A Hardy type inequality for $W_0^{2,1}(\Omega)$ functions

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Abstract

We consider functions $u \in W_0^{2,1}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. We prove that $\frac{u(x)}{d(x)} \in W_0^{1,1}(\Omega)$ with

$$\left\| \nabla \left(\frac{u(x)}{d(x)} \right) \right\|_{L^1(\Omega)} \le C \|u\|_{W^{2,1}(\Omega)},$$

where d is a smooth positive function which coincides with $\mathrm{dist}(x, \partial\Omega)$ near $\partial\Omega$ and C is a constant depending only on Ω.

Résumé

Une inégalité de type Hardy pour les fonctions de $W_0^{2,1}(\Omega)$. Nous considèrons des fonctions $u \in W_0^{2,1}(\Omega)$, où $\Omega \subset \mathbb{R}^N$ est un domaine régulier borné. Nous prouvons que $\frac{u(x)}{d(x)} \in W_0^{1,1}(\Omega)$ avec

$$\left\| \nabla \left(\frac{u(x)}{d(x)} \right) \right\|_{L^1(\Omega)} \le C \|u\|_{W^{2,1}(\Omega)},$$

où d est une fonction régulière positive qui coïncide avec $\mathrm{dist}(x, \partial\Omega)$ près de $\partial\Omega$ et C est une constante ne dépendant que de Ω.

1. Introduction

In [4], the following one dimensional Hardy type inequality was proven (see Theorem 1.2 in [4]): Suppose that $u \in W^{2,1}(0,1)$ satisfies $u(0) = u'(0) = 0$, then $\frac{u(x)}{x} \in W^{1,1}(0,1)$ with $\left.\frac{u(x)}{x}\right|_0 = 0$ and

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$$\left\| \left(\frac{u(x)}{x} \right)' \right\|_{L^1(0,1)} \le \|u''\|_{L^1(0,1)} . \tag{1}$$

As explained [4], this inequality is somehow unexpected because one can construct a function $u \in W^{2,1}(0,1)$ such that $u(0) = u'(0) = 0$ and that neither $\frac{u'(x)}{x}$ nor $\frac{u(x)}{x^2}$ belong to $L^1(0,1)$; however, as (1) shows, for such function u, the difference $\frac{u'(x)}{x} - \frac{u(x)}{x^2} = \left(\frac{u(x)}{x}\right)'$ is in fact an L^1 function, reflecting a "magical" cancelation of the non-integrable terms.

The purpose of this work is to present the complete analog of the estimate (1) in dimension $N \ge 2$. We have the following:

Theorem 1.1 *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Given $x \in \Omega$, we denote by $\delta(x)$ the distance from x to the boundary $\partial\Omega$. Let $d : \Omega \to (0, +\infty)$ be a smooth function such that $d(x) = \delta(x)$ near $\partial\Omega$. Then for every $u \in W_0^{2,1}(\Omega)$, we have $\frac{u(x)}{d(x)} \in W_0^{1,1}(\Omega)$ with*

$$\left\| \nabla \left(\frac{u(x)}{d(x)} \right) \right\|_{L^1(\Omega)} \le C \|u\|_{W^{2,1}(\Omega)} , \tag{2}$$

where $C > 0$ is a constant depending only on Ω.

In Section 2 we present the notation and in Section 3 we sketch the proof of Theorem 1.1.

2. Notation and preliminaries

Throughout this work, we denote $\tilde{y} = (y_1, \ldots, y_{N-1})$, $\mathbb{R}_+^N := \{y_N > 0\}$, and $B_r^N := \left\{ y \in \mathbb{R}^N \ : \ |y| < r \right\}$; $\Omega \subset \mathbb{R}^N$ is always a bounded domain with smooth boundary $\partial\Omega$; we denote by $\delta(x) := \mathrm{dist}(x, \partial\Omega)$. Using Lemma 14.16 in [6], one can construct a smooth change of coordinates $\Phi : B_r^{N-1} \times (-\epsilon_0, \epsilon_0) \to \mathbb{R}^N$, where

$$\Phi(\tilde{y}, t) := \tilde{\Phi}(\tilde{y}) + y_N \cdot \nu_{\partial\Omega}(\tilde{\Phi}(\tilde{y})), \tag{3}$$

and $\tilde{\Phi} : B_r^{N-1} \to \mathcal{V}(\tilde{x}_0)$ is a smooth coordinate chart at $\tilde{x}_0 \in \partial\Omega$. If we denote

$$\mathcal{N}(\tilde{x}_0) := \Phi\left(B_r^{N-1} \times (-\epsilon_0, \epsilon_0) \right) , \tag{4}$$

then the map $\Phi|_{B_r^{N-1} \times (0,\epsilon_0)}$ is a diffeomorphism and

$$\mathcal{N}_+(\tilde{x}_0) := \{ x \in \Omega_{\epsilon_0} \ : \ y_x \in \mathcal{V}(\tilde{x}_0) \} = \Phi\left(B_r^{N-1} \times (0, \epsilon_0) \right) . \tag{5}$$

This type of coordinates are sometimes called *flow coordinates* (see e.g. [3] and [7]). From now on, $C > 0$ will denote a constant only depending on Ω.

3. The proof of the Theorem

The key ingredient in the proof is the following lemma

Lemma 3.1 *Suppose $u \in C_0^\infty(\mathbb{R}_+^N)$. Then for all $i = 1, \ldots, N$ we have*

$$\left\| \partial_i \left(\frac{u(y)}{y_N} \right) \right\|_{L^1(\mathbb{R}_+^N)} \le C \|u\|_{W^{2,1}(\mathbb{R}_+^N)} .$$

Proof. We first notice that when $i = N$, the result is essentially contained in the proof of Theorem 1.2 of [4] when $j = 0$, $k = 1$ and $m = 2$. We refer the reader to [4] for the details. When $1 \leq i \leq N-1$, define $v(x) = u(\Psi(x))$ where $\Psi(x_1, \ldots, x_i, \ldots, x_N) = (x_1, \ldots, x_i + x_N, \ldots, x_N)$. We have

$$\frac{1}{x_N}\frac{\partial u}{\partial y_i}(\Psi(x)) = \frac{\partial}{\partial x_N}\left(\frac{v(x)}{x_N}\right) - \frac{\partial}{\partial y_N}\left(\frac{u(y)}{y_N}\right)\Big|_{y=\Psi(x)},$$

hence the estimate is reduced to the estimate for $i = N$.

□

Next we use Lemma 3.1 together with the straightening of the boundary given by Φ in Section 2 to obtain

Lemma 3.2 *Let $\tilde{x}_0 \in \partial\Omega$ and $\mathcal{N}_+(\tilde{x}_0)$ be given by (5). Suppose $u \in C_0^\infty(\mathcal{N}_+(\tilde{x}_0))$. Then for all $i = 1, \ldots, N$ we have*

$$\left\|\partial_i\left(\frac{u(x)}{\delta(x)}\right)\right\|_{L^1(\mathcal{N}_+(\tilde{x}_0))} \leq C\,\|u\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_0))}.$$

Proof. Let $v(\tilde{y}, y_N); = u(\Phi(\tilde{y}, y_N))$. Using that Φ is a smooth diffeomorphism gives

$$\int_{\mathcal{N}_+(\tilde{x}_0)}\left|\partial_i\left(\frac{u(x)}{\delta(x)}\right)\right| dx \leq C\sum_{j=1}^{N}\int_{B_r^{N-1}}\int_0^{\epsilon_0}\left|\partial_j\left(\frac{v(\tilde{y}, y_N)}{y_N}\right)\right| dy_N d\tilde{y}. \tag{6}$$

Since $v \in C_0^\infty(B_r^{N-1} \times (0, \epsilon_0)) \subset C_0^\infty(\mathbb{R}_+^N)$, we can apply Lemma 3.1 and obtain

$$\int_{B_r^{N-1}}\int_0^{\epsilon_0}\left|\partial_j\left(\frac{v(\tilde{y}, y_N)}{y_N}\right)\right| dy_N d\tilde{y} \leq C\,\|v\|_{W^{2,1}(B_r^{N-1}\times(0,\epsilon_0))}.$$

Notice that by the chain rule and the fact that Φ is a smooth diffeomorphism, we get

$$\|v\|_{W^{2,1}(B_r^{N-1}\times(0,\epsilon_0))} \leq C\,\|u\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_0))}.$$

□

Proof of Theorem 1.1. Applying Lemma 3.2 and a partition of unity (see e.g. Lemma 9.3 in [2] and Theorem 3.15 in [1]), one can obtain that

$$\left\|\partial_i\left(\frac{u(x)}{\delta(x)^{m-1}}\right)\right\|_{L^1(\Omega)} \leq C\,\|u\|_{W^{m,1}(\Omega)}.$$

for $u \in C_0^\infty(\Omega)$ and and $i = 1, \ldots, N$. Then one can complete the proof of Theorem 1.1 using a standard density argument.

□

Remark 1 In fact, we have a full generalization of Theorem 1.1 for functions in $W_0^{m,1}(\Omega)$ for all the integers $m \geq 2$, which is presented in [5].

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