A HARDY TYPE INEQUALITY FOR $W_0^{m,1}(\Omega)$ FUNCTIONS

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Abstract. We consider functions $u \in W_0^{m,1}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, and $m \geq 2$ is an integer. For all $j \geq 0$, $1 \leq k \leq m-1$, such that $1 \leq j+k \leq m$, we prove that $\frac{\partial^j u(x)}{d(x)^{m-j-k}} \in W_0^{k,1}(\Omega)$ with

$$\left\| \partial^k \left(\frac{\partial^j u(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)} \leq C \left\| u \right\|_{W^{m,1}(\Omega)},$$

where d is a smooth positive function which coincides with $\mathrm{dist}(x, \partial\Omega)$ near $\partial\Omega$, and ∂^l denotes any partial differential operator of order l.

1. Introduction

In [4], the following one dimensional Hardy type inequality was proven (see Theorem 1.2 in [4]): Suppose that $u \in W^{2,1}(0,1)$ satisfies $u(0) = u'(0) = 0$, then $\frac{u(x)}{x} \in W^{1,1}(0,1)$ with $\left.\frac{u(x)}{x}\right|_0 = 0$ and

$$\left\| \left(\frac{u(x)}{x} \right)' \right\|_{L^1(0,1)} \leq \left\| u'' \right\|_{L^1(0,1)}. \tag{1}$$

As explained [4], this inequality is somehow unexpected because one can construct a function $u \in W^{2,1}(0,1)$ such that $u(0) = u'(0) = 0$ and that neither $\frac{u'(x)}{x}$ nor $\frac{u(x)}{x^2}$ belong to $L^1(0,1)$; however, as (1) shows, for such function u, the difference $\frac{u'(x)}{x} - \frac{u(x)}{x^2} = \left(\frac{u(x)}{x}\right)'$ is in fact an L^1 function, reflecting a "magical" cancelation of the non-integrable terms.

With estimate (1) already proven, it was natural to raise the following question: Assume Ω is a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 2$ and let u be in $W_0^{2,1}(\Omega)$. For $x \in \Omega$, denote by $\delta(x) = d(x, \partial\Omega)$ the distance from x to the boundary of Ω, and let $d : \Omega \to (0, +\infty)$ be a smooth function such that $d(x) = \delta(x)$ near $\partial\Omega$. Is it true that $\frac{u}{d} \in W_0^{1,1}(\Omega)$? If so, can one obtain the corresponding Hardy-type estimate

$$\int_\Omega \left| \nabla \left(\frac{u(x)}{d(x)} \right) \right| dx \leq C \left\| \nabla^2 u \right\|_{L^1(\Omega)},$$

for some constant C?

The purpose of this work is to give a positive answer to the above question. In fact, this is a special case of the following:

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Given $x \in \Omega$, we denote by $\delta(x)$ the distance from x to the boundary $\partial\Omega$. Let $d : \Omega \to (0, +\infty)$ be a smooth function such that*

Date: May 31, 2011.

Key words and phrases. Hardy inequality, Sobolev spaces.

H.C. was partially supported by NSF Grant DMS 0802958.

J.D. was partially supported by CAPDE-Anillo ACT-125 and Fondo Basal CMM.

H.W. was supported by the European Commission under the Initial Training Network-FIRST, agreement No. PITN-GA-2009-238702, and by NSF Grant DMS 0802958.

$d(x)=\delta(x)$ near $\partial\Omega$. Suppose $m\geq 2$ and let j,k be non-negative integers such that $1\leq k\leq m-1$ and $1\leq j+k\leq m$. Then for every $u\in W_0^{m,1}(\Omega)$, we have $\frac{\partial^j u(x)}{d(x)^{m-j-k}}\in W_0^{k,1}(\Omega)$ with

$$\left\|\partial^k\left(\frac{\partial^j u(x)}{d(x)^{m-j-k}}\right)\right\|_{L^1(\Omega)}\leq C\left\|u\right\|_{W^{m,1}(\Omega)}, \tag{2}$$

where ∂^l denotes any partial differential operator of order l and $C>0$ is a constant depending only on Ω and m.

The rest of this paper is organized into three sections: In Section 2 we introduce the notation used throughout this work and give some preliminary results. In order to present the main ideas used to prove Theorem 1, we begin in Section 3 with the proof of Theorem 1 for the special case $m=2$, then in Section 4 we provide the proof of Theorem 1 for the general case $m\geq 2$.

2. Notation and preliminaries

Throughout this work, we denote by $\mathbb{R}^N_+:=\left\{(y_1,\dots,y_{N-1},y_N)\in\mathbb{R}^N\ :\ y_N>0\right\}$ the upper half-space, and $B_r^N(x_0):=\left\{x\in\mathbb{R}^N\ :\ |x-x_0|<r\right\}$, also, when $x_0=0$, we write $B_r^N:=B_r^N(0)$.

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Given $x\in\Omega$, we denote by $\delta(x)$ the distance from x to the boundary $\partial\Omega$, that is

$$\delta(x):=\operatorname{dist}(x,\partial\Omega)=\inf\left\{|x-y|\ :\ y\in\partial\Omega\right\}.$$

For $\epsilon>0$, the tubular neighborhood of $\partial\Omega$ in Ω is the set

$$\Omega_\epsilon:=\left\{x\in\Omega\ :\ \delta(x)<\epsilon\right\}.$$

The following is a well known result (see e.g. Lemma 14.16 in [5]) and it shows that δ is smooth in some neighborhood of $\partial\Omega$.

Lemma 2.1. *Let Ω and $\delta:\Omega\to(0,\infty)$ be as above. Then there exists $\epsilon_0>0$ only depending on Ω, such that $\delta|_{\Omega_{\epsilon_0}}:\Omega_{\epsilon_0}\to(0,\infty)$ is smooth. Moreover, for every $x\in\Omega_{\epsilon_0}$ there exists a unique $y_x\in\partial\Omega$ so that*

$$x=y_x+\delta(x)\nu_{\partial\Omega}(y_x),$$

where $\nu_{\partial\Omega}$ denotes the unit inward normal vector field associated to $\partial\Omega$.

Since $\partial\Omega$ is smooth, for fixed $\tilde{x}_0\in\partial\Omega$, there exists a neighborhood $\mathcal{V}(\tilde{x}_0)\subset\partial\Omega$, a radius $r>0$ and a map

$$\tilde{\Phi}:B_r^{N-1}\to\mathcal{V}(\tilde{x}_0) \tag{3}$$

which defines a smooth diffeomorphism. Define

$$\mathcal{N}_+(\tilde{x}_0):=\left\{x\in\Omega_{\epsilon_0}\ :\ y_x\in\mathcal{V}(\tilde{x}_0)\right\}, \tag{4}$$

where ϵ_0 and y_x are as in Lemma 2.1. We denote by $\Phi:B_r^{N-1}\times(-\epsilon_0,\epsilon_0)\to\mathbb{R}^N$ the map defined as

$$\Phi(\tilde{y},t):=\tilde{\Phi}(\tilde{y})+y_N\cdot\nu_{\partial\Omega}(\tilde{\Phi}(\tilde{y})), \tag{5}$$

where $\tilde{y}=(y_1,\dots,y_{N-1})$, and we write

$$\mathcal{N}(\tilde{x}_0):=\Phi\left(B_r^{N-1}\times(-\epsilon_0,\epsilon_0)\right). \tag{6}$$

About the map Φ we have the following:

Lemma 2.2. *The map $\Phi|_{B_r^{N-1}\times(0,\epsilon_0)}$ is a diffeomorphism and*

$$\mathcal{N}_+(\tilde{x}_0)=\Phi\left(B_r^{N-1}\times(0,\epsilon_0)\right).$$

Proof. This is a direct corollary of the definition of Φ through $\tilde{\Phi}$, and Lemma 2.1. $\square$

Remark 2.1. The map $\Phi|_{B_r^{N-1}\times(0,\epsilon_0)}$ gives a local coordinate chart which straightens the boundary near $\tilde{x}_0$. This type of coordinates are sometimes called *flow coordinates* (see e.g. [3] and [6]).

From now on, $C>0$ will always denote a constant only depending on Ω and possibly the integer $m\geq 2$. The following is a direct, but very useful, corollary.

Corollary 2.1. *Let $f\in L^1(\mathcal{N}_+(\tilde{x}_0))$ and Φ be given by* (5)*. Then*

$$\frac{1}{C}\int_{B_r^{N-1}}\int_0^{\epsilon_0}|f(\Phi(\tilde{y},y_N))|\,dy_N d\tilde{y}\leq\int_{\mathcal{N}_+(\tilde{x}_0)}|f(x)|\,dx\leq C\int_{B_r^{N-1}}\int_0^{\epsilon_0}|f(\Phi(\tilde{y},y_N))|\,dy_N d\tilde{y}$$

Proof. Since $\Phi|_{B_r^{N-1}\times(0,\epsilon_0)}$ is a diffeomorphism, we know that for all $(\tilde{y},y_N)\in B_r^{N-1}\times(0,\epsilon_0)$ we have

$$\frac{1}{C}\leq|\det D\Phi(\tilde{y},y_N)|\leq C.$$

The result then follows from the change of variables formula. □

The following lemma provides us with a partition of unity in $\mathbb{R}^N$, constructed from the neighborhoods $\mathcal{N}(\tilde{x}_0)$. Consider the open cover of $\partial\Omega$ given by $\{\mathcal{V}(\tilde{x})\,:\,\tilde{x}\in\partial\Omega\}$, where $\mathcal{V}(\tilde{x})\subset\partial\Omega$ is defined in (3). By the compactness of $\partial\Omega$, there exists $\{\tilde{x}_1,\ldots,\tilde{x}_M\}\subset\partial\Omega$, so that $\partial\Omega=\cup_{l=1}^M\mathcal{V}(\tilde{x}_l)$. Notice that by the definition of $\mathcal{N}(\tilde{x}_0)$ in (6) we also have that $\cup_{l=1}^M\mathcal{N}(\tilde{x}_l)$ is an open cover of $\partial\Omega$ in $\mathbb{R}^N$. The following is a classical result (see e.g. Lemma 9.3 in [2] and Theorem 3.15 in [1]).

Lemma 2.3 (partition of unity)**.** *There exist functions $\rho_0,\rho_1,\ldots,\rho_M\in C^\infty(\mathbb{R}^N)$ such that*

(i) $0\leq\rho_l\leq 1$ *for all* $l=0,1,\ldots,M$ *and* $\sum_{l=0}^M\rho_i(x)=1$ *for all* $x\in\mathbb{R}^N$*,*
(ii) $\operatorname{supp}\rho_l\subset\mathcal{N}(\tilde{x}_l)$*, for all* $l=1,\ldots,M$*,*
(iii) $\rho_0|_\Omega\in C_0^\infty(\Omega)$*.*

In order to simplify the notation, we will denote by ∂^l any partial differential operator of order l where l is a positive integer[1]. Also, ∂_i will denote the partial derivative with respect to the i-th variable, and $\partial^2_{ij}=\partial_i\circ\partial_j$.

Remark 2.2. We conclude this section by showing that, to prove Theorem 1, it is enough to prove estimate (2) for smooth functions with compact support. Suppose $u\in W_0^{m,1}(\Omega)$, then there exists a sequence $\{u_n\}\subset C_0^\infty(\Omega)$, so that $\|u-u_n\|_{W^{m,1}(\Omega)}\to 0$ as $n\to\infty$. In particular, after maybe extracting a subsequence, one can assume that

$$\partial^l u_n\to\partial^l u \text{ a.e. in } \Omega,\ \text{for all } 0\leq l\leq m.$$

Since d is smooth, the above implies that for a.e $x\in\Omega$ and all $j\geq 0$, $1\leq k\leq m-1$ and $1\leq j+k\leq m$:

$$\begin{aligned}\partial^k\left(\frac{\partial^j u(x)}{d(x)^{m-j-k}}\right)&=\frac{\partial^{j+k}u(x)}{d(x)^{m-j-k}}+\partial^j u(x)\partial^k\left(\frac{1}{d(x)^{m-j-k}}\right)\\&=\lim_{n\to\infty}\frac{\partial^{j+k}u_n(x)}{d(x)^{m-j-k}}+\partial^j u_n(x)\partial^k\left(\frac{1}{d(x)^{m-j-k}}\right)\\&=\lim_{n\to\infty}\partial^k\left(\frac{\partial^j u_n(x)}{d(x)^{m-j-k}}\right).\end{aligned}$$

Therefore, Fatou's Lemma applies and we obtain

$$\left\|\partial^k\left(\frac{\partial^j u(x)}{d(x)^{m-j-k}}\right)\right\|_{L^1(\Omega)}\leq\liminf_{n\to\infty}\left\|\partial^k\left(\frac{\partial^j u_n(x)}{d(x)^{m-j-k}}\right)\right\|_{L^1(\Omega)}.$$

[1]In general, one would say: "For a given multi-index $\alpha=(\alpha_1,\ldots,\alpha_N)$, we denote by ∂^α the partial differential operator of order $l=|\alpha|=\alpha_1+\ldots+\alpha_N$". Since we only care about the order of the operator, it makes sense to abuse the notation and identify α with its order $|\alpha|=l$.

Once (2) has been proven for $u_n \in C_0^\infty(\Omega)$, we get

$$\left\| \partial^k \left(\frac{\partial^j u_n(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)} \leq C \left\| u_n \right\|_{W^{m,1}(\Omega)},$$

and thus we can conclude that

$$\left\| \partial^k \left(\frac{\partial^j u(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)} \leq C \liminf_{n\to\infty} \left\| u_n \right\|_{W^{m,1}(\Omega)} = C \left\| u \right\|_{W^{m,1}(\Omega)}.$$

Finally estimate (2) together with the fact that $\frac{\partial^j u_n(x)}{d(x)^{m-j-k}} \in C_0^\infty(\Omega)$ and $\overline{C_0^\infty(\Omega)}^{W^{k,1}(\Omega)} = W_0^{k,1}(\Omega)$ gives that $\frac{\partial^j u(x)}{d(x)^{m-j-k}} \in W_0^{k,1}(\Omega)$.

3. The case $m = 2$

We begin this section by proving estimate (2) in Theorem 1 for $\Omega = \mathbb{R}_+^N$, $m = 2$, $j = 0$ and $k = 1$.

Lemma 3.1. *Suppose that $u \in C_0^\infty(\mathbb{R}_+^N)$. Then for all $i = 1, \dots, N$*

$$\left\| \partial_i \left(\frac{u(y)}{y_N} \right) \right\|_{L^1(\mathbb{R}_+^N)} \leq 2 \left\| u \right\|_{W^{2,1}(\mathbb{R}_+^N)}.$$

Proof. Consider first the case $i = N$. This is similar to (1), but for the sake of completeness, we will provide the proof. Notice that we can write

$$\frac{\partial}{\partial y_N} \left(\frac{u(\tilde{y}, y_N)}{y_N} \right) = \frac{1}{y_N^2} \int_0^{y_N} \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) t dt,$$

hence by integrating the above we obtain

$$\begin{aligned}
\int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial}{\partial y_N} \left(\frac{u(\tilde{y}, y_N)}{y_N} \right) \right| dy_N d\tilde{y} &\leq \int_{\mathbb{R}^{N-1}} \int_0^\infty \frac{1}{y_N^2} \int_0^{y_N} \left| \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) \right| t dt dy_N d\tilde{y} \\
&= \int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) \right| t \int_t^\infty \frac{1}{y_N^2} dy_N dt d\tilde{y} \\
&= \int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) \right| t \int_t^\infty \frac{1}{y_N^2} dy_N dt d\tilde{y} \\
&= \int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) \right| dt d\tilde{y},
\end{aligned}$$

hence

$$\int_{\mathbb{R}_+^N} \left| \frac{\partial}{\partial y_N} \left(\frac{u(y)}{y_N} \right) \right| dy \leq \int_{\mathbb{R}_+^N} \left| \frac{\partial^2 u(y)}{\partial y_N^2} \right| dy. \tag{7}$$

When $1 \leq i \leq N - 1$, we need to estimate $\int_{\mathbb{R}_+^N} \frac{1}{y_N} \left| \frac{\partial u}{\partial y_i}(y) \right| dy$. To do so, consider the change of variables $y = \Psi(x)$, where

$$\Psi(x_1, \dots, x_i, \dots, x_N) = (x_1, \dots, x_i + x_N, \dots, x_N). \tag{8}$$

Notice that $\det D\Psi(x) = 1$, hence

$$\int_{\mathbb{R}_+^N} \frac{1}{y_N} \left| \frac{\partial u(y)}{\partial y_i} \right| dy = \int_{\mathbb{R}_+^N} \frac{1}{x_N} \left| \frac{\partial u}{\partial y_i}(\Psi(x)) \right| dx.$$

Observe that if we let $v(x) = u(\Psi(x))$, we can write

$$\frac{1}{x_N}\frac{\partial u}{\partial y_i}(\Psi(x)) = \frac{\partial}{\partial x_N}\left(\frac{v(x)}{x_N}\right) - \frac{\partial}{\partial y_N}\left(\frac{u(y)}{y_N}\right)\Big|_{y=\Psi(x)}. \tag{9}$$

Applying estimate (7) to u and v yields

$$\begin{aligned}\int_{\mathbb{R}^N_+}\frac{1}{x_N}\left|\frac{\partial u}{\partial y_i}(\Psi(x))\right|dx &\le \int_{\mathbb{R}^N_+}\left|\frac{\partial}{\partial x_N}\left(\frac{v(x)}{x_N}\right)\right|dx + \int_{\mathbb{R}^N_+}\left|\frac{\partial}{\partial y_N}\left(\frac{u(y)}{y_N}\right)\Big|_{y=\Psi(x)}\right|dx\\ &= \int_{\mathbb{R}^N_+}\left|\frac{\partial}{\partial x_N}\left(\frac{v(x)}{x_N}\right)\right|dx + \int_{\mathbb{R}^N_+}\left|\frac{\partial}{\partial y_N}\left(\frac{u(y)}{y_N}\right)\right|dy\\ &\le \int_{\mathbb{R}^N_+}\left|\frac{\partial^2 v(x)}{\partial x_N^2}\right|dx + \int_{\mathbb{R}^N_+}\left|\frac{\partial^2 u(y)}{\partial y_N^2}\right|dy.\end{aligned}$$

Finally, notice that

$$\frac{\partial^2 v(x)}{\partial x_N^2} = \frac{\partial^2 u(y)}{\partial y_N^2}\Big|_{y=\Psi(x)} + 2\,\frac{\partial^2 u(y)}{\partial y_i\partial y_N}\Big|_{y=\Psi(x)} + \frac{\partial^2 u(y)}{\partial y_i^2}\Big|_{y=\Psi(x)}. \tag{10}$$

Thus, after reversing the change of variables when needed, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N_+}\frac{1}{y_N}\left|\frac{\partial u(y)}{\partial y_i}\right|dy &= \int_{\mathbb{R}^N_+}\frac{1}{x_N}\left|\frac{\partial u}{\partial y_i}(\Psi(x))\right|dx\\ &\le 2\int_{\mathbb{R}^N_+}\left|\frac{\partial^2 u(y)}{\partial y_N^2}\right|dy + 2\int_{\mathbb{R}^N_+}\left|\frac{\partial^2 u(y)}{\partial y_i\partial y_N}\right|dy + \int_{\mathbb{R}^N_+}\left|\frac{\partial^2 u(y)}{\partial y_i^2}\right|dy\\ &\le 2\,\|u\|_{W^{2,1}(\mathbb{R}^N_+)}.\end{aligned}$$

□

Recall (see Section 2) that for every $\tilde{x}_0 \in \partial\Omega$, there exist the neighborhood $\mathcal{N}_+(\tilde{x}_0) \subset \Omega$ given by (4) and the diffeomorphism $\Phi : B_r^{N-1} \times (0,\epsilon_0) \to \mathcal{N}_+(\tilde{x}_0)$ given by (5). Moreover, we know that $\delta(x)$ is smooth over $\mathcal{N}_+(\tilde{x}_0)$. Hence we have

Lemma 3.2. *Let $\tilde{x}_0 \in \partial\Omega$ and $\mathcal{N}_+(\tilde{x}_0)$ be given by (4), and suppose $u \in C_0^\infty(\mathcal{N}_+(\tilde{x}_0))$. Then for all $i = 1,\dots,N$*

$$\left\|\partial_i\left(\frac{u(x)}{\delta(x)}\right)\right\|_{L^1(\mathcal{N}_+(\tilde{x}_0))} \le C\,\|u\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_0))}.$$

Proof. We first use Corollary 2.1 and obtain

$$\int_{\mathcal{N}_+(\tilde{x}_0)}\left|\partial_i\left(\frac{u(x)}{\delta(x)}\right)\right|dx \le C\int_{B_r^{N-1}}\int_0^{\epsilon_0}\left|\partial_i\left(\frac{u(x)}{\delta(x)}\right)\Big|_{x=\Phi(\tilde{y},y_N)}\right|dy_N\,d\tilde{y}.$$

Let $v(\tilde{y},y_N) = u(\Phi(\tilde{y},y_N))$. We claim that

$$\int_{B_r^{N-1}}\int_0^{\epsilon_0}\left|\partial_i\left(\frac{u(x)}{\delta(x)}\right)\Big|_{x=\Phi(\tilde{y},y_N)}\right|dy_N\,d\tilde{y} \le C\sum_{j=1}^N\int_{B_r^{N-1}}\int_0^{\epsilon_0}\left|\partial_j\left(\frac{v(\tilde{y},y_N)}{y_N}\right)\right|dy_N\,d\tilde{y}. \tag{11}$$

We will prove (11) at the end, so that we can conclude the argument. Since $v \in C_0^\infty(B_r^{N-1}\times(0,\epsilon_0)) \subset C_0^\infty(\mathbb{R}^N_+)$, we can apply Lemma 3.1 and obtain

$$\int_{B_r^{N-1}}\int_0^{\epsilon_0}\left|\partial_j\left(\frac{v(\tilde{y},y_N)}{y_N}\right)\right|dy_N\,d\tilde{y} \le C\,\|v\|_{W^{2,1}(B_r^{N-1}\times(0,\epsilon_0))}.$$

Notice that by the chain rule and the fact that Φ is a diffeomorphism, we get that for all $1 \leq i, j \leq N$

$$\left|\partial^2_{ij} v(\tilde{y}, y_N)\right| \leq C \left(\sum_{p,q=1}^{N} \left|\partial^2_{pq} u(x)|_{x=\Phi(\tilde{y},y_N)}\right| + \sum_{p=1}^{N} \left|\partial_p u(x)|_{x=\Phi(\tilde{y},y_N)}\right| \right),$$

so we with the aid of Corollary 2.1, we can write

$$\begin{aligned}
\|v\|_{W^{2,1}(B_r^{N-1}\times(0,\epsilon_0))} &\leq C \int_{B_r^{N-1}} \int_0^{\epsilon_0} \left(\sum_{p,q} \left|\partial^2_{pq} u|_{x=\Phi(\tilde{y},y_N)}\right| + \sum_{p} \left|\partial_p u|_{x=\Phi(\tilde{y},y_N)}\right| \right) dy_N d\tilde{y} \\
&\leq C \int_{\mathcal{N}_+(\tilde{x}_0)} \left(\sum_{p,q} \left|\partial^2_{pq} u(x)\right| + \sum_{p} \left|\partial_p u(x)\right| \right) dx \\
&\leq C \|u\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_0))}.
\end{aligned}$$

To conclude, we need to prove (11). To do so, notice that $u(x) = v(\Phi^{-1}(x))$, and $\delta(x) = c(\Phi^{-1}(x))$, where $c(\tilde{y}, y_N) = y_N$. Thus, by using the chain rule we obtain

$$\partial_i \left(\frac{u(x)}{\delta(x)} \right)\bigg|_{x=\Phi(\tilde{y},y_N)} = \sum_{j=1}^{N} \partial_j \left(\frac{v(y)}{c(y)} \right)\bigg|_{y=(\tilde{y},y_N)} \cdot \partial_i(\Phi^{-1})_j(\Phi(\tilde{y}, y_N)),$$

and since Φ is a diffeomorphism, we obtain

$$\left| \partial_i \left(\frac{u(x)}{\delta(x)} \right)\bigg|_{x=\Phi(\tilde{y},y_N)} \right| \leq C \sum_{j=1}^{N} \left| \partial_j \left(\frac{v(y)}{c(y)} \right)\bigg|_{y=(\tilde{y},y_N)} \right|.$$

Estimate (11) then follows by integrating the above inequality. □

We end this section with the proof of the main result when $m = 2$.

Proof of Theorem 1 when $m = 2$. When $j = 1$ and $k = 1$ the estimate (2) is trivial. Taking into account Remark 2.2, we only need to prove

$$\left\| \partial_i \left(\frac{u(x)}{d(x)} \right) \right\|_{L^1(\Omega)} \leq C \|u\|_{W^{2,1}(\Omega)} \tag{12}$$

for $u \in C_0^\infty(\Omega)$ and $i = 1, 2, \dots, N$. To do so, we use the partition of unity given by Lemma 2.3 to write $u(x) = \sum_{l=0}^{M} u_l(x)$ on Ω where $u_l(x) := \rho_l(x)u(x)$, $l = 0, 1, \dots, M$. Now, without loss of generality, we can assume that $d(x) = \delta(x)$ for all $x \in \Omega_{\epsilon_0}$, and that $d(x) \geq C > 0$ for all $x \in \operatorname{supp} \rho_0 \cap \Omega$. Notice that in $\operatorname{supp} \rho_0 \cap \Omega$, we have

$$\frac{u_0}{d} \in C^\infty(\overline{\operatorname{supp} \rho_0 \cap \Omega}), \text{ with } \left\| \frac{u_0}{d} \right\|_{W^{1,1}(\operatorname{supp} \rho_0 \cap \Omega)} \leq C \|u_0\|_{W^{1,1}(\sup \rho_0 \cap \Omega)}.$$

To take care of the boundary part, notice that $u_l \in C_0^\infty(\mathcal{N}_+(\tilde{x}_l))$ for $l = 1, \dots, M$, so Lemma 3.2 applies and we obtain

$$\left\| \partial_i \left(\frac{u_l(x)}{\delta(x)} \right) \right\|_{L^1(\mathcal{N}_+(\tilde{x}_l))} \leq C \|u_l\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_l))}, \text{ for all } l = 1, \dots, M.$$

To conclude, notice that $\partial_i\left(\frac{u(x)}{d(x)}\right) = \sum_{l=1}^M \partial_i\left(\frac{u_l(x)}{\delta(x)}\right) + \partial_i\left(\frac{u_0(x)}{d(x)}\right)$ on Ω and that $|\rho_l(x)|, |\partial_i\rho_l(x)|$ and $\left|\partial^2_{ij}\rho_l(x)\right|$ are uniformly bounded for all $l = 0, 1, \ldots, M$, therefore

$$\begin{aligned}\left\|\partial_i\left(\frac{u(x)}{d(x)}\right)\right\|_{L^1(\Omega)} &\le \sum_{l=1}^M \left\|\partial_i\left(\frac{u_l(x)}{\delta(x)}\right)\right\|_{L^1(\mathcal{N}_+(\tilde{x}_l))} + \left\|\partial_i\left(\frac{u_0(x)}{d(x)}\right)\right\|_{L^1(\mathrm{supp}\rho_0\cap\Omega)} \\ &\le C\left(\sum_{l=1}^M \|u_l\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_l))} + \|u_0\|_{W^{1,1}(\mathrm{supp}\rho_0\cap\Omega)}\right) \\ &\le C\left(\sum_{l=1}^M \|u\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_l))} + \|u\|_{W^{1,1}(\mathrm{supp}\rho_0\cap\Omega)}\right) \\ &\le C\,\|u\|_{W^{2,1}(\Omega)},\end{aligned}$$

thus completing the proof. □

4. The general case $m \ge 2$

To prove the general case, we need to generalize Lemma 3.1 in the following way

Lemma 4.1. *Suppose $u \in C_0^\infty(\mathbb{R}^N_+)$. Then for all $m \ge 1$ and $i = 1, \ldots, N$ we have*

$$\left\|\partial_i\left(\frac{u(y)}{y_N^{m-1}}\right)\right\|_{L^1(\mathbb{R}^N_+)} \le C\,\|u\|_{W^{m,1}(\mathbb{R}^N_+)}.$$

Proof. The case $m = 1$ is a trivial statement, whereas $m = 2$ is exactly what we proved in Lemma 3.1. So from now on we suppose $m \ge 3$. We first notice that when $i = N$, the result follows from the proof of Theorem 1.2 of [4] when $j = 0$ and $k = 1$. We refer the reader to [4] for the details.

When $1 \le i \le N-1$, we can proceed as in the proof of Lemma 3.1. Define $v(x) = u(\Psi(x))$ where Ψ is given by (8). Notice that when $m \ge 3$, instead of equation (9) we have

$$\frac{1}{x_N^{m-1}}\frac{\partial u}{\partial y_i}(\Psi(x)) = \frac{\partial}{\partial x_N}\left(\frac{v(x)}{x_N^{m-1}}\right) - \frac{\partial}{\partial y_N}\left(\frac{u(y)}{y_N^{m-1}}\right)\bigg|_{y=\Psi(x)},$$

and instead of (10) we have

$$\frac{\partial^m v(x)}{\partial x_N^m} = \sum_{l=0}^m \binom{m}{l}\frac{\partial^m u(y)}{\partial y_i^{m-l}\partial y_N^l}\bigg|_{y=\Psi(x)}.$$

Hence the estimate is reduced to the already proven result for $i = N$. We omit the details. □

We also have the analog of Lemma 3.2.

Lemma 4.2. *Let $\tilde{x}_0 \in \partial\Omega$ and $\mathcal{N}_+(\tilde{x}_0)$ as in Lemma 3.2. Let $u \in C_0^\infty(\mathcal{N}_+(\tilde{x}_0))$. Then for all $m \ge 1$ and $i = 1, \ldots, N$ we have*

$$\left\|\partial_i\left(\frac{u(x)}{\delta(x)^{m-1}}\right)\right\|_{L^1(\mathcal{N}_+(\tilde{x}_0))} \le C\,\|u\|_{W^{m,1}(\mathcal{N}_+(\tilde{x}_0))}.$$

Proof. The proof involves only minor modifications from the proof of Lemma 3.2, which we provide in the next few lines. Corollary 2.1 gives

$$\int_{\mathcal{N}_+(\tilde{x}_0)}\left|\partial_i\left(\frac{u(x)}{\delta(x)^{m-1}}\right)\right|dx \le C\int_{B_r^{N-1}}\int_0^{\epsilon_0}\left|\partial_i\left(\frac{u(x)}{\delta(x)^{m-1}}\right)\bigg|_{x=\Phi(\tilde{y},y_N)}\right|dy_N\,d\tilde{y}.$$

If $v(\tilde{y}, y_N) = u(\Phi(\tilde{y}, y_N))$, then

$$\int_{B_r^{N-1}} \int_0^{\epsilon_0} \left| \partial_i \left(\frac{u(x)}{\delta(x)^{m-1}} \right) \Big|_{x=\Phi(\tilde{y},y_N)} \right| dy_N d\tilde{y} \le C \sum_{j=1}^{N} \int_{B_r^{N-1}} \int_0^{\epsilon_0} \left| \partial_j \left(\frac{v(\tilde{y}, y_N)}{y_N^{m-1}} \right) \right| dy_N d\tilde{y}. \tag{13}$$

Just as for (11), estimate (13) follows from the fact that Φ is a smooth diffeomorphism. Since $v \in C_0^\infty(B_r^{N-1} \times (0, \epsilon_0)) \subset C_0^\infty(\mathbb{R}_+^N)$, we can apply Lemma 4.1 and obtain

$$\int_{B_r^{N-1}} \int_0^{\epsilon_0} \left| \partial_j \left(\frac{v(\tilde{y}, y_N)}{y_N^{m-1}} \right) \right| dy_N d\tilde{y} \le C \|v\|_{W^{m,1}(B_r^{N-1} \times (0,\epsilon_0))}.$$

Notice that by the chain rule and the fact that Φ is a smooth diffeomorphism, we get

$$|\partial^m v(\tilde{y}, y_N)| \le C \sum_{l \le m} \left| \partial^l u(x)|_{x=\Phi(\tilde{y},y_N)} \right|,$$

where the left hand side is a fixed m-th order partial derivative, and in the right hand side the summation contains all partial differential operators of order $l \le m$. Again with the aid of Corollary 2.1, we can write

$$\begin{aligned} \|v\|_{W^{m,1}(B_r^{N-1} \times (0,\epsilon_0))} &\le C \sum_{l \le m} \int_{B_r^{N-1}} \int_0^{\epsilon_0} \left(\left| \partial^l u|_{x=\Phi(\tilde{y},y_N)} \right| \right) dy_N d\tilde{y} \\ &\le C \sum_{l \le m} \int_{\mathcal{N}_+(\tilde{x}_0)} \left| \partial^l u(x) \right| dx \\ &\le C \|u\|_{W^{m,1}(\mathcal{N}_+(\tilde{x}_0))}. \end{aligned}$$

□

And of course we have

Lemma 4.3. *Suppose $u \in C_0^\infty(\Omega)$. Then for all $m \ge 1$ and $i = 1, \dots, N$ we have*

$$\left\| \partial_i \left(\frac{u(x)}{\delta(x)^{m-1}} \right) \right\|_{L^1(\Omega)} \le C \|u\|_{W^{m,1}(\Omega)}.$$

We omit the proof of the above lemma, because it is almost a line by line copy of the proof of the estimate (12) in Section 3 using the partition of unity. We are now ready to prove Theorem 1.

Proof Theorem 1. For any fixed integer $m \ge 3$, just as what we did for the case $m = 2$, it is enough to prove the estimate (2) for $u \in C_0^\infty(\Omega)$. Notice that since

$$\left\| \partial^j u \right\|_{W^{m-j,1}(\Omega)} \le \|u\|_{W^{m,1}(\Omega)} \text{ for all } 0 \le j \le m,$$

it is enough to show

$$\left\| \partial^k \left(\frac{u(x)}{d(x)^{m-k}} \right) \right\|_{L^1(\Omega)} \le C \|u\|_{W^{m,1}(\Omega)}, \tag{14}$$

for $u \in C_0^\infty(\Omega)$ and $1 \le k \le m-1$. We proceed by induction in k. The case $k = 1$ corresponds exactly to Lemma 4.3. If one assumes the result for k, then we have to estimate for $i = 1, \dots, N$

$$\partial_i \partial^k \left(\frac{u(x)}{d(x)^{m-k-1}} \right) = \partial^k \left(\frac{\partial_i u(x)}{d(x)^{m-k-1}} \right) - (m-k-1) \partial^k \left(\frac{u(x) \partial_i d(x)}{d(x)^{m-k}} \right).$$

Using the induction hypothesis for $\tilde{m} = m-1$ yields

$$\left\| \partial^k \left(\frac{\partial_i u(x)}{d(x)^{(m-1)-k}} \right) \right\|_{L^1(\Omega)} \le C \|\partial_i u\|_{W^{m-1,1}(\Omega)} \le C \|u\|_{W^{m,1}(\Omega)},$$

on the other hand, by using the induction hypothesis and the fact that d is smooth in $\overline{\Omega}$, we obtain

$$\left\|\partial^k\left(\frac{u(x)\partial_i d(x)}{d(x)^{m-k}}\right)\right\|_{L^1(\Omega)} \leq C\left\|u\partial_i d\right\|_{W^{m,1}(\Omega)} \leq C\left\|u\right\|_{W^{m,1}(\Omega)}.$$

Therefore

$$\left\|\partial_i\partial^k\left(\frac{u(x)}{d(x)^{m-k-1}}\right)\right\|_{L^1(\Omega)} \leq C\left\|u\right\|_{W^{m,1}(\Omega)},$$

thus concluding the proof. □

Acknowledgment. We would like to thank M. Marcus for suggesting the use of flow coordinates. Also we thank H. Brezis for his valuable suggestions in the elaboration of this article.

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