A HARDY TYPE INEQUALITY FOR $W^{m,1}(0,1)$ FUNCTIONS

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Abstract. In this paper, we consider functions $u \in W^{m,1}(0,1)$ where $m \geq 2$ and $u(0) = Du(0) = \ldots = D^{m-1}u(0) = 0$. Although it is not true in general that $\frac{D^j u(x)}{x^{m-j}} \in L^1(0,1)$ for $j \in \{0, 1, \ldots, m-1\}$, we prove that $\frac{D^j u(x)}{x^{m-j-k}} \in W^{k,1}(0,1)$ if $k \geq 1$ and $1 \leq j+k \leq m$, with j,k integers. Furthermore, we have the following Hardy type inequality,

$$\left\| D^k \left(\frac{D^j u(x)}{x^{m-j-k}} \right) \right\|_{L^1(0,1)} \leq \frac{(k-1)!}{(m-j-1)!} \left\| D^m u \right\|_{L^1(0,1)},$$

where the constant is optimal.

1. Introduction

It is well known ([1]) that if $u \in W^{1,p}(0,1)$ and $u(0) = 0$ then the so called Hardy inequality holds for $p > 1$, that is

$$\int_0^1 \left| \frac{u(x)}{x} \right|^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^1 \left| u'(x) \right|^p dx. \tag{1}$$

The constant $\frac{p}{p-1}$ is optimal for this inequality and it blows up as p goes to 1. This behaviour is confirmed by the fact that no such inequality can be proved when $p = 1$, as we can consider (see e.g. [2]) the non-negative function on $(0,1)$ defined by

$$v(x) = \frac{1}{1 - \log x}. \tag{2}$$

A simple computation shows that this function belongs to $W^{1,1}(0,1), u(0) = 0$, but $\frac{u(x)}{x}$ is not integrable.

When we turn to functions $u \in W^{2,p}(0,1)$, $p \geq 1$, with $u(0) = u'(0) = 0$, there are three natural quantities to consider: $\frac{u(x)}{x^2}$, $\frac{u'(x)}{x}$ and $\left(\frac{u(x)}{x}\right)' = \frac{u'(x)}{x} - \frac{u(x)}{x^2}$. If $p > 1$, it is clear that both $\frac{u'(x)}{x}$ and $\frac{u(x)}{x^2} = \frac{u'(x)}{x} - \frac{1}{x^2}\int_0^x tu''(t)dt$ belong to $L^p(0,1)$. Thus $\left(\frac{u(x)}{x}\right)' \in L^p(0,1)$. If $p = 1$ one can no longer assert that $\frac{u(x)}{x^2}, \frac{u'(x)}{x}$ belong to $L^1(0,1)$, but surprisingly $\left(\frac{u(x)}{x}\right)' \in L^1(0,1)$. This reflects a "magic" cancellation of the non-integrable terms in the difference $\left(\frac{u(x)}{x}\right)' = \frac{u'(x)}{x} - \frac{u(x)}{x^2}$.

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The same phenomenon remains valid when we keep increasing the number of derivatives, and this is the main result of this paper.

Definition 1.1. *We say that u has the property (P_m) if*

$$u \in W^{m,1}(0,1) \text{ and } u(0) = Du(0) = \ldots = D^{m-1}u(0) = 0,$$

where $D^i u$ denotes the i-th derivative of u.

Theorem 1.2. *If u has the property (P_m) and j, k are non-negative integers, then*

(1) *If $k \geq 1$ and $1 \leq j + k \leq m$ then $\frac{D^j u(x)}{x^{m-j-k}}$ has the property (P_k) and*

$$\left\| D^k \left(\frac{D^j u(x)}{x^{m-j-k}} \right) \right\|_{L^1(0,1)} \leq \frac{(k-1)!}{(m-j-1)!} \left\| D^m u \right\|_{L^1(0,1)} . \tag{3}$$

The constant being the best possible.

(2) *There exists w having the property (P_m) such that*

$$\frac{D^j w(x)}{x^{m-j}} \notin L^1(0,1) \quad \text{for all } j \in \{0, 1, \ldots, m-1\} . \tag{4}$$

Remark 1.1. For functions $u \in W^{2,p}(0,1)$, $p > 1$, with $u(0) = u'(0) = 0$, a slightly stronger result holds, namely, when we estimate the L^p norms of the three quantities $\frac{u(x)}{x^2}$, $\frac{u'(x)}{x}$ and $\left(\frac{u(x)}{x}\right)'$, we obtain

$$\left\| \frac{u(x)}{x^2} \right\|_p \leq \alpha_p \left\| u'' \right\|_p , \quad \left\| \frac{u'(x)}{x} \right\|_p \leq \beta_p \left\| u'' \right\|_p , \text{ and } \left\| \left(\frac{u(x)}{x} \right)' \right\|_p \leq \gamma_p \left\| u'' \right\|_p , \tag{5}$$

with $\alpha_p, \beta_p, \gamma_p$ be the best possible constants. It is easy to see that $\alpha_p \to \infty$, $\beta_p \to \infty$ when p approaches 1. However, a similar "magic" cancellation appears and γ_p remains bounded as p goes to 1. A proof of this latter fact is presented in Section 3.

2. Proof of the Theorem

We begin with the following observation.

Lemma 2.1 (Representation formula)**.** *If u has property (P_m), then*

$$u(x) = \frac{1}{(m-1)!} \int_0^x D^m u(s)(x-s)^{m-1} ds.$$

Proof. We proceed by induction. The case $m = 1$ is immediate since $u \in W^{1,1}(0,1)$ if and only if u is absolutely continuous. Now notice that

$$D^{m-1}u(x) = \int_0^x D^m u(s) ds,$$

if we use the induction hypothesis, we obtain

$$u(x) = \frac{1}{(m-2)!} \int_0^x \left(\int_0^s D^m u(t) dt \right) (x-s)^{m-2} ds.$$

The proof is completed after using Fubini's Theorem. $\square$

Based on the function defined by (2), we have

Lemma 2.2. *There exists a function w having property (P_m), such that*

$$\frac{D^{m-1}w(x)}{x}, \frac{D^{m-2}w(x)}{x^2}, \dots, \frac{Dw(x)}{x^{m-1}}, \frac{w(x)}{x^m} \notin L^1. \tag{6}$$

Proof. In order to construct the function w, consider the function v defined in (2). As we said, v is a non-negative function on $(0,1)$, it has the property (P_1), but $\frac{v(x)}{x}$ does not belong to $L^1(0,1)$. Define $w(x)$ as

$$w(x) = \frac{1}{(m-2)!}\int_0^x v(s)(x-s)^{m-2}ds,$$

so w solves the equation $D^{m-1}w(x) = v(x)$, with initial condition $w(0) = Dw(0) = \ldots = D^{m-2}w(0) = 0$. Notice that w has the property (P_m), $D^k w(x) \geq 0$, $D^k w(1) < \infty$ and

$$\lim_{s \to 0} \frac{D^{m-k}w(s)}{s^{k-1}} = 0,$$

for all $k = 1, \dots, m-1$. We now show that w satisfies (6). Notice that

$$\begin{aligned} +\infty &= \int_0^1 \frac{v(x)}{x}dx \\ &= \int_0^1 \frac{D^{m-1}w(x)}{x}dx \\ &= D^{m-2}w(1) + \int_0^1 \frac{D^{m-2}w(x)}{x^2}dx, \end{aligned}$$

thus $\int_0^1 \frac{D^{m-2}w(x)}{x^2}dx = +\infty$. Similarly, if we keep integrating by parts we conclude that

$$\left\| \frac{D^{m-j}w(x)}{x^j} \right\|_{L^1(0,1)} = \int_0^1 \frac{D^{m-j}w(x)}{x^j} = \infty, \quad \forall\, j = 1, \dots, m.$$

□

We can proceed to prove the theorem

Proof of Theorem 1.2. The second part was proved in Lemma 2.2, so we will only prove the first part. Since the result is immediate when $j + k = m$, in the following we always assume that $j + k \leq m - 1$.

To prove that $\frac{D^j u(x)}{x^{m-j-k}}$ has the property (P_k), we proceed by induction. For $k = 1$ and any $j = 0, \dots, m-1$, $\frac{D^j u(x)}{x^{m-j-1}}$ has the property (P_1) because

$$\left.\frac{D^j u(x)}{x^{m-j-1}}\right|_{x=0} = (m-j-1)! D^{m-1}u(0) = 0.$$

Now assume the result holds for some k. Notice that if $j+k+1 \le m-1$ then

$$D\left(\frac{D^j u(x)}{x^{m-j-k-1}}\right) = \frac{D^{j+1}u(x)}{x^{m-(j+1)-k}} - (m-j-k-1)\frac{D^j u(x)}{x^{m-j-k}},$$

the righthand side of which has property (P_k) by the induction assumption. Thus we conclude that $D\left(\frac{D^j u(x)}{x^{m-j-k-1}}\right)$ has the property (P_k), completing the induction step.

Now we prove the estimate (3). Notice that

$$D^k\left(\frac{D^j u(x)}{x^{m-j-k}}\right) = \sum_{i=0}^{k}\binom{k}{i} D^{j+i}u(x) D^{k-i}\left(\frac{1}{x^{m-j-k}}\right), \tag{7}$$

and that

$$D^{k-i}\left(\frac{1}{x^{m-j-k}}\right) = (-1)^{k-i}\frac{(m-j-i-1)!}{(m-j-k-1)!}\frac{1}{x^{m-j-i}}. \tag{8}$$

Using the representation formula for u from Lemma 2.1, we obtain

$$D^{i+j}u(x) = \frac{1}{(m-j-i-1)!}\int_0^x D^m u(s)(x-s)^{m-j-i-1}ds. \tag{9}$$

By combining (7), (8) and (9) we obtain

$$
\begin{aligned}
D^k\left(\frac{D^j u(x)}{x^{m-j-k}}\right) &= \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\frac{1}{(m-j-k-1)!}\int_0^x D^m u(s)\frac{(x-s)^{m-j-i-1}}{x^{m-j-i}}ds \\
&= \frac{1}{(m-j-k-1)!}\int_0^x D^m u(s)\frac{(x-s)^{m-j-1}}{x^{m-j}}\left(\sum_{i=0}^{k}\binom{k}{i}\left(\frac{x}{x-s}\right)^i(-1)^{k-i}\right)ds \\
&= \frac{1}{(m-j-k-1)!}\int_0^x D^m u(s)\frac{(x-s)^{m-j-1}}{x^{m-j}}\left(\frac{s}{x-s}\right)^k ds. \\
&= \frac{1}{(m-j-k-1)!}\int_0^x D^m u(s)\left(1-\frac{s}{x}\right)^{m-j-k-1}\left(\frac{s}{x}\right)^{k-1}\frac{s}{x^2}ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_0^1\left|D^k\left(\frac{D^j u(x)}{x^{m-j-k}}\right)\right|dx &\le \frac{1}{(m-j-k-1)!}\int_0^1|D^m u(s)|\left(\int_s^1\left(1-\frac{s}{x}\right)^{m-j-k-1}\left(\frac{s}{x}\right)^{k-1}\frac{s}{x^2}dx\right)ds \\
&= \frac{1}{(m-j-k-1)!}\int_0^1|D^m u(s)|\left(\int_s^1(1-t)^{m-j-k-1}t^{k-1}dt\right)ds \\
&\le \frac{1}{(m-j-k-1)!}\left\|D^m u\right\|_{L^1(0,1)}\int_0^1(1-t)^{m-j-k-1}t^{k-1}dt \\
&= \frac{(k-1)!}{(m-j-1)!}\left\|D^m u\right\|_{L^1(0,1)}.
\end{aligned}
$$

The optimality of the constant is guaranteed by the optimality of Hölder's inequality. The proof of the theorem is now completed. □

In view of the above results it is natural to ask whether a similar estimate holds in higher dimension. More precisely we raise

Open Problem. Assume Ω is a smooth bounded domain in $\mathbb{R}^N$ with $N \ge 2$. Let $u(x)$ be in $W_0^{2,1}(\Omega)$. For $x \in \Omega$, denote by $\delta(x) = d(x,\partial\Omega)$, the distance from x to the boundary of Ω. Let $d(x)$ be a positive smooth function in Ω such that $d(x) = \delta(x)$ near $\partial\Omega$. Is it true that $\frac{u(x)}{d(x)} \in W^{1,1}(\Omega)$? If so, can one obtain the corresponding Hardy-type estimate

$$
\int_\Omega\left|D\left(\frac{u(x)}{d(x)}\right)\right|dx \le C\left\|D^2 u\right\|_{L^1(\Omega)},
$$

for some constant C?

The difficulty arises when one considers, for example, $N = 2$ and the domain $\Omega = \mathbb{R}^2_+ = \{(x_1,x_2) : x_2 \ge 0, x_1 \in \mathbb{R}\}$. From Theorem 1.2 it is clear that for $u \in C_c^\infty([0,1]\times[0,1])$

$$
\int_\Omega\left|\frac{\partial}{\partial x_2}\left(\frac{u(x_1,x_2)}{x_2}\right)\right|dx_1dx_2 \le C\int_\Omega\left|\frac{\partial^2 u(x_1,x_2)}{\partial x_2^2}\right|dx_1dx_2.
$$

However we do not know if the following is true,

$$\int_\Omega \left| \frac{\partial}{\partial x_1} \left(\frac{u(x_1,x_2)}{x_2} \right) \right| dx_1 dx_2 \le C \left\| D^2 u \right\|_{L^1(\Omega)} .$$

3. $W^{m,p}$ functions

We begin by proving the result stated in Remark 1.1. Notice that for $u \in W^{2,p}(0,1)$ satisfying $u(0) = u'(0) = 0$, we can write

$$\left(\frac{u(x)}{x} \right)' = \frac{1}{x^2} \int_0^x s u''(s) ds.$$

For $p > 1$, we can apply Hölder's inequality and Fubini's theorem to obtain,

$$\begin{aligned} \int_0^1 \left| \left(\frac{u(x)}{x} \right)' \right|^p dx &\le \int_0^1 \frac{x^{\frac{p}{p'}}}{x^{2p}} \int_0^x s^p \left| u''(s) \right|^p ds dx \\ &= \int_0^1 s^p \left| u''(s) \right|^p \left(\int_s^1 \frac{1}{x^{p+1}} dx \right) ds \\ &= \frac{1}{p} \int_0^1 \left| u''(s) \right|^p (1 - s^p) ds \\ &\le \frac{1}{p} \int_0^1 \left| u''(s) \right|^p ds, \end{aligned}$$

where p' and p are given by $\frac{1}{p} + \frac{1}{p'} = 1$. Hence

$$\left\| \left(\frac{u(x)}{x} \right)' \right\|_p \le p^{-\frac{1}{p}} \left\| u'' \right\|_p .$$

Thus, if we define γ_p as in (5), we have proved that $\gamma_p \le p^{-\frac{1}{p}}$, that is γ_p remains bounded as p goes to 1.

As one might expect, an analougous to Theorem 1.2 can be proved for $W^{m,p}$ functions. The result reads as follows:

Theorem 3.1. *If u belongs to $W^{m,p}(0,1)$, $p \ge 1$ and satisfies $u(0) = Du(0) = \ldots = D^{m-1}u(0) = 0$. Then for $k \ge 1$ and $1 \le j + k \le m$,*

$$\left\| D^k \left(\frac{D^j u(x)}{x^{m-j-k}} \right) \right\|_{L^p(0,1)} \le \frac{B(pk, p(m-j-k-1)+1)^{\frac{1}{p}}}{(m-j-k-1)!} \left\| D^m u \right\|_{L^p(0,1)} , \tag{10}$$

where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}$ denotes Euler's Beta function.

Proof. From the proof of Theorem 1.2, we have

$$D^k \left(\frac{D^j u(x)}{x^{m-j-k}} \right) = \frac{1}{(m-j-k-1)!} \int_0^x D^m u(s) \left(1 - \frac{s}{x} \right)^{m-j-k-1} \left(\frac{s}{x} \right)^{k-1} \frac{s}{x^2} ds.$$

After applying Hölder's inequality, Fubini's theorem and a change of variables one obtains that

$$\begin{aligned}\int_0^1 \left|D^k\left(\frac{D^j u(x)}{x^{m-j-k}}\right)\right|^p dx &\leq \left(\frac{1}{(m-j-k-1)!}\right)^p \int_0^1 |D^m u(s)|^p \left(\int_s^1 (1-t)^{p(m-j-k-1)}\, t^{pk-1} dt\right) ds \\ &\leq \left(\frac{1}{(m-j-k-1)!}\right)^p \int_0^1 |D^m u(s)|^p \left(\int_0^1 (1-t)^{p(m-j-k-1)}\, t^{pk-1} dt\right) ds \\ &= B(pk, p(m-j-k-1)+1)\left(\frac{1}{(m-j-k-1)!}\right)^p \int_0^1 |D^m u(s)|^p\, ds.\end{aligned}$$

□

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References

[1] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge, at the University Press, 1952. 2d ed.
[2] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, 2010.

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