HARDY-SOBOLEV-TYPE INEQUALITIES WITH MONOMIAL WEIGHTS

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Abstract. We give an elementary proof of a family of Hardy-Sobolev-type inequalities with monomial weights. As a corollary we obtain a weighted trace inequality related to the fractional Laplacian.

1. Introduction

Let $N \geq 1$ and $p > 1$, the famous result of Sobolev [16] says that there exists a constant $C > 0$ depending only on N and p such that

$$\left(\int_{\mathbb{R}^N} |u(x)|^{p^*} \,\mathrm{d}x\right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}, \tag{1}$$

where $p^* = \frac{Np}{N-p}$ and u is any function in $C_c^1(\mathbb{R}^N)$. Later, Gagliardo [6] and Nirenberg [14] independently found a proof of (1) that also works for $p = 1$, giving us the now classical Sobolev-Gagliardo-Nirenberg (SGN) inequality.

Another classical inequality is the Hardy inequality [8], namely, for $p < N$ there exists a constant $C > 0$ such that

$$\left(\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} \,\mathrm{d}x\right)^{\frac{1}{p}} \leq C \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}, \tag{2}$$

for all $u \in C_c^1(\mathbb{R}^N)$. These two classical inequalities can be placed inside a more general inequality: the so called Caffarelli-Kohn-Nirenberg (CKN) inequality [5]. A particular case of this inequality says that if $a, b \in \mathbb{R}$ satisfy $1 - \frac{N}{p} \leq a - b \leq 1$ then

$$\left(\int_{\mathbb{R}^N} \left||x|^b\, u(x)\right|^{p^*} \,\mathrm{d}x\right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{R}^N} \left||x|^a\, \nabla u(x)\right|^p \,\mathrm{d}x\right)^{\frac{1}{p}}, \tag{3}$$

where $p^* \geq 1$ is given by

$$\frac{1}{p^*} + \frac{b+1}{N} = \frac{1}{p} + \frac{a}{N}. \tag{4}$$

Observe that if $a = b = 0$ we recover (1), and if $a = 0$ and $b = -1$ we obtain (2).

Using the CKN inequality as an inspiration, is that we are concerned with the validity of Hardy-Sobolev type inequalities with monomial weights of the form x^A, where $A = (a_1, \ldots, a_N) \in \mathbb{R}^N$ and

$$x^A := |x_1|^{a_1} \cdot |x_2|^{a_2} \cdot \ldots \cdot |x_N|^{a_N}.$$

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In particular we would like to give conditions on $A, B \in \mathbb{R}^N$ and $p \geq 1$, for the existence of $p^* \geq 1$ and a constant $C > 0$ such that

$$\left(\int_{\mathbb{R}^N} \left|x^B u(x)\right|^{p^*} \,\mathrm{d}x\right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{R}^N} \left|x^A \nabla u(x)\right|^{p} \,\mathrm{d}x\right)^{\frac{1}{p}} \tag{5}$$

for all $u \in C_c^\infty(\mathbb{R}^N)$.

The subject of Hardy-Sobolev (or weighted Sobolev inequalities) like the one above has been vastly studied in the past. We do not plan to give a comprehensive survey on such results, but the interested reader might want to check [10, 12, 15] and the citations therein for further reference. One important observation about weighted Sobolev inequalities is that the results vary from very general results dealing with large families of weights (like $\mathcal{A}_p$ weights), and very specific results like the CKN inequality.

In the case of general results it is usual to find hypotheses that sometimes are restrictive, because of this is that we believe that applying such general results to specific cases tend to hide some key features that might appear for each particular case. To illustrate this, let us mention a recent work about general weights that applies to the monomial case we are studying. Meyries and Veraar [13] proved general embedding theorems between weighted Sobolev spaces, where the weights belong to the Muckenhoupt classes $\mathcal{A}_p$, and in this context monomial weights of the form $w(x) = x^A$ appear as an application of their result (see [13, Proposition 4.3]). This result has the great advantage of applying to general weights as they give necessary and sufficient conditions for the validity of several Sobolev embeddings between weighted spaces. However, working in such generality has a disadvantage: it does not give other relevant information on the result itself. For example, when one applies the result from [13] to obtain (5) we do not receive an answer to other questions pertinent to (5), namely, what is the best constant C? are there extremals to the inequality?, what happens when one changes the domain of integration to other subsets of $\mathbb{R}^N$?

If one focuses on a particular case like (5) instead of relying on the general results one might obtain alternate proofs and additional insights that might answer some other relevant questions. For example, Cabré and Ros-Oton [2] established a particular case of (5) in dimension two in their study of the regularity of stable solutions to reaction-diffusion problems in domains with double revolution symmetry. In a follow up paper [3], they generalized their previous inequality to higher dimension, namely for $A = (a_1, \ldots, a_N)$, $a_i \geq 0$ $\forall i = 1, \ldots, N$, they showed

$$\left(\int_{\mathbb{R}^N_A} |u(x)|^{p^*} x^A \,\mathrm{d}x\right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{R}^N_A} |\nabla u(x)|^{p} x^A \,\mathrm{d}x\right)^{\frac{1}{p}}, \tag{6}$$

where

$$\mathbb{R}^N_A = \{ (x_1, \ldots, x_N) \in \mathbb{R}^N : x_i \geq 0 \text{ when } a_i > 0 \},$$

and

$$p^* = \frac{Dp}{D - p},$$

where $D = N + a_1 + \ldots + a_N$. Their proof of (6) is based on first proving an isoperimetric inequality for the measure $\mathrm{d}\mu = x^A \,\mathrm{d}x$ by using the Alexandroff-Bakelman-Pucci (ABP) method applied to an associated elliptic equation. As we already mentioned, one of the advantages of finding alternative proofs to the same results is that each proof provides different insights on the result itself. In the case of the proof in [3], they are able to expand the parameter range that the general theory of Muckenhoupt weights gives. Additionally, they also provide the extremals for (6) and the best constant C, which in turn gave them the possibility to prove the following Trudinger type inequality for $u \in C_c^1(\Omega)$

$$\int_\Omega \exp\left[\left(\frac{c_1 |u(x)|}{\|\nabla u\|_{L^D(\Omega, x^A \,\mathrm{d}x)}}\right)^{\frac{D}{D-1}}\right] x^A \,\mathrm{d}x \leq c_2 \int_\Omega x^A \,\mathrm{d}x,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain.

As we mentioned before, the main purpose of this article is to prove (5) for suitable $A, B \in \mathbb{R}^N$, but in addition, we would like to provide an alternative technique to prove such inequality, namely a proof that uses elementary calculus tools instead of the ABP method, the theory of Muckenhoupt weights, or other general techniques appearing in the literature about weighted Sobolev inequalities.

One advantage of the method we use in comparison with the ABP method used in [3] is that it works not only for domains of the form $\mathbb{R}^N_A$ - that is, a domain depending on the weight - but for any set of the form

$$\mathbb{R}^N_{I,+} = \{ (x_1, \dots, x_N) \in \mathbb{R}^N : x_i \geq 0 \text{ for } i \in I \},$$

or

$$\mathbb{R}^N_{I,-} = \{ (x_1, \dots, x_N) \in \mathbb{R}^N : x_i \leq 0 \text{ for } i \in I \},$$

where I is any subset of $\{ 1, 2, \dots, N \}$. The trade-off is that our proof will not give the best constant nor the extremals for the inequality.

The main result of this paper is the following

Theorem 1. *Consider $N \geq 1$, $p \geq 1$, $A = (a_1, \dots, a_N), B = (b_1, \dots, b_N) \in \mathbb{R}^N$. Let $a := a_1 + \dots + a_N$ and $b := b_1 + \dots + b_N$, for $p^* \geq 1$ defined by*

$$\frac{1}{p^*} + \frac{b+1}{N} = \frac{1}{p} + \frac{a}{N}, \tag{7}$$

suppose

(1) $\frac{1}{p^*} a_i + \left(1 - \frac{1}{p}\right) b_i > 0$ for all $i = 1, \dots, N$,
(2) $0 \leq a_i - b_i < 1$ for all $i = 1, \dots, N$.
(3) $1 - \frac{N}{p} < a - b \leq 1$.

then there exists a constant $C > 0$ such that for all $u \in C^1_c(\mathbb{R}^N)$

$$\left(\int_{\mathbb{R}^N} \left| x^B u(x) \right|^{p^*} \, \mathrm{d}x \right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{R}^N} \left| x^A \nabla u(x) \right|^{p} \, \mathrm{d}x \right)^{\frac{1}{p}}. \tag{8}$$

Remark 1. Some remarks are relevant at this point.

- The conditions (1)-(3) are not optimal. For instance if $p = 1$, condition (1) says that $a_i > 0$ for all i, however one can allow some of the a_i's to be equal to 0 if the respective b_i is also 0. See section 2 for an example of this, and section 5 for additional remarks.
- As we announced earlier, inequality (8) remains valid if one changes the domain of integration from $\mathbb{R}^N$ to $\mathbb{R}^N_{I,+}$ or $\mathbb{R}^N_{I,-}$. We will comment on this later in section 5.

The rest of this paper is divided as follows. In section 2 we establish some preliminary simplifications for the proof of Theorem 1. In section 3 we prove Theorem 1 in a case that is not covered in (1)-(3) but that illustrates the main idea behind the proof. In section 4 we give the proof of the Theorem and later in section 5 we make some comments regarding the generalizations mentioned in remark 1. Finally, in section 6 we prove a weighted Sobolev trace inequality related to the fractional Laplacian.

2. Prelimaries

Let us begin by saying that in what follows $C > 0$ will represent various constants that are universal, in the sense that they might depend on the structural parameters like the dimension N, the vectors $A, B \in \mathbb{R}^N$ or the exponent p, but C will not depend on the functions $u \in C^1_c(\mathbb{R}^N)$.

Our first remark is that it is enough to prove Theorem 1 for $u \geq 0$. Indeed, for any $u \in C^1_c(\mathbb{R}^N)$ and $\delta > 0$ consider the function $w_\delta := \sqrt{\delta^2 + u^2} - \delta$ and observe that $w_\delta \geq 0$ and $w_\delta \in C^1_c(\mathbb{R}^N)$ with

$$\nabla w_\delta = \frac{u}{\sqrt{u^2 + \delta^2}} \nabla u,$$

moreover, the support of w_δ is contained in the support of u. Now, if we have proved (8) for non-negative functions we can apply it to w_δ. Since $|\nabla w_\delta| \leq |\nabla u|$, we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^N} \left|x^B w_\delta(x)\right|^{p^*} \mathrm{d}x\right)^{\frac{1}{p^*}} &\leq C\left(\int_{\mathbb{R}^N} \left|x^A \nabla w_\delta(x)\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \\ &\leq C\left(\int_{\mathbb{R}^N} \left|x^A \nabla u(x)\right|^p \mathrm{d}x\right)^{\frac{1}{p}}.\end{aligned}$$

Finally, we use Fatou's lemma when letting δ go to zero to obtain the desired inequality for u.

Another simplification one can perform is to observe that it is enough to prove Theorem 1 for $p = 1$ as if this case is handled, then for $p > 1$ one can consider $v = |u|^{\gamma-1} u$ where

$$\gamma = 1 + \frac{N(p-1)}{N + (a-b-1)p}.$$

Observe that $\nabla v = \gamma |u|^{\gamma-1} \nabla u$ and since $\gamma > 1$ we deduce that $v \in C_c^1(\mathbb{R}^N)$. Define the vectors $\tilde{A} = (\tilde{a}_1, \ldots, \tilde{a}_N)$, $\tilde{B} = (\tilde{b}_1, \ldots, \tilde{b}_N) \in \mathbb{R}^N$ in terms of $A = (a_1, \ldots, a_N), B = (b_1, \ldots, b_N) \in \mathbb{R}^N$ as

$$\tilde{a}_i = a_i + \frac{N(p-1)}{N + (a-b-1)p} b_i,$$

and

$$\tilde{b}_i = \frac{p(N+a-b-1)}{N + (a-b-1)p} b_i.$$

Observe that if $\tilde{a} = \tilde{a}_1 + \ldots + \tilde{a}_N$ and $\tilde{b} = \tilde{b}_1 + \ldots + \tilde{b}_N$, then $\tilde{a}_i - \tilde{b}_i = a_i - b_i$ and $\tilde{a} - \tilde{b} = a - b$, hence if A and B satisfy the conditions of Theorem 1 for $p > 1$ then $\tilde{A}$ and $\tilde{B}$ satisfy the conditions for $p = 1$. Thus we can apply Theorem 1 for this particular $\tilde{A}$, $\tilde{B}$, and v as above, that is

$$\left(\int_{\mathbb{R}^N} \left|x^{\tilde{B}} v(x)\right|^{\frac{N}{N+\tilde{a}-\tilde{b}-1}} \mathrm{d}x\right)^{\frac{N+\tilde{a}-\tilde{b}-1}{N}} \leq C \int_{\mathbb{R}^N} \left|x^{\tilde{A}} \nabla v(x)\right| \mathrm{d}x. \tag{9}$$

Thanks to the choice of γ, $\tilde{A}$ and $\tilde{B}$ we observe that the left hand side can be written as

$$\left(\int_{\mathbb{R}^N} \left|x^{\tilde{B}} v(x)\right|^{\frac{N}{N+\tilde{a}-\tilde{b}-1}} \mathrm{d}x\right)^{\frac{N+\tilde{a}-\tilde{b}-1}{N}} = \left(\int_{\mathbb{R}^N} \left|x^{B} u(x)\right|^{\frac{Np}{N+(a-b-1)p}} \mathrm{d}x\right)^{\frac{N+\tilde{a}-\tilde{b}-1}{N}},$$

and thanks to Hölder inequality we see that the integral in the right hand of (9) side can be bounded by

$$\begin{aligned}\int_{\mathbb{R}^N} \left|x^{\tilde{A}} \nabla v(x)\right| \mathrm{d}x &= \gamma \int_{\mathbb{R}^N} |u(x)|^{\gamma-1} \left|x^{\tilde{A}} \nabla u(x)\right| \mathrm{d}x \\ &\leq C\left(\int_{\mathbb{R}^N} \left|x^A \nabla u(x)\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \\ &\quad \times \left(\int_{\mathbb{R}^N} \left|x^{\tilde{A}-A} |u(x)|^{\gamma-1}\right|^{\frac{p}{p-1}} \mathrm{d}x\right)^{1-\frac{1}{p}} \\ &= C\left(\int_{\mathbb{R}^N} \left|x^A \nabla u(x)\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \\ &\quad \times \left(\int_{\mathbb{R}^N} \left|x^B u(x)\right|^{\frac{Np}{N+(a-b-1)p}} \mathrm{d}x\right)^{1-\frac{1}{p}}\end{aligned}$$

and the inequality for $p > 1$ follows as

$$\frac{N+\tilde{a}-\tilde{b}-1}{N} + \frac{1}{p} - 1 = \frac{N+(a-b-1)p}{Np}.$$

3. A particular case

As we mentioned in section 2 in what follows we will only focus on the case $p = 1$ and non-negative functions $u \in C_c^1(\mathbb{R}^N)$.

In this section we will begin with a very particular case of Theorem 1, namely the 1-D version of the theorem. It is important to mention that some of the results we will present here have been known for a long time (see for instance [1, 9], or the book by Kufner and Persson about Hardy type inequalities [11] for a vast survey on similar and more general inequalities), but for the sake of completeness we will give the proofs of each result.

Proposition 1. *Let $a > 0$ and $b \in \mathbb{R}$ such that $0 \leq a - b \leq 1$. If*

$$p^* = \frac{1}{a-b},$$

then there exists a constant $C > 0$ such that

$$\left(\int_{\mathbb{R}} \left||y|^b\, u(y)\right|^{p^*} \, \mathrm{d}y\right)^{\frac{1}{p^*}} \leq C \int_{\mathbb{R}} \left||y|^a\, u'(y)\right| \, \mathrm{d}y, \tag{10}$$

for all $u \in C_c^1(\mathbb{R})$.

Proof. Take $u \in C_c^1(\mathbb{R})$ such that $u \geq 0$. And consider the following cases:
Case $b = a - 1$: In this case $p^* = 1$. By using the compact support of u we can integrate by parts to obtain

$$\begin{aligned}
\int_{\mathbb{R}} |y|^{a-1}\, u(y)\, \mathrm{d}y &= \frac{1}{a}\int_{\mathbb{R}} \left(|y|^{a-1}\, y\right)' u(y)\, \mathrm{d}y \\
&= -\frac{1}{a}\int_{\mathbb{R}} |y|^{a-1}\, y u'(y)\, \mathrm{d}y \\
&\leq \frac{1}{a}\int_{\mathbb{R}} |y|^a\, |u'(y)|\, \mathrm{d}y.
\end{aligned}$$

Case $b = a$: In this case $p^* = \infty$ and for $y \in \mathbb{R}$ we have

$$\begin{aligned}
|y|^a\, u(y) &= -\int_y^\infty \left(|y|^a\, u(y)\right)' \, \mathrm{d}y \\
&= -\int_y^\infty \left(a\, |y|^{a-2}\, y u(y) + |y|^a\, u'(y)\right) \mathrm{d}y \\
&\leq a\int_{\mathbb{R}} \left||y|^{a-1}\, u(y)\right| \, \mathrm{d}y + \int_{\mathbb{R}} \left||y|^a\, u'(y)\right| \, \mathrm{d}y
\end{aligned}$$

but by the case $b = a - 1$ we have $\int_{\mathbb{R}} \left||y|^{a-1}\, u(y)\right| \, \mathrm{d}y \leq \frac{1}{a}\int_{\mathbb{R}} \left||y|^a\, u'(y)\right| \, \mathrm{d}y$ so the first term on the right hand side can be estimated by the second and we obtain

$$\sup_{y\in\mathbb{R}} \left||y|^a\, u(y)\right| \leq 2\int_{\mathbb{R}} \left||y|^a\, u'(y)\right| \, \mathrm{d}y.$$

Case $0 < a-b < 1$: Observe that $p^* = \frac{1}{a-b}$, hence $bp^*+1 = ap^* > 0$. If we integrate by parts over $\mathbb{R}$ to obtain

$$\begin{aligned}\int_{\mathbb{R}} |y|^{bp^*} u(y)^{p^*}\,\mathrm{d}y &= \int_{\mathbb{R}} \left(\frac{|y|^{bp^*} y}{bp^*+1}\right)' u(y)^{p^*}\,\mathrm{d}y \\ &= -\frac{1}{a}\int_{\mathbb{R}} |y|^{bp^*} y u(y)^{p^*-1} u'(y)\,\mathrm{d}y \\ &\le \frac{1}{a}\int_{\mathbb{R}} |y|^{bp^*+1} |u(y)|^{p^*-1} |u'(y)|\,\mathrm{d}y \\ &= \frac{1}{a}\int_{\mathbb{R}} \left| |y|^{\frac{bp^*+1-a}{p^*-1}} u(y)\right|^{p^*-1} \left||y|^a u'(y)\right|\,\mathrm{d}y\end{aligned}$$

but $bp^*+1-a = a(p^*-1)$, and since we already established that

$$\sup_{y\in\mathbb{R}} \left||y|^a u(y)\right| \le 2\int_{\mathbb{R}} \left||y|^a u'(y)\right|\,\mathrm{d}y$$

we conclude

$$\begin{aligned}\int_{\mathbb{R}} |y|^{bp^*} u(y)^{p^*}\,\mathrm{d}y &\le \frac{1}{a}\left(\sup_{y\in\mathbb{R}} \left||y|^a u(y)\right|\right)^{p^*-1} \int_{\mathbb{R}} \left||y|^a u'(y)\right|\,\mathrm{d}y \\ &\le \frac{2^{p^*-1}}{a}\left(\int_{\mathbb{R}} \left||y|^a u'(y)\right|\,\mathrm{d}y\right)^{p^*-1}\left(\int_{\mathbb{R}} \left||y|^a u'(y)\right|\,\mathrm{d}y\right) \\ &\le C\left(\int_{\mathbb{R}} \left||y|^a u'(y)\right|\,\mathrm{d}y\right)^{p^*}.\end{aligned}$$

■

This 1-D result is one of the main ingredients in the proof of Theorem 1. To illustrate the idea, let us prove first a simplified version, namely we have

Theorem 2. *Let $N \ge 1$, $a > 0$ and $b \in \mathbb{R}$ such that $0 \le a-b \le 1$, then for $p^* \ge 1$ satisfying*

$$\frac{1}{p^*} + \frac{b+1}{N} = 1 + \frac{a}{N}. \tag{11}$$

there exists a universal constant $C > 0$ such that for all $u \in C_c^1(\mathbb{R}^N)$

$$\left(\int_{\mathbb{R}^N} \left||y|^b u(\bar{x},y)\right|^{p^*} \mathrm{d}\bar{x}\,\mathrm{d}y\right)^{\frac{1}{p^*}} \le C\int_{\mathbb{R}^N} \left||y|^a \nabla u(\bar{x},y)\right| \mathrm{d}\bar{x}\,\mathrm{d}y, \tag{12}$$

where $\bar{x} = (x_1,\dots,x_{N-1})$ and $y = x_N$.

Remark 2. Observe that this theorem corresponds to the case $A = (0,\dots,0,a)$ and $B = (0,\dots,0,b)$ in Theorem 1. As we mentioned in remark 1 the restrictions on the vectors A and B given in Theorem 1 are not optimal as one can allow some of the coordinates of A to be zero if the respective coordinate in B is also zero as this result shows.

Proof. We only need to worry about the case $N \ge 2$ as Proposition 1 corresponds exactly to the case $N = 1$. Observe that the exponent p^* is given by

$$p^* = \frac{N}{N+a-b-1}.$$

Case $b = a-1$: In this case $p^* = 1$. We use the 1-D result to write for fixed $\bar{x} \in \mathbb{R}^{N-1}$

$$\int_{\mathbb{R}} \left||y|^{a-1} u(\bar{x},y)\right|\,\mathrm{d}y \le \frac{1}{a}\int_{\mathbb{R}} \left||y|^a \partial_y u(\bar{x},y)\right|\,\mathrm{d}y,$$

hence the result follows by integrating with respect to $\bar{x} \in \mathbb{R}^{N-1}$.

Case $b = a$: In this case $p^* = \frac{N}{N-1}$, and the proof is just applying the classical Sobolev inequality to the function $|y|^a u(\bar{x}, y)$ and the previous case, that is

$$\begin{aligned}\left(\int_{\mathbb{R}^N} \left||y|^a u(\bar{x}, y)\right|^{\frac{N}{N-1}} \,\mathrm{d}y\,\mathrm{d}\bar{x}\right)^{\frac{N}{N-1}} &\le C \int_{\mathbb{R}^N} \left|\nabla \left(|y|^a u(\bar{x}, y)\right)\right| \,\mathrm{d}y\,\mathrm{d}\bar{x} \\ &= C \int_{\mathbb{R}^N} \left|a\,|y|^{a-2}\, y u(\bar{x}, y) + |y|^a \nabla u(\bar{x}, y)\right| \,\mathrm{d}y\,\mathrm{d}\bar{x} \\ &\le C \left(a \int_{\mathbb{R}^N} \left||y|^{a-1} u(\bar{x}, y)\right| \,\mathrm{d}y\,\mathrm{d}\bar{x}\right. \\ &\quad \left. + \int_{\mathbb{R}^N} \left||y|^a \nabla u(\bar{x}, y)\right| \,\mathrm{d}y\,\mathrm{d}\bar{x}\right) \\ &\le C \int_{\mathbb{R}^N} \left||y|^a \nabla u(\bar{x}, y)\right| \,\mathrm{d}y\,\mathrm{d}\bar{x}\end{aligned}$$

Case $0 < a - b < 1$: The key is to write the following identity

$$\begin{aligned}\left||y|^b u(\bar{x}, y)\right|^{\frac{N}{N+a-b-1}} = \left||y|^a u(\bar{x}, y)\right|^{\frac{(1-a+b)(N-1)}{N+a-b-1}} &\times \left||y|^{a-1} u(\bar{x}, y)\right|^{\frac{(a-b)(N-1)}{N+a-b-1}} \\ &\times \left||y|^b u(\bar{x}, y)\right|^{\frac{1}{N+a-b-1}}.\end{aligned}$$

Observe that $\frac{(1+b-a)(N-1)}{N+a-b} + \frac{(a-b)(N-1)}{N+a-b-1} + \frac{a-b}{N+a-b-1} = 1$, and that each term is positive since we are assuming $0 < a - b < 1$. After integrating with respect to the y variable over $\mathbb{R}$ and using the generalized Hölder's inequality we obtain

$$\begin{aligned}\int_{\mathbb{R}} \left||y|^b u(\bar{x}, y)\right|^{\frac{N}{N+a-b-1}} \,\mathrm{d}y \le &\left(\int_{\mathbb{R}} \left||y|^a u(\bar{x}, y)\right| \,\mathrm{d}y\right)^{\frac{(1-a+b)(N-1)}{N+a-b-1}} \\ &\times \left(\int_{\mathbb{R}} \left||y|^{a-1} u(\bar{x}, y)\right| \,\mathrm{d}y\right)^{\frac{(a-b)(N-1)}{N+a-b-1}} \\ &\times \left(\int_{\mathbb{R}} \left||y|^b u(\bar{x}, y)\right|^{\frac{1}{a-b}} \,\mathrm{d}y\right)^{\frac{a-b}{N+a-b-1}}.\end{aligned}$$

On the one hand, for $\bar{x}_i = (\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_{N-1})$ we can write

$$\left||y|^a u(\bar{x}, y)\right| \le \prod_{i=1}^{N-1} \left(\int_{\mathbb{R}} \left||y|^a \partial_{x_i} u(\bar{x}_i, y)\right| \,\mathrm{d}x_i\right)^{\frac{1}{N-1}},$$

hence by the generalized Hölder inequality we obtain

$$\left(\int_{\mathbb{R}} \left||y|^a u(\bar{x}, y)\right| \,\mathrm{d}y\right)^{\frac{(1-a+b)(N-1)}{N+a-b-1}} \le \prod_{i=1}^{N-1} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \left||y|^a \partial_{x_i} u(\bar{x}_i, y)\right| \,\mathrm{d}x_i\,\mathrm{d}y\right)^{\frac{1-a+b}{N+a-b-1}}.$$

On the other hand, using Proposition 1 we know that there exists a constant C, such that for all $\bar{x} \in \mathbb{R}^{N-1}$

$$\int_{\mathbb{R}} \left||y|^{a-1} u(\bar{x}, y)\right| \,\mathrm{d}y \le C \int_{\mathbb{R}} \left||y|^a \partial_y u(\bar{x}, y)\right| \,\mathrm{d}y \tag{13}$$

$$\left(\int_{\mathbb{R}} \left||y|^b u(\bar{x}, y)\right|^{\frac{1}{a-b}} \,\mathrm{d}y\right)^{a-b} \le C \int_{\mathbb{R}} \left||y|^a \partial_y u(\bar{x}, y)\right| \,\mathrm{d}y, \tag{14}$$

thus

$$\left(\int_{\mathbb{R}} \left||y|^{a-1} u(\bar{x},y)\right| \,\mathrm{d}y\right)^{\frac{(a-b)(N-1)}{N+a-b-1}} \left(\int_{\mathbb{R}} \left||y|^{b} u(\bar{x},y)\right|^{\frac{1}{a-b}} \,\mathrm{d}y\right)^{\frac{a-b}{N+a-b-1}} \leq C \left(\int_{\mathbb{R}} \left||y|^{a} \partial_y u(\bar{x},y)\right| \,\mathrm{d}y\right)^{\frac{1+(a-b)(N-1)}{N+a-b-1}},$$

and as a consequence we obtain

$$\int_{\mathbb{R}} \left||y|^{b} u(\bar{x},y)\right|^{\frac{N}{N+a-b-1}} \,\mathrm{d}y \leq C \left(\int_{\mathbb{R}} \left||y|^{a} \partial_y u(\bar{x},y)\right| \,\mathrm{d}y\right)^{\frac{1+(a-b)(N-1)}{N+a-b-1}} \times \prod_{i=1}^{N-1} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left||y|^{a} \partial_{x_i} u(\bar{x}_i,y)\right| \,\mathrm{d}x_i \,\mathrm{d}y\right)^{\frac{1-a+b}{N+a-b-1}}. \tag{15}$$

Integrating (15) with respect to the x_1 variable yields

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left||y|^{b} u(\bar{x},y)\right|^{\frac{N}{N+a-b-1}} \,\mathrm{d}y \,\mathrm{d}x_1 \leq C \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left||y|^{a} \partial_{x_1} u(\bar{x},y)\right| \,\mathrm{d}x_1 \,\mathrm{d}y\right)^{\frac{1-a+b}{N+a-b-1}} \times \int_{\mathbb{R}} \left[\left(\int_{\mathbb{R}} \left||y|^{a} \partial_y u(\bar{x},y)\right| \,\mathrm{d}y\right)^{\frac{1+(a-b)(N-1)}{N+a-b-1}} \times \prod_{i=2}^{N-1} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left||y|^{a} \partial_{x_i} u(\bar{x}_i,y)\right| \,\mathrm{d}x_i \,\mathrm{d}y\right)^{\frac{1-a+b}{N+a-b-1}}\right] \mathrm{d}x_1,$$

but $\frac{(1-a+b)(N-2)}{N+a-b-1} + \frac{1+(a-b)(N-1)}{N+a-b-1} = 1$, therefore we can apply the generalized Hölder inequality once again to obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left||y|^{b} u(\bar{x},y)\right|^{\frac{N}{N+a-b-1}} \,\mathrm{d}y \,\mathrm{d}x_1 \leq C \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left||y|^{a} \partial_{x_1} u(\bar{x},y)\right| \,\mathrm{d}x_1 \,\mathrm{d}y\right)^{\frac{1-a+b}{N+a-b-1}} \times \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left||y|^{a} \partial_y u(\bar{x},y)\right| \,\mathrm{d}y \,\mathrm{d}x_1\right)^{\frac{1+(a-b)(N-1)}{N+a-b-1}} \times \prod_{i=2}^{N-1} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left||y|^{a} \partial_{x_i} u(\bar{x}_i,y)\right| \,\mathrm{d}x_i \,\mathrm{d}y \,\mathrm{d}x_1\right)^{\frac{1-a+b}{N+a-b-1}}.$$

If we continue integrating with respect to the remaining variables $x_2,\dots,x_N$ and using the generalized Hölder inequality accordingly we obtain

$$\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}} \left||y|^{b} u(\bar{x},y)\right|^{\frac{N}{N+a-b-1}} \,\mathrm{d}y \,\mathrm{d}\bar{x} \leq C \left(\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}} \left||y|^{a} \partial_y u(\bar{x},y)\right| \,\mathrm{d}y\right)^{\frac{1+(a-b)(N-1)}{N+a-b-1}} \times \prod_{i=1}^{N-1} \left(\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}} \left||y|^{a} \partial_{x_i} u(\bar{x},y)\right| \,\mathrm{d}y \,\mathrm{d}\bar{x}\right)^{\frac{1-a+b}{N+a-b-1}} \leq C \left(\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}} \left||y|^{a} \nabla u(\bar{x},y)\right| \,\mathrm{d}y \,\mathrm{d}\bar{x}\right)^{\frac{N}{N+a-b-1}},$$

and the result is proved. ■

At this point we would like to remark that the key idea behind this proof is to "split" the integrand into three parts: two corresponding to borderline cases $b = a$ and $b = a - 1$, and one to the case $0 < a - b < 1$ in dimension less than N. This is the idea we will use throughout this paper.

As a different, but related application of this idea, is to give an alternative proof of a result of Maz'ya [12, Section 2.1.7] for weights that are radially symmetric only with respect to part of the vector $x \in \mathbb{R}^N$, namely

Theorem 3. *Let $N \geq 1$, $1 \leq k \leq N$, $a > 1-k$ and $b \in \mathbb{R}$ such that $0 \leq a-b \leq 1$. If $p^* \geq 1$ satisfies*

$$\frac{1}{p^*} + \frac{b+1}{N} = 1 + \frac{a}{N}, \tag{16}$$

then there exists a universal constant $C > 0$ such that for all $u \in C_c^1(\mathbb{R}^N)$

$$\left(\int_{\mathbb{R}^{N-k}} \int_{\mathbb{R}^k} \left| |y|^b \, u(\bar{x}, y) \right|^{p^*} \, \mathrm{d}y \, \mathrm{d}\bar{x} \right)^{\frac{1}{p^*}} \leq C \int_{\mathbb{R}^{N-k}} \int_{\mathbb{R}^k} \left| |y|^a \, \nabla u(\bar{x}, y) \right| \, \mathrm{d}y \, \mathrm{d}\bar{x}, \tag{17}$$

where $\bar{x} = (x_1, \ldots, x_{N-k})$ and $y = (x_{N-k+1}, \ldots, y_N)$.

Proof. The proof of this result is completely analogous to the previous one. We only show the main differences. The case $k = N$ corresponds to the CKN inequality [5, Theorem 1], and the case $k = 1$ is the previous result. So in what follows $N > 1$ and $1 < k < N$.

Case $b = a - 1$: Here we have $p^* = 1$, and the inequality follows by using Green's theorem: for fixed $(\bar{x}, y) \in \mathbb{R}^{N-k} \times \mathbb{R}^k$ we have the identity

$$\begin{aligned} 0 &= \int_{\mathbb{R}^k} \operatorname{div}_k \left(|y|^{a-1} \, u(\bar{x}, y) y \right) \, \mathrm{d}y \\ &= \int_{\mathbb{R}^k} \left((a-1+k) \, |y|^{a-1} \, u(\bar{x}, y) + |y|^{a-1} \, y \cdot \nabla_k u(\bar{x}, y) \right) \, \mathrm{d}y, \end{aligned}$$

where div_k denotes the divergence operator and ∇_k is the gradient operator with respect to the k variables of y. This identity implies for $a > 1-k$

$$\int_{\mathbb{R}^{N-k}} \int_{\mathbb{R}^k} |y|^{a-1} \, u(\bar{x}, y) \, \mathrm{d}y \, \mathrm{d}\bar{x} \leq \frac{1}{a-1+k} \int_{\mathbb{R}^{N-k}} \int_{\mathbb{R}^k} |y|^a \, |\nabla_k u(\bar{x}, y)| \, \mathrm{d}y \, \mathrm{d}\bar{x},$$

and the result follows.

Case $b = a$: Here $p^* = \frac{N}{N-1}$. As in Theorem 2 this case follows directly by applying the Sobolev inequality to the function $v(\bar{x}, y) = |y|^a \, u(\bar{x}, y)$ with the aid of the previous case, that is

$$\begin{aligned} \left(\int_{\mathbb{R}^{N-k}} \int_{\mathbb{R}^k} \left| |y|^a \, u(\bar{x}, y) \right|^{\frac{N}{N-1}} \, \mathrm{d}y \, \mathrm{d}\bar{x} \right)^{\frac{N-1}{N}} &\leq C \int_{\mathbb{R}^{N-k}} \int_{\mathbb{R}^k} \left| \nabla(|y|^a \, u(\bar{x}, y)) \right| \, \mathrm{d}y \, \mathrm{d}\bar{x} \\ &\leq C \int_{\mathbb{R}^{N-k}} \int_{\mathbb{R}^k} \left| |y|^a \, \nabla u(\bar{x}, y) \right| \, \mathrm{d}y \, \mathrm{d}\bar{x} \\ &\quad + \int_{\mathbb{R}^{N-k}} \int_{\mathbb{R}^k} \left| a \, |y|^{a-1} \, u(\bar{x}, y) \right| \, \mathrm{d}y \, \mathrm{d}\bar{x} \\ &\leq C \int_{\mathbb{R}^{N-k}} \int_{\mathbb{R}^k} \left| |y|^a \, \nabla u(\bar{x}, y) \right| \, \mathrm{d}y \, \mathrm{d}\bar{x}, \end{aligned}$$

and this case is done.

Case $0 < a - b < 1$: Here $p^* = \frac{N}{N+a-b-1}$, hence we write for $(\bar{x}, y) \in \mathbb{R}^{N-k} \times \mathbb{R}^k$

$$\begin{aligned} \left| |y|^b \, u(\bar{x}, y) \right|^{\frac{N}{N+a-b-1}} &= \left| |y|^a \, u(\bar{x}, y) \right|^{\frac{(b+1-a)(N-k)}{N+a-b-1}} \times \left| |y|^{a-1} \, u(\bar{x}, y) \right|^{\frac{(a-b)(N-k)}{N+a-b-1}} \\ &\quad \times \left| |y|^b \, u(\bar{x}, y) \right|^{\frac{k}{N+a-b-1}}. \end{aligned}$$

Hence, by integrating over $\mathbb{R}^k$ and using the generalized Hölder inequality, we obtain

$$\int_{\mathbb{R}^k} \left| |y|^b u(\bar{x},y) \right|^{\frac{N}{N-1-b+a}} \mathrm{d}y \le \left(\int_{\mathbb{R}^k} |y|^a |u(\bar{x},y)| \, \mathrm{d}y \right)^{\frac{(1+b-a)(N-k)}{N+a-b-1}} \times \left(\int_{\mathbb{R}^k} |y|^{a-1} |u(\bar{x},y)| \, \mathrm{d}y \right)^{\frac{(a-b)(N-k)}{N+a-b-1}} \times \left(\int_{\mathbb{R}^k} \left| |y|^b u(\bar{x},y) \right|^{\frac{k}{k+a-b-1}} \mathrm{d}y \right)^{\frac{k+a-b-1}{N+a-b-1}}.$$

From the CKN inequality [5, Theorem 1] applied for fixed $x \in \mathbb{R}^{N-k}$ we deduce that

$$\left(\int_{\mathbb{R}^k} |y|^{a-1} |u(\bar{x},y)| \, \mathrm{d}y \right)^{\frac{(a-b)(N-k)}{N+a-b-1}} \left(\int_{\mathbb{R}^k} \left| |y|^b u(\bar{x},y) \right|^{\frac{k}{k+a-b-1}} \mathrm{d}y \right)^{\frac{k+a-b-1}{N+a-b-1}} \le C \left(\int_{\mathbb{R}^k} |y|^a |\nabla_k u(\bar{x},y)| \, \mathrm{d}y \right)^{\frac{(a-b)(N-k)+k}{N+a-b-1}}.$$

In addition, we also have

$$\left(\int_{\mathbb{R}^k} |y|^a |u(\bar{x},y)| \, \mathrm{d}y \right)^{\frac{(1+b-a)(N-k)}{N+a-b-1}} \le \prod_{i=1}^{N-k} \left(\int_{\mathbb{R}^{k+1}} |y|^a |\partial_{x_i} u(\bar{x},y)| \, \mathrm{d}x_i \, \mathrm{d}y \right)^{\frac{(1+b-a)}{N+a-b-1}}.$$

The result follows by successive integrations over the variables x_i, $i = 1, \ldots, N-k$, and several applications of the generalized Hölder inequality, where an important observation is that

$$(N-k-1)\frac{1+b-a}{N+a-b-1} + \frac{(a-b)(N-k)+k}{N+a-b-1} = 1.$$

∎

4. Proof of the main theorem

Having illustrated the idea behind the proof in the previous section, we are ready to give the proof of Theorem 1 for $p = 1$.

Proof of Theorem 1. We consider for e_i, the standard basis element in $\mathbb{R}^N$, the following factorization

$$\left| x^B u(x) \right|^{\frac{N}{N+a-b-1}} = \prod_{i=1}^{N} \left| x^{A-e_i} u(x) \right|^{\frac{\gamma_i}{N+a-b-1}} \times \prod_{i=1}^{N} \left| x^{A-a_i e_i} x_i^{b_i} u(x) \right|^{\frac{\delta_i}{N+a-b-1}}, \tag{18}$$

where γ_i and δ_i are chosen so that

$$(a_i - 1)\gamma_i + a_i \sum_{j \ne i} \gamma_j + b_i \delta_i + a_i \sum_{j \ne i} \delta_j = N b_i, \quad i = 1, \ldots, N \tag{19}$$

and

$$\sum_{j=1}^{N} \gamma_j + (a_i - b_i)\delta_i + \sum_{j \ne i} \delta_j = N + a - b - 1, \quad i = 1, \ldots, N. \tag{20}$$

Observe that by adding (19) + (20) for $i = 1, \ldots, N$ we deduce

$$\sum_{i=1}^{N} \gamma_i + \sum_{i=1}^{N} \delta_i = N. \tag{21}$$

To solve the system of equations (19)-(21), we subtract (21) from (20), which gives (recall we are assuming $a_i - b_i < 1$)

$$\delta_i = \frac{1+b-a}{1+b_i-a_i}, \quad i = 1, \ldots, N. \tag{22}$$

To find γ_i, use (21) in (19) to find

$$\gamma_i = (a_i - b_i)(N - \delta_i),$$

hence, using (22) gives

$$\gamma_i = (a_i - b_i)\left(N - \frac{1+b-a}{1+b_i-a_i}\right), \quad i = 1, \dots, N. \tag{23}$$

We observe that if (1)-(2) are satisfied then $\delta_i > 0$ and $\gamma_i > 0$ for all i. To continue the proof, we integrate (18) in the x_1 variable to obtain

$$\int_{\mathbb{R}} \left|x^B u(x)\right|^{\frac{N}{N+a-b-1}} \,\mathrm{d}x_1 = \int_{\mathbb{R}} \left(\prod_{i=1}^N \left|x^{A-e_i} u(x)\right|^{\frac{\gamma_i}{N+a-b-1}} \times \prod_{i=1}^N \left|x^{A-a_i e_i} x_i^{b_i} u(x)\right|^{\frac{\delta_i}{N+a-b-1}}\right) \mathrm{d}x_1$$

But from (20) for $i = 1$ we see that the exponents satisfy the condition to use the generalized Hölder inequality, that is

$$\begin{aligned}
\int_{\mathbb{R}} \left|x^B u(x)\right|^{\frac{N}{N+a-b-1}} \,\mathrm{d}x_1 \le & \prod_{i=2}^N \left(\int_{\mathbb{R}} \left|x^{A-e_i} u(x)\right| \mathrm{d}x_1\right)^{\frac{\gamma_i}{N+a-b-1}} \\
& \times \prod_{i=2}^N \left(\int_{\mathbb{R}} \left|x^{A-a_i e_i} x_i^{b_i} u(x)\right| \mathrm{d}x_1\right)^{\frac{\delta_i}{N+a-b-1}} \\
& \times \left(\int_{\mathbb{R}} \left|x^{A-e_1} u(x)\right| \mathrm{d}x_1\right)^{\frac{\gamma_1}{N+a-b-1}} \\
& \times \left(\int_{\mathbb{R}} \left|x^{A-a_1 e_1} x_1^{b_1} u(x)\right|^{\frac{1}{a_1-b_1}} \mathrm{d}x_1\right)^{\frac{(a_1-b_1)\delta_1}{N+a-b-1}}.
\end{aligned}$$

Observe that the second to last term can be estimated using Proposition 1 for $b_1 = a_1 - 1$, that is

$$\begin{aligned}
\int_{\mathbb{R}} \left|x^{A-e_1} u(x)\right| \mathrm{d}x_1 &= x^{A-a_1 e_1} \int_{\mathbb{R}} \left|x_1^{a_1-1} u(x)\right| \mathrm{d}x_1 \\
&\le C x^{A-a_1 e_1} \int_{\mathbb{R}} \left|x_1^{a_1} \partial_{x_1} u(x)\right| \mathrm{d}x_1 \\
&= C \int_{\mathbb{R}} \left|x^A \partial_{x_1} u(x)\right| \mathrm{d}x_1.
\end{aligned}$$

The last term also corresponds to Proposition 1, this time for $0 < a_1 - b_1 < 1$, that is

$$\begin{aligned}
\left(\int_{\mathbb{R}} \left|x^{A-a_1 e_1} x_1^{b_1} u(x)\right|^{\frac{1}{a_1-b_1}} \mathrm{d}x_1\right)^{a_1-b_1} &= x^{A-a_1 e_1} \left(\int_{\mathbb{R}} \left|x_1^{b_1} u(x)\right|^{\frac{1}{a_1-b_1}} \mathrm{d}x_1\right)^{a_1-b_1} \\
&\le C x^{A-a_1 e_1} \left(\int_{\mathbb{R}} \left|x_1^{a_1} \partial_{x_1} u(x)\right| \mathrm{d}x_1\right) \\
&= C \left(\int_{\mathbb{R}} \left|x^A \partial_{x_1} u(x)\right| \mathrm{d}x_1\right).
\end{aligned}$$

Summarizing, these two estimates yield

$$\begin{aligned}
\int_{\mathbb{R}} \left|x^B u(x)\right|^{\frac{N}{N+a-b-1}} \,\mathrm{d}x_1 \le & C \prod_{i=2}^N \left(\int_{\mathbb{R}} \left|x^{A-e_i} u(x)\right| \mathrm{d}x_1\right)^{\frac{\gamma_i}{N+a-b-1}} \\
& \times \prod_{i=2}^N \left(\int_{\mathbb{R}} \left|x^{A-a_i e_i} x_i^{b_i} u(x)\right| \mathrm{d}x_1\right)^{\frac{\delta_i}{N+a-b-1}} \\
& \times \left(\int_{\mathbb{R}} \left|x^A \partial_{x_1} u(x)\right| \mathrm{d}x_1\right)^{\frac{\gamma_1+\delta_1}{N+a-b-1}}.
\end{aligned} \tag{24}$$

If we now integrate (24) with respect to the x_2 variable and use Hölder's inequality once again we obtain

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} \left|x^B u(x)\right|^{\frac{N}{N+a-b-1}}\,\mathrm{d}x_1\,\mathrm{d}x_2 \leq{}& C\prod_{i=3}^{N}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|x^{A-e_i}u(x)\right|\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{\frac{\gamma_i}{N+a-b-1}}\\ &\times\prod_{i=3}^{N}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|x^{A-a_ie_i}x_i^{b_i}u(x)\right|\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{\frac{\delta_i}{N+a-b-1}}\\ &\times\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|x^A\partial_{x_1}u(x)\right|\,\mathrm{d}x_1\right)^{\frac{\gamma_1+\delta_1}{N+a-b-1}}\\ &\times\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|x^{A-e_2}u(x)\right|\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{\frac{\gamma_2}{N+a-b-1}}\\ &\times\left(\int_{\mathbb{R}}\left[\int_{\mathbb{R}}\left|x^{A-a_2e_2}x_2^{b_2}u(x)\right|\,\mathrm{d}x_1\right]^{\frac{1}{a_2-b_2}}\,\mathrm{d}x_2\right)^{\frac{(a_2-b_2)\delta_2}{N+a-b-1}}.\end{aligned}$$

The second to last term corresponds to Proposition 1, this time in the x_2 variable for $b_2 = a_2 - 1$, therefore we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left|x^{A-e_2}u(x)\right|\,\mathrm{d}x_1\,\mathrm{d}x_2 \leq C\int_{\mathbb{R}}\int_{\mathbb{R}}\left|x^A\partial_{x_2}u(x)\right|\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

For the last term, we first use Minkowski's inequality for integrals, and use 1-D version of Proposition 1 for $0 < a_2 - b_2 < 1$ in the x_2 variable to write

$$\begin{aligned}&\left(\int_{\mathbb{R}}\left[\int_{\mathbb{R}}\left|x^{A-a_2e_2}x_2^{b_2}u(x)\right|\,\mathrm{d}x_1\right]^{\frac{1}{a_2-b_2}}\,\mathrm{d}x_2\right)^{a_2-b_2}\\ &\qquad\leq\left(\int_{\mathbb{R}}x^{A-a_2e_2}\left[\int_{\mathbb{R}}\left|x_2^{b_2}u(x)\right|^{\frac{1}{a_2-b_2}}\,\mathrm{d}x_2\right]^{a_2-b_2}\,\mathrm{d}x_1\right)\\ &\qquad\qquad\leq C\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|x^A\partial_{x_2}u(x)\right|\,\mathrm{d}x_2\,\mathrm{d}x_1\right).\end{aligned}$$

Summarizing, we have obtained

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} \left|x^B u(x)\right|^{\frac{N}{N+a-b-1}}\,\mathrm{d}x_1\,\mathrm{d}x_2 \leq{}& C\prod_{i=3}^{N}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|x^{A-e_i}u(x)\right|\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{\frac{\gamma_i}{N+a-b-1}}\\ &\times\prod_{i=3}^{N}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|x^{A-a_ie_i}x_i^{b_i}u(x)\right|\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{\frac{\delta_i}{N+a-b-1}}\\ &\times\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|x^A\partial_{x_1}u(x)\right|\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{\frac{\gamma_1+\delta_1}{N+a-b-1}}\\ &\times\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|x^A\partial_{x_2}u(x)\right|\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{\frac{\gamma_2+\delta_2}{N+a-b-1}}.\end{aligned}\tag{25}$$

The rest of the proof consist in integrating with respect to the remaining variables and using both Hölder and Minkowski inequalities accordingly, together with Proposition 1. We omit the details.

■

5. Comments about the main Theorem

Let us first discuss the fact that when $p = 1$ the condition $a_i > 0$ for all $i = 1, \ldots, N$ in Theorem 1 is not optimal. As we mentioned in Remark 1 and in Theorem 2, it is possible to have the Sobolev type inequality (8) for vectors A if some of the coordinates a_i are 0, if the respective b_i are also 0. To see this, let $1 < k < N$ and consider $A = (0, \bar{A}) \in \mathbb{R}^{N-k} \times \mathbb{R}^k$ and $B = (0, \bar{B}) \in \mathbb{R}^{N-k} \times \mathbb{R}^k$. Recall that for $x \in \mathbb{R}^N$ the identity (18) is

$$\left|x^B u(x)\right|^{\frac{N}{N+a-b-1}} = \prod_{i=1}^{N} \left|x^{A-e_i} u(x)\right|^{\frac{\gamma_i}{N+a-b-1}} \times \prod_{i=1}^{N} \left|x^{A-a_i e_i} x_i^{b_i} u(x)\right|^{\frac{\delta_i}{N+a-b-1}},$$

but since $a_i = b_i = 0$ for all $i = 1, \ldots, N-k$ we deduce from (22) and (23) that $\gamma_i = 0$ and $\delta_i = 1+b-a$ for all $i = 1, \ldots, N-k$, hence identity (18) becomes

$$\begin{aligned} \left|x^B u(z)\right|^{\frac{N}{N+a-b-1}} = & \prod_{i=N-k+1}^{N} \left|x^{A-e_i} u(\bar{x}, y)\right|^{\frac{\gamma_i}{N+a-b-1}} \times \prod_{i=N-k+1}^{N} \left|x^{A-a_i e_i} x_i^{b_i} u(\bar{x}, y)\right|^{\frac{\delta_i}{N+a-b-1}} \\ & \times \left|y^{\bar{A}} u(\bar{x}, y)\right|^{\frac{\delta_i (N-k)}{N+a-b-1}}, \end{aligned} \tag{26}$$

where $\bar{x} = (x_1, \ldots, x_{N-k})$ and $y = (x_{N-k+1}, \ldots, x_N)$. The proof then continues in the same fashion as in the proof of Theorem 1 with the observation that the last term in (26) is bounded by

$$\left|y^{\bar{A}} u(\bar{x}, y)\right| \leq \prod_{i=1}^{N-k} \left(\int_{\mathbb{R}} \left|y^{\bar{A}} \partial_{x_i} u(\bar{x}_i, y)\right| \mathrm{d}x_i\right)^{\frac{1}{N-k}},$$

where $\bar{x}_i = (\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_{N-k})$. ■

Another topic we announced in Remark 1 is the changes one can make to the domain of integration in (8). So far we have only stated and proved Theorem 1 when the domain of integration is $\mathbb{R}^N$, however we claimed that Theorem remains valid if one changes the domain of integration to $\mathbb{R}^N_{I,+}$ or $\mathbb{R}^N_{I,-}$.

To see this, we first consider the corresponding 1-D result, that is

Proposition 2. *Let $a > 0$ and $b \in \mathbb{R}$ such that $0 \leq a - b \leq 1$, then for*

$$p^* = \frac{1}{a-b}$$

there exists a constant $C > 0$ such that

$$\left(\int_{\mathbb{R}_+} \left||y|^b u(y)\right|^{p^*} \mathrm{d}y\right)^{\frac{1}{p^*}} \leq C \int_{\mathbb{R}_+} \left||y|^a \nabla u(y)\right| \mathrm{d}y. \tag{27}$$

Proof. The proof is analogous to the proof of Proposition 1. We only need to be careful with the integrations by parts we performed, as we now integrate over $(0, \infty)$ instead of over $\mathbb{R}$.
<u>*Case $b = a - 1$*</u>: When we integrate by parts we do it first over (ε, ∞) for $\varepsilon > 0$ to obtain

$$\begin{aligned} \int_{\varepsilon}^{\infty} |y|^{a-1} u(y)\,\mathrm{d}y &= \frac{1}{a} \int_{\varepsilon}^{\infty} \left(|y|^{a-1} y\right)' u(y)\,\mathrm{d}y \\ &= -\frac{1}{a} \varepsilon^a u(\varepsilon) - \frac{1}{a} \int_{\varepsilon}^{\infty} |y|^{a-1} y u'(y)\,\mathrm{d}y. \end{aligned}$$

Since $a > 0$ and $u(\varepsilon) \geq 0$ we can drop the first term in the last line to obtain

$$\int_{\varepsilon}^{\infty} |y|^{a-1} u(y)\,\mathrm{d}y \leq \frac{1}{a} \int_{\mathbb{R}_+} \left||y|^a u'(y)\right| \mathrm{d}y,$$

and we conclude by taking the limit as ε goes to 0.
<u>*Case $b = a$*</u>: Analogous to the proof of the respective case in Proposition 1, we only need to consider $y \geq 0$ instead of $y \in \mathbb{R}$.

Case $0 < a - b < 1$: We again integrate over (ε, ∞) for $\varepsilon > 0$ using $bp^* + 1 = ap^* > 0$ to drop the boundary term at ε, that is

$$\begin{aligned}\int_{\varepsilon}^{\infty} |y|^{bp^*} u(y)^{p^*}\, \mathrm{d}y &= \int_{\varepsilon}^{\infty} \left(\frac{|y|^{bp^*} y}{bp^* + 1}\right)' u(y)^{p^*}\, \mathrm{d}y \\ &= -\frac{1}{bp^* + 1}\varepsilon^{bp^*+1} u(\varepsilon)^{p^*+1} - \frac{1}{a}\int_{\varepsilon}^{\infty} |y|^{bp^*} y u(y)^{p^*-1} u'(y)\, \mathrm{d}y \\ &\leq \frac{1}{a}\int_{\varepsilon}^{\infty} |y|^{bp^*+1} |u(y)|^{p^*-1} |u'(y)|\, \mathrm{d}y \\ &\leq \frac{1}{a}\int_{0}^{\infty} \left| |y|^{\frac{bp^*+1-a}{p^*-1}} u(y)\right|^{p^*-1} ||y|^{a} u'(y)|\, \mathrm{d}y\end{aligned}$$

and we conclude using the same idea from Proposition 1 and then taking the limit $\varepsilon \to 0$. ■

By performing a change of variables $y \mapsto -y$ we obtain immediately

Corollary 1. *Let $a > 0$ and $b \in \mathbb{R}$ such that $0 \leq a - b \leq 1$, then for*

$$p^* = \frac{1}{a - b}$$

there exists a constant $C > 0$ such that

$$\left(\int_{\mathbb{R}_-} \left||y|^{b} u(y)\right|^{p^*} \mathrm{d}y\right)^{\frac{1}{p^*}} \leq C \int_{\mathbb{R}_-} ||y|^{a} \nabla u(y)|\, \mathrm{d}y. \tag{28}$$

Using these two results instead of Proposition 1 in the proof of Theorem 1 yield the generalization where the domain is $\mathbb{R}^N_{I,+}$ or $\mathbb{R}^N_{I,-}$, that is we have

Theorem 4. *Let $N \geq 1$, $p \geq 1$, $A = (a_1, \ldots, a_N), B = (b_1, \ldots, b_N) \in \mathbb{R}^N$ satisfying*

(1) $a_i > 0$ for all $i = 1, \ldots, N$,
(2) $0 \leq a_i - b_i < 1$ for all $i = 1, \ldots, N$,
(3) $0 \leq 1 + b - a \leq \frac{N}{p}$, where $a := a_1 + \ldots + a_N$ and $b := b_1 + \ldots + b_N$,

If $p^* \geq 1$ is such that

$$\frac{1}{p^*} + \frac{b+1}{N} = \frac{1}{p} + \frac{a}{N}. \tag{29}$$

Then there exists a universal constant $C > 0$ such that for all $u \in C^1_c(\mathbb{R}^N)$ and all $I \subseteq \{1, 2, \ldots, N\}$ we have

$$\left(\int_{\mathbb{R}^N_{I,\pm}} \left|x^B u(x)\right|^{p^*} \mathrm{d}x\right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{R}^N_{I,\pm}} \left|x^A \nabla u(x)\right|^{p} \mathrm{d}x\right)^{\frac{1}{p}}, \tag{30}$$

where $\mathbb{R}^N_{I,\pm}$ is either $\mathbb{R}^N_{I,+}$ or $\mathbb{R}^N_{I,-}$.

We will not write the proof of this generalization as it is completely analogous to the proof of Theorem 1. We just emphasize that whenever $i \in I$ and we integrate with respect to the x_i variable we use Proposition 2 or Corollary 1 to obtain the respective estimates, and if $i \notin I$ we use Proposition 1.

6. A Sobolev type trace inequality

In [3] the authors also proved the following Morrey type inequality: If

$$\mathbb{R}^N_A := \{ x \in \mathbb{R} : x_i > 0 \text{ if } a_i > 0 \},$$

then we have

Theorem 5 (Theorem 1.6 in [3])**.** *Let $N \geq 1$, $p \geq 1$, $A \in \mathbb{R}^N$ satisfying $a < 1-\frac{N}{p}$ for $a = a_1+\ldots+a_N$ and $a_i \geq 0$ for all $i = 1, \ldots, N$. Then there exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^N$ and all $u \in C_c^1(\mathbb{R}^N)$*

$$|u(x) - u(y)| \leq C\, |x-y|^{1-a-\frac{N}{p}} \left\| x^A \nabla u \right\|_{L^p(\mathbb{R}^N_A)}.$$

A corollary of Theorems 1 and 5 is the following trace theorem

Theorem 6. *Let $N \geq 1$, $p \geq 1$ and $A \in \mathbb{R}^k$ such that $a < 1-\frac{k}{p}$ for $a = a_1+\ldots+a_k$ with $a_i > 0$ for all $i = 1, \ldots, k$. Then for*

$$\frac{1}{p_*} + \frac{1}{N} = \frac{1}{p}\left(\frac{N+k}{N}\right) + \frac{a}{N}$$

there exists a constant $C > 0$ such that for all $u \in C_c^1(\mathbb{R}^{N+k})$

$$\left(\int_{\mathbb{R}^N} |u(x,0)|^{p_*}\, \mathrm{d}x\right)^{\frac{1}{p_*}} \leq C \left(\int_{\mathbb{R}^N}\int_{(\mathbb{R}_+)^k} \left|y^A \nabla u(x,y)\right|^p \mathrm{d}y\, \mathrm{d}x\right)^{\frac{1}{p}},$$

where $x = (x_1, \ldots, x_N)$ and $y = (x_{N+1}, \ldots, x_{N+k})$.

Proof. Observe that for every $(x,y) \in \mathbb{R}^N \times (\mathbb{R}_+)^k$ and $A \in \mathbb{R}^k$ satisfying $a < 1-\frac{k}{p}$ we can use Theorem 5 in $\mathbb{R}^k$ to obtain

$$|u(x,y)| \geq |u(x,0)| - C_0 \left(\int_{(\mathbb{R}_+)^k} \left|s^A \nabla_k u(x,s)\right|^p \mathrm{d}s\right)^{\frac{1}{p}} |y|^{1-a-\frac{k}{p}}. \tag{31}$$

Define

$$\rho(x) = \left(\frac{|u(x,0)|}{C_0\left(\int_{(\mathbb{R}_+)^k} |s^A \nabla_k u(x,s)|^p \mathrm{d}s\right)^{\frac{1}{p}}}\right)^{\frac{p}{(1-a)p-k}},$$

and for any $q \geq 1$ raise (31) to the power q and integrate over the y variable to obtain

$$\begin{aligned}
\int_{(\mathbb{R}_+)^k} |u(x,y)|^q\, \mathrm{d}y &\geq \int_{B_{\rho(x)}(0)\cap(\mathbb{R}_+)^k} |u(x,y)|^q\, \mathrm{d}y \\
&\geq \int_{B_{\rho(x)}(0)\cap(\mathbb{R}_+)^k} \Bigg(|u(x,0)| \\
&\quad -C_0\left(\int_{(\mathbb{R}_+)^k} \left|s^A \nabla_k u(x,s)\right|^p \mathrm{d}s\right)^{\frac{1}{p}} |y|^{1-a-\frac{k}{p}}\Bigg)^q \mathrm{d}y \\
&= C_0^q \left(\int_{(\mathbb{R}_+)^k} |s^a \nabla_k u(x,s)|^p\, \mathrm{d}s\right)^{\frac{q}{p}} \\
&\quad \times \int_{B_{\rho(x)}(0)\cap(\mathbb{R}_+)^k} \left(\rho(x)^{1-a-\frac{k}{p}} - |y|^{1-a-\frac{k}{p}}\right)^q \mathrm{d}y \\
&= C \left(\int_{(\mathbb{R}_+)^k} |s^a \nabla_k u(x,s)|^p\, \mathrm{d}s\right)^{\frac{q}{p}} \rho(x)^{k+q\left(1-a-\frac{k}{p}\right)} \\
&\quad \times \int_0^1 \left(1-\tau^{1-a-\frac{k}{p}}\right)^q \tau^{k-1}\, \mathrm{d}\tau \\
&= C\, |u(x,0)|^{q+\frac{kp}{(1-a)p-k}} \left(\int_{(\mathbb{R}_+)^k} \left|s^A \nabla_k u(x,s)\right|^p \mathrm{d}s\right)^{-\frac{k}{(1-a)p-k}},
\end{aligned}$$

hence

$$|u(x,0)|^{q+\frac{kp}{(1-a)p-k}} \leq C\left(\int_{(\mathbb{R}_+)^k} |u(x,y)|^q \,\mathrm{d}y\right)\left(\int_{(\mathbb{R}_+)^k} \left|s^A \nabla_k u(x,s)\right|^p \,\mathrm{d}s\right)^{\frac{k}{(1-a)p-k}}, \tag{32}$$

for all $x \in \mathbb{R}^N$. We use (32) for $q = p^* = \frac{(N+k)p}{N+k+(a-1)p}$: let

$$q_0 = p^* + \frac{kp}{(1-a)p-k} = \frac{(N+k)p}{N+k+(a-1)p} + \frac{kp}{(1-a)p-k},$$

then

$$|u(x,0)|^{p_*} \leq C\left(\int_{(\mathbb{R}_+)^k} |u(x,y)|^{p^*} \,\mathrm{d}y\right)^{\frac{p_*}{q_0}}\left(\int_{(\mathbb{R}_+)^k} \left|s^A \partial_y u(x,s)\right|^p \,\mathrm{d}s\right)^{\frac{p_* k}{q_0((1-a)p-k)}},$$

and observe that

$$\frac{p_*}{q_0} + \frac{p_* k}{q_0((1-a)p-k)} = 1,$$

therefore we can integrate over $x \in \mathbb{R}^N$ and use Hölder inequality to obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} |u(x,0)|^{p_*} \,\mathrm{d}x &\leq C\left(\int_{\mathbb{R}^N}\int_{(\mathbb{R}_+)^k} |u(x,y)|^{p^*} \,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{p_*}{q_0}} \\
&\quad \times \left(\int_{\mathbb{R}^N}\int_{(\mathbb{R}_+)^k} \left|y^A \partial_y u(x,y)\right|^p \,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{p_*}{q_0(p(1-a)-1)}} \\
&\leq C\left(\int_{\mathbb{R}^N}\int_{(\mathbb{R}_+)^k} \left|y^A \nabla u(x,y)\right|^p \,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{p^* p_*}{p q_0}} \\
&\quad \times \left(\int_{\mathbb{R}^N}\int_{(\mathbb{R}_+)^k} \left|y^A \nabla u(x,y)\right|^p \,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{p_*}{q_0(p(1-a)-1)}} \\
&\leq C\left(\int_{\mathbb{R}^N}\int_{(\mathbb{R}_+)^k} \left|y^A \nabla u(x,y)\right|^p \,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{p_*}{p q_0}\left(p^* + \frac{p}{p(1-\alpha)-1}\right)} \\
&\leq C\left(\int_{\mathbb{R}^N}\int_{(\mathbb{R}_+)^k} \left|y^A \nabla u(x,y)\right|^p \,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{p_*}{p}}
\end{aligned}$$

thanks to Theorem 1. ■

An important case of the trace theorem above occurs when $k = 1$, that is

$$\left(\int_{\mathbb{R}^N} |u(x,0)|^{p_*} \,\mathrm{d}x\right)^{\frac{1}{p_*}} \leq C\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}_+} |y^a \nabla u(x,y)|^p \,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{p}}, \tag{33}$$

for all $0 \leq a < 1 - \frac{1}{p}$. This kind of inequality is relevant in the context of the localization of the fractional Laplacian obtained by Caffarelli and Silvestre in [4]. They show that if u solves

$$\begin{cases} \operatorname{div}(y^a \nabla u(x,y)) = 0 & (x,y) \in \mathbb{R}^N \times \mathbb{R}_+, \\ u(x,0) = f(x) & x \in \mathbb{R}^N \end{cases}$$

then

$$\lim_{y \to 0^+} -y^a \frac{\partial u}{\partial y}(x,y) = C_{N,a}(-\Delta)^{\frac{1-a}{2}} f(x),$$

that is, trace operators like $u(x,y) \mapsto u(x,0)$ are meaningful for functions u satisfying the integrability condition $\|y^a \nabla u\|_{L^p(\mathbb{R}^N \times \mathbb{R}_+)} < \infty$.

Let us mention at this point that inequality (33) for $p = 2$ can be deduced with no major effort from the standard Sobolev inequality in the fractional Sobolev spaces H^s as one can prove

$$\|u(x,0)\|_{H^\sigma(\mathbb{R}^N)} \leq C\left(\left\|y^{1-2\sigma}u(x,y)\right\|_{L^2(\mathbb{R}^N\times\mathbb{R}_+)} + \left\|y^{1-2\sigma}\nabla u(x,y)\right\|_{L^2(\mathbb{R}^N\times\mathbb{R}_+)}\right)$$

For the case $p > 1$ we can mention the work of Grisvard [7] who did a more general study of the embeddings and interpolation spaces between $L^p(\mathbb{R}_+^{N+1},\,\mathrm{d}\mu)$ and $W^{1,p}(\mathbb{R}_+^{N+1},\,\mathrm{d}\mu)$ for $\mathrm{d}\mu = x_N^a\,\mathrm{d}x$, and the traces in $L^p(\mathbb{R}^N,\,\mathrm{d}x)$ of function from $W^{1,p}(\mathbb{R}_+^{N+1},\,\mathrm{d}\mu)$.

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