POINCARÉ'S INEQUALITY AND SOBOLEV SPACES WITH MONOMIAL WEIGHTS

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Abstract. In this article we use a weighted version of Poincaré's inequality to study density and extension properties of weighted Sobolev spaces over some open set $\Omega \subseteq \mathbb{R}^N$. Additionally, we study the specific case of monomial weights

$$w(x_1, \ldots, x_N) = \prod_{i=1}^{N} |x_i|^{a_i}, \quad a_i \geq 0,$$

showing properties of the associated weighed Sobolev spaces.

1. Introduction

For given $N \geq 1$ and $p \geq 1$, one version of the the classical (local) Poincaré inequality asserts the existence of a constant $C > 0$ for which

$$\int_Q |u - u_Q|^p \, \mathrm{d}x \leq C l(Q)^p \int_Q |\nabla u|^p \, \mathrm{d}x \tag{1}$$

holds for any $u \in C^1(\mathbb{R}^N)$ with average $u_Q = \frac{1}{|Q|} \int_Q u \, \mathrm{d}x$ over the cube $Q \subseteq \mathbb{R}^N$ of edge length $l(Q)$. The Poincaré inequality is central in the study of Sobolev spaces, which in turn are of great importance in the analysis of partial differential equations (see for instance [14] and the many references therein). By using this classical result as an inspiration one can ask oneself for the validity of weighted versions of (1), namely

$$\int_Q |u - u_{Q,w}|^p \, w \, \mathrm{d}x \leq C l(Q)^p \int_Q |\nabla u|^p \, w \, \mathrm{d}x, \tag{2}$$

where

$$u_{Q,w} = \frac{1}{\int_Q w \, \mathrm{d}x} \int_Q u w \, \mathrm{d}x$$

is the weighted average of u over Q, and w is some locally integrable non-negative function. In general it is difficult to characterize the weights for which (2) is valid, but there are a some relevant results that are worth mentioning. On the one hand in dimension $N = 1$ the question was completely answered by Chua and Wheeden [11] in a vast more general setting. One of their results reads that (2) is valid for $p > 1$ if and only if w satisfies

$$\frac{1}{w[a,b]} \left\{ \sup_{a<x<b} \left[w[x,b]^{\frac{1}{p}} \left(\int_a^x w[a,t]^{p'} w(t)^{1-p'} \, \mathrm{d}t \right)^{\frac{1}{p'}} \right] + \sup_{a<x<b} \left[w[a,x]^{\frac{1}{p}} \left(\int_x^b w[t,b]^{p'} w(t)^{1-p'} \, \mathrm{d}t \right)^{\frac{1}{p'}} \right] \right\} < \infty, \tag{3}$$

where $w[x,y] = \int_x^y w(s) \, \mathrm{d}s$ and p' is the Hölder conjugate exponent of p. A similar result is valid for $p = 1$.

On the other hand, if $N \geq 2$ the question is in general open, however there are a few classes of weights that are worth mentioning:

Date: January 28, 2022.

2010 *Mathematics Subject Classification.* 26D10, 35A23, 46E35.

Key words and phrases. Degenerate elliptic, monomial weights, Poincaré inequality, weighted Sobolev spaces.

This research has been partially funded by Fondecyt Regular 1180516.

- The Muckenhoupt class A_p. A weight w belongs to the class A_p for $p>1$ if there exists a constant $C>0$ such that
$$\left(\int_Q w\,\mathrm{d}x\right)\left(\int_Q w^{1-p'}\,\mathrm{d}x\right)^{p-1}\leq Cl(Q)^{Np}$$
holds for every cube $Q\subseteq\mathbb{R}^N$. As one can see for instance in [15, Chapter 15] if $w\in A_p$ then w satisfies (2).
- A function $f:\mathbb{R}^N\to\mathbb{R}^N$ is a quasi-conformal map if $f=(f_1,f_2,\ldots,f_N)$ is a homeomorphism satisfying $f_k\in W^{1,p}_{loc}(\mathbb{R}^N)$ and there is a constant $C>0$ such that
$$\max_{|h|=1}|DF(x)h|^N\leq C\left|\det DF(x)\right|\quad\text{for a.e. } x\in\mathbb{R}^N.$$
It can also be seen in [15, Chapter 15] that if f is quasi-conformal then $w=\left|\det DF\right|^{1-\frac{p}{N}}$ satisfies (2).
- Chanillo and Wheeden showed in [6] the validity of (2) for some weights that are neither A_p nor obtained from quasi-conformal maps: they consider weights w which can be written as
$$w(x)=(1+|x|)^\delta\prod_{i=1}^m\left(\frac{|x-a_i|}{1+|x-a_i|}\right)^{\gamma_i}v(x)$$
where $v\in A_p$, $\delta\geq 0$, $\gamma_i\geq 0$.

We are interested in weights for which the weighted Poincaré inequality (2) holds, but that do not necessarily belong to any of the above classes. One of the main reasons for only focusing on such weights, is that having a (weighted) Poincaré inequality opens the door to several other important results, but in order to make this statement precise we need to give a few definitions.

In what follows we will call a function w a *weight over* $\Omega\subseteq\mathbb{R}^N$ if w is a locally integrable function and non-negative a.e. in Ω. We will denote the measure $w\,\mathrm{d}x$ by $\mathrm{d}w$ or $\mathrm{d}w(x)$, and for any Lebesgue measurable set $E\subseteq\Omega$ we will write $w(E)=\int_E w\,\mathrm{d}x$. For a measurable set E and $p\geq 1$ we will denote by $L^{p,w}(E)=L^p(E,\mathrm{d}w)$ the set of Lebesgue measurable functions satisfying
$$\|u\|_{p,w}^p=\int_E|u|^p\,\mathrm{d}w<\infty,$$
additionally for a measurable set $E\subseteq\Omega$ satisfying $0<w(E)<\infty$ we will write

$$u_{E,w}=\frac{1}{w(E)}\int_E u\,\mathrm{d}w,\tag{4}$$

the weighted average of the function u over E.

In what follows we will consider w a weight over $\Omega\subseteq\mathbb{R}^N$ satisfying the weighted Poincaré inequality (2) for all cubes $Q\subseteq\Omega$,

$$\begin{aligned}w^{1-p'}&\in L^1_{loc}(\Omega)\quad\text{if } p>1,\\ w^{-1}&\in L^\infty_{loc}(\Omega)\quad\text{if } p=1,\end{aligned}\tag{5}$$

and that w is a *doubling weight* in Ω, that is there exists a constant $C_0>0$ such that

$$w(2Q)\leq C_0 w(Q)\tag{6}$$

for every cube $Q\subseteq\Omega$. Here λQ denotes the cube with the same center as Q but with its edge length multiplied by $\lambda>0$.

As we mentioned before the validity of a Poincaré inequality opens the door to several results in the theory of weighted Sobolev spaces, in particular we are interested in two aspect of the theory: the density of smooth functions and the extension problem. It is known (see for instance [15, Chapter 20]) that a doubling weight satisfying in addition the *weighted* $(1,p)$*-Poincaré inequality*
$$\frac{1}{w(Q)}\int_Q|u-u_{Q,w}|\,\mathrm{d}w\leq C_1 l(Q)\left(\frac{1}{w(\lambda Q)}\int_{\lambda Q}|\nabla u|^p\,\mathrm{d}w\right)^{\frac{1}{p}}$$
for some $\lambda\geq 1$, then it also satisfies the following two properties:

(PI) *Uniqueness of the gradient*: If $(u_n)_{n\in\mathbb{N}} \subseteq C^1(\Omega)$ satisfy

$$\int_\Omega |u_n|^p \, \mathrm{d}w \underset{n\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \int_\Omega |\nabla u_n - v|^p \, \mathrm{d}w \underset{n\to\infty}{\longrightarrow} 0$$

for some $v : \Omega \to \mathbb{R}^N$, then $v = 0$.

(PII) *Sobolev inequality*: There exists a constant $C_2 > 0$ and $k > 1$ such that

$$\left(\frac{1}{w(Q)}\int_Q |u|^{kp} \, \mathrm{d}w\right)^{\frac{1}{kp}} \leq C_2 l(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla u|^p \, \mathrm{d}w\right)^{\frac{1}{p}}$$

for $u \in C^1_c(Q)$.

On the one hand, property (PI) allows us to properly define the weighted Sobolev space

$$H^{1,p,w}(\Omega) = \text{the completion of } \left\{ u \in C^1(\Omega) : u, \frac{\partial u}{\partial x_i} \in L^{p,w}(\Omega) \text{ for all } i \right\} \tag{7}$$

under the norm

$$\|u\|^p_{1,p,w} = \|u\|^p_{p,w} + \|\nabla u\|^p_{p,w} \tag{8}$$

as one can see in [15, Section 1.9]. On the other hand the Sobolev inequality (PII) is one of the cornerstones when studying the regularity of solutions of PDEs which can be located in such weighted Sobolev spaces (see for example [12]).

The space $H^{1,p,w}(\Omega)$ is not the only weighted Sobolev space that one can define, in fact, one can consider the Banach space

$$W^{1,p,w}(\Omega) = \text{the completion of } \left\{ u \in W^{1,1}_{loc}(\Omega) : u, \frac{\partial u}{\partial x_i} \in L^{p,w}(\Omega) \quad \text{for all } i \right\}, \tag{9}$$

under the norm (8). Notice that $H^{1,p,w}(\Omega) \subseteq W^{1,p,w}(\Omega)$, and observe that if the weight satisfies (5) then the class defining the space $W^{1,p,w}(\Omega)$ is already complete, and that the definition of $H^{1,p,w}(\Omega)$ becomes simpler as one can see that

$$H^{1,p,w}(\Omega) = \text{the closure of } C^1(\Omega) \cap W^{1,p,w}(\Omega) \text{ in } W^{1,p,w}(\Omega).$$

In the unweighted case $w = 1$ it is known that $H^{1,p}(\Omega) = W^{1,p}(\Omega)$ for *any* open set Ω: this is the classical result of Meyers and Serrin [18], nonetheless for $N \geq 2$ one can construct a weight $w \not\equiv 1$ for which $H^{1,p,w}(\Omega) \subsetneq W^{1,p,w}(\Omega)$. The complete description of the weights for which one has equality is a non trivial task (see for instance [23]).

An interesting fact discovered by Serra Cassano [20] is that if $\Omega \subseteq \mathbb{R}^N$ is bounded and w is a weight satisfying (2) and (5) for $p = 2$ then $H^{1,2,w}(\Omega) = W^{1,2,w}(\Omega)$. This result can be generalized for every $p \geq 1$ and Ω not necessarily bounded to obtain

Theorem 1 (Weighted $H = W$)**.** *Let $p \geq 1$, $\Omega \subseteq \mathbb{R}^N$ a open set, and w a weight over Ω satisfying* (5) *and* (2) *for every cube $Q \subseteq \Omega$ then*

$$H^{1,p,w}(\Omega) = W^{1,p,w}(\Omega).$$

Having a precise ambient space leads to other questions, in particular we are also interested in the so called extension problem, that is to find open sets $\Omega \subseteq \mathbb{R}^N$ for which it is possible to define a bounded linear operator $\mathcal{E} : W^{1,p,w}(\Omega) \to W^{1,p,w}(\mathbb{R}^N)$ satisfying $\mathcal{E}u = u$ inside Ω. A set Ω for which such extension operator can be found is called a (weighted) Sobolev extension domain, and in the unweighted case the class of such domains has been vastly studied: it has been show to include smooth domains [16], Lipschitz domains [3, 21], and the so called (ε, δ)-domains [17]. Let us recall the definition of (ε, δ)-domains: an open connected set $\Omega \subseteq \mathbb{R}^N$ is an (ε, δ)-domain if for all $x, y \in \Omega$ satisfying $|x - y| < \delta$ there exists a rectifiable path $\gamma \subseteq \Omega$ from x to y such that

$$L(\gamma) < \frac{|x-y|}{\varepsilon} \quad \text{and} \quad \operatorname{dist}(z, \partial\Omega) > \frac{\varepsilon|x-z||y-z|}{|x-y|} \quad \forall\, z \in \gamma,$$

where $L(\gamma)$ is the length of γ. If we denote by

$$\operatorname{rad}(\Omega) = \inf_{x\in\Omega}\sup_{y\in\Omega} |x-y| = \sup\left\{ r > 0 : \partial B(x,s) \cap \Omega \neq \varnothing \quad \forall\, x \in \Omega,\ \forall\, 0 \leq s < r \right\},$$

then the result of Jones [17] says that an (ε, δ)-domain with $\mathrm{rad}(\Omega) > 0$ is an unweighted Sobolev extension domain. The result of Jones has been generalized to the weighted case by Chua in a series of articles [7–10] under different hypotheses over the weight w, in particular Chua shows that if Ω is an (ε, δ)-domain with $\mathrm{rad}(\Omega) > 0$ and w is a weight satisfying (2) for all cubes in Ω, and also (5) and (6), then Ω is a weighted Sobolev extension domain.

A further generalization can be made to the results of Jones and Chua, namely we can localize the extension in the sense that for a given open set $\tilde{\Omega} \supseteq \Omega$ we can construct an extension operator $\mathcal{E} : W^{1,p,w}(\Omega) \to W^{1,p,w}(\tilde{\Omega})$ as the following theorem shows.

Theorem 2 (Localized Extension)**.** *Let Ω be an (ε, δ)-domain with $rad(\Omega) > 0$, and let $\tilde{\Omega} \subseteq \mathbb{R}^N$ be an open set such that $\Omega \subseteq \tilde{\Omega}$. If $1 \le p < \infty$ and w is a weight over Ω for which* (2) *holds for every cube $Q \subseteq \Omega$. If in addition the weight satisfies* (5) *and* (6) *then there exists a bounded linear operator*

$$\mathcal{E} : \mathrm{Lip}_{\mathrm{loc}}(\Omega) \to \mathrm{Lip}_{\mathrm{loc}}(\tilde{\Omega})$$

such that $\mathcal{E}u = u$ almost everywhere in Ω and

$$\|\mathcal{E}u\|_{W^{1,p,w}(\tilde{\Omega})} \le C \|u\|_{W^{1,p,w}(\Omega)}$$

for every $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$.

This localized extension theorem is useful if the weight that we are working with satisfies (2), (5), and (6) *locally*. Consider for instance a set $\tilde{\Omega} \subsetneq \mathbb{R}^N$ and a weight w satisfying (2),(5), and (6) *only* in $\tilde{\Omega}$ (not necessarily in all $\mathbb{R}^N$), then $H^{1,p,w}(\tilde{\Omega}) = W^{1,p,w}(\tilde{\Omega})$ by (1) and therefore the extension operator induces an extension operator

$$\mathcal{E} : W^{1,p,w}(\Omega) \to W^{1,p,w}(\tilde{\Omega}).$$

The situation described in the above paragraph appears clearly when dealing with *monomial weights*: For each $k \in \{1, 2, \ldots, M\}$ we consider $\Omega_k \subseteq \mathbb{R}^{N_k}$ and w_k weights over Ω_k. We can define a weight over $\Omega := \prod_{k=1}^M \Omega_k \subseteq \mathbb{R}^N$ by

$$w(x) = w(x_1, x_2, \ldots, x_N) := \prod_{k=1}^{M} w_k(x_k), \tag{10}$$

where $N = N_1 + N_2 + \ldots + N_M$. These weights satisfy (2) provided each of the functions w_k also satisfy it as the following result shows:

Theorem 3. *For every $k \in \{1, 2, \ldots, M\}$ suppose that w_k is a weight for which Poincaré inequality* (2) *holds in $\Omega_k \subseteq \mathbb{R}^{N_k}$, that is there exists a constant $C_k > 0$ such that*

$$\int_{Q_k} |u - u_{Q_k,w_k}|^p \, \mathrm{d}w_k \le C_k l(Q_k)^p \int_{Q_k} |\nabla u|^p \, \mathrm{d}w_k,$$

holds for every cube $Q_k \subseteq \Omega_k$ and every $u \in C^1(\mathbb{R}^{N_k})$.

Then for the weight $w(x) = \prod_{k=1}^M w_k(x_k)$ there exists $C > 0$ such that

$$\int_{Q} |u - u_{Q,w}|^p \, \mathrm{d}w \le C l(Q)^p \int_{Q} |\nabla u|^p \, \mathrm{d}w$$

holds for every cube $Q \subseteq \Omega = \prod_{k=1}^M \Omega_k$ and every $u \in C^1(\mathbb{R}^N)$.

Weights of the form (10) have recently attracted attention as one can see in [1, 2, 4, 5, 13, 19, 22] and references therein. In those works, weighted Sobolev-type inequalities were studied for the monomial weight $w : \mathbb{R}^N \to \mathbb{R}_+$ defined as

$$w(x_1, \ldots, x_N) = x^A := |x_1|^{a_1} \cdot |x_2|^{a_2} \cdot \ldots \cdot |x_N|^{a_N}, \tag{11}$$

where $A = (a_1, \ldots, a_N) \in \mathbb{R}^N$ satisfies $a_i \ge 0$ for all i. One can easily see that this kind of weight belongs to the class A_p if it satisfies $a_i < p - 1$ for all i and as a consequence it satisfies (2) for every $Q \subseteq \mathbb{R}^N$. However if one of the a_i's verifies $a_i \ge p - 1$ then the weight is no longer A_p and does not satisfy (5) in $\mathbb{R}^N$, thus one cannot deduce the validity of the Poincaré inequality (2) nor its consequences. However if we restrict the analysis to the cone

$$\mathbb{R}^N_A = \{ (x_1, \ldots, x_N) \in \mathbb{R}^N : x_i > 0 \text{ whenever } a_i > 0 \}$$

instead of the whole space we can see that if $a_i \geq 0$ for all i then x^A satisfies (5) and additionally it follows from [2, 4] that the weighted Sobolev inequality

$$\left(\int_{\mathbb{R}^N_A} |u|^{\frac{Dp}{D-p}} \, x^A \, \mathrm{d}x \right)^{\frac{D-p}{Dp}} \leq C \left(\int_{\mathbb{R}^N_A} |\nabla u|^p \, x^A \, \mathrm{d}x \right)^{\frac{1}{p}} \tag{12}$$

holds for any $u \in C^1_c(\mathbb{R}^N)$, and $D = N + a_1 + \ldots + a_N$. We will see in Sections 5 and 6 that monomial weights verify a stronger version of (2) thus allowing us to use Theorems 1 and 2 to properly defined weighted Sobolev spaces for $\Omega \subseteq \mathbb{R}^N_A$. To make the relevance of this precise suppose for example that $\Omega = (0,1)^N$, then Theorem 2 and (12) induce an embedding

$$W^{1,p,x^A}(\Omega) \hookrightarrow L^{q,x^A}(\Omega)$$

for every $1 \leq q \leq \frac{Dp}{D-p}$ which in complete analogy with the unweighted case we prove to be compact if $q < \frac{Dp}{D-p}$ (see Theorem 5 in Section 6 below). Other important properties regarding the space W^{1,p,x^A} and inspired by the unweighted case will be given in Section 6. For example, we will prove the existence of a bounded trace operator $\gamma_i : W^{1,p,x^A}(\Omega) \mapsto L^{q,w_i}(\{ x_i = 0 \})$ if a_i is sufficiently small (Theorem 6), and the density of smooth function with support away from the face $\{ x_i = 0 \}$ if a_i is large (Theorem 8).

The rest of this article is organized as follows: Sections 2 and 3 are devoted to the proof of Theorem 1 and Theorem 2 respectively. In Section 4 we prove Theorem 3. We use Section 5 to introduce a special type of weights on the real line for which a stronger version of (2) holds, and finally in Section 6 we analyze the case of monomial weights x^A, where we study embeddings into weighted $L^{p^*}(\Omega)$, $L^{p_*}(\partial\Omega)$, and an additional density results for $W^{1,p,x^A}(\mathbb{R}^N_A)$ when $a_i \geq p-1$.

2. Proof of Theorem 1

We begin this section by recalling some facts about the Whitney cover of a set (see for instance [21, Chapter VI]). Given an open domain Ω there exists a countable collection $\mathcal{W}$ of closed cubes satisfying the following properties:

- (W1) $\Omega = \bigcup_{Q \in \mathcal{W}} Q$.
- (W2) The sides of each $Q \in \mathcal{W}$ are parallel to the coordinate axes.
- (W3) The edge length of $Q \in \mathcal{W}$ satisfies $l(Q) = 2^{-k}$ for some $k \in \mathbb{Z}$.
- (W4) If Q_1 touches Q_2, that is if $Q_1 \cap Q_2 \neq \varnothing$ then $\frac{l}{4} l(Q_2) \leq l(Q_1) \leq 4 l(Q_2)$.
- (W5) $\mathrm{int}(Q_1) \cap \mathrm{int}(Q_2) = \varnothing$ if $Q_1 \neq Q_2$, where $\mathrm{int}(Q)$ denotes the interior of the cube Q.
- (W6) For every $Q \in \mathcal{W}$ one has $\sqrt{N} l(Q) \leq \mathrm{dist}(Q, \Omega^c) \leq 4\sqrt{N} l(Q)$, where $\mathrm{dist}(Q, \Omega^c)$ denotes the distance from Q to the complement of Ω.

An important fact following from (W6) is that for every $Q \in \mathcal{W}$ we have that $\frac{17}{16} Q \subseteq \Omega$.

For a given open set Ω and $\mathcal{W}$ its Whitney cover we will construct a partition of unity in the following fashion: for $n \in \mathbb{N}$ we partition $\mathcal{W}$ as follows

$$\mathcal{W}_{\leq} = \{ Q \in \mathcal{Q} : l(Q) \leq 2^{-n} \},$$
$$\mathcal{W}_{>} = \{ Q \in \mathcal{Q} : l(Q) > 2^{-n} \}.$$

and notice that if $Q \in \mathcal{W}_{>}$ then $l(Q) = 2^k$ for some $k > -n$. Hence each $Q \in \mathcal{W}_{>}$ can be bisected $k+n$ times to obtain $2^{(k+n)N}$ cubes $\hat{Q}$ of edge length $l(\hat{Q}) = 2^{-n}$ covering Q. We denote by $\mathcal{W}'_{>}(Q)$ the collection of all the resulting bisected cubes $\hat{Q}$ of the cube Q. Therefore if we denote by $\mathcal{W}_n = \mathcal{W}_{\leq} \cup \bigcup_{Q \in \mathcal{W}_{>}} \mathcal{W}'_{>}(Q)$ then we obviously get

$$\Omega = \bigcup_{Q \in \mathcal{W}_n} Q,$$

and because $\frac{17}{16} Q \subseteq \Omega$ for all $Q \in \mathcal{W}$ then we also have $\frac{17}{16} Q \subseteq \Omega$ for all $Q \in \mathcal{W}_n$. Additionally it is not difficult to see that there exists a constant $C > 0$, depending only on the dimension N, such that for each $x \in \Omega$ there are at most C cubes in $\mathcal{W}_n$ satisfying $x \in \frac{17}{16} Q$ (this is due to the fact that the bisection process produces at most 3^N touching cubes and properties (W4) and (W6)).

We consider the following partition of the unity: if $Q \in \mathcal{W}_n$ let $\tilde{Q} = \frac{17}{16}Q$ and $\tilde{\varphi}_{\tilde{Q}} \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \tilde{\varphi}_{\tilde{Q}} \leq 1$, $\tilde{\varphi}_{\tilde{Q}} \equiv 1$ in Q, $\operatorname{supp} \tilde{\varphi}_{\tilde{Q}} \subseteq \tilde{Q}$, and $\left|\nabla \tilde{\varphi}_{\tilde{Q}}\right| \leq Cl(\tilde{Q})^{-1}$. Finally let

$$\varphi_{\tilde{Q}} := \frac{\tilde{\varphi}_{\tilde{Q}}}{\sum\limits_{Q \in \mathcal{W}_n} \tilde{\varphi}_{\tilde{Q}}}$$

and we observe that because the sum in the denominator is locally finite we have

$$\left|\nabla \varphi_{\tilde{Q}}\right| \leq \frac{C}{l(\tilde{Q})} \quad \text{and that} \quad \sum_{Q \in \mathcal{W}_n} \varphi_{\tilde{Q}} = 1.$$

Proof of Theorem 1. Let $u \in W^{1,p,w}(\Omega)$ and $n \in \mathbb{N}$, consider the partition of unity constructed above $\{\varphi_{\tilde{Q}}\}_{Q \in \mathcal{W}_n}$ and define

$$u_n = \sum_{Q \in \mathcal{W}_n} u_{\tilde{Q},w} \varphi_{\tilde{Q}},$$

which clearly belongs to $C^\infty(\Omega)$ because the sum is locally finite. We firstly claim that $u_n \underset{n \to \infty}{\longrightarrow} u$ in $L^{p,w}(\Omega)$, indeed the Poincaré inequality (2) over each $\tilde{Q}$ gives

$$\begin{aligned}
\int_\Omega |u_n - u|^p \, \mathrm{d}w &= \int_\Omega \left| \sum_{Q \in \mathcal{W}_n} \varphi_{\tilde{Q}} (u_{\tilde{Q},w} - u) \right|^p \mathrm{d}w \\
&\leq C \sum_{Q \in \mathcal{W}_n} \int_{\tilde{Q}} \left| u_{\tilde{Q},w} - u \right|^p \mathrm{d}w \\
&\leq C \sum_{Q \in \mathcal{W}_n} l(\tilde{Q})^p \int_{\tilde{Q}} |\nabla u|^p \, \mathrm{d}w \\
&\leq C 2^{-np} \sum_{Q \in \mathcal{W}_n} \int_\Omega |\nabla u|^p \chi_{\tilde{Q}} \, \mathrm{d}w \\
&\leq C 2^{-np} \int_\Omega |\nabla u|^p \, \mathrm{d}w,
\end{aligned}$$

hence $u_n \underset{n\to\infty}{\longrightarrow} u$ in $L^{p,w}(\Omega)$. Similarly we have

$$\begin{aligned}
\int_\Omega |\nabla(u_n - u)|^p \, \mathrm{d}w &= \int_\Omega \left| \nabla \left(\sum_{Q\in\mathcal{W}_n} \varphi_{\tilde{Q}}(u_{\tilde{Q},w} - u) \right) \right|^p \mathrm{d}w \\
&= \int_\Omega \left| \left(\sum_{Q\in\mathcal{W}_n} \nabla\varphi_{\tilde{Q}}(u_{\tilde{Q},w} - u) \right) - \sum_{Q\in\mathcal{W}_n} \varphi_{\tilde{Q}} \nabla u \right|^p \mathrm{d}w \\
&\le C \int_\Omega \left| \sum_{Q\in\mathcal{W}_n} \nabla\varphi_{\tilde{Q}}(u_{\tilde{Q},w} - u) \right|^p \mathrm{d}w \\
&\qquad + C \int_\Omega \left| \sum_{Q\in\mathcal{W}_n} \varphi_{\tilde{Q}} \nabla u \right|^p \mathrm{d}w \\
&\le C \int_\Omega \left| \sum_{Q\in\mathcal{W}_n} \frac{(u_{\tilde{Q},w} - u)\chi_{\tilde{Q}}}{l(\tilde{Q})} \right|^p \mathrm{d}w + C \int_\Omega |\nabla u|^p \, \mathrm{d}w \\
&\le C \sum_{Q\in\mathcal{W}_n} \frac{1}{l(\tilde{Q})^p} \int_\Omega \left| u_{\tilde{Q},w} - u \right|^p \chi_{\tilde{Q}} \, \mathrm{d}w + C \int_\Omega |\nabla u|^p \, \mathrm{d}w \\
&\le C \int_\Omega |\nabla u|^p \, \mathrm{d}w,
\end{aligned}$$

where χ_A denotes the characteristic function of the set A.

The above tells us that the sequence $\{ u_n \}_{n\in\mathbb{N}}$ is bounded in $W^{1,p,w}(\Omega)$ and that $u_n \underset{n\to\infty}{\longrightarrow} u$ in $L^{p,w}(\Omega)$. Since for $1 < p < \infty$ $W^{1,p,w}(\Omega)$ is a reflexive Banach space, by passing to a sub-sequence (denoted the same) we may suppose that $u_n \rightharpoonup u$ weakly in $W^{1,p,w}(\Omega)$. Finally, Mazur's lemma tells us there is a sequence consisting of convex combinations of u_n converging strongly to u in $W^{1,p,w}(\Omega)$.

In the case $p = 1$ we can proceed in the same fashion to obtain in addition that for every measurable set $A \subseteq \mathbb{R}^N$ one has

$$\int_{\Omega\cap A} |\nabla(u_n - u)| \, \mathrm{d}w \le C \int_{\Omega\cap A} |\nabla u| \, \mathrm{d}w,$$

and since $\nabla u \in (L^{1,w}(\Omega))^N$ we conclude that

$$\sup_{n\in\mathbb{N}} \int_{\Omega\cap A} |\nabla(u_n - u)| \, \mathrm{d}w \underset{A\searrow\varnothing}{\longrightarrow} 0,$$

therefore the sequence $\{ \nabla(u_n - u) \}_{n\in\mathbb{N}}$ is equi-integrable and uniformly bounded in $(L^{1,w}(\Omega))^N$, hence the Dunford-Pettis theorems tells us that up to a subsequence (denoted the same) we obtain that $\nabla(u_n - u) \rightharpoonup v$ weakly for some some $v \in (L^{1,w}(\Omega))^N$. As before, Mazur's lemma tells us that there is a sequence of convex combinations of $u_n - u$ denoted by f_n such that ∇f_n converges strongly to v in $(L^{1,w}(\Omega))^N$. However, because $u_n \to u$ in $L^{1,w}(\Omega)$ we also have that $f_n \underset{n\to\infty}{\longrightarrow} 0$ strongly in $L^{1,w}(\Omega)$, so by recalling that w satisfies the weighted Poincaré inequality, we can use (PI) to conclude that $v = 0$ and as a consequence $f_n \underset{n\to\infty}{\longrightarrow} 0$ strongly in $W^{1,1,w}(\Omega)$. ■

3. Proof of Theorem 2

The proof of this theorem is an adaptation of the original proof of Jones [17] and its modification by Chua [7–10] to weighted spaces. Most of the calculations are similar to the aforementioned works so we will only highlight the key differences in the proof.

In the classical proof of Jones the extension operator is built from Ω to the whole space $\mathbb{R}^N$ and the initial step is to consider the Whitney covers $\mathcal{W}_1$, $\mathcal{W}_2$ of Ω and of $\mathbb{R}^N \setminus \overline{\Omega}$ respectively. Then by using the fact that Ω is an (ε, δ) domain one can select a subset of $\mathcal{W}_3 \subseteq \mathcal{W}_2$ of cubes Q that are near $\partial\Omega$ and having a "reflection" $Q^* \in \mathcal{W}_1$. In our case the situation is similar, we consider $\mathcal{W}_1$ the Whitney cover of Ω and $\mathcal{W}_2$ the Whitney cover of $\tilde{\Omega} \setminus \overline{\Omega}$. The set $\mathcal{W}_3$ of Jones' would be the set

$$\mathcal{W}_3 = \{ Q \in W_2 : l(Q) \leq \frac{\varepsilon \tilde{\delta}}{16N} \},$$

where $\tilde{\delta} = \min\{ \delta, \mathrm{rad}(\Omega) \}$ however, depending on the choice of $\tilde{\Omega}$ the cubes in $\mathcal{W}_3$ could be far from $\partial\Omega$ (and near $\partial\tilde{\Omega}$) and no appropriate "reflected" cube can be found. To fix this, we instead consider

$$\tilde{\mathcal{W}}_3 = \{ Q \in \mathcal{W}_2 : \mathrm{dist}(Q, \partial\Omega) \leq \frac{\varepsilon \tilde{\delta}}{16\sqrt{N}} \}$$

and we observe that for each $Q \in \tilde{\mathcal{W}}_3$ we have that

$$l(Q) \leq \frac{\mathrm{dist}(Q, \partial(\tilde{\Omega} \setminus \overline{\Omega}))}{\sqrt{N}} \leq \frac{\mathrm{dist}(Q, \partial\Omega)}{\sqrt{N}} \leq \frac{\varepsilon \tilde{\delta}}{16N}$$

and therefore $\tilde{\mathcal{W}}_3 \subseteq \mathcal{W}_3$. As a consequence the following properties of $\tilde{\mathcal{W}}_3$ follow almost verbatim from [17, Lemmas 2.4-2.8]: There exists a constant $C > 0$ such that

(J1) For every cube $Q \in \tilde{\mathcal{W}}_3$ there exists a cube $S \in \mathcal{W}_1$ such that

$$l(Q) \leq l(S) \leq 4l(Q) \quad \text{and} \quad \mathrm{dist}(S, Q) \leq Cl(Q).$$

We will refer any $S \in \mathcal{W}_1$ satisfying mentioned properties as the Jones' reflection of Q, and we will denote it as Q^*.

(J2) If $Q \in \tilde{\mathcal{W}}_3$ and if $S_1, S_2 \in \mathcal{W}_1$ are Jones' reflections of Q, then $\mathrm{dist}(S_1, S_2) \leq Cl(Q)$.

(J3) For all cubes $S \in \mathcal{W}_1$ there are at most C cubes $Q \in \tilde{\mathcal{W}}_3$ such that S is a Jones' reflection of Q.

(J4) If $Q_1, Q_2 \in \tilde{\mathcal{W}}_3$ touch then $\mathrm{dist}(Q_1^*, Q_2^*) \leq Cl(Q_1)$.

(J5) If $Q_1, Q_2 \in \tilde{\mathcal{W}}_3$ touch then there exists a chain $F = \{Q_1^* = S_0, S_1, \ldots, S_m = Q_2^*\}$ of touching cubes in $\mathcal{W}_1$ connecting Q_1^* and Q_2^*, with $m \leq C$.

The second step is using the above construction to define the extension operator. We consider a partition of the unity subordinated to $\tilde{\mathcal{W}}_3$ in the following fashion: for every $Q \in \tilde{\mathcal{W}}_3$ we choose a function $\psi_Q \in C_c^\infty(\mathbb{R}^n)$ such that $\psi_Q = 1$ on Q with support in $\frac{17}{16}Q$, $|\nabla \psi_Q| \leq Cl(Q)^{-1}$ and

$$\sum_{Q \in \tilde{\mathcal{W}}_3} \psi_Q = 1 \quad \text{in} \quad \bigcup_{Q \in \tilde{\mathcal{W}}_3} Q, \qquad \text{and} \quad 0 \leq \sum_{Q \in \tilde{\mathcal{W}}_3} \psi_Q \leq 1 \quad \text{everywhere.}$$

Recalling that for an (ε, δ) domain Ω one has $|\partial\Omega| = 0$ (see [17, Lemma 2.3]) then one can define the extension operator for almost every $x \in \tilde{\Omega}$ as follows, for $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ we define

$$\mathcal{E}u(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ \sum\limits_{Q \in \tilde{\mathcal{W}}_3} u_{Q,\omega} \psi_Q(x) & \text{if } x \in \tilde{\Omega} \setminus \overline{\Omega}. \end{cases} \tag{13}$$

The final steps consists on showing that this operator satisfies the required properties, but using the fact that w satisfies (2),(5), and (6) allows us to follow almost verbatim the proof of [8, Theorem 1.1] to obtain

$$\|\mathcal{E}u\|_{W^{1,p,w}(\tilde{\Omega})} \leq C \|u\|_{W^{1,p,w}(\Omega)},$$

and that $\mathcal{E}u$ is locally Lipschitz, we omit the details. ■

Remark 1. *We chose to present Theorem 2 only for the space* $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$*, but as the reader can see, the technique introduced by Jones and adapted by Chua allows us to obtain a similar theorem for the space* $\mathrm{Lip}_{\mathrm{loc}}^{k-1}(\Omega)$*,* $k \geq 1$.

4. Proof of Theorem 3

The proof of Theorem 3 is a direct consequence of the following

Proposition 1. *For $i = 1, 2$ suppose that μ_i is a measure satisfying the following Poincaré inequality for a cube $Q_i \subseteq \mathbb{R}^{N_i}$ of edge length l_i:*

$$\int_{Q_i} \left| u - \frac{1}{\mu_i(Q_i)} \int_{Q_i} u \, \mathrm{d}\mu_i \right|^p \mathrm{d}\mu_i \leq C_i l_i^p \int_{Q_i} |\nabla u|^p \, \mathrm{d}\mu_i,$$

for $u \in C^1(\mathbb{R}^{N_i})$. Then the product measure $\mu = \mu_1 \times \mu_2$ satisfies

$$\int_{Q_1 \times Q_2} \left| u - \frac{1}{\mu(Q_1 \times Q_2)} \int_{Q_1 \times Q_2} u \, \mathrm{d}\mu \right|^p \mathrm{d}\mu \leq C \max\{ l_1^p, l_2^p \} \int_{Q_1 \times Q_2} |\nabla u|^p \, \mathrm{d}\mu,$$

for $u \in C^1(\mathbb{R}^{N_1} \times \mathbb{R}^{N_2})$.

Proof. Suppose that $\frac{1}{\mu(Q_1 \times Q_2)} \int_{Q_1 \times Q_2} u \, \mathrm{d}\mu = 0$ and for $u \in C^1(\mathbb{R}^{N_1} \times \mathbb{R}^{N_2})$ consider

$$g(x) = \frac{1}{\mu_2(Q_2)} \int_{Q_2} u(x, y) \, \mathrm{d}\mu_2(y).$$

Observe that $\frac{1}{\mu_1(Q_1)} \int_{Q_1} g(x) \, \mathrm{d}\mu_1(x) = 0$ therefore

$$\int_{Q_1} |g(x)|^p \, \mathrm{d}\mu_1(x) \leq C l_1^p \int_{Q_1} |\nabla g(x)|^p \, \mathrm{d}\mu_1(x),$$

but on the one hand Minkowski's inequality for integrals tells us that

$$\begin{aligned}
\left(\int_{Q_1} |\nabla g(x)|^p \, \mathrm{d}\mu_1(x) \right)^{\frac{1}{p}} &= \frac{1}{\mu_2(Q_2)} \left(\int_{Q_1} \left| \int_{Q_2} \nabla_x u(x, y) \, \mathrm{d}\mu_2(y) \right|^p \mathrm{d}\mu_1(x) \right)^{\frac{1}{p}} \\
&\leq \frac{1}{\mu_2(Q_2)} \int_{Q_2} \left(\int_{Q_1} |\nabla_x u(x, y)|^p \, \mathrm{d}\mu_1(x) \right)^{\frac{1}{p}} \mathrm{d}\mu_2(y) \\
&\leq \left(\frac{1}{\mu_2(Q_2)} \int_{Q_2} \int_{Q_1} |\nabla_x u(x, y)|^p \, \mathrm{d}\mu_1(x) \, \mathrm{d}\mu_2(y) \right)^{\frac{1}{p}},
\end{aligned}$$

and on the other hand

$$\begin{aligned}
\int_{Q_1 \times Q_2} |u|^p \, \mathrm{d}\mu &\leq 2^{p-1} \int_{Q_1 \times Q_2} |u - g|^p \, \mathrm{d}\mu + 2^{p-1} \int_{Q_1 \times Q_2} |g|^p \, \mathrm{d}\mu \\
&\leq 2^{p-1} \int_{Q_1} \int_{Q_2} \left| u(x, y) - \frac{1}{\mu_2(Q_2)} \int_{Q_2} u(x, y) \, \mathrm{d}\mu_2(y) \right|^p \mathrm{d}\mu_2(y) \, \mathrm{d}\mu_1(x) \\
&\quad + 2^{p-1} l_1^p \int_{Q_2} \int_{Q_1} |\nabla g(x)|^p \, \mathrm{d}\mu_1(x) \, \mathrm{d}\mu_2(y) \\
&\leq C_2 2^{p-1} l_2^p \int_{Q_1} \int_{Q_2} |\nabla_y u(x, y)|^p \, \mathrm{d}\mu_2(y) \, \mathrm{d}\mu_1(x) \\
&\quad + C_1 2^{p-1} l_1^p \mu_2(Q_2) \int_{Q_1} |\nabla g(x)|^p \, \mathrm{d}\mu_1(x) \\
&\leq C_2 2^{p-1} l_2^p \int_{Q_1} \int_{Q_2} |\nabla_y u(x, y)|^p \, \mathrm{d}\mu_2(y) \, \mathrm{d}\mu_1(x) \\
&\quad + C_1 2^{p-1} l_1^p \int_{Q_2} \int_{Q_1} |\nabla_x u(x, y)|^p \, \mathrm{d}\mu_1(x) \, \mathrm{d}\mu_2(y) \\
&\leq C \max\{ C_1 l_1^p, C_2 l_2^p \} \int_{Q_1 \times Q_2} |\nabla u|^p \, \mathrm{d}\mu.
\end{aligned}$$

■

5. A special class of weights

In the introduction we mentioned that the problem of classifying the weights for which the Poincaré inequality (2) is valid in dimension $N=1$ is completely solved in [11]. In this section we introduce a special class of weights with a condition that is a bit easier to verify than (3) yet producing a slightly stronger version of the Poincaré inequality (2) in dimension $N=1$.

Definition 1. *For a function $w:I\to[0,\infty)$ and $p>1$, one says that w satisfies the condition $(\dagger_p)$ on the interval I if there exists a constant $K\geq 0$ such that for every sub-interval $(\alpha,\beta)\subseteq I$ one has*

$$\int_\alpha^\beta \left[\left|\int_\alpha^y w(s)^{1-p'}\,\mathrm{d}s\right|^{p-1}+\left|\int_y^\beta w(s)^{1-p'}\,\mathrm{d}s\right|^{p-1}\right] w(y)\,\mathrm{d}y\leq K\left|\beta-\alpha\right|^p. \tag{$\dagger_p$}$$

Remark 2. *Observe that if w is A_p in I then one could write for $(\alpha,\beta)\subseteq I$*

$$\begin{aligned}\int_\alpha^\beta w(y)\left|\int_\alpha^y w(s)^{1-p'}\,\mathrm{d}s\right|^{p-1}\,\mathrm{d}y&\leq\left(\int_\alpha^\beta w(y)\,\mathrm{d}y\right)\cdot\left(\int_\alpha^\beta w(s)^{1-p'}\,\mathrm{d}s\right)^{p-1}\\&\leq K\left|\beta-\alpha\right|^p,\end{aligned}$$

and

$$\begin{aligned}\int_\alpha^\beta w(y)\left|\int_y^\beta w(s)^{1-p'}\,\mathrm{d}s\right|^{p-1}\,\mathrm{d}y&\leq\left(\int_\alpha^\beta w(y)\,\mathrm{d}y\right)\cdot\left(\int_\alpha^\beta w(s)^{1-p'}\,\mathrm{d}s\right)^{p-1}\\&\leq K\left|\beta-\alpha\right|^p,\end{aligned}$$

because w is A_p in I, therefore every A_p weight satisfies $(\dagger_p)$.

On the other hand, it is known that if $-1<a<p-1$ then $w(y)=|y|^a$ is A_p in $\mathbb{R}$ (see for instance [15]) and in particular for every $I\subseteq\mathbb{R}$, but if $a\geq p-1$ then w is not A_p in $\mathbb{R}_+$ (and a fortiori *in $\mathbb{R}$). Indeed, if $a>p-1$, $0<\alpha<\beta$ then for $0<x=\frac{\alpha}{\beta}<1$ we could write*

$$\frac{\left(\int_\alpha^\beta w(y)\,\mathrm{d}y\right)\left(\int_\alpha^\beta w(y)^{1-p'}\,\mathrm{d}y\right)^{p-1}}{(\beta-\alpha)^p}\sim\frac{(1-x^{1+a})(1-x^{\frac{a}{p-1}-1})^{p-1}}{x^{a-p+1}(1-x)^p}\underset{x\to0^+}{\longrightarrow}+\infty.$$

Similarly, if $a=p-1$ we have

$$\frac{\int_\alpha^\beta w(y)\,\mathrm{d}y\left|\int_\alpha^\beta w(y)^{1-p'}\,\mathrm{d}y\right|^{p-1}}{(\beta-\alpha)^p}\sim\frac{(1-x^p)(-\ln x)^{p-1}}{(1-x)^p}\underset{x\to0^+}{\longrightarrow}+\infty,$$

however, we will see that $|y|^a$ does satisfy $(\dagger_p)$ in $\mathbb{R}_+$ for all $a>-1$.

Weights satisfying $(\dagger_p)$ satisfy a stronger Poincaré inequality in dimension $N=1$ as the following result shows:

Proposition 2. *For an interval $I\subseteq\mathbb{R}$ suppose $w:I\to[0,\infty)$ satisfies $(\dagger_p)$ in I. Then for every $p>1$ there exists a constant $C>0$ such that for every interval $Q\subseteq I$ of length $l>0$ and every $u\in C^1(\mathbb{R})$ one has*

$$\int_Q\int_Q\left|u(x)-u(y)\right|^p\,\mathrm{d}w(x)\,\mathrm{d}w(y)\leq Cl^pw(Q)\int_Q\left|u'(x)\right|^p\,\mathrm{d}w(x).$$

Proof. For $x,y\in Q$ write

$$|u(x)-u(y)|\leq\int_y^x|u'(s)|\,\mathrm{d}s\leq\left(\int_Q|u'(s)|^p\,\mathrm{d}w(s)\right)^{\frac1p}\left(\int_y^x w^{1-p'}(s)\,\mathrm{d}s\right)^{1-\frac1p},$$

hence

$$\begin{aligned}\int_Q\int_Q|u(x)-u(y)|^p\,\mathrm{d}w(y)\,\mathrm{d}w(x)\leq&\left(\int_Q|u'(s)|^p\,\mathrm{d}w(s)\right)\\&\times\int_Q\int_Q\left|\int_y^x w^{1-p'}(s)\,\mathrm{d}s\right|^{p-1}\,\mathrm{d}w(y)\,\mathrm{d}w(x),\end{aligned}$$

but if $Q = (\alpha, \beta)$ then

$$\begin{aligned}
\int_Q \int_Q \left| \int_y^x w^{1-p'}(s)\,\mathrm{d}s \right|^{p-1} \mathrm{d}w(x)\,\mathrm{d}w(y) \\
&= \int_Q \int_\alpha^\beta \left| \int_y^x w^{1-p'}(s)\,\mathrm{d}s \right|^{p-1} \mathrm{d}w(x)\,\mathrm{d}w(y) \\
&= \int_Q \left(\int_\alpha^x \left| \int_y^x w^{1-p'}(s)\,\mathrm{d}s \right|^{p-1} \mathrm{d}w(y) \right) \mathrm{d}w(x) \\
&\quad + \int_Q \left(\int_x^\beta \left| \int_y^x w^{1-p'}(s)\,\mathrm{d}s \right|^{p-1} \mathrm{d}w(y) \right) \mathrm{d}w(x) \\
&\le \int_Q \left(K\,|x-\alpha|^p\right) \mathrm{d}w(x) + \int_Q \left(K\,|\beta - x|^p\right) \mathrm{d}w(x) \\
&\le 2Kl^p w(Q),
\end{aligned}$$

hence

$$\int_Q \int_Q |u(x) - u(y)|^p \,\mathrm{d}w(y)\,\mathrm{d}w(x) \le 2Kl^p w(Q) \int_Q |u'(s)|^p \,\mathrm{d}w(s).$$

■

Corollary 1. *With the same hypotheses as Proposition 2 we can find a constant $C > 0$ such that for every interval $Q \subseteq I$ and every $u \in C^1(\mathbb{R})$ one has*

$$\int_Q |u - u_{Q,w}|^p \,\mathrm{d}w \le Kl(Q)^p \int_Q |\nabla u|^p \,\mathrm{d}w.$$

Proof. Observe that for any $x \in Q$ one can write

$$|u(x) - u_{Q,w}|^p \le \frac{1}{w(Q)} \int_Q |u(x) - u(y)|^p \,\mathrm{d}w(y)$$

thanks to Hölder's inequality. The result follows by integrating with respect to $\mathrm{d}w(x)$ and using Proposition 2. ■

6. The case of Monomial weights

In this section we use the previous results to study the case of monomials weights of the form

$$w(x) = w(x_1, x_2, \ldots, x_N) = \prod_{i=1}^{k} |x_i|^{a_i}$$

where $1 \le k \le N$ and $a_i \ge 0$. We begin by proving that the weight function $w(s) = |s|^a$ satisfies $(\dagger_p)$ in $\mathbb{R}_+$ for all $a > -1$, and moreover any function resembling $|s|^a$ does satisfy $(\dagger_p)$.

In what follows, we will say that $f \sim g$ for $f, g : U \to \mathbb{R}$ if there exists $C \ge 1$ such that

$$\frac{1}{C} g(s) \le f(s) \le C g(s) \quad \forall\, s \in U.$$

Lemma 1. *Let $p > 1$, $a > -1$, and $w : [0, \infty) \to [0, \infty)$ satisfying*

$$w(s) \sim |s|^a \quad \forall\, s > 0,$$

then w satisfies $(\dagger_p)$ in $\mathbb{R}_+$.

Proof. For $0 < b < c < \infty$ define $I_b, J_c : (b, c) \to \mathbb{R}$ as

$$I_b(x) = \int_b^x w(y) \left| \int_y^x w(s)^{1-p'} \,\mathrm{d}s \right|^{p-1} \mathrm{d}y,$$

$$J_c(x) = \int_x^c w(y) \left| \int_y^x w(s)^{1-p'} \,\mathrm{d}s \right|^{p-1} \mathrm{d}y.$$

By the hypothesis over w we have

$$\left|\int_x^y w(s)^{1-p'}\,\mathrm{d}s\right| \sim \left|\int_x^y |s|^{-\frac{a}{p-1}}\,\mathrm{d}s\right| \sim \begin{cases} \left||y|^{-\frac{a}{p-1}}y - |x|^{-\frac{a}{p-1}}x\right| & \text{if } a \neq p-1, \\ \left|\ln\left(\frac{y}{x}\right)\right| & \text{if } a = p-1. \end{cases}$$

We separate the study into three cases: $-1 < a < p-1$, $a = p-1$, and $a > p-1$. Although the first case follows directly by noticing that $|s|^a$ is an A_p weight we will provide a proof for the sake completeness.

Case 1 $-1 < a < p-1$: Observe that

$$I_b(x) \sim x^{p-1-a}\int_b^x y^a\left(1-\left(\frac{y}{x}\right)^{\frac{p-1-a}{p-1}}\right)^{p-1}\mathrm{d}y$$

if $y = t^{\frac{p-1}{p-1-a}}x$ then

$$I_b(x) \sim x^p\int_{\left(\frac{b}{x}\right)^{\frac{p-1-a}{p-1}}}^1 t^{\frac{ap}{p-1-a}}(1-t)^{p-1}\,\mathrm{d}t$$

but if $\tilde{I}(s) = \int_s^1 t^{\frac{ap}{p-1-a}}(1-t)^{p-1}\,\mathrm{d}t$ then we have

$$\tilde{I}(s) \le K_1(1-s^{\frac{p-1}{p-1-a}})^p \quad \forall\, s\in[0,1],$$

because L'Hôpital's theorem tells us that if $s\sim 1$ then

$$\frac{\tilde{I}(s)}{(1-s^{\frac{p-1}{p-1-a}})^p} \sim \left(\frac{1-s}{1-s^{\frac{p-1}{p-1-a}}}\right)^{p-1} \sim 1,$$

and because $-1 < a < p-1$ we also know that

$$\tilde{I}(0) = \int_0^1 t^{\frac{ap}{p-1-a}}(1-t)^{p-1}\,\mathrm{d}t < \infty$$

therefore we deduce that $K_1 = \sup\limits_{s\in[0,1]}\frac{\tilde{I}(s)}{(1-s^{\frac{p-1}{p-1-a}})^p}$ is finite. This immediately gives

$$I_b(x) \sim x^p\tilde{I}\left(\left(\frac{b}{x}\right)^{\frac{p-1-a}{p-1}}\right) \le K_1(x-b)^p.$$

Similarly

$$J_c(x) \sim x^{p-1-a}\int_x^c y^a\left(\left(\frac{y}{x}\right)^{\frac{p-1-a}{p-1}}-1\right)^{p-1}\mathrm{d}y$$

if $y = t^{\frac{p-1}{p-1-a}}x$ then

$$J_c(x) \sim x^p\int_1^{\left(\frac{c}{x}\right)^{\frac{p-1-a}{p-1}}} t^{\frac{ap}{p-1-a}}(t-1)^{p-1}\,\mathrm{d}t.$$

If we let $\tilde{J}(s) = \int_1^s t^{\frac{ap}{p-1-a}}(t-1)^{p-1}\,\mathrm{d}t$ then

$$\tilde{J}(s) \le K_2(s^{\frac{p-1}{p-1-a}}-1)^p, \quad \forall\, s\ge 1.$$

Indeed, observe that for $s>1$ large L'Hôpital's rule gives us that

$$\frac{\tilde{J}(s)}{(s^{\frac{p-1}{p-1-a}}-1)^p} \sim \left(\frac{s^{\frac{a}{p-1-a}}(s-1)}{s^{\frac{p-1}{p-1-a}}-1}\right)^{p-1} = \left(\frac{1-s^{-1}}{1-s^{-\frac{p-1}{p-1-a}}}\right)^{p-1} \sim 1,$$

and similarly for $s\sim 1$ we have

$$\frac{\tilde{J}(s)}{(s^{\frac{p-1}{p-1-a}}-1)^p} \sim \left(\frac{s-1}{s^{\frac{p-1}{p-1-a}}-1}\right)^{p-1} \sim 1,$$

therefore $K_2 = \sup\limits_{s\ge 1}\frac{\tilde{J}(s)}{(s^{\frac{p-1}{p-1-a}}-1)^p}$ is finite and we conclude that

$$J_c(x) \le K_2(c-x)^p.$$

Case 2 $a = p-1$: In this case we find that

$$I_b(x) \sim x^p \int_0^{\ln\left(\frac{x}{b}\right)} e^{-pt} t^{p-1}\, \mathrm{d}t$$

but if $H(s) = \int_0^s e^{-pt} t^{p-1}$ it is not difficult to see that

$$H(s) \le K_3(1-e^{-s}) \quad \forall\, s \ge 0,$$

which immediately gives

$$I_b(x) \le K_3(x-b)^p.$$

Proceeding in the same fashion gives $J_c(x) \le K_4(c-x)^p$, we omit the details.

Case 3 $a > p-1$: As above we obtain that

$$I_b(x) \sim x^p \int_1^{\left(\frac{x}{b}\right)^{\frac{a-p+1}{p-1}}} t^{\frac{ap}{p-1-a}} (t-1)^{p-1}\ \mathrm{d}t$$

but if $\tilde{I}(s) = \int_1^s t^{\frac{ap}{p-1-a}} (t-1)^{p-1}\ \mathrm{d}t$, and just as in the previous cases it is not difficult to see that

$$\tilde{I}(s) \le K(1 - s^{\frac{p-1}{p-1-a}})^p \quad \forall\, s \ge 1,$$

which immediately gives

$$I_b(x) \le K(x-b)^p.$$

Likewise we obtain $J_c(x) \le K(c-x)^p$.

■

Thanks to Lemma 1 we are able to apply Theorems 1 to 3 to the weight

$$w(x) = w(x_1, x_2, \ldots, x_N) = \prod_{i=1}^k |x_i|^{a_i} = x^A$$

and conclude that the weighted Poincaré inequality (2) is valid in every open subset of $\mathbb{R}^N_A = (\mathbb{R}_+)^k \times \mathbb{R}^{N-k}$ whenever $A = (a_1, \ldots, a_k, 0) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ satisfies $a_i > -1$ for all $i \in \{1, \ldots, k\}$. In addition we can define the spaces

$$L^{p,A}(\Omega) = L^{p,x^A}(\Omega),$$
$$W^{1,p,A}(\Omega) = W^{1,p,x^A}(\Omega),$$
$$H^{1,p,A}(\Omega) = H^{1,p,x^A}(\Omega),$$

and that for each (ε,δ) domain $\Omega \subseteq \mathbb{R}^N_A$ we have the existence of an extension operator $\mathcal{E} : W^{1,p,A}(\Omega) \to W^{1,p,A}(\mathbb{R}^N_A)$.

Remark 3. *Observe that if $\overline{\Omega} \subset \mathbb{R}^N_A$ then $x^A \ge \kappa > 0$ for all $x \in \Omega$ and as a consequence*

$$W^{1,p,A}(\Omega) \hookrightarrow W^{1,p}(\Omega)$$

continuously. In order to avoid this situation and to "see" the behavior of the weight x^A over $\overline{\Omega}$ we will suppose that Ω contains a set of the form $(0,1)^k \times \tilde{\Omega}$ for some $\tilde{\Omega} \subseteq \mathbb{R}^{N-k}$.

As we mentioned in the introduction, and with the aid of [2, Theorem 1.3] or [4, Theorem 1], the above construction allows us to obtain a Sobolev embedding theorem of such domains, that is

Theorem 4. *Suppose that $A \in (\mathbb{R}_+)^k \times \{0\}^{N-k}$ and that $\Omega \subseteq \mathbb{R}^N_A$, then*

$$W^{1,p,A}(\Omega) \hookrightarrow L^{r,A}(\Omega),$$

for every $p \le r \le \frac{Dp}{N-p}$, where $D = N + a_1 + \ldots + a_k$.

In particular one has that

$$W^{1,p,A}(\Omega) \hookrightarrow L^{r,A}(\Omega)$$

for all $1 \le r \le \frac{Dp}{D-p}$ for bounded Ω, and just as in the unweighted case this embedding is compact in the subcritical case.

Theorem 5. *Let $A \in (\mathbb{R}_+)^k \times \{ 0 \}^{N-k}$ and Ω a bounded domain in $\mathbb{R}^N_A$. The inclusion $W^{1,p,A}(\Omega) \hookrightarrow L^{r,A}(\Omega)$ is compact when $1 \le r < \frac{Dp}{D-p}$.*

Proof. Observe that it is enough to prove the result for $r = 1$, as if we know that the embedding $W^{1,p,A}(\Omega) \hookrightarrow L^{1,A}(\Omega)$ is compact when for $1 < r < \frac{Dp}{D-p}$ one can use the interpolation inequality to obtain $\theta \in (0,1)$ such that

$$\|u\|_{L^{r,A}(\Omega)} \le \|u\|^{\theta}_{L^{\frac{Dp}{D-p},A}(\Omega)} \|u\|^{1-\theta}_{L^{1,A}(\Omega)} \le C \|u\|^{\theta}_{W^{1,p,A}(\Omega)} \|u\|^{1-\theta}_{L^1(K)}$$

so if $(u_n)_{n\in\mathbb{N}} \subseteq W^{1,p,A}(\Omega)$ is a bounded sequence then for any subsequence (denoted the same) such that (u_n) is Cauchy in $L^{1,A}(\Omega)$, then $(u_n)_{n\in\mathbb{N}}$ is also Cauchy in $L^{r,A}(\Omega)$.

Let $\mathcal{B}$ be the unit ball in $W^{1,p,A}(\Omega)$, we will show that that $\mathcal{B}$ is totally bounded in $L^{1,A}(\Omega)$. Let $\varepsilon > 0$ and define

$$\Omega_m = \bigcup_{i=1}^{k} \{ x \in \Omega : \ |x_i| < \frac{2}{m} \} .$$

Observe that either Ω_m is empty or $|\Omega_m| = o(1)$ as $m \to \infty$ (since $|\Omega| < \infty$). Using the Hardy-Sobolev inequality [4, Theorem 1] we deduce

$$\begin{aligned}
\|u\|_{L^{1,A}(\Omega_m)} &\le \|u\|_{L^{\frac{Dp}{D-p},A}(\Omega)} |\Omega_m|^{1-\frac{D-p}{Dp}} \\
&\le C \|u\|_{W^{1,p,A}(\Omega)} |\Omega_m|^{1-\frac{D-p}{Dp}} \\
&\le C |\Omega_m|^{1-\frac{D-p}{Dp}} , \quad \forall\, u \in \mathcal{B},
\end{aligned}$$

therefore we can find $m > 0$ large enough such that

$$\|u\|_{L^{1,A}(\Omega_m)} < \frac{\varepsilon}{3}, \quad \forall\, u \in \mathcal{B}.$$

Now consider $\phi \in C^\infty(\mathbb{R})$ with $0 \le \phi \le 1$, $|\phi'| \le L$ such that

$$\phi(t) = \begin{cases} 0 & \text{if } t \le 1 \\ 1 & \text{if } t \ge 2, \end{cases}$$

and define $\Phi_m(x) = \prod_{i=1}^N \phi(mx_i)$, which satisfies $0 \le \Phi_m \le 1$ and $|\nabla \Phi_m| \le Lm$. Clearly the set

$$\Phi_m\mathcal{B} = \{ \Phi_m u : u \in \mathcal{B} \}$$

is bounded in $W^{1,p}(\Omega)$. Indeed, if $x \in \operatorname{supp} \Phi_m u$ then $m^a x^A \ge 1$, therefore

$$\begin{aligned}
\int_\Omega |\nabla (\Phi_m u)|^p &\le m^a \int_\Omega |\nabla (\Phi_m u)|^p x^A \,\mathrm{d}x \\
&\le C_p m^a \left(\int_\Omega |\nabla u|^p x^A \,\mathrm{d}x + L^p m^p \int_\Omega |u|^p x^A \,\mathrm{d}x \right) \\
&\le C(\Omega, m) \|u\|^p_{W^{1,p,A}(\Omega)} .
\end{aligned}$$

Similarly $\int_\Omega |\Phi_m u|^p \,\mathrm{d}x \le C(\Omega, m) \|u\|^p_{W^{1,p,A}(\Omega)}$ and we can use Rellich theorem to conclude that $\Phi_m\mathcal{B}$ is totally bounded in $L^1(\Omega)$. We claim that since $a_i \ge 0$ for all i, then $\Phi_m\mathcal{B}$ is also totally bounded in $L^{1,A}(\Omega)$. Indeed, observe that by Hölder's inequality we have

$$\int_\Omega |x^A v| \le \left(\max_{x\in\Omega} x^A \right) \int_\Omega |v| \le C(\Omega, A) \int_\Omega |v| ,$$

thus if we have an δ-cover of $\Phi_m\mathcal{B}$ in $L^1(\Omega)$, then we have a $\delta C(\Omega, A)$-cover of $\Phi_m\mathcal{B}$ in $L^{1,A}(\Omega)$.

Hence we may cover $\phi_m\mathcal{B}$ by a finite number of balls of radius $\varepsilon > 0$ in $L^{1,A}(\Omega)$, that is, there exist $\{ g_1, \dots, g_M \} \subseteq L^{1,A}(\Omega)$ such that for any $u \in \mathcal{B}$ there is $i \in \{ 1, \dots, M \}$ such that

$$\|\Phi_m u - g_i\|_{L^{1,A}(\Omega)} < \frac{\varepsilon}{3},$$

from here we can write

$$\begin{aligned}\|u-g_i\|_{L^{1,A}(\Omega)} &\leq \|\Phi_m u - g_i\|_{L^{1,A}(\Omega)} + \|u-\Phi_m u\|_{L^{1,A}(\Omega)} \\ &< \frac{\varepsilon}{3} + 2\,\|u\|_{L^{1,A}(\Omega_m)} \\ &\leq \varepsilon.\end{aligned}$$

This implies that we can construct a ε-cover of $\mathcal{B}$ in $L^{1,A}(\Omega)$. ■

In analogy with the unweighted case, one can try to define traces of functions in $W^{1,p,A}(\Omega)$ onto part of the boundary $\{\, x_i = 0 \,\} \cap \partial\Omega$. A simple computation involving functions of the form $x_i^\gamma \ln x_i$ tells us that it is impossible to define the trace of a function u when $a_i \geq p-1$ and $\{\, x_i = 0 \,\} \cap \partial\Omega$ contains an $(N-1)$-dimensional open set. However, if the parameter satisfies $a_i < p-1$ one has the following

Theorem 6. *Suppose $A \in (\mathbb{R}_+)^k \times \{\, 0 \,\}^{N-k}$ satisfies $a_1 < p-1$. For $D = N + a_1 + \ldots + a_k$ define*

$$p_1 = \frac{(D-a_1-1)p}{D-p}$$

and $\hat{A}_1 = (a_2, \ldots, a_k, 0) \in (\mathbb{R}_+)^{k-1} \times \mathbb{R}^{N-k}$. Then, there exists a constant $C > 0$ such that for all $u \in C_c^1(\mathbb{R}^N)$

$$\|u(0,\cdot)\|_{L^{p_1,\hat{A}_1}((\mathbb{R}_+)^{k-1}\times\mathbb{R}^{N-k})} \leq C\,\|\nabla u\|_{L^{p,A}(\mathbb{R}^N_A)}\,.$$

Proof. We write $x = (x_1, y)$ and we begin with inequality (32) in [4, Theorem 6] (which follows from [2, Theorem 1.6]) for $k = 1$ to obtain for each $y \in \mathbb{R}^{N-1}$, $q = \frac{(D-a_1)p}{D-p}$ and $\alpha = q + \frac{p}{p-a_1-1}$

$$|u(0,y)|^\alpha \leq C\left(\int_{\mathbb{R}_+} |u(x_1,y)|^q \,\mathrm{d}x_1\right)\left(\int_{\mathbb{R}_+} |\partial_1 u(x_1,y)|^p\, x_1^{a_1}\,\mathrm{d}x_1\right)^{\frac{1}{p-a_1-1}},$$

since $a_1 < p-1$. We raise this inequality to the power $\dfrac{p_1}{\alpha}$, we multiply it by $y^{\hat{A}_1}$ and we integrate it over the y variable over $(\mathbb{R}_+)^{k-1} \times \mathbb{R}^{N-k}$ to obtain

$$\begin{aligned}&\int_{(\mathbb{R}_+)^{k-1}\times\mathbb{R}^{N-k}} |u(0,y)|^{p_1}\, y^{\hat{A}_1}\,\mathrm{d}y \\ &\qquad\leq C\int_{(\mathbb{R}_+)^{k-1}\times\mathbb{R}^{N-k}}\left[\left(\int_{\mathbb{R}_+} |u(x_1,y)|^q\, y^{\hat{A}_1}\,\mathrm{d}x_1\right)^{\frac{p_1}{\alpha}}\right. \\ &\qquad\qquad\left.\times\left(\int_{\mathbb{R}_+} |\partial_1 u(x_1,y)|^p\, x_1^{a_1} y^{\hat{A}_1}\,\mathrm{d}x_1\right)^{\frac{p_1}{\alpha(p-a_1-1)}}\right]\mathrm{d}y,\end{aligned}$$

because $\frac{p_1}{\alpha} + \frac{p_1}{\alpha(p-a_1-1)} = 1$. Using Hölder's inequality yields

$$\begin{aligned}&\int_{(\mathbb{R}_+)^{k-1}\times\mathbb{R}^{N-k}} |u(0,y)|^{p_1}\, y^{\hat{A}_1}\,\mathrm{d}y \\ &\qquad\leq C\left(\int_{(\mathbb{R}_+)^{k-1}\times\mathbb{R}^{N-k}}\int_{\mathbb{R}_+} |u(x_1,y)|^q\, y^{\hat{A}_1}\,\mathrm{d}x_1\,\mathrm{d}y\right)^{\frac{p_1}{\alpha}} \\ &\qquad\qquad\times\left(\int_{(\mathbb{R}_+)^{k-1}\times\mathbb{R}^{N-k}}\int_{\mathbb{R}_+} |\partial_1 u(x_1,y)|^p\, x_1^{a_1} y^{\hat{A}_1}\,\mathrm{d}x_1\,\mathrm{d}y\right)^{\frac{p_1}{\alpha(p-a_1-1)}}.\end{aligned}$$

The choice of $q = \frac{(D-a_1)p}{D-p}$ also allows us to use [4, Theorem 1] to say that

$$\int_{(\mathbb{R}_+)^{k-1}\times\mathbb{R}^{N-k}}\int_{\mathbb{R}_+} |u(x_1,y)|^q\, y^{\hat{A}_1}\,\mathrm{d}x_1\,\mathrm{d}y \leq C\left(\int_{(\mathbb{R}_+)^{k-1}\times\mathbb{R}^{N-k}}\int_{\mathbb{R}_+} |\nabla u(x_1,y)|^p\, x_1^{a_1} y^{\hat{A}_1}\,\mathrm{d}x_1\,\mathrm{d}y\right)^{\frac{q}{p}}$$

and as a consequence

$$\int_{(\mathbb{R}_+)^{k-1}\times\mathbb{R}^{N-k}} |u(0,y)|^{p_1}\, y^{\hat{A}_1}\,\mathrm{d}y \leq C\left(\int_{(\mathbb{R}_+)^{k-1}\times\mathbb{R}^{N-k}}\int_{\mathbb{R}_+} |\nabla u(x_1,y)|^p\, x_1^{a_1} y^{\hat{A}_1}\,\mathrm{d}x_1\,\mathrm{d}y\right)^{\frac{qp_1}{p\alpha}+\frac{p_1}{\alpha(p-a_1-1)}},$$

this concludes the proof when noticing that

$$\alpha = q + \frac{p}{p-a_1-1} \Rightarrow \frac{qp_1}{\alpha p} + \frac{p_1}{\alpha(p-a_1-1)} = \frac{p_1}{p}.$$

■

The above theorem tells us that it is convenient to suppose (after possibly relabeling the coordinates) that the vector $A = (a_1,\dots,a_k,0)$ satisfies

$$a_i < p-1 \quad \text{if } i \leq k_1 \tag{14}$$

$$a_i \geq p-1 \quad \text{if } k_1 < i \leq k \tag{15}$$

for some $0 \leq k_1 \leq k$. With this convention and for $i \leq k$ we can consider the sets

$$\Gamma_i = \{\, x \in \mathbb{R}^N : x_j > 0\ \forall j \in \{1,\dots,k\}\setminus\{\,i\,\},\ x_i = 0\,\}$$

which correspond to each of the faces of the boundary of $\mathbb{R}^N_A$. Theorem 6 above tells us that if $i \leq k_1$ there is a bounded trace operator $\gamma_i : W^{1,p,A}(\mathbb{R}^N_A) \to L^{p_i,\hat{A}_i}(\Gamma_i)$, where

$$p_i = \frac{(D-a_i-1)p}{D-p},$$

$$\hat{A}_i = (a_1,\dots,a_{i-1},a_{i+1},\dots,a_k) \in \mathbb{R}^{k-1},$$

and consequently we can define the space

$$W_0^{1,p,A}(\mathbb{R}^N_A) = \bigcap_{i=1}^{k_1} \ker \gamma_i,$$

which serves as the analog of the unweighted space $W_0^{1,p}$ of traceless functions. This space can also be defined as the closure of the set of smooth functions with compact support in the cone $(\mathbb{R}_+)^{k_1}\times\mathbb{R}^{N-k_1}$, that is if we consider

$$H_0^{1,p,A}(\mathbb{R}^N_A) = \overline{C_c^1((\mathbb{R}_+)^{k_1}\times\mathbb{R}^{N-k_1})}^{W^{1,p,A}(\Omega)}$$

then we have

Theorem 7. $W_0^{1,p,A}((\mathbb{R}_+)^{k_1}\times\mathbb{R}^{N-k_1}) = H_0^{1,p,A}((\mathbb{R}_+)^{k_1}\times\mathbb{R}^{N-k_1})$

Proof. In order to make the notation simpler, we will suppose that $k_1 = k = N$. Since $C_c^\infty((\mathbb{R}_+)^N) \subseteq W_0^{1,p,A}((\mathbb{R}_+)^N)$ we only need to show the inclusion $W_0^{1,p,A}((\mathbb{R}_+)^N) \subseteq H_0^{1,p,A}((\mathbb{R}_+)^N)$.

Let $u \in W_0^{1,p,A}((\mathbb{R}_+)^N)$, then $u \in W^{1,p,A}((\mathbb{R}_+)^N)$ and $\gamma_i u = 0$ in $L^{p_i,\hat{A}_i}((\mathbb{R}_+)^{N-1})$ for all i. Consider a smooth function ρ such that $\rho(s) = 0$ if $|s| \leq 1$, $\rho(s) = 1$ if $|s| \geq 2$ and $0 \leq \rho(s) \leq 1$. Define $\rho_n(x) = \prod_{i=1}^N \rho(nx_i)$ and for $u \in W^{1,p,A}(\Omega)$ consider $u_n = \rho_n u$. We claim that if $u \in W_0^{1,p,A}((\mathbb{R}_+)^N)$ then $u_n \underset{n\to\infty}{\longrightarrow} u$ in $W^{1,p,A}((\mathbb{R}_+)^N)$. Observe that

$$\nabla u_n = \rho_n \nabla u + \nabla\rho_n u,$$

and as a consequence

$$\int_{(\mathbb{R}_+)^N} |\nabla(u-u_n)|^p\, x^A \,\mathrm{d}x \le C\int_{(\mathbb{R}_+)^N} |(1-\rho_n)\nabla u|^p\, x^A\,\mathrm{d}x + \int_{(\mathbb{R}_+)^N} |u\nabla\rho_n|^p\, x^A\,\mathrm{d}x.$$

Since $\rho_n \underset{n\to\infty}{\longrightarrow} 1$ in $(\mathbb{R}_+)^N$ we deduce by dominated convergence that the first term tends to zero as n tends to ∞. For the second term, we observe that $|\nabla\rho_n| \le Cn$ and that it has support on the set $\bigcup_{i=1}^N \{\, x\in\mathbb{R}^N : 0\le x_i \le \frac{2}{n}\,\}$ and as a consequence we only need to show that

$$n^p \int_0^{\frac{2}{n}} \int_{(\mathbb{R}_+)^{N-1}} |u(\hat{x}_i + x_i e_i)|^p\, x^A\,\mathrm{d}\hat{x}_i\,\mathrm{d}x_i \underset{n\to\infty}{\longrightarrow} 0,$$

or equivalently

$$\frac{1}{m^p}\int_0^m \int_{(\mathbb{R}_+)^{N-1}} |u(\hat{x}_i + x_i e_i)|^p\, x^A\,\mathrm{d}\hat{x}_i\,\mathrm{d}x_i \underset{m\to 0}{\longrightarrow} 0,$$

for all i.

Observe that for $x_i>0$ and almost every $\hat{x}_i \in (\mathbb{R}_+)^{N-1}$ we can write

$$\begin{aligned}|u(\hat{x}_i + x_i e_i) - u(\hat{x}_i)| &\le \int_0^{x_i} \left|\frac{\partial u}{\partial x_i}(\hat{x}_i + s e_i)\right| \mathrm{d}s\\ &\le C x_i^{\frac{p-1-a_i}{p}} \left(\int_0^{x_i} \left|\frac{\partial u}{\partial x_i}(\hat{x}_i + s e_i)\right|^p s^{a_i}\,\mathrm{d}s\right)^{\frac{1}{p}}\end{aligned}$$

and as a consequence

$$|u(\hat{x}_i + x_i e_i) - u(\hat{x}_i)|^p\, x^A \le C x_i^{p-1}\int_0^{x_i} |\nabla u(\hat{x}_i + s e_i)|^p\, s_i^a \hat{x}_i^{\hat{A}_i}\,\mathrm{d}s.$$

Now, consider $B\subseteq(\mathbb{R}_+)^{N-1}$ a bounded set and integrate the above inequality over B, to obtain

$$\int_B |u(\hat{x}_i + x_i e_i) - u(\hat{x}_i)|^p\, x^A\,\mathrm{d}\hat{x}_i \le C x_i^{p-1}\int_0^{x_i}\int_B |\nabla u(\hat{x}_i + s e_i)|^p\, \hat{x}_i^{\hat{A}_i} s_i^a\,\mathrm{d}\hat{x}_i\,\mathrm{d}s.$$

We integrate the above in the x_i variable over the interval $(0,m)$ and we get

$$\begin{aligned}&\int_0^m\int_B |u(\hat{x}_i + x_i e_i) - u(\hat{x}_i)|^p x^A\,\mathrm{d}\hat{x}_i\,\mathrm{d}x_i\\ &\qquad\le C\int_0^m x_i^{p-1}\int_0^{x_i}\int_B |\nabla u(\hat{x}_i + s e_i)|^p\, \hat{x}_i^{\hat{A}_i} s^{a_i}\,\mathrm{d}\hat{x}_i\,\mathrm{d}s\,\mathrm{d}x_i\\ &\qquad\le C\int_0^m x_i^{p-1}\int_0^{m}\int_B |\nabla u(\hat{x}_i + s e_i)|^p\, \hat{x}_i^{\hat{A}_i} s^{a_i}\,\mathrm{d}\hat{x}_i\,\mathrm{d}s\,\mathrm{d}x_i\\ &\qquad= C m^p\int_0^m\int_B |\nabla u(\hat{x}_i + x_i e_i)|^p\, x^A\,\mathrm{d}\hat{x}_i\,\mathrm{d}x_i.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\frac{1}{m^p}\int_0^m\int_B |u(\hat{x}_i+x_ie_i)|^p\, x^A\,\mathrm{d}\hat{x}_i\,\mathrm{d}x_i &\le \frac{C}{m^{p-1-a_i}}\int_B |u(\hat{x}_i)|^p\,\hat{x}_i^{\hat{A}_i}\,\mathrm{d}\hat{x}_i\\
&\quad + C\int_0^m\int_B |\nabla u(\hat{x}_i+x_ie_i)|^p\, x^A\,\mathrm{d}\hat{x}_i\,\mathrm{d}x_i\\
&\le \frac{C\,|B|^{1-\frac{p}{p_i}}}{m^{p-1-a_i}}\left(\int_B |u(\hat{x}_i)|^{p_i}\,\hat{x}_i^{\hat{A}_i}\,\mathrm{d}\hat{x}_i\right)^{\frac{p}{p_i}}\\
&\quad + C\int_0^m\int_B |\nabla u(\hat{x}_i+x_ie_i)|^p\, x^A\,\mathrm{d}\hat{x}_i\,\mathrm{d}x_i\\
&\le \frac{C\,|B|^{1-\frac{p}{p_i}}}{m^{p-1-a_i}}\left(\int_{(\mathbb{R}_+)^{N-1}} |\gamma_i u(\hat{x}_i)|^{p_i}\,\hat{x}_i^{\hat{A}_i}\,\mathrm{d}\hat{x}_i\right)^{\frac{p}{p_i}}\\
&\quad + C\int_0^m\int_B |\nabla u(\hat{x}_i+x_ie_i)|^p\, x^A\,\mathrm{d}\hat{x}_i\,\mathrm{d}x_i\\
&= C\int_0^m\int_B |\nabla u(\hat{x}_i+x_ie_i)|^p\, x^A\,\mathrm{d}\hat{x}_i\,\mathrm{d}x_i\\
&\le C\int_0^m\int_{(\mathbb{R}_+)^{N-1}} |\nabla u(\hat{x}_i+x_ie_i)|^p\, x^A\,\mathrm{d}\hat{x}_i\,\mathrm{d}x_i
\end{aligned}$$

because we are supposing $\gamma_i u = 0$ on $L^{p_i,\hat{A}_i}((\mathbb{R}_+)^{N-1})$. By taking $B \nearrow (\mathbb{R}_+)^{N-1}$ and using monotone convergence we deduce that

$$\begin{aligned}
\frac{1}{m^p}\int_0^m\int_{(\mathbb{R}_+)^{N-1}} &|u(\hat{x}_i+x_ie_i)|^p\, x^A\,\mathrm{d}\hat{x}_i\,\mathrm{d}x_i\\
&\le C\int_0^m\int_{(\mathbb{R}_+)^{N-1}} |\nabla u(\hat{x}_i+x_ie_i)|^p\, x^A\,\mathrm{d}\hat{x}_i\,\mathrm{d}x_i\\
&\underset{m\to 0}{\longrightarrow} 0,
\end{aligned}$$

by dominated convergence.

■

A more surprising fact occurs if we define

$$\tilde{H}_0^{1,p,A}(\mathbb{R}^N_A) := \overline{C_c^1(\mathbb{R}^{k_1}\times(\mathbb{R}_+)^{k-k_1}\times\mathbb{R}^{N-k})}^{W^{1,p,A}(\Omega)}$$

which in some sense is the space of functions "vanishing" on Γ_i when $k_1 < i \le k$ (so that $a_i \ge p-1$ and no bounded trace operator can exist). It turns out that this space coincides with $W^{1,p,A}(\mathbb{R}^N_A)$ as it can be seen in the following

Theorem 8. *$C_c^1(\mathbb{R}^{k_1}\times(\mathbb{R}_+)^{k-k_1}\times\mathbb{R}^{N-k})$ is dense $W^{1,p,A}(\mathbb{R}^N_A)$.*

To prove this result we need the following is a version of a critical Hardy inequality that it is well known but we provide its proof for the reader's convenience.

Lemma 2. *Let $p>1$ and $R>0$, then for every $u\in C^1(\mathbb{R})$ satisfying $\operatorname{supp} u \subseteq (-R,R)$ one has*

$$\int_0^R \frac{1}{x}\left|\frac{u(x)}{1-\ln\left(\frac{x}{R}\right)}\right|^p \mathrm{d}x \le \left(\frac{p}{p-1}\right)^p \int_0^R x^{p-1}\,|u'(x)|^p\,\mathrm{d}x$$

Proof. Observe that for $\delta > 0$ we can integrate by parts to get

$$\begin{aligned}\int_\delta^R \frac{|u|^p}{x(1-\ln\left(\frac{x}{R}\right))^{p-1}} &= -\frac{|u(\delta)|^p}{(1-\ln\left(\frac{\delta}{R}\right))^{p-1}} - \int_\delta^R x\left(\frac{|u|^p}{x(1-\ln\left(\frac{x}{R}\right))^{p-1}}\right)' \mathrm{d}x \\ &\leq -\int_\delta^R x\left(\frac{|u|^p}{x(1-\ln\left(\frac{x}{R}\right))^{p-1}}\right)' \mathrm{d}x \\ &= -p\int_\delta^R \frac{p\,|u|^{p-2}\,uu'}{(1-\ln\left(\frac{x}{R}\right))^{p-1}}\,\mathrm{d}x \\ &\quad + \int_\delta^R \frac{|u|^p}{x(1-\ln\left(\frac{x}{R}\right))^{p-1}}\,\mathrm{d}x \\ &\quad - (p-1)\int_\delta^R \frac{|u|^p}{x(1-\ln\left(\frac{x}{R}\right))^{p}}\,\mathrm{d}x,\end{aligned}$$

hence

$$(p-1)\int_\delta^R \frac{|u|^p}{x(1-\ln\left(\frac{x}{R}\right))^{p}}\,\mathrm{d}x \leq -p\int_\delta^R \frac{p\,|u|^{p-2}\,uu'}{(1-\ln\left(\frac{x}{R}\right))^{p-1}}\,\mathrm{d}x$$

and the result follows from Hölder's inequality and by letting δ decrease to zero. ∎

Proof of Theorem 8. We suppose without loss of generality that $k_1 = 0$ and that $k = N$ so that the notation becomes less crowded.

We will show that $C_c^1((\mathbb{R}_+)^N)$ is dense in $H^{1,p,A}((\mathbb{R}_+)^N)$, so we take $u \in H^{1,p,A}((\mathbb{R}_+)^N)$ and $\varepsilon > 0$. By definition we know that $u, \partial_{x_i} u \in L^{p,A}((\mathbb{R}_+)^N)$ and there exists $u_\varepsilon \in C^1((\mathbb{R}_+)^N)$ with support contained in $B(0,R)$ for some $R = R_\varepsilon > 1$ such that

$$\begin{gathered}\|u_\varepsilon\|_{W^{1,p,A}((\mathbb{R}_+)^N)} \leq C\,\|u\|_{W^{1,p,A}((\mathbb{R}_+)^N)}, \text{ and} \\ \int_{(\mathbb{R}_+)^N} |u-u_\varepsilon|^p\, x^A\,\mathrm{d}x + \int_{(\mathbb{R}_+)^N} |\nabla(u-u_\varepsilon)|^p\, x^A\,\mathrm{d}x \leq \varepsilon.\end{gathered}$$

For $0 < \delta < \frac{1}{eR^4}$, we consider a cut-off function $\rho_\delta \in C^1(\mathbb{R})$ satisfying $\operatorname{supp}\rho_\delta \subset [\delta^4, \infty)$ and

$$\begin{aligned}\rho_\delta(y) &= 1 \text{ for all } y \geq \delta, \\ |\rho_\delta(y)| &\leq C, \\ |\rho_\delta'(y)| &\leq \frac{C}{y\,|\ln\delta|},\end{aligned}$$

which can be done by mollifying the continuous function

$$\rho(y) = \begin{cases} 0 & \text{if } y < \delta^3, \\ 3 - \dfrac{\ln y}{\ln\delta} & \text{if } \delta^3 \leq y < \delta^2, \\ 1 & \text{if } y \geq \delta^2. \end{cases}$$

Define $\varphi_\delta(x) = \prod_{i=1}^N \rho_\delta(x_i)$ and consider $u_{\varepsilon,\delta} = \varphi_\delta u_\varepsilon$. We claim that for $\delta > 0$ small we have

$$\int_{(\mathbb{R}_+)^N} |u_{\varepsilon,\delta}-u_\varepsilon|^p\, x^A\,\mathrm{d}x + \int_{(\mathbb{R}_+)^N} |\nabla(u_{\varepsilon,\delta}-u_\varepsilon)|^p\, x^A\,\mathrm{d}x \leq \varepsilon.$$

Indeed, we notice that

$$\int_{(\mathbb{R}_+)^N} |u_{\varepsilon,\delta}-u_\varepsilon|^p\, x^A\,\mathrm{d}x = \int_{(\mathbb{R}_+)^N} |(\varphi_\varepsilon - 1)u_\varepsilon|^p\, x^A\,\mathrm{d}x$$

and that

$$\begin{aligned}\int_{(\mathbb{R}_+)^N} |\nabla(u_{\varepsilon,\delta} - u_\varepsilon)|^p\, x^A \,\mathrm{d}x &= \int_{(\mathbb{R}_+)^N} |(\varphi_\delta - 1)\nabla u_\varepsilon + \nabla\varphi_\delta u_\varepsilon|^p\, x^A \,\mathrm{d}x \\ &\le C\left[\int_{(\mathbb{R}_+)^N} |(\varphi_\delta - 1)\nabla u_\varepsilon|^p\, x^A \,\mathrm{d}x \right.\\ &\qquad \left. + \sum_{i=1}^N \int_{(\mathbb{R}_+)^N} |\rho_\delta'(x_i) u_\varepsilon|^p\, x^A \,\mathrm{d}x\right].\end{aligned}$$

If $a_i > p-1$, thanks to the Hardy-Sobolev inequality [4, Theorem 1] we have

$$\begin{aligned}\int_{(\mathbb{R}_+)^N} |\rho_\delta'(x_i) u_\varepsilon|^p\, x^A \,\mathrm{d}x &\le \frac{1}{|\ln\delta|^p}\int_{(\mathbb{R}_+)^{N-1}} \hat{x}_i^{\hat{A}_i} \int_0^\delta |u_\varepsilon(x_i e_i + \hat{x}_i)|^p\, x_i^{a_i - p} \,\mathrm{d}x_i \,\mathrm{d}\hat{x}_i \\ &\le \frac{C}{|\ln\delta|^p}\int_{(\mathbb{R}_+)^{N-1}} \hat{x}_i^{\hat{A}_i} \int_0^\infty |\partial_i u_\varepsilon(x_i e_i + \hat{x}_i)|^p\, x_i^{a_i} \,\mathrm{d}x_i \,\mathrm{d}\hat{x}_i \\ &= \frac{C}{|\ln\delta|^p}\int_{(\mathbb{R}_+)^N} |\partial_i u_\varepsilon(x)|^p\, x^A \,\mathrm{d}x\end{aligned}$$

On the other hand if $a_i = p-1$ we use Lemma 2 to obtain that

$$\begin{aligned}\int_{(\mathbb{R}_+)^{N-1}} \hat{x}_i^{\hat{A}_i} \int_0^R \frac{1}{x_i}\left|\frac{u_\varepsilon(x_i e_i + \hat{x}_i)}{1 - \ln\frac{x_i}{R}}\right|^p \mathrm{d}x_i \,\mathrm{d}\hat{x}_i &\le \left(\frac{p}{p-1}\right)^p \int_{(\mathbb{R}_+)^N} |\partial_i u_\varepsilon(x)|^p\, x^A \,\mathrm{d}x \\ &\le C\, \|u\|^p_{W^{1,p,A}((\mathbb{R}_+)^N)}. \end{aligned} \tag{16}$$

From the assumption that $\delta < e^{-1}R^{-4}$ we deduce that $1 - 4\ln\left(\frac{\delta}{R}\right) \le -5\ln\delta$ and

$$\begin{aligned}\int_{(\mathbb{R}_+)^N} |\rho_\delta'(x_i) u_\varepsilon|^p\, x^A \,\mathrm{d}x &= \int_{(\mathbb{R}_+)^{N-1}} \hat{x}_i^{\hat{A}_i} \int_{\delta^4}^\delta \frac{1}{x_i}\left|\frac{u_\varepsilon(x_i e_i + \hat{x}_i)}{\ln\delta}\right|^p \mathrm{d}x_i \,\mathrm{d}\hat{x}_i \\ &\le 5^p \int_{(\mathbb{R}_+)^{N-1}} \hat{x}_i^{\hat{A}_i} \int_{\delta^4}^\delta \frac{1}{x_i}\left|\frac{u_\varepsilon(x_i e_i + \hat{x}_i)}{1 - 4\ln\frac{\delta}{R}}\right|^p \mathrm{d}x_i \,\mathrm{d}\hat{x}_i \\ &\le 5^p \int_{(\mathbb{R}_+)^{N-1}} \hat{x}_i^{\hat{A}_i} \int_{\delta^4}^\delta \frac{1}{x_i}\left|\frac{u_\varepsilon(x_i e_i + \hat{x}_i)}{1 - \ln\frac{x_i}{R}}\right|^p \mathrm{d}x_i \,\mathrm{d}\hat{x}_i \\ &\underset{\delta\to 0}{\longrightarrow} 0.\end{aligned}$$

In any case we conclude that

$$\int_{(\mathbb{R}_+)^N} |u_{\varepsilon,\delta} - u_\varepsilon|^p\, x^A \,\mathrm{d}x + \int_{(\mathbb{R}_+)^N} |\nabla(u_{\varepsilon,\delta} - u_\varepsilon)|^p\, x^A \,\mathrm{d}x \underset{\delta\to 0}{\longrightarrow} 0,$$

therefore we can select $\delta_\varepsilon > 0$ small so that

$$\int_{(\mathbb{R}_+)^N} |u_{\varepsilon,\delta_\varepsilon} - u_\varepsilon|^p\, x^A \,\mathrm{d}x + \int_{(\mathbb{R}_+)^N} |\nabla(u_{\varepsilon,\delta_\varepsilon} - u_\varepsilon)|^p\, x^A \,\mathrm{d}x \le \varepsilon$$

hence for given $\varepsilon > 0$ we have found a function $u_{\varepsilon,\delta_\varepsilon} \in C_c^1((\mathbb{R}_+)^N)$ such that

$$\int_{(\mathbb{R}_+)^N} |u - u_{\varepsilon,\delta_\varepsilon}|^p\, x^A \,\mathrm{d}x + \int_{(\mathbb{R}_+)^N} |\nabla(u - u_{\varepsilon,\delta_\varepsilon})|^p\, x^A \,\mathrm{d}x \le C\varepsilon.$$

■

References

1. Brock, F., Chiacchio, F. & Mercaldo, A. A weighted isoperimetric inequality in an orthant. *Potential Anal.* **41,** 171–186. ISSN: 0926-2601. `https://doi.org/10.1007/s11118-013-9367-4` (2014).
2. Cabré, X. & Ros-Oton, X. Sobolev and isoperimetric inequalities with monomial weights. *J. Differential Equations* **255,** 4312–4336. ISSN: 0022-0396. `http://dx.doi.org/10.1016/j.jde.2013.08.010` (2013).

3. Calderón, A.-P. *Lebesgue spaces of differentiable functions and distributions* in *Proc. Sympos. Pure Math., Vol. IV* (American Mathematical Society, Providence, R.I., 1961), 33–49.
4. Castro, H. Hardy-Sobolev-type inequalities with monomial weights. *Ann. Mat. Pura Appl. (4)* **196,** 579–598. ISSN: 0373-3114. `http://dx.doi.org/10.1007/s10231-016-0587-2` (2017).
5. Castro, H. Extremals for Hardy-Sobolev type inequalities with monomial weights. *J. Math. Anal. Appl.* **494,** 124645, 31. ISSN: 0022-247X. `https://doi.org/10.1016/j.jmaa.2020.124645` (2021).
6. Chanillo, S. & Wheeden, R. L. Poincaré inequalities for a class of non-A_p weights. *Indiana Univ. Math. J.* **41,** 605–623. ISSN: 0022-2518. `https://doi.org/10.1512/iumj.1992.41.41033` (1992).
7. Chua, S.-K. Extension theorems on weighted Sobolev spaces. *Indiana Univ. Math. J.* **41,** 1027–1076. ISSN: 0022-2518. `https://doi.org/10.1512/iumj.1992.41.41053` (1992).
8. Chua, S.-K. Some remarks on extension theorems for weighted Sobolev spaces. *Illinois J. Math.* **38,** 95–126. ISSN: 0019-2082. `http://projecteuclid.org/euclid.ijm/1255986890` (1994).
9. Chua, S.-K. On weighted Sobolev spaces. *Canad. J. Math.* **48,** 527–541. ISSN: 0008-414X. `https://doi.org/10.4153/CJM-1996-027-5` (1996).
10. Chua, S.-K. Extension theorems on weighted Sobolev spaces and some applications. *Canad. J. Math.* **58,** 492–528. ISSN: 0008-414X. `https://doi.org/10.4153/CJM-2006-021-0` (2006).
11. Chua, S.-K. & Wheeden, R. L. Sharp conditions for weighted 1-dimensional Poincaré inequalities. *Indiana Univ. Math. J.* **49,** 143–175. ISSN: 0022-2518. `https://doi.org/10.1512/iumj.2000.49.1754` (2000).
12. Fabes, E. B., Kenig, C. E. & Serapioni, R. P. The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations* **7,** 77–116. ISSN: 0360-5302. `https://doi.org/10.1080/03605308208820218` (1982).
13. Feo, F., Martín, J. & Posteraro, M. R. Sobolev anisotropic inequalities with monomial weights. *J. Math. Anal. Appl.* **505,** Paper No. 125557, 30. ISSN: 0022-247X. `https://doi.org/10.1016/j.jmaa.2021.125557` (2022).
14. Hajłasz, P. & Koskela, P. Sobolev met Poincaré. *Mem. Amer. Math. Soc.* **145,** x+101. ISSN: 0065-9266. `https://doi.org/10.1090/memo/0688` (2000).
15. Heinonen, J., Kilpeläinen, T. & Martio, O. *Nonlinear potential theory of degenerate elliptic equations* Unabridged republication of the 1993 original, xii+404. ISBN: 0-486-45050-3 (Dover Publications, Inc., Mineola, NY, 2006).
16. Hestenes, M. R. Extension of the range of a differentiable function. *Duke Math. J.* **8,** 183–192. ISSN: 0012-7094. `http://projecteuclid.org/euclid.dmj/1077492502` (1941).
17. Jones, P. W. Quasiconformal mappings and extendability of functions in Sobolev spaces. *Acta Math.* **147,** 71–88. ISSN: 0001-5962. `https://doi.org/10.1007/BF02392869` (1981).
18. Meyers, N. G. & Serrin, J. $H = W$. *Proc. Nat. Acad. Sci. U.S.A.* **51,** 1055–1056. ISSN: 0027-8424. `https://doi.org/10.1073/pnas.51.6.1055` (1964).
19. Nguyen, V. H. Sharp weighted Sobolev and Gagliardo-Nirenberg inequalities on half-spaces via mass transport and consequences. *Proc. Lond. Math. Soc. (3)* **111,** 127–148. ISSN: 0024-6115. `https://doi.org/10.1112/plms/pdv026` (2015).
20. Serra Cassano, F. On the local boundedness of certain solutions for a class of degenerate elliptic equations. *Boll. Un. Mat. Ital. B (7)* **10,** 651–680 (1996).
21. Stein, E. M. *Singular integrals and differentiability properties of functions* xiv+290 (Princeton University Press, Princeton, N.J., 1970).
22. Wang, J. Weighted Hardy-Sobolev, Log-Sobolev and Moser-Onofri-Beckner Inequalities with Monomial Weights. *Potential Analysis.* ISSN: 1572-929X. `https://doi.org/10.1007/s11118-021-09938-9` (Mar. 2021).
23. Zhikov, V. V. On weighted Sobolev spaces. *Mat. Sb.* **189,** 27–58. ISSN: 0368-8666. `https://doi.org/10.1070/SM1998v189n08ABEH000344` (1998).

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