>0$, hence u would have infinitely many sing changes near the origin, reaching a contradiction. From lemma 2.5 we obtain

$$E_\lambda(u)(x)=E_{\lambda,0}(u)(x)=\frac{1}{2}u'(1)^2-\lambda(1-\alpha)\int_x^1 u(s)^2ds - \left(\frac{1}{2}-\alpha+\frac{1}{p+1}\right)\int_x^1 u(s)^{p+1}ds.$$

But since $\lambda\leq 0$ and $p=2_\alpha-1$, we obtain that

$$E_\lambda(u)(x)\geq\frac{1}{2}u'(1)^2>0,$$

for every non-trivial solution. Now, by Remark 2.2 we have that

$$x^{\alpha-\frac{1}{2}}u(x)=x^{\alpha+\frac{1}{2}}u'(x)=O(1)$$

near the origin, so one can apply corollary 4.6 to conclude.

The case $\frac{1}{2}<\alpha<\frac{3}{4}$ ***and*** $0<\lambda\leq\lambda^*_\alpha$***:***

In order to prove this theorem, we need a better Pohozaev type identity that the one given by lemma 2.5. However, we will still use corollary 4.6, and show that $E_\lambda(u)(x)\geq a>0$ for all $x\sim 0$ (as we pointed out earlier, from Remark 2.2 one has that every solution u of equation (1) satisfies (48)).

Suppose that we have a function $\psi:(0,1)\to\mathbb{R}$ satisfying

$$\psi(x)\in C^2(0,1]\cap C^0[0,1] \text{ and } x\psi'(x)\in C^0[0,1]. \tag{49}$$

Multiply equation (1) by $u(x)\psi(x)$ and integrate over $[\varepsilon,1]$ to obtain

$$\begin{aligned}\lambda\int_\varepsilon^1 u(x)^2\psi(x)dx+\int_\varepsilon^1 u(x)^{p+1}\psi(x)dx &= \int_\varepsilon^1 x^{2\alpha}u'(x)\left(u(x)\psi(x)\right)'dx - x^{2\alpha}u'(x)u(x)\psi(x)\Big|_\varepsilon^1\\
&=\int_\varepsilon^1 x^{2\alpha}u'(x)^2\psi(x)dx+\int_\varepsilon^1 x^{2\alpha}u(x)u'(x)\psi'(x)dx - x^{2\alpha}u'(x)u(x)\psi(x)\Big|_\varepsilon^1\\
&=\int_\varepsilon^1 x^{2\alpha}u'(x)^2\psi(x)dx-\frac{1}{2}\int_\varepsilon^1\left(x^{2\alpha}\psi'(x)\right)'u(x)^2dx\\
&\quad+\frac{1}{2}x^{2\alpha}\psi'(x)u(x)^2\Big|_\varepsilon^1 - x^{2\alpha}u'(x)u(x)\psi(x)\Big|_\varepsilon^1.\end{aligned}$$

Since $u(1)=0$, we obtain

$$\begin{aligned}\int_\varepsilon^1 x^{2\alpha}\psi(x)u'(x)^2dx &= \int_\varepsilon^1 u(x)^2\left[\lambda\psi(x)+\frac{1}{2}\left(x^{2\alpha}\psi'(x)\right)'\right]dx\\
&\quad+\int_\varepsilon^1 u(x)^{p+1}\psi(x)dx-\varepsilon^{2\alpha}u'(\varepsilon)u(\varepsilon)\psi(\varepsilon)+\frac{1}{2}\varepsilon^{2\alpha}\psi'(\varepsilon)u(\varepsilon)^2.\end{aligned}\tag{50}$$

Suppose now that $\phi : (0,1) \to \mathbb{R}$ satisfies

(51) $$\phi \in C^1(0,1) \text{ and } x^{-1}\phi(x) \in C[0,1].$$

Multiply equation (1) by $u'(x)\phi(x)$ and integrate over $[\varepsilon, 1]$ to obtain

$$L.H.S = R.H.S,$$

where

$$L.H.S. = \frac{\lambda}{2}\int_\varepsilon^1 \left(u(x)^2\right)' \phi(x)dx + \frac{1}{p+1}\int_\varepsilon^1 \left(u(x)^{p+1}\right)' \phi(x)dx$$

and

$$R.H.S. = \int_\varepsilon^1 x^{2\alpha}u'(x)\left(u'(x)\phi(x)\right)' dx - x^{2\alpha}u'(x)^2\phi(x)\Big|_\varepsilon^1$$

For the right hand side one has

$$\begin{aligned}\int_\varepsilon^1 x^{2\alpha}u'(x)\left(u'(x)\phi(x)\right)' dx &= \int_\varepsilon^1 x^{2\alpha}u'(x)^2\phi'(x)dx + \frac{1}{2}\int_\varepsilon^1 x^{2\alpha}\phi(x)\left(u'(x)^2\right)' dx \\ &= \int_\varepsilon^1 u'(x)^2\left[x^{2\alpha}\phi'(x) - \frac{1}{2}\left(x^{2\alpha}\phi(x)\right)'\right] dx \\ &\quad + \frac{1}{2}x^{2\alpha}u'(x)^2\phi(x)\Big|_\varepsilon^1.\end{aligned}$$

so we have

(52)
$$R.H.S. = \int_\varepsilon^1 u'(x)^2\left[x^{2\alpha}\phi'(x) - \frac{1}{2}\left(x^{2\alpha}\phi(x)\right)'\right] dx - \frac{1}{2}u'(1)^2\phi(1) + \frac{1}{2}\varepsilon^{2\alpha}u'(\varepsilon)^2\phi(\varepsilon).$$

Whereas for the left hand side

(53)
$$\begin{aligned}L.H.S. &= -\frac{\lambda}{2}\int_\varepsilon^1 u(x)^2\phi'(x)dx - \frac{1}{p+1}\int_\varepsilon^1 u(x)^{p+1}\phi'(x)dx + \frac{\lambda}{2}u(x)^2\phi(x)\Big|_\varepsilon^1 \\ &\quad + \frac{1}{p+1}u(x)^{p+1}\phi(x)\Big|_\varepsilon^1 \\ &= -\frac{\lambda}{2}\int_\varepsilon^1 u(x)^2\phi'(x)dx - \frac{1}{p+1}\int_\varepsilon^1 u(x)^{p+1}\phi'(x)dx - \frac{\lambda}{2}u(\varepsilon)^2\phi(\varepsilon) \\ &\quad - \frac{1}{p+1}u(\varepsilon)^{p+1}\phi(\varepsilon).\end{aligned}$$

Putting together (52) and (53) gives

(54)
$$\begin{aligned}\int_\varepsilon^1 u'(x)^2\left[x^{2\alpha}\phi'(x) - \frac{1}{2}\left(x^{2\alpha}\phi(x)\right)'\right] dx &= \frac{1}{2}u'(1)^2\phi(1) \\ &\quad - \frac{\lambda}{2}\int_\varepsilon^1 u(x)^2\phi'(x)dx - \frac{1}{p+1}\int_\varepsilon^1 u(x)^{p+1}\phi'(x)dx \\ &\quad - \varepsilon^{-1}\phi(\varepsilon)\left(\frac{1}{2}\varepsilon^{2\alpha+1}u'(\varepsilon)^2 + \frac{\lambda}{2}\varepsilon u(\varepsilon)^2 + \frac{1}{p+1}\varepsilon u(\varepsilon)^{p+1}\right)\end{aligned}$$

Finally, suppose there exist ψ and ϕ satisfying (49) and (51) respectively, which also satisfy the following system of ODEs

$$\begin{cases} x^{2\alpha}\phi'(x) - \frac{1}{2}\left(x^{2\alpha}\phi(x)\right)' - x^{2\alpha}\psi(x) = 0, \\ \lambda\psi(x) + \frac{1}{2}\left(x^{2\alpha}\psi'(x)\right)' + \frac{\lambda}{2}\phi'(x) = 0, \end{cases} \tag{55}$$

then from (50) and (54) we deduce

$$\int_\varepsilon^1 u(x)^{p+1}\left[\psi(x) + \frac{1}{p+1}\phi'(x)\right]dx = \frac{1}{2}u'(1)^2\phi(1) + \varepsilon^{2\alpha}u'(\varepsilon)u(\varepsilon)\psi(\varepsilon) \tag{56}$$
$$- \frac{1}{2}\varepsilon^{2\alpha}\psi'(\varepsilon)u(\varepsilon)^2 - \varepsilon^{-1}\phi(\varepsilon)\left(\frac{1}{2}\varepsilon^{2\alpha+1}u'(\varepsilon)^2 + \frac{\lambda}{2}\varepsilon u(\varepsilon)^2 + \frac{1}{p+1}\varepsilon u(\varepsilon)^{p+1}\right).$$

In order to continue, we need to prove the existence of the functions ψ and ϕ and understand their behavior near 0, and this is content of the following

Lemma 4.7. *Let* $\frac{1}{2} < \alpha < \frac{3}{4}$ *and* $0 < \lambda \le \lambda_\alpha^*$. *Define*

$$\phi(x) := xJ_\nu\left(\frac{\sqrt{\lambda}}{1-\alpha}x^{1-\alpha}\right)J_{-\nu}\left(\frac{\sqrt{\lambda}}{1-\alpha}x^{1-\alpha}\right), \tag{57}$$

where ν *and* J_ν *are defined by* (18) *and* (20) *respectively. Let*

$$\psi(x) := \frac{1}{2}\phi'(x) - \frac{\alpha}{x}\phi(x). \tag{58}$$

Then ψ, ϕ *satisfy* (49),(51) *and* (55), *moreover we have that for* $p \ge 2_\alpha - 1$

$$\psi(x) + \frac{1}{p+1}\phi'(x) < 0 \text{ for all } 0 < x < 1, \tag{59}$$

$$\phi(1) \ge 0. \tag{60}$$

Also, there exist constants $A > 0$ *and* $B \in \mathbb{R}$, *such that for* $x \sim 0$

$$\phi(x) = Ax + O(x^{3-2\alpha})$$
$$\psi(x) = \left(\frac{1}{2} - \alpha\right)A + Bx^{2-2\alpha} + O(x^{4-4\alpha}).$$

We postpone the proof of this lemma for the end if this section. The proof of Theorem 8 continues in the following way: using ψ, ϕ from lemma 4.7 in (56) gives

$$0 > \int_\varepsilon^1 u(x)^{p+1}\left[\psi(x) + \frac{1}{p+1}\phi'(x)\right]dx = \frac{1}{2}u'(1)^2\phi(1) - AE_\lambda(u)(\varepsilon) + R(\varepsilon),$$

where

$$R(\varepsilon) = AE_\lambda(u)(\varepsilon) - \varepsilon^{-1}\phi(\varepsilon)\left(\frac{1}{2}\varepsilon^{2\alpha+1}u'(\varepsilon)^2 + \frac{\lambda}{2}\varepsilon u(\varepsilon)^2 + \frac{1}{p+1}\varepsilon u(\varepsilon)^{p+1}\right)$$
$$+ \varepsilon^{2\alpha}u'(\varepsilon)u(\varepsilon)\psi(\varepsilon) - \frac{1}{2}\varepsilon^{2\alpha}\psi'(\varepsilon)u(\varepsilon)^2.$$

If we can prove that $R(\varepsilon) = o(1)$ for every u solution of equation (1), then the above inequality would imply

$$E_\lambda(u)(\varepsilon) > \frac{1}{2A}u'(1)^2\phi(1) - o(1),$$

so $E = \lim_{\varepsilon\to 0^+} E_\lambda(u)(\varepsilon) > \frac{1}{2A}u'(1)^2\phi(1) \geq 0$ for every solution, then by corollary 4.6 u would have infinitely many sign changes. Hence equation (1) has no solution.

So everything reduces to prove that $R(\varepsilon) = o(1)$, which follows directly from Remark 2.2 and the expansions of ϕ and ψ from lemma 4.7. We omit the details. □

Proof of Lemma 4.7. A tedious but straightforward computation shows that ϕ and ψ, defined by (57) and (58) respectively, indeed solve the system (55). From (57) and a formula from [52, p. 147] we obtain that

$$\phi(x) = \sum_{m=0}^{\infty} \frac{(-1)^m(2m)!\lambda^m}{4^m m!^2\,\Gamma(m+1+\nu)\Gamma(m+1-\nu)(1-\alpha)^{2m}} x^{1+2m(1-\alpha)}, \tag{61}$$

which readily gives (49) and (51). To prove (59), notice that we can write

$$\psi(x) + \frac{1}{p+1}\phi'(x) = \left(\frac{1}{2} - \alpha + \frac{1}{p+1}\right) J_\nu(y)J_{-\nu}(y) + (1-\alpha)\left(\frac{1}{2} + \frac{1}{p+1}\right) y\left[J_\nu(y)J'_{-\nu}(y) + J'_\nu(y)J_{-\nu}(y)\right],$$

where $y = \frac{\sqrt{\lambda}}{1-\alpha}x^{1-\alpha}$. Since $\frac{1}{2} - \alpha + \frac{1}{p+1} \leq 0$ for all $p \geq 2_\alpha - 1$, it is enough to prove that $J_\nu(y)J_{-\nu}(y) > 0$ for $y \in (0, j_{-\nu 1})$ (which is obviously true since $j_{-\nu 1} < j_{\nu 1}$), and that

$$J_\nu(y)J'_{-\nu}(y) + J'_\nu(y)J_{-\nu}(y) < 0 \text{ for } y \in (0, j_{-\nu 1}).$$

To do this, notice that using recurrence formulas from [52, p. 45] gives

$$J_\nu(y)J'_{-\nu}(y) + J'_\nu(y)J_{-\nu}(y) = -\left(J_\nu(y)J_{1-\nu}(y) + J_{-\nu}(y)J_{1+\nu}(y)\right),$$

but

- ⋄ $J_{-\nu}(y) > 0$, because $y \leq j_{-\nu 1}$;
- ⋄ $J_{1-\nu}(y) > 0$, because $y \leq j_{-\nu 1} < j_{(1-\nu)1}$;
- ⋄ $J_\nu(y) > 0$, because $y \leq j_{-\nu 1} < j_{\nu 1}$; and
- ⋄ $J_{1+\nu}(y) > 0$, because $y \leq j_{-\nu 1} < j_{(1+\nu)1}$;

thus every term inside the parentheses is positive. Observe that

$$\frac{\sqrt{\lambda}}{1-\alpha} \leq \frac{\sqrt{\lambda^*_\alpha}}{1-\alpha} = j_{-\nu 1} < \frac{\sqrt{\lambda_1}}{1-\alpha} = j_{\nu 1},$$

so $J_\nu\left(\frac{\sqrt{\lambda}}{1-\alpha}\right) > 0$ and $J_{-\nu}\left(\frac{\sqrt{\lambda}}{1-\alpha}\right) \geq 0$, which implies $\phi(1) \geq 0$, with equality if and only if $\lambda = \lambda^*_\alpha$.

Finally, the expansions near the origin of ϕ and ψ follow directly from (61), we just need to verify that $A > 0$, which is true since

$$A = \frac{1}{\Gamma(1+\nu)\Gamma(1-\nu)} > 0.$$

□

5. The super-critical case: $p > 2_\alpha - 1$

Proof of Theorem 9. Suppose u solves equation (1). With the aid of lemma 2.5, with $\beta = 0$, and lemma 2.13, we obtain

$$\lambda(1-\alpha)\int_0^1 u(x)^2dx + \left(\frac{1}{2} - \alpha + \frac{1}{p+1}\right)\int_0^1 u(x)^{p+1}dx = \frac{1}{2}u'(1)^2 > 0,$$

but $\frac{1}{2} - \alpha + \frac{1}{p+1} < 0$, so the above gives

$$\int_0^1 u(x)^{p+1}dx < \frac{\lambda(1-\alpha)}{\alpha - \frac{1}{2} - \frac{1}{p+1}}\int_0^1 u(x)^2dx.$$

Now, notice that

$$\begin{aligned}\lambda_1\int_0^1 u(x)^2dx &< \int_0^1 x^{2\alpha}u'(x)^2dx\\ &= \lambda\int_0^1 u(x)^2 + \int_0^1 u(x)^{p+1}dx\\ &\leq \left[\lambda + \frac{\lambda(1-\alpha)}{\alpha - \frac{1}{2} - \frac{1}{p+1}}\right]\int_0^1 u(x)^2dx,\end{aligned}$$

thus for every solution of equation (1) one has

$$\lambda > \lambda_1\left(\frac{\alpha - \frac{1}{2} - \frac{1}{p+1}}{\frac{1}{2} - \frac{1}{p+1}}\right).$$

The above shows that if $\lambda \leq \lambda_1\left(\frac{\alpha - \frac{1}{2} - \frac{1}{p+1}}{\frac{1}{2} - \frac{1}{p+1}}\right)$, then there is no solution. □

6. The case $\alpha \geq 1$

Proof of Theorem 10. We again use lemma 2.5, with $\beta = 0$, and lemma 2.13 to obtain, for $p > 1$,

$$\lambda(1-\alpha)\int_0^1 u(x)^2dx + \left(\frac{1}{2} - \alpha + \frac{1}{p+1}\right)\int_0^1 u(x)^{p+1}dx = \frac{1}{2}u'(1)^2 > 0.$$

Notice that if $\alpha = 1$, then the above yields

$$\left(\frac{1}{2} - \alpha + \frac{1}{p+1}\right)\int_0^1 u(x)^{p+1}dx > 0$$

which is impossible for $p > 1$, hence no solution exists if $\alpha = 1$ and $\lambda \in \mathbb{R}$. On the other hand, if $\alpha > 1$ and $\lambda \geq 0$ we obtain

$$0 > \lambda(1-\alpha)\int_0^1 u(x)^2dx + \left(\frac{1}{2} - \alpha + \frac{1}{p+1}\right)\int_0^1 u(x)^{p+1}dx > 0,$$

also impossible. Finally, if $\alpha > 1$ and $\lambda < 0$, the above gives

$$\int_0^1 u(x)^{p+1}dx < \frac{\lambda(1-\alpha)}{\alpha - \frac{1}{2} - \frac{1}{p+1}}\int_0^1 u(x)^2dx.$$

Now, multiply equation (1) by u, integrate by parts with the aid of Remark 2.2 to obtain

$$\begin{aligned}\int_0^1 x^{2\alpha}u'(x)^2dx &= \lambda\int_0^1 u(x)^2dx + \int_0^1 u(x)^{p+1}dx\\ &< \lambda\left(1+\frac{(1-\alpha)}{\alpha-\frac{1}{2}-\frac{1}{p+1}}\right)\int_0^1 u(x)^2dx,\end{aligned}$$

but, since $\lambda < 0$, $p > 1$ and $\alpha > 1$ we obtain

$$\lambda\left(1+\frac{(1-\alpha)}{\alpha-\frac{1}{2}-\frac{1}{p+1}}\right) = \frac{\lambda(p-1)}{2\left(\alpha-\frac{1}{2}-\frac{1}{p+1}\right)(p+1)} < 0.$$

Therefore

$$0 < \int_0^1 x^{2\alpha}u'(x)^2dx < \lambda\left(1+\frac{(1-\alpha)}{\alpha-\frac{1}{2}-\frac{1}{p+1}}\right)\int_0^1 u(x)^2dx < 0,$$

impossible. □

7. The case $0 < \alpha < \frac{1}{2}$

Proof of Theorem 3. The proof of the existence of a minimizer v_0 of

$$S_{\lambda,\alpha,0} := \inf_{v\in\mathcal{M}_0} I_{\lambda,\alpha}(v).$$

is a line by line copy of the proof of Theorems 1 and 5, where the only change is that instead of minimizing $I_{\alpha,\lambda}$ over $\mathcal{M} = X_0^\alpha \cap \left\{\|u\|_{p+1} = 1\right\}$, we do it over $\mathcal{M}_0 = X_{00}^\alpha \cap \left\{\|u\|_{p+1} = 1\right\}$. Then if one defines $u_0(x) = S_{\lambda,\alpha,0}^{\frac{1}{p-2}}|v_0(x)|$, we obtain a solution of

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + u^p & \text{in } (0,1),\\ u > 0 & \text{in } (0,1),\\ u(1) = u(0) = 0. \end{cases}$$

The regularity properties follow immediately from the fact that $X_0^\alpha \hookrightarrow C[0,1]$ for all $\alpha < \frac{1}{2}$, which implies that $u \in C[0,1]$ and as a consequence $x^{2\alpha}u' \in C^1[0,1]$ and $x^{2\alpha-1}u \in C[0,1]$. The details are left to the reader. □

Proof of Theorem 4. To prove this theorem we assume we have a solution and we multiply equation (1) by $\varphi_{1,0}$, the first eigenfunction of equation (6), and we integrate by parts over $[\varepsilon, 1]$ to obtain

$$(\lambda-\lambda_{1,0})\int_\varepsilon^1 u(x)\varphi_{1,0}(x)dx + \int_\varepsilon^1 u(x)^p\varphi_{1,0}(x)dx = \varepsilon^{2\alpha}u'(\varepsilon)\varphi_{1,0}(\varepsilon) - \varepsilon^{2\alpha}\varphi_{1,0}'(\varepsilon)u(\varepsilon).$$

To reach a contradiction, we need to understand what happens to the boundary terms. Since $\lambda \geq \lambda_{1,0} > 0$, we obtain from equation (1) that

$$-(x^{2\alpha}u'(x))' = \lambda u + u^{p+1} \geq 0.$$

If we integrate twice we get

$$u(x) \leq -u'(1)\left(\frac{1-x^{1-2\alpha}}{1-2\alpha}\right),$$

which implies, since $\alpha < \frac{1}{2}$, that $0 < u(x) \leq C = C(u'(1))$ for all $0 < x < 1$, thus $-\lambda C - C^{p+1} \leq (x^{2\alpha}u')' \leq 0$, and we conclude that $\left|x^{2\alpha}u'\right|$ is bounded. Therefore, since $\varphi_{1,0}(\varepsilon) = o(1)$, we can write $\varepsilon^{2\alpha}u'(\varepsilon)\varphi_{1,0}(\varepsilon) = o(1)$ as $\varepsilon \to 0^+$.

On the other hand, it can be seen from the definition of $\varphi_{1,0}$ that $x^{2\alpha}\varphi_{1,0}'(x) \geq 0$ for all $x \sim 0$, so we have $\varepsilon^{2\alpha}\varphi_{1,0}'(\varepsilon)u(\varepsilon) \geq 0$. Therefore

$$(\lambda - \lambda_{1,0}) \int_\varepsilon^1 u(x)\varphi_{1,0}(x)dx + \int_\varepsilon^1 u(x)^p\varphi_{1,0}(x)dx \leq o(1), \text{ for all } \varepsilon > 0$$

but since $\lambda \geq \lambda_{1,0}$, $\varphi_{1,0} > 0$ and $u > 0$, we reach a contradiction when we send ε to 0.

□

Acknowledgments

This work was completed during the author's Ph.D. research at Rutgers University, to which I am grateful for its stimulating atmosphere.

The author would like to express his gratitude to Prof. H. Brezis for suggesting the problem and for his valuable comments during the elaboration of this article. I also would like to thank Prof. L. A. Peletier for the enlightening discussions regarding section 1.5.

Work partially supported by NSF Grant DMS 1207793.

References

[1] Frederick V. Atkinson and Lambertus A. Peletier, *Emden-Fowler equations involving critical exponents*, Nonlinear Anal. **10** (1986), no. 8, 755–776. MR: 851145 (87j:34039)

[2] ———, *Large solutions of elliptic equations involving critical exponents*, Asymptotic Anal. **1** (1988), no. 2, 139–160. MR: 950011 (89i:35004)

[3] Catherine Bandle and Rafael Benguria, *The Brézis-Nirenberg problem on* $\mathbb{S}^3$, J. Differential Equations **178** (2002), no. 1, 264–279. MR: 1878530 (2003c:35045)

[4] Catherine Bandle and Lambertus A. Peletier, *Best Sobolev constants and Emden equations for the critical exponent in* S^3, Math. Ann. **313** (1999), no. 1, 83–93. MR: 1666821 (2001c:35030)

[5] Rafael D. Benguria, Jean Dolbeault, and Maria J. Esteban, *Classification of the solutions of semilinear elliptic problems in a ball*, J. Differential Equations **167** (2000), no. 2, 438–466. MR: 1793200 (2001k:35083)

[6] Rafael D. Benguria, Rupert L. Frank, and Michael Loss, *The sharp constant in the Hardy-Sobolev-Maz'ya inequality in the three dimensional upper half-space*, Math. Res. Lett. **15** (2008), no. 4, 613–622. MR: 2424899 (2009j:46074)

[7] Henri Berestycki, *On some nonlinear Sturm-Liouville problems*, J. Differential Equations **26** (1977), no. 3, 375–390. MR: 0481230 (58 #1358)

[8] Henri Berestycki and Maria J. Esteban, *Existence and bifurcation of solutions for an elliptic degenerate problem*, J. Differential Equations **134** (1997), no. 1, 1–25. MR: 1429089 (97k:34052)

[9] Haim Brezis and Elliott Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), no. 3, 486–490. MR: 699419 (84e:28003)

[10] Haim Brezis and Louis Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), no. 4, 437–477. MR: 709644 (84h:35059)

[11] Haim Brezis and Lambertus A. Peletier, *Asymptotics for elliptic equations involving critical growth*, Partial differential equations and the calculus of variations, Vol. I, 1989, pp. 149–192. MR: 1034005 (91a:35030)

[12] ———, *Elliptic equations with critical exponent on spherical caps of* $\mathbb{S}^3$, J. Anal. Math. **98** (2006), 279–316.

[13] Chris J. Budd and John Norbury, *Semilinear elliptic equations and supercritical growth*, J. Differential Equations **68** (1987), no. 2, 169–197. MR: 892022 (88i:35056)

[14] Chris J. Budd and Lambertus A. Peletier, *Asymptotics for semilinear elliptic equations with supercritical nonlinearities in annular domains*, Asymptotic Anal. **6** (1993), no. 3, 219–239. MR: 1201194 (94b:35110)

[15] Luis A. Caffarelli, Robert V. Kohn, and Louis Nirenberg, *First order interpolation inequalities with weights*, Compositio Math. **53** (1984), no. 3, 259–275. MR: 768824 (86c:46028)

[16] Paolo Caldiroli and Roberta Musina, *On the existence of extremal functions for a weighted Sobolev embedding with critical exponent*, Calc. Var. Partial Differential Equations **8** (1999), no. 4, 365–387. MR: 1700269 (2000d:35064)

[17] ———, *Existence and nonexistence results for a class of nonlinear, singular Sturm-Liouville equations*, Adv. Differential Equations **6** (2001), no. 3, 303–326. MR: 1799488 (2001i:34024)

[18] Daomin Cao and YanYan Li, *Results on positive solutions of elliptic equations with a critical Hardy-Sobolev operator*, Methods Appl. Anal. **15** (2008), no. 1, 81–95. MR: 2482211 (2010g:35111)

[19] Alberto Capozzi, Donato Fortunato, and Giuliana Palmieri, *An existence result for nonlinear elliptic problems involving critical Sobolev exponent*, Ann. Inst. H. Poincaré Anal. Non Linéaire **2** (1985), no. 6, 463–470. MR: 831041 (87j:35126)

[20] Alfonso Castro and Alan C. Lazer, *Critical point theory and the number of solutions of a nonlinear Dirichlet problem*, Ann. Mat. Pura Appl. (4) **120** (1979), 113–137. MR: 551063 (81d:58022)

[21] Hernán Castro, *Uniqueness results for a singular non-linear Sturm-Liouville equation*, 2012. Preprint.

[22] Hernán Castro and Hui Wang, *A singular Sturm-Liouville equation under homogeneous boundary conditions*, J. Funct. Anal. **261** (2011), no. 6, 1542–1590. MR: 2813481 (2012f:34056)

[23] Florin Catrina and Zhi-Qiang Wang, *On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions*, Comm. Pure Appl. Math. **54** (2001), no. 2, 229–258. MR: 1794994 (2001k:35028)

[24] Giovanna Cerami, Donato Fortunato, and Michael Struwe, *Bifurcation and multiplicity results for nonlinear elliptic problems involving critical Sobolev exponents*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), no. 5, 341–350. MR: 779872 (86e:35016)

[25] Giovanna Cerami, Sergio Solimini, and Michael Struwe, *Some existence results for superlinear elliptic boundary value problems involving critical exponents*, J. Funct. Anal. **69** (1986), no. 3, 289–306. MR: 867663 (88b:35074)

[26] Michael G. Crandall and Paul H. Rabinowitz, *Bifurcation from simple eigenvalues*, J. Functional Analysis **8** (1971), 321–340. MR: 0288640 (44 #5836)

[27] Colette De Coster, Maria do Rosário Grossinho, and Patrick Habets, *On pairs of positive solutions for a singular boundary value*, Appl. Anal. **59** (1995), no. 1-4, 241–256. MR: 1378039 (97d:34016)

[28] Colette De Coster and Michel Willem, *Density, spectral theory and homoclinics for singular Sturm-Liouville systems*, J. Comput. Appl. Math. **52** (1994), no. 1-3, 45–70. Oscillations in nonlinear systems: applications and numerical aspects. MR: 1310122 (95m:34048)

[29] Djairo Guedes de Figueiredo, Pierre-Louis Lions, and Roger D. Nussbaum, *A priori estimates and existence of positive solutions of semilinear elliptic equations*, J. Math. Pures Appl. (9) **61** (1982), no. 1, 41–63. MR: 664341 (83h:35039)

[30] Michel Duhoux, *Maximum and anti-maximum principles for singular Sturm-Liouville problems*, Proc. Roy. Soc. Edinburgh Sect. A **128** (1998), no. 3, 525–547. MR: 1632831 (99e:34028)

[31] ———, *Nonlinear singular Sturm-Liouville problems*, Nonlinear Anal. **38** (1999), no. 7, Ser. A: Theory Methods, 897–918. MR: 1711754 (2000i:34045)

[32] Dennis R. Dunninger and J. Cleo Kurtz, *Existence of solutions for some nonlinear singular boundary value problems*, J. Math. Anal. Appl. **115** (1986), no. 2, 396–405. MR: 836234 (87e:34027)

[33] Árpád. Elbert, *Concavity of the zeros of Bessel functions*, Studia Sci. Math. Hungar. **12** (1977), no. 1-2, 81–88 (1980). MR: 568466 (81d:33004)

[34] Maria J. Esteban, *Multiple solutions of semilinear elliptic problems in a ball*, J. Differential Equations **57** (1985), no. 1, 112–137. MR: 788425 (87f:35100)

[35] Basilis Gidas, Wei Ming Ni, and Louis Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), no. 3, 209–243. MR: 544879 (80h:35043)

[36] Pierre-Louis Lions, *On the existence of positive solutions of semilinear elliptic equations*, SIAM Rev. **24** (1982), no. 4, 441–467. MR: 678562 (84a:35093)
[37] Gianni Mancini and Kunnath Sandeep, *On a semilinear elliptic equation in* $\mathbb{H}^n$, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **7** (2008), no. 4, 635–671. MR: 2483639 (2011d:35170)
[38] Frank Merle and Lambertus A. Peletier, *Positive solutions of elliptic equations involving supercritical growth*, Proc. Roy. Soc. Edinburgh Sect. A **118** (1991), no. 1-2, 49–62. MR: 1113842 (92g:35074)
[39] Frank Merle, Lambertus A. Peletier, and James Serrin, *A bifurcation problem at a singular limit*, Indiana Univ. Math. J. **43** (1994), no. 2, 585–609. MR: 1291530 (95j:35024)
[40] Bohumír Opic and Alois Kufner, *Hardy-type inequalities*, Pitman Research Notes in Mathematics Series, vol. 219, Longman Scientific & Technical, Harlow, 1990. MR: 1069756 (92b:26028)
[41] Zhao Peihao and Zhong Chengkui, *On the infinitely many positive solutions of a supercritical elliptic problem*, Nonlinear Anal. **44** (2001), no. 1, Ser. A: Theory Methods, 123–139. MR: 1815695 (2002b:35066)
[42] Robert Piessens, *A series expansion for the first positive zero of the Bessel functions*, Math. Comp. **42** (1984), no. 165, 195–197. MR: 725995 (84m:33014)
[43] Patrizia Pucci and James Serrin, *A general variational identity*, Indiana Univ. Math. J. **35** (1986), no. 3, 681–703. MR: 855181 (88b:35072)
[44] ———, *Critical exponents and critical dimensions for polyharmonic operators*, J. Math. Pures Appl. (9) **69** (1990), no. 1, 55–83. MR: 1054124 (91i:35065)
[45] Paul H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Functional Analysis **7** (1971), 487–513. MR: 0301587 (46 #745)
[46] ———, *Variational methods for nonlinear elliptic eigenvalue problems*, Indiana Univ. Math. J. **23** (1973/74), 729–754. MR: 0333442 (48 #11767)
[47] Charles A. Stuart, *Bifurcation from the essential spectrum*, Equadiff 82 (Würzburg, 1982), 1983, pp. 575–596. MR: 726615 (85f:35163)
[48] ———, *Bifurcation from the essential spectrum*, Topological nonlinear analysis, II (Frascati, 1995), 1997, pp. 397–443. MR: 1453894 (98f:47070)
[49] ———, *Buckling of a heavy tapered rod*, J. Math. Pures Appl. (9) **80** (2001), no. 3, 281–337. MR: 1826347 (2002b:74023)
[50] Charles A. Stuart and Grégory Vuillaume, *Buckling of a critically tapered rod: global bifurcation*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **459** (2003), no. 2036, 1863–1889. MR: 1993662 (2004e:47108)
[51] ———, *Buckling of a critically tapered rod: properties of some global branches of solutions*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **460** (2004), no. 2051, 3261–3282. MR: 2098717 (2005g:34046)
[52] George N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, England, 1944. MR: 0010746 (6,64a)

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