OSCILLATIONS IN A SEMI-LINEAR SINGULAR STURM-LIOUVILLE EQUATION

HERNÁN CASTRO

Instituto de Matemática y Física,
Universidad de Talca, Casilla 747, Talca, Chile

Abstract. We study the semi-linear Sturm-Liouville equation

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + |u|^{p-1}\, u & \text{in } (0,1), \\ \qquad u(1) = 0, \end{cases}$$

where $\alpha \geq 1$, $p > 1$, and λ are real parameters. We prove that all non-trivial solutions are oscillatory and unbounded as x approaches 0. Moreover, there exist $\gamma > 0$ and $\delta > 0$ such that any solution $u(x)$ resembles $x^{-\gamma}\eta(x^{-\delta})$ near the origin, where η is a non-trivial periodic solution to the Emden-Fowler equation $\delta^2\eta'' + |\eta|^{p-1}\,\eta = 0$ in $[0,\infty)$.

1. Introduction

Consider the semi-linear Sturm-Liouville equation

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + |u|^{p-1}\, u & \text{in } (0,1), \\ \qquad u(1) = 0, \end{cases} \tag{1}$$

where $\alpha \geq 1$, $p > 1$, and λ are real parameters.

In [5] we performed a detailed study of the existence and non-existence of *positive* solutions to (1) when $\alpha > 0$. One important feature of (1) that appears when $0 < \alpha < 1$ is that the spectrum of the differential operator $Lu := -(x^{2\alpha}u'(x))'$ consists solely of isolated eigenvalues (see [6, Theorem 1.17]), therefore classical bifurcation theory (see for example [7, 8]) tells us that there exists a branch of positive solutions to (1) emanating from the first eigenvalue of L. The results in [5] give us some detailed information on how that branch behaves in the space $\mathbb{R} \times C^0[0,1]$ for different values of α, λ, and p. For instance, it is shown that for $0 < \alpha < 1$ there exists a critical exponent p_α for which (1) behaves quite similarly to the classical Brezis-Nirenberg problem in the unit ball [3], in the sense that the parameter $0 < \alpha < 1$ plays a role comparable to the dimension in the case of the Brezis-Nirenberg problem (see [5, Section 1.4] for the details).

E-mail address: `hcastro@inst-mat.utalca.cl`.

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On the other hand equation (1) changes dramatically when $\alpha \geq 1$. Firstly, the spectrum of the differential operator L has no eigenvalues for $\alpha \geq 1$, in fact the spectrum is purely essential and it can be computed to be the continuum $[\frac{1}{4}, \infty)$ when $\alpha = 1$, and $[0, \infty)$ when $\alpha > 1$ (see [6]), therefore classical bifurcation theory does not apply. However, one can still ask whether bifurcation occurs form the bottom of the spectrum, situation that has been studied vastly in the past. We refer to the very good survey paper written by Stuart [9] (and the references therein) where a general framework for this situation is discussed, as well as the papers by Stuart and Stuart-Vuillaume [10–13] where the buckling of a tapered rod is studied and the operator L appears naturally. Regarding (1) when $\alpha \geq 1$, we established that (1) has no positive solutions, for all values of $p > 1$ and $\lambda \in \mathbb{R}$ (see [5, Theorem 1.10]). We also established in [5] that solutions with finitely many zeros do not exist, in fact, any non-trivial solution u to (1) must be oscillatory in (0,1), in the sense that there exists a sequence $\{ z_n \}_{n \geq 1} \subset (0, 1)$ of zeros of u such that $z_n \to 0$ as $n \to \infty$.

Aside from the oscillatory behavior mentioned above (which can also be deduced from the results in [4]), to our knowledge there is no further literature for equation (1) when $\alpha \geq 1$. A related work is the one of Berestycki and Esteban [2], who study the bifurcation phenomena for the equation

$$\begin{cases} -x^{2\alpha} u'' = \lambda u + |u|^{p-1} u & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{2}$$

for $\alpha = 1$. One of the results they prove is that for each positive integer k and $\lambda < \frac{1}{4}$, there exists a solution in $C^0[0, 1]$ to (2) with exactly k zeros in $(0, 1)$: this fact alone shows us that (1) and (2) have very different features, as no such solutions can be found for (1). Equation (2) in the case $\alpha > 1$ is considered in [1].

With the above in mind, the purpose of this paper is to answer some of the questions raised in [5] regarding equation (1) when $\alpha \geq 1$, more precisely, we would like to answer the following two questions:

Question 1. Do bounded solutions exist to the equation (1) when $\alpha \geq 1$? If so, how many are there?

Question 2. What is the rate of convergence of $z_n \to 0$ as $n \to \infty$?

To answer these questions we use a shooting argument, that is we consider the "final" value problem

$$\begin{cases} -(x^{2\alpha} u')' = \lambda u + |u|^{p-1} u & \text{in } (0,1), \\ u(1) = 0, \\ u'(1) = \theta. \end{cases} \tag{3}$$

for $\theta \neq 0$. From the Cauchy-Lipschitz-Picard theorem we know that (3) has a unique solution in a neighborhood of $x = 1$, which we denote by $u(x; \theta)$. It is not difficult to show that there exists a positive constant $C = C(\theta, \alpha, p, \lambda)$ such that

$$|u(x; \theta)| \leq C x^{-2\alpha}, \text{ and } |u'(x; \theta)| \leq C x^{-1-2\alpha}, \quad \text{for all } 0 < x < 1, \tag{4}$$

from where we deduce that blow up, if any, can only occur at the origin. We are able to prove the following

Theorem 1. *Given $\alpha > 1$, $p > 1$, $\lambda \in \mathbb{R}$, and $\theta \neq 0$, consider $u(x; \theta)$ the solution of (3).*

(i) Let $\gamma := \frac{2\alpha}{p+3}$*, then there exists a sequence* $x_n \to 0$ *and a constant* $C_0 > 0$ *such that*

$$|u(x_n;\theta)| \geq \frac{C_0}{x_n^{\gamma}}.$$

(ii) For $\delta := \alpha - 1 + \frac{\alpha(p-1)}{p+3}$*, the sequence of zeros* $\{ z_n \}$ *of* $u(x;\theta)$ *satisfies*

$$z_n = O\left(n^{-\frac{1}{\delta}}\right)$$

as $n \to \infty$*.*

Remark 1. The result of Theorem 1 follows from Proposition 1 in Section 2, where we show that for x near 0, $u(x;\theta)$ resembles $x^{-\gamma}\eta_\infty(x^{-\delta})$, where γ and δ are as in Theorem 1. The function η_∞ is a periodic solution to the Emden-Fowler equation

$$\delta^2\ddot{\eta}_\infty(t) + |\eta_\infty(t)|^{p-1}\,\eta_\infty(t) = 0, \quad t \geq 0.$$

The rest of this paper is organized as follows: in Section 2 we establish the notation and some preliminary results, and in Section 3 we present the proof of Theorem 1.

2. Preliminaries

In what follows, we will assume that $\alpha \geq 1$, $p > 1$ and $\lambda \in \mathbb{R}$ are given, but we will omit the dependence on these parameters to make the notation simpler. For $\theta \neq 0$, consider $u(x) = u(x;\theta)$ the unique solution of (3) which we know can be extended to all $(0,1)$ (because of (4)). To prove our result, we consider the following change of variables $t = x^{-\delta}$ and

$$\eta(t) = x^{\gamma}u(x),$$

where δ and γ are to be chosen. In what follows we will use the "prime" notation to denote derivatives with respect to the variable x, and the "dot" notation to denote derivatives with respect to the variable t. Observe that

$$u'(x) = -\gamma x^{-\gamma-1}\eta(t) - \delta x^{-\delta-\gamma-1}\dot{\eta}(t),$$

and

$$\begin{aligned}\left(x^{2\alpha}u'(x)\right)' = \delta^2 x^{2\alpha-2\delta-\gamma-2}\ddot{\eta}(t) - \delta\left(2\alpha-\delta-2\gamma-1\right)x^{2\alpha-\delta-\gamma-2}\dot{\eta}(t)\\ -\gamma\left(2\alpha-\gamma-1\right)x^{2\alpha-\gamma-2}\eta(t),\end{aligned}$$

therefore, η satisfies

$$\begin{aligned}\delta^2\ddot{\eta}(t) - \delta\left(2\alpha-\delta-2\gamma-1\right)t^{-1}\dot{\eta}(t) + \left(\lambda t^{\frac{2\alpha-2}{\delta}-2} - \gamma\left(2\alpha-\gamma-1\right)t^{-2}\right)\eta(t)\\ + t^{\frac{-2+2\alpha-2\delta+\gamma(p-1)}{\delta}}\,|\eta(t)|^{p-1}\,\eta(t) = 0.\end{aligned}$$

If we choose γ and δ so that $2\alpha - \delta - 2\gamma - 1 = 0$ and $-2 + 2\alpha - 2\delta + \gamma(p-1) = 0$, that is

$$\gamma := \frac{2\alpha}{p+3} > 0, \tag{5}$$

and

$$\delta := \alpha - 1 + \frac{\alpha}{p+3}(p-1) > 0, \tag{6}$$

we obtain that η is a solution of

$$
\begin{cases} -\delta^2 \ddot{\eta}(t) = g(t)\eta(t) + |\eta(t)|^{p-1}\,\eta(t), \quad t \in (1,\infty), \\ \eta(1) = 0, \\ \dot{\eta}(1) = -\dfrac{\theta}{\delta}, \end{cases} \tag{7}
$$

where $g(t) := (\lambda t^{\frac{2\alpha-2}{\delta}} - \lambda_c)t^{-2}$ and $\lambda_c := \gamma(2\alpha - \gamma - 1) > 0$.

For η solution to (7), consider the following "energy" functional

$$
E_\eta(t) := \frac{\delta^2}{2}\dot{\eta}(t)^2 + \frac{1}{2}g(t)\eta(t)^2 + \frac{|\eta(t)|^{p+1}}{p+1}. \tag{8}
$$

Observe that

$$
\dot{E}_\eta(t) = \left(\delta^2\ddot{\eta}(t) + g(t)\eta(t) + |\eta(t)|^{p-1}\,\eta(t)\right)\dot{\eta}(t) + \frac{1}{2}\dot{g}(t)\eta(t)^2,
$$

therefore, using (7) we obtain

$$
\dot{E}_\eta(t) = \frac{1}{2}\dot{g}(t)\eta(t)^2, \tag{9}
$$

where

$$
\dot{g}(t) := -2\left[\lambda\left(1 - \frac{\alpha-1}{\delta}\right)t^{\frac{2\alpha-2}{\delta}} - \lambda_c\right]t^{-3}. \tag{10}
$$

Remark 2. Notice that from (9) and (10), we deduce that $E_\eta(t)$ is *eventually* non-decreasing (resp. non-increasing), in the sense that there exists $T_0 \geq 1$ such that $\dot{E}_\eta(t) \geq 0$ (resp. ≤ 0) for all $t \geq T_0$. More precisely if $\alpha > 1$ we can define T_0 as the unique zero of $\dot{g}(t)$. We will be using this later.

With the aid of this energy functional, we can prove the following

Lemma 1. *Let η be a solution of* (7)*, then there exists a constant $C > 0$ such that*

$$
|\eta(t)| \leq C \quad \textit{for all } t \geq 1.
$$

Although the proof can be carried out directly for all $\alpha \geq 1$, for the sake of clarity we will present the cases $\alpha = 1$ and $\alpha > 1$ separately.

Proof of Lemma 1 for $\alpha = 1$. Observe that for $\alpha = 1$ we have $g(t) = (\lambda - \lambda_c)t^{-2}$, thus (8) and (9) become

$$
\begin{aligned} E_\eta(t) &= \frac{\delta^2}{2}\dot{\eta}(t)^2 + \frac{\lambda-\lambda_c}{2t^2}\eta(t)^2 + \frac{|\eta(t)|^{p+1}}{p+1}, \\ \dot{E}_\eta(t) &= -\left(\lambda - \lambda_c\right)t^{-3}\eta(t)^2, \end{aligned}
$$

respectively. With this in mind, we study the cases $\lambda \geq \lambda_c$, and $\lambda < \lambda_c$.

If $\lambda \geq \lambda_c$, we obtain that $\dot{E}_\eta(t) = -\left(\lambda - \lambda_c\right)t^{-3}\eta(t)^2 \leq 0$ for all $t \geq 1$, therefore E_η must be a non-increasing function, that is $0 \leq E_\eta(t) \leq E_\eta(1) = \frac{\theta^2}{2}$.

In particular, for all $t \geq 1$ we have

$$
\frac{|\eta(t)|^{p+1}}{p+1} \leq \frac{\delta^2}{2}\eta'(t)^2 + \frac{\lambda-\lambda_c}{2t^2}\eta(t)^2 + \frac{|\eta(t)|^{p+1}}{p+1} = E_\eta(t) \leq \frac{\theta^2}{2},
$$

therefore η is bounded.

To prove that $\eta(t)$ is bounded when $\lambda < \lambda_c$ we argue by contradiction, that is, we will suppose that there exists a sequence $t_n \to \infty$ such that

$$|\eta(t_n)| = \max\{\, |\eta(t)| : 1 \le t \le t_n \,\} \underset{n\to\infty}{\longrightarrow} +\infty.$$

For $n > 1$ we have

$$E_\eta(t_n) - E_\eta(t_1) = \int_{t_1}^{t_n} \dot{E}_\eta(t)\,\mathrm{d}t. \tag{11}$$

On the one hand, we can write

$$\begin{aligned} E_\eta(t_n) - E_\eta(t_1) &= \frac{\delta^2}{2}\dot{\eta}(t_n)^2 + \frac{|\eta(t_n)|^{p+1}}{p+1} + \frac{\lambda-\lambda_c}{2t_n^2}\eta(t_n)^2 \\ &\qquad - \frac{\delta^2}{2}\dot{\eta}(t_1)^2 - \frac{|\eta(t_1)|^{p+1}}{p+1} - \frac{\lambda-\lambda_c}{2t_1^2}\eta(t_1)^2 \\ &\ge \frac{|\eta(t_n)|^{p+1}}{p+1} + \frac{\lambda-\lambda_c}{2t_n^2}\eta(t_n)^2 \\ &\qquad - \frac{\delta^2}{2}\dot{\eta}(t_1)^2 - \frac{|\eta(t_1)|^{p+1}}{p+1} - \frac{\lambda-\lambda_c}{2t_1^2}\eta(t_1)^2. \end{aligned}$$

On the other hand, since $|\eta(t_n)| \ge |\eta(t)|$ for $t_1 \le t \le t_n$ we obtain

$$\begin{aligned} \int_{t_1}^{t_n} \dot{E}_\eta(t)\,\mathrm{d}t &= -(\lambda-\lambda_c)\int_{t_1}^{t_n} t^{-3}\eta(t)^2\,\mathrm{d}t \\ &\le -(\lambda-\lambda_c)\eta(t_n)^2\int_{t_1}^{t_n} t^{-3}\,\mathrm{d}t \\ &= (\lambda-\lambda_c)\eta(t_n)^2\left(\frac{1}{2t_n^2} - \frac{1}{2t_1^2}\right). \end{aligned}$$

To conclude, notice that if we define

$$f(x) := \frac{|x|^{p+1}}{p+1} + \frac{\lambda-\lambda_c}{2t_1^2}\,|x|^2,$$

then identity (11) together with the above estimates tell us that

$$f(\eta(t_n)) \le f(\eta(t_1)) + \frac{\delta^2}{2}\dot{\eta}(t_1)^2,$$

but since $f(x) \to \infty$ when $|x| \to \infty$ we reach a contradiction by letting $n \to \infty$. $\square$

Proof of Lemma 1 when $\alpha > 1$. Recall that

$$\dot{E}_\eta(t) = \frac{1}{2}\dot{g}(t)\eta(t)^2,$$

and observe that by our choice of δ in (6) we have $\delta > \alpha - 1$, therefore if $\lambda > 0$ then $\dot{g}(t) < 0$ for sufficiently large t. Thus $\dot{E}_\eta(t) \le 0$ for sufficiently large t. Hence E_η is non-increasing for large t, and as a consequence $E_\eta(t) \le C_1$ for some constant $C_1 > 0$. Also, for every large t we have $g(t) = t^{-2}\left(\lambda t^{\frac{2\alpha-2}{\delta}} - \lambda_c\right) \ge 0$. In particular, for every sufficiently large t we have

$$\frac{1}{p+1}\,|\eta(t)|^{p+1} \le \frac{\delta^2}{2}\dot{\eta}(t)^2 + \frac{1}{2}g(t)\eta(t)^2 + \frac{1}{p+1}\,|\eta(t)|^{p+1} = E_\eta(t) \le C_1,$$

hence $|\eta(t)|$ is bounded.

If $\lambda \leq 0$ we argue by contradiction. As before, we use the sequence $t_n \to \infty$ satisfying

$$|\eta(t_n)| = \max\left\{ |\eta(t)| : 1 \leq t \leq t_n \right\} \underset{n\to\infty}{\longrightarrow} +\infty.$$

On the one hand we have

$$\begin{aligned} E_\eta(t_n) - E_\eta(t_1) &= \frac{\delta^2}{2}\dot{\eta}(t_n)^2 + \frac{1}{p+1}\left|\eta(t_n)\right|^{p+1} + \frac{1}{2}g(t_n)\eta(t_n)^2 \\ &\qquad - \frac{\delta^2}{2}\dot{\eta}(t_1)^2 - \frac{1}{p+1}\left|\eta(t_1)\right|^{p+1} - \frac{1}{2}g(t_n)\eta(t_1)^2 \\ &\geq \frac{1}{p+1}\left|\eta(t_n)\right|^{p+1} + \frac{1}{2}g(t_n)\eta(t_n)^2 \\ &\qquad - \frac{\delta^2}{2}\dot{\eta}(t_1)^2 - \frac{1}{p+1}\left|\eta(t_1)\right|^{p+1} - \frac{1}{2}g(t_n)\eta(t_1)^2. \end{aligned}$$

On the other hand, since $\lambda \leq 0$ we obtain that $\dot{g}(t) > 0$, for all $t \geq 1$, in addition, $|\eta(t_n)| \geq |\eta(t)|$ for $t_1 \leq t \leq t_n$ therefore

$$\begin{aligned} \int_{t_1}^{t_n} \dot{E}_\eta(t)\,\mathrm{d}t &= \frac{1}{2}\int_{t_1}^{t_n} \dot{g}(t)\eta(t)^2\,\mathrm{d}t \\ &\leq \frac{1}{2}\eta(t_n)^2 \int_{t_1}^{t_n} \dot{g}(t)\,\mathrm{d}t \\ &= \frac{1}{2}\eta(t_n)^2\left(g(t_n) - g(t_1)\right). \end{aligned}$$

Notice that setting

$$f(x) := \frac{|x|^{p+1}}{p+1} + \frac{1}{2}g(t_1)\left|x\right|^2,$$

then the conclusion follows as in the case $\alpha = 1$. We omit the details. □

Remark 3. When $\alpha = 1$ and $\lambda = \lambda_c$, the proof of Lemma 1 tells us that $E_\eta(t)$ is in fact *constant*, that is $E_\eta(t) = E_\eta(1) = \frac{\theta^2}{2}$ for all $t \geq 1$, so (8) becomes

$$\frac{\delta^2}{2}\dot{\eta}(t)^2 + \frac{1}{p+1}\left|\eta(t)\right|^{p+1} = \frac{\theta^2}{2}.$$

Observe that this implies that η is a non-trivial periodic function, which gives a rather explicit description of u for $\lambda = \lambda_c$.

Lemma 2.

$$\dot{E}_\eta(t) \underset{t\to\infty}{\longrightarrow} 0$$

and there exists $E_\infty > 0$ *such that*

$$E_\eta(t) \underset{t\to\infty}{\longrightarrow} E_\infty.$$

Proof. Lemma 1 tells us that η is uniformly bounded, therefore

$$\left|\dot{E}_\eta(t)\right| = \frac{1}{2}\eta(t)^2\left|\dot{g}(t)\right| \leq C\left|\dot{g}(t)\right|,$$

and from (10) we obtain that $|\dot{g}(t)| \to 0$ as $t \to \infty$, thus $\dot{E}_\eta(t) \to 0$.

By Remarks 2 and 3 we know that $E_\eta(t)$ is either constant or *eventually* non-increasing/non-decreasing, therefore

$$E_\infty := \lim_{t\to\infty} E_\eta(t)$$

exists in the extended sense (it might be infinite). However, we have shown that η is bounded, therefore by considering the sequence of critical points of η, we conclude that $E_\infty < \infty$. Also, by taking the sequence of zeros of η we deduce that $E_\infty \geq 0$.

To conclude we need to show that in fact $E_\infty > 0$. From Remark 2 we know that either E_η is eventually non-decreasing or eventually non-increasing. In the non-decreasing case we have $E_\infty \geq E_\eta(T_0) > 0$. In the the case when E_η is eventually non-increasing we need to be a little more careful. Since we have shown that η is bounded, we distinguish two cases:

Case 1: $\|\eta\|_\infty$ is not achieved in (T_0, ∞). In this case, and because η is oscillatory (by [5, Remark 1.10] u must be oscillatory in $(0,1)$), one can construct a sequence $t_n \to \infty$ of critical points of η in (T_0, ∞) such that

$$|\eta(t_n)| = \max\left\{ |\eta(t)| : 1 \leq t \leq t_n \right\}.$$

Since $\dot{g}(t) \leq 0$ for all $t \geq T_0$, we have

$$\begin{aligned} E_\eta(t_n) - E_\eta(T_0) &= \int_{T_0}^{t_n} \dot{E}_\eta(t)\,\mathrm{d}t \\ &= \frac{1}{2}\int_{T_0}^{t_n} \dot{g}(t)\eta(t)^2\,\mathrm{d}t \\ &\geq \frac{\eta(t_n)^2}{2}\int_{T_0}^{t_n} \dot{g}(t)\,\mathrm{d}t \\ &= \frac{1}{2}g(t_n)\eta(t_n)^2 - \frac{1}{2}g(T_0)\eta(t_n)^2. \end{aligned}$$

In addition, and because t_n is a critical point of η, we have

$$\begin{aligned} \frac{1}{p+1}|\eta(t_n)|^{p+1} + \frac{1}{2}g(T_0)\eta(t_n)^2 &\geq E_\eta(T_0) \\ &= \frac{\delta^2}{8}\dot{\eta}(T_0)^2 + \frac{1}{2}g(T_0)\eta(T_0)^2 + \frac{|\eta(T_0)|^{p+1}}{p+1} \\ &> 0, \end{aligned}$$

because $g(T_0) > 0$ and $\eta \not\equiv 0$. As a consequence we deduce the existence of a positive constant C such that $|\eta(t_n)| \geq C$ for all sufficiently large n, therefore,

$$E_\infty = \lim_{n\to\infty} E(t_n) > 0.$$

Case 2: There exists $T_1 \geq T_0$, a critical point of η, such that

$$\|\eta\|_{L^\infty(T_0,\infty)} = |\eta(T_1)| > 0.$$

If we consider $t_n \to \infty$, the sequence of zeros of η in (T_1, ∞) and we use the fact $\dot{g}(t) \leq 0$ for $t \geq T_0$, we have

$$\begin{aligned} E_\eta(t_n) - E_\eta(T_1) &= \int_{T_1}^{t_n} \dot{E}_\eta(t)\,\mathrm{d}t \\ &= \frac{1}{2}\int_{T_1}^{t_n} \dot{g}(t)\eta(t)^2\,\mathrm{d}t \\ &\geq \frac{\eta(T_1)^2}{2}\int_{T_1}^{t_n} \dot{g}(t)\,\mathrm{d}t \\ &= \frac{1}{2}g(t_n)\eta(T_1)^2 - \frac{1}{2}g(T_1)\eta(T_1)^2, \end{aligned}$$

thus obtaining

$$E(t_n) \geq E_\eta(T_1) + \frac{1}{2}g(t_n)\eta(T_1)^2 - \frac{1}{2}g(T_1)\eta(T_1)^2.$$

Recalling that $g(t) \to 0$ as $t \to \infty$ and that T_1 is a critical point, if we let $n \to \infty$ we conclude that

$$E_\infty \geq E_\eta(T_1) - \frac{1}{2}g(T_1)\eta(T_1)^2 = \frac{1}{p+1}\left|\eta(T_1)\right|^{p+1} > 0,$$

and the proof is complete. □

Proposition 1. *Given any sequence $t_n \geq 1$ going to infinity, there exists a periodic function $\eta_\infty \in C^2(1,\infty)$ and a sequence $n_k \to \infty$ such that $\eta_{n_k}(t) := \eta(t+t_{n_k})$ converges in the C^1-norm over compact subsets of $[0,\infty)$ to η_∞ as $k \to \infty$. Moreover, η_∞ is a non-trivial periodic solution to the Emden-Fowler equation*

$$\delta^2\ddot{\eta}_\infty(t) + \left|\eta_\infty(t)\right|^{p-1}\eta_\infty(t) = 0. \tag{12}$$

Proof. Given $\{ t_n \}$, define $\eta_n(t) := \eta(t+t_n)$. Observe that for n sufficiently large, η_n satisfies the equation

$$\frac{\delta^2}{2}\dot{\eta}_n(t)^2 = E_\eta(t+t_n) - \frac{1}{p+1}\left|\eta_n(t)\right|^{p+1} - g(t+t_n)\eta_n(t), \quad t \in [0,\infty). \tag{13}$$

From Lemmas 1 and 2 we deduce the existence of $C > 0$ independent of n such that $E_\eta(t+t_n) \leq C$, and that $|\eta(t+t_n)| \leq C$ for all $t \geq 0$. In addition,

$$g(t+t_n) \to 0 \text{ as } n \to \infty,$$

hence from (13) we deduce that $|\dot{\eta}_n(t)|$ is bounded independently of n. Therefore, by the Arzela-Ascoli theorem, we obtain the existence of a function $\eta_\infty \in C^0[0,\infty)$ and a subsequence $n_k \to \infty$ such that

$$\eta_{n_k}(t) \underset{k\to\infty}{\longrightarrow} \eta_\infty(t)$$

uniformly over compact subsets of $[0,\infty)$. This allows us to pass to the limit in (13) and deduce that in fact $\eta_\infty \in C^1[0,\infty)$ and that it is a solution to

$$\frac{\delta^2}{2}\dot{\eta}_\infty(t)^2 + \frac{1}{p+1}\left|\eta_\infty(t)\right|^{p+1} = E_\infty, \quad t \in [0,\infty). \tag{14}$$

Moreover, by differentiating (13) we also obtain that $|\ddot{\eta}_n(t)|$ is uniformly bounded, therefore by passing to a further subsequence if necessary, we deduce that the convergence is in fact in the C^1-norm over compact subsets of $[0,\infty)$, and that η_∞ is solution to the Emden-Fowler equation (12). Since $E_\infty > 0$ and by observing that the non-trivial trajectories in equation (14) are closed curves, thus we conclude that η_∞ is non-trivial and periodic. □

Remark 4. The function η_∞ depends on the sequence t_n in the following fashion: if we denote by ξ the unique solution to (12) satisfying in addition

$$\xi(0) = 0 \quad \text{and} \quad \dot{\xi}(0) = \frac{2E_\infty}{\delta^2},$$

then $\eta_\infty(t) = \xi(t + t_\infty)$, where $t_\infty \geq 0$ depends on the sequence t_n used in the definition of η_n. In particular, if t_n is the sequence of zeros of η, then $t_\infty = 0$ and $\eta_\infty = \xi$.

3. Proof of Theorem 1

Notice that Proposition 1 tells us that for every $\theta \neq 0$, and any sequence $t_n \to \infty$, the function $\eta_n(t) = \eta(t + t_n)$ is close to the periodic function η_∞. In particular, this implies that between two consecutive large zeros of η the maximum value of $|\eta|$ must be close to the maximum value of $|\eta_\infty|$ in the same interval. By Remark 4, we have that $\|\eta_\infty\|_\infty = \|\xi\|_\infty$, so with this in mind let

$$C_0 := \frac{1}{2}\|\xi\|_\infty > 0$$

and denote by $\{\tilde{z}_n\}$ the sequence of zeros of η. Observe that we can construct a sequence $\{\tilde{m}_n\}$, satisfying for n sufficiently large

$$\tilde{z}_n < \tilde{m}_n < \tilde{z}_{n+1} \quad \text{and} \quad |\eta(\tilde{m}_n)| \geq C_0.$$

By taking the sequence $m_n \in (0,1)$, defined by $m_n := \tilde{m}_n^{-\frac{1}{\delta}} \to 0$ as $n \to \infty$, we obtain

$$|u(m_n;\theta)| = \frac{|\eta(\tilde{m}_n)|}{m_n^\gamma} \geq \frac{C_0}{m_n^\gamma},$$

thus proving the first part of the theorem.

In addition, from Proposition 1 we deduce that for n large, the sequence of zeros $\{\tilde{z}_n\}$ of η must become very close to the sequence of zeros of η_∞, denoted $\{\tilde{z}_n^\infty\}$. Observe that the sequence $\{\tilde{z}_n^\infty\}$ grows at order n, in fact the sequence satisfies

$$\tilde{z}_n^\infty = a + bn,$$

for some constants $a, b > 0$ (this follows from the fact that η_∞ is a periodic solution of (14)). As a consequence we deduce that the sequence $\{\tilde{z}_n\}$ must satisfy an estimate of the form

$$a_1 + b_1 n \leq \tilde{z}_n \leq a_2 + b_2 n, \qquad \text{for all sufficiently large } n,$$

and a_i, b_i constants close to a, b. Recalling that $u(x) = x^{-\gamma}\eta(x^{-\delta})$, we conclude that $\{z_n\}$, the sequence of zeros of $u(x;\theta)$, must tend to zero at a rate of order $n^{-\frac{1}{\delta}}$ as $n \to \infty$, or more precisely

$$\left(\frac{1}{a_2 + b_2 n}\right)^{\frac{1}{\delta}} \leq z_n \leq \left(\frac{1}{a_1 + b_1 n}\right)^{\frac{1}{\delta}}.$$

□

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