A SINGULAR STURM-LIOUVILLE EQUATION UNDER HOMOGENEOUS BOUNDARY CONDITIONS

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Abstract. Given $\alpha > 0$ and $f \in L^2(0,1)$, we are interested in the equation

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = f(x) \text{ on } (0,1], \\ u(1) = 0. \end{cases}$$

We prescribe appropriate (weighted) homogeneous boundary conditions at the origin and prove the existence and uniqueness of $H^2_{loc}(0,1]$ solutions. We study the regularity at the origin of such solutions. We perform a spectral analysis of the differential operator $\mathcal{L}u := -\left(x^{2\alpha}u'\right)' + u$ under those appropriate homogenous boundary conditions.

1. Introduction

This paper concerns the following Sturm-Liouvile equation

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = f(x) \text{ on } (0,1], \\ u(1) = 0, \end{cases} \tag{1}$$

where α is a positive real number and $f \in L^2(0,1)$ is given. In this work we will study the existence, uniqueness and regularity of solutions of equation (1), under suitable homogeneous boundary data. We also discuss spectral properties of the differential operator $\mathcal{L}u := -\left(x^{2\alpha}u'\right)' + u$.

The classical ODE theory says that if for instance the right hand side f is a continuous function on $(0,1]$, then the solution set of equation (1) is a one parameter family of $C^2(0,1]$-functions. As we already mentioned, the first goal of this work is to select "distinguished" elements of that family by prescribing (weighted) *homogeneous* boundary conditions at the origin. In a subsequent paper, [3], we will study the equation (1) under non-homogeneous boundary conditions at the origin.

When $0 < \alpha < \frac{1}{2}$, we have both a Dirichlet and a (weighted) Neumann problem. When $\alpha \geq \frac{1}{2}$, we only have a "Canonical" solution obtained by prescribing either a (weighted) Dirichlet or a (weighted) Neumann condition; as we are going to explain in Remark 19, the two boundary conditions yield the same solution.

1.1. The case $0 < \alpha < \frac{1}{2}$.

We first consider the Dirichlet problem.

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Theorem 1.1 (Existence for Dirichlet Problem)**.** *Given $0<\alpha<\frac{1}{2}$ and $f\in L^2(0,1)$, there exists a function $u\in H^2_{loc}(0,1]$ satisfying* (1) *together with the following properties:*

(i) $\lim_{x\to 0^+} u(x)=0$.
(ii) $u\in C^{0,1-2\alpha}[0,1]$ *with* $\|u\|_{C^{0,1-2\alpha}}\le C\|f\|_{L^2}$.
(iii) $x^{2\alpha}u'\in H^1(0,1)$ *with* $\left\|x^{2\alpha}u'\right\|_{H^1}\le C\|f\|_{L^2}$.
(iv) $x^{2\alpha-1}u\in H^1(0,1)$ *with* $\left\|x^{2\alpha-1}u\right\|_{H^1}\le C\|f\|_{L^2}$.
(v) $x^{2\alpha}u\in H^2(0,1)$ *with* $\left\|x^{2\alpha}u\right\|_{H^2}\le C\|f\|_{L^2}$.

Here the constant C only depends on α.

Before stating the uniqueness result, we would like to give a few remarks of about this Theorem.

Remark 1. There exists a function $f\in C_0^\infty(0,1)$ such that near the origin the solution given by Theorem 1.1 can be expanded in the following way

$$u(x)=a_1x^{1-2\alpha}+a_2x^{3-4\alpha}+a_3x^{5-6\alpha}+\cdots \tag{2}$$

where $a_1\neq 0$. See Section 3.1 for the proof.

Remark 2. Theorem 1.1 only says $(x^{2\alpha}u')'=x^{2\alpha}u''+2\alpha x^{2\alpha-1}u'$ is in $L^2(0,1)$. A natural question is whether each term on the right-hand side belongs to $L^2(0,1)$. The answer is that, in general, neither of them is in $L^2(0,1)$; in fact, they are not even in $L^1(0,1)$. One can see this phenomenon in equation (2), where we have that $x^{2\alpha-1}u'(x)\sim x^{2\alpha}u''(x)\sim x^{-1}\notin L^1(0,1)$.

Remark 3. Part (iii) in Theorem 1.1 implies that $u\in W^{1,p}(0,1)$ for all $1\le p<\frac{1}{2\alpha}$ with $\|u'\|_{L^p}\le C\|f\|_{L^2}$, where C is a constant only depending on α. However, one *cannot* expect that $u\in W^{1,\frac{1}{2\alpha}}(0,1)$ even if $f\in C_0^\infty(0,1)$, as the power series expansion (2) shows that $u'\sim x^{-2\alpha}$ near the origin.

Remark 4. Concerning the assertions in Theorem 1.1, we have the following implications: (i) and (iii) $\Rightarrow$ (iv); (iv) $\Rightarrow$ (ii); (iii) and (iv) $\Rightarrow$ (v). Those implications can be found in the proof of Theorem 1.1.

Remark 5. The assertions in Theorem 1.1 are optimal in the following sense: there exists $f\in L^2(0,1)$ such that $u\notin C^{0,\beta}[0,1]$ $\forall\beta>1-2\alpha$; and one can find another $f\in L^2(0,1)$ such that $x^{2\alpha-1}u\notin H^2(0,1)$, $x^{2\alpha}u'\notin H^2(0,1)$, and $x^{2\alpha}u\notin H^3(0,1)$. See Section 3.1 for the counterexamples.

Remark 6. Theorem 1.1 tells us that both $x^{2\alpha}u'$ and $x^{2\alpha-1}u$ belong to $H^1(0,1)$, so in particular they are continuous up to the origin. It is natural to examine their values at the origin and how they are related to the right-hand side $f\in L^2(0,1)$. We actually have

$$\lim_{x\to 0^+} x^{2\alpha}u'(x)=\int_0^1 f(x)g(x)dx, \tag{3}$$

and

$$\lim_{x\to 0^+} x^{2\alpha-1}u(x)=\frac{1}{1-2\alpha}\int_0^1 f(x)g(x)dx, \tag{4}$$

where the function g is the solution of

$$\begin{cases} -(x^{2\alpha}g'(x))'+g(x)=0 \text{ on } (0,1], \\ g(1)=0, \\ \lim_{x\to 0^+} g(x)=1. \end{cases}$$

See Section 3.1 for the proof of this Remark. The existence and regularity of such function g is the main topic of the subsequent paper [3] (the uniqueness of such g comes from Theorem 1.2 below).

Theorem 1.2 (Uniqueness for the Dirichlet problem)**.** *Let $0<\alpha<\frac{1}{2}$. Assume that $u\in H^2_{loc}(0,1]$ satisfies*

$$\begin{cases} -(x^{2\alpha}u'(x))'+u(x)=0 \text{ on } (0,1], \\ u(1)=0, \\ \lim_{x\to 0^+} u(x)=0. \end{cases} \tag{5}$$

Then $u\equiv 0$.

In order to simplify the terminology, we denote by u_D the unique solution to (1) given by Theorem 1.1. Next we consider the regularity property of the solution u_D when the right-hand side f has a better regularity.

Theorem 1.3. *Let $0<\alpha<\frac{1}{2}$ and $f\in W^{1,\frac{1}{2\alpha}}(0,1)$. Let u_D be the solution to (1) given by Theorem 1.1. Then $x^{2\alpha-1}u_D\in W^{2,p}(0,1)$ for all $1\le p<\frac{1}{2\alpha}$ with $\left\|x^{2\alpha-1}u_D\right\|_{W^{2,p}}\le C\left\|f\right\|_{W^{1,p}}$, where C is a constant only depending on p and α.*

Remark 7. One *cannot* expect that $x^{2\alpha-1}u_D\in W^{2,\frac{1}{2\alpha}}(0,1)$ even if $f\in C_0^\infty(0,1)$, as the power series expansion (2) shows that $(x^{2\alpha-1}u_D(x))''\sim x^{-2\alpha}$ near the origin.

Remark 8. When $\alpha\ge\frac{1}{2}$, we *cannot* prescribe the Dirichlet boundary condition $\lim_{x\to 0^+}u(x)=0$. Actually, for $\alpha\ge\frac{1}{2}$, there is no $H^2_{loc}(0,1]$-solution of

$$\begin{cases} -(x^{2\alpha}u'(x))'+u(x)=f \text{ on } (0,1], \\ u(1)=0, \\ \lim_{x\to 0^+} u(x)=0, \end{cases} \tag{6}$$

for either $f\equiv 1$ or some $f\in C_0^\infty(0,1)$. See Section 3.1 for the proof.

Next we consider the case $0<\alpha<\frac{1}{2}$ together with a weighted Neumann condition.

Theorem 1.4 (Existence for Neumann Problem)**.** *Given $0<\alpha<\frac{1}{2}$ and $f\in L^2(0,1)$, there exists a function $u\in H^2_{loc}(0,1]$ satisfying (1) together with the following properties:*

(i) *$u\in H^1(0,1)$ with $\left\|u\right\|_{H^1}\le C\left\|f\right\|_{L^2}$.*
(ii) *$\lim_{x\to 0^+}x^{2\alpha-\frac{1}{2}}u'(x)=0$.*
(iii) *$x^{2\alpha-1}u'\in L^2(0,1)$ and $x^{2\alpha}u''\in L^2(0,1)$, with $\left\|x^{2\alpha-1}u'\right\|_{L^2}+\left\|x^{2\alpha}u''\right\|_{L^2}\le C\left\|f\right\|_{L^2}$. In particular, $x^{2\alpha}u'\in H^1(0,1)$.*

Here the constant C only depends on α.

Remark 9. Notice the difference between Dirichlet and Neumann with respect to property (iii) of Theorem 1.4. See Remark 2.

Remark 10. The boundary behavior $\lim_{x\to 0^+}x^{2\alpha-\frac{1}{2}}u'(x)=0$ is optimal in the following sense: for any $0<x\le\frac{1}{2}$, define

$$K_\alpha(x)=\sup_{\|f\|_{L^2}\le 1}\left|x^{2\alpha-\frac{1}{2}}u'(x)\right|.$$

Then $0<\delta\le K_\alpha(x)\le 2$, for some constant δ only depending on α. See Section 3.2 for the proof.

Remark 11. Theorem 1.4 implies that $u\in C^0[0,1]$, so it is natural to consider the dependence on f of the quantity $\lim_{x\to 0^+}u(x)$. One has

$$\lim_{x\to 0^+}u(x)=\int_0^1 f(x)h(x)dx, \tag{7}$$

where h is the solution of

$$\begin{cases} -(x^{2\alpha}h'(x))' + h(x) = 0 \text{ on } (0,1], \\ h(1) = 0, \\ \lim_{x\to 0^+} x^{2\alpha}h'(x) = 1. \end{cases}$$

In particular, equation (7) implies that the quantity $\lim_{x\to 0^+} u(x)$ is not necessarily 0. See Section 3.2 for the proof of this Remark. The existence and regularity of h is part of [3], but the uniqueness of h comes from Theorem 1.5 below.

Theorem 1.5 (Uniqueness for the Neumann Problem)**.** *Let $0 < \alpha < \frac{1}{2}$. Assume that $u \in H^2_{loc}(0,1]$ satisfies*

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 \text{ on } (0,1], \\ u(1) = 0, \\ \lim_{x\to 0^+} x^{2\alpha}u'(x) = 0. \end{cases} \tag{8}$$

Then $u \equiv 0$.

We denote by u_N the unique solution of (1) given by Theorem 1.4. We now state the following regularity result.

Theorem 1.6. *Let $0 < \alpha < \frac{1}{2}$ and $f \in L^2(0,1)$. Let u_N be the solution of (1) given by Theorem 1.4.*

(i) *If $f \in W^{1,\frac{1}{2\alpha}}(0,1)$, then $u_N \in W^{2,p}(0,1)$ for all $1 \le p < \frac{1}{2\alpha}$ with*

$$\|u_N\|_{W^{2,p}(0,1)} \le C\|f\|_{W^{1,p}}.$$

(ii) *If $f \in W^{2,\frac{1}{2\alpha}}(0,1)$, then $x^{2\alpha-1}u_N' \in W^{2,p}(0,1)$ for all $1 \le p < \frac{1}{2\alpha}$, with*

$$\left\|x^{2\alpha-1}u_N'\right\|_{W^{2,p}(0,1)} \le C\|f\|_{W^{2,p}}.$$

Here the constant C depends only on p and α.

Remark 12. One *cannot* expect that $u_N \in W^{2,\frac{1}{2\alpha}}(0,1)$ nor $x^{2\alpha-1}u_N' \in W^{2,\frac{1}{2\alpha}}(0,1)$. Actually, there exists an $f \in C_0^\infty(0,1)$ such that, $u_N \notin W^{2,\frac{1}{2\alpha}}(0,1)$ and $x^{2\alpha-1}u_N' \notin W^{2,\frac{1}{2\alpha}}(0,1)$. See Section 3.2 for the proof.

We now turn to the case $\alpha \ge \frac{1}{2}$. It is convenient to divide this case into three subcases. As we already pointed out, we only have a "Canonical" solution obtained by prescribing either a (weighted) Dirichlet or a (weighted) Neumann condition.

1.2. The case $\frac{1}{2} \le \alpha < \frac{3}{4}$.

Theorem 1.7 (Existence for the "Canonical" Problem)**.** *Given $\frac{1}{2} \le \alpha < \frac{3}{4}$ and $f \in L^2(0,1)$, there exists $u \in H^2_{loc}(0,1]$ satisfying (1) together with the following properties:*

(i) *$u \in C^{0,\frac{3}{2}-2\alpha}$ with $\|u\|_{C^{0,\frac{3}{2}-2\alpha}} \le C\|f\|_{L^2}$. In particular, $\lim_{x\to 0^+}(1-\ln x)^{-\frac{1}{2}}u(x) = 0$.*
(ii) *$\lim_{x\to 0^+} x^{2\alpha-\frac{1}{2}}u'(x) = 0$.*
(iii) *$x^{2\alpha-1}u' \in L^2(0,1)$ and $x^{2\alpha}u'' \in L^2(0,1)$, with $\left\|x^{2\alpha-1}u'\right\|_{L^2} + \left\|x^{2\alpha}u''\right\|_{L^2} \le C\|f\|_{L^2}$. In particular, $x^{2\alpha}u' \in H^1(0,1)$.*

Here the constant C depends only on α.

Remark 13. The same conclusions as in Remark 9–11 still hold for the solution given by Theorem 1.7.

Theorem 1.8 (Uniqueness for the "Canonical" Problem)**.** *Let $\frac{1}{2} \leq \alpha < \frac{3}{4}$. Assume $u \in H^2_{loc}(0,1]$ satisfies*

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 \text{ on } (0,1], \\ u(1) = 0. \end{cases}$$

If in addition one of the following conditions is satisfied

(i) $\lim_{x\to 0^+} x^{2\alpha}u'(x) = 0$,
(ii) $\lim_{x\to 0^+} (1-\ln x)^{-1} u(x) = 0$ *when* $\alpha = \frac{1}{2}$,
(iii) $u \in L^{\frac{1}{2\alpha-1}}(0,1)$ *when* $\frac{1}{2} < \alpha < \frac{3}{4}$,
(iv) $\lim_{x\to 0^+} x^{2\alpha-1}u(x) = 0$ *when* $\frac{1}{2} < \alpha < \frac{3}{4}$,

then $u \equiv 0$.

Again, to simplify the terminology, we call the unique solution of (1) given by Theorem 1.7 the "Canonical" solution and denote it by u_C. We now state the following regularity result.

Theorem 1.9. *Let $\alpha = \frac{1}{2}$, k be an positive integer, and $f \in H^k(0,1)$. Let u_C be the solution to* (1) *given by Theorem 1.7. Then $u_C \in H^{k+1}(0,1)$ and $xu_C \in H^{k+2}(0,1)$ with*

$$\|u_C\|_{H^{k+1}} + \|xu_C\|_{H^{k+2}} \leq C\|f\|_{H^k},$$

where C is a constant depending only on k.

Remark 14. A variant of Theorem 1.9 is already known. For instance in [4], the authors study the Legendre operator $Lu = -\left((1-x^2)u'\right)'$ in the interval $(-1,1)$, and they prove that the operator $A = L + I$ defines an isomorphism from $D^k(A) := \left\{u \in H^{k+1}(-1,1) \,:\, (1-x^2)u(x) \in H^{k+2}(-1,1)\right\}$ to $H^k(-1,1)$ for all $k \in \mathbb{N}$.

Theorem 1.10. *Let $\frac{1}{2} < \alpha < \frac{3}{4}$ and $f \in W^{1,\frac{1}{2\alpha-1}}(0,1)$. Let u_C be the solution to* (1) *given by Theorem 1.7. Then both $u_C \in W^{1,p}(0,1)$ and $x^{2\alpha-1}u_C' \in W^{1,p}(0,1)$ for all $1 \leq p < \frac{1}{2\alpha-1}$ with*

$$\|u_C\|_{W^{1,p}} + \left\|x^{2\alpha-1}u_C'\right\|_{W^{1,p}} \leq C\|f\|_{W^{1,p}},$$

where C is a constant depending only on p and α.

Remark 15. One *cannot* expect that $u_C \in W^{1,\frac{1}{2\alpha-1}}(0,1)$ nor $x^{2\alpha-1}u_C' \in W^{1,\frac{1}{2\alpha-1}}(0,1)$. Actually, there exists an $f \in C_0^\infty(0,1)$ such that $u_C \notin W^{1,\frac{1}{2\alpha-1}}(0,1)$ and $x^{2\alpha-1}u_C' \notin W^{1,\frac{1}{2\alpha-1}}(0,1)$. See Section 3.2 for the proof.

1.3. The case $\frac{3}{4} \leq \alpha < 1$.

Theorem 1.11 (Existence for the "Canonical" Problem)**.** *Given $\frac{3}{4} \leq \alpha < 1$ and $f \in L^2(0,1)$, there exists a function $u \in H^2_{loc}(0,1]$ satisfying* (1) *together with the following properties:*

(i) $u \in L^p(0,1)$ *with* $\|u\|_{L^p} \leq C\|f\|_{L^2}$, *where p is any number in $[1,\infty)$ if $\alpha = \frac{3}{4}$, and $p = \frac{2}{4\alpha-3}$ if $\frac{3}{4} < \alpha < 1$.*
(ii) $\lim_{x\to 0^+} (1-\ln x)^{-\frac{1}{2}} u(x) = 0$ *if* $\alpha = \frac{3}{4}$; $\lim_{x\to 0^+} x^{2\alpha-\frac{3}{2}}u(x) = 0$ *if* $\frac{3}{4} < \alpha < 1$.
(iii) $\lim_{x\to 0^+} x^{2\alpha-\frac{1}{2}}u'(x) = 0$.
(iv) $x^{2\alpha-1}u' \in L^2(0,1)$ *and* $x^{2\alpha}u'' \in L^2(0,1)$, *with* $\left\|x^{2\alpha-1}u'\right\|_{L^2} + \left\|x^{2\alpha}u''\right\|_{L^2} \leq C\|f\|_{L^2}$. *In particular,* $x^{2\alpha}u' \in H^1(0,1)$.

Here the constant C depends only on α.

Remark 16. The boundary behavior in assertion (ii) of Theorem 1.11 is optimal in the following sense: for any $0 < x \leq \frac{1}{2}$ and $\frac{3}{4} \leq \alpha < 1$, define

$$\widetilde{K}_\alpha(x) = \begin{cases} \sup\limits_{\|f\|_{L^2} \leq 1} \left| (1 - \ln x)^{-\frac{1}{2}} \, u(x) \right|, \text{ when } \alpha = \dfrac{3}{4}, \\ \sup\limits_{\|f\|_{L^2} \leq 1} \left| x^{2\alpha - \frac{3}{2}} u(x) \right|, \text{ when } \dfrac{3}{4} < \alpha < 1. \end{cases}$$

Then $0 < \delta \leq \widetilde{K}_\alpha(x) \leq C$, for some constants δ and C only depending on α. See Section 3.2 for the proof.

Remark 17. The same conclusions as in Remark 9 and 10 hold for the solution given by Theorem 1.11.

Theorem 1.12 (Uniqueness for the "Canonical" Problem)**.** *Let $\frac{3}{4} \leq \alpha < 1$. Assume that $u \in H^2_{loc}(0, 1]$ satisfies*

$$\begin{cases} -(x^{2\alpha} u'(x))' + u(x) = 0 \text{ on } (0, 1], \\ u(1) = 0. \end{cases}$$

If in addition one of the following conditions is satisfied

(i) $\lim_{x \to 0^+} x^{2\alpha} u'(x) = 0$,
(ii) $\lim_{x \to 0^+} x^{2\alpha - 1} u(x) = 0$,
(iii) $u \in L^{\frac{1}{2\alpha - 1}}(0, 1)$,

then $u \equiv 0$.

We still call the unique solution of (1) given by Theorem 1.11 the "Canonical" solution and denote it by u_C. Concerning the regularity of u_C for $\frac{3}{4} \leq \alpha < 1$ we have the following

Theorem 1.13. *Let $\frac{3}{4} \leq \alpha < 1$ and $f \in W^{1, \frac{1}{2\alpha - 1}}(0, 1)$. Let u_C be the solution to (1) given by Theorem 1.11. Then both $u_C \in W^{1,p}(0, 1)$ and $x^{2\alpha - 1} u_C' \in W^{1,p}(0, 1)$ for all $1 \leq p < \frac{1}{2\alpha - 1}$ with*

$$\|u_C\|_{W^{1,p}} + \left\|x^{2\alpha - 1} u_C'\right\|_{W^{1,p}} \leq C \|f\|_{W^{1,p}},$$

where C is a constant depending only on p and α.

Remark 18. The same conclusion as in Remark 15 holds here.

1.4. The case $\alpha \geq 1$.

Theorem 1.14 (Existence for the "Canonical" Problem)**.** *Given $\alpha \geq 1$ and $f \in L^2(0, 1)$, there exists a function $u \in H^2_{loc}(0, 1]$ satisfying (1) together with the following properties:*

(i) *$u \in L^2(0, 1)$ with $\|u\|_{L^2} \leq \|f\|_{L^2}$.*
(ii) *$\lim_{x \to 0^+} x^{\frac{\alpha}{2}} u(x) = 0$.*
(iii) *$\lim_{x \to 0^+} x^{\frac{3\alpha}{2}} u'(x) = 0$.*
(iv) *$x^\alpha u' \in L^2(0, 1)$ and $x^{2\alpha} u'' \in L^2(0, 1)$ with $\|x^\alpha u'\|_{L^2} + \left\|x^{2\alpha} u''\right\|_{L^2} \leq C \|f\|_{L^2}$, where C is a constant depending only on α. In particular, $x^{2\alpha} u' \in H^1(0, 1)$.*

Theorem 1.15 (Uniqueness for the "Canonical" Problem)**.** *Let $\alpha \geq 1$. Assume that $u \in H^2_{loc}(0, 1]$ satisfies*

$$\begin{cases} -(x^{2\alpha} u'(x))' + u(x) = 0 \text{ on } (0, 1], \\ u(1) = 0. \end{cases}$$

If in addition one of the following conditions is satisfied

(i) $\lim_{x\to 0^+} x^{\frac{3+\sqrt{5}}{2}} u'(x) = 0$ *when* $\alpha = 1$,
(ii) $\lim_{x\to 0^+} x^{\frac{1+\sqrt{5}}{2}} u(x) = 0$ *when* $\alpha = 1$,
(iii) $\lim_{x\to 0^+} x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u'(x) = 0$ *when* $\alpha > 1$,
(iv) $\lim_{x\to 0^+} x^{\frac{\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u(x) = 0$ *when* $\alpha > 1$,
(v) $u \in L^1(0,1)$,

then $u \equiv 0$.

As before, we call the solution of (1) given by Theorem 1.14 the "Canonical" solution and still denote it by u_C.

Remark 19. For $\alpha \geq \frac{1}{2}$, the existence results (Theorem 1.7, 1.11, 1.14) and the uniqueness results (Theorem 1.8, 1.12, 1.15) guarantee that the weighted Dirichlet and Neumann conditions yield the same "Canonical" solution u_C.

1.5. **Connection with the variational formulation.**

Next we give a variational characterization of the unique solutions u_D, u_N and u_C given by Theorem 1.1, 1.4, 1.7, 1.11, 1.14. We begin by defining the underlying space

$$X^\alpha = \left\{ u \in H^1_{loc}(0,1) : u \in L^2(0,1) \text{ and } x^\alpha u' \in L^2(0,1) \right\}, \ \alpha > 0. \tag{9}$$

For $u,\ v \in X^\alpha$ define

$$a(u,v) = \int_0^1 x^{2\alpha} u'(x) v'(x) dx + \int_0^1 u(x) v(x) dx$$

and

$$I(u) = a(u,u).$$

The space X^α becomes a Hilbert space under the inner product $a(\cdot,\cdot)$. See Appendix A for a detailed analysis of the space X^α.

Notice that the elements of X^α are continuous away from 0 (in fact they are in $H^1_{loc}(0,1]$), so the following is a well-defined (closed) subspace

$$X^\alpha_0 = \{ u \in X^\alpha : u(1) = 0 \}. \tag{10}$$

Also, as it is shown in the Appendix A, when $0 < \alpha < \frac{1}{2}$, the functions in X^α are continuous at the origin, making

$$X^\alpha_{00} = \{ u \in X^\alpha_0 : u(0) = 0 \} \tag{11}$$

a well defined subspace.

Let $0 < \alpha < \frac{1}{2}$ and $f \in L^2(0,1)$. Then the Dirichlet solution u_D given by Theorem 1.1 is characterized by the following property:

$$u_D \in X^\alpha_{00}, \text{ and } \min_{v \in X^\alpha_{00}} \left\{ \frac{1}{2} I(v) - \int_0^1 f(x) v(x) dx \right\} = \frac{1}{2} I(u_D) - \int_0^1 f(x) u_D(x) dx, \tag{12}$$

while the Neumann solution u_N given by Theorem 1.4 is characterized by:

$$u_N \in X^\alpha_0, \text{ and } \min_{v \in X^\alpha_0} \left\{ \frac{1}{2} I(v) - \int_0^1 f(x) v(x) dx \right\} = \frac{1}{2} I(u_N) - \int_0^1 f(x) u_N(x) dx. \tag{13}$$

Let $\alpha \geq \frac{1}{2}$ and $f \in L^2(0,1)$. Then the "Canonical" solution u_C given by Theorem 1.7, 1.11, or 1.14 is characterized by the following property:

$$u_C \in X_0^\alpha, \text{ and } \min_{v \in X_0^\alpha} \left\{ \frac{1}{2} I(v) - \int_0^1 f(x)v(x)dx \right\} = \frac{1}{2} I(u_C) - \int_0^1 f(x)u_C(x)dx. \tag{14}$$

The variational formulations (12), (13) and (14) will be established at the beginning of Section 3, which is the starting point for the proofs of all the existence results.

1.6. **The spectrum.**

Now we proceed to state the spectral properties of the differential operator $\mathcal{L}u := -\left(x^{2\alpha}u'\right)' + u$. We can define two bounded operators associated with it: when $0 < \alpha < \frac{1}{2}$, we define the Dirichlet operator T_D,

$$\begin{aligned} T_D : L^2(0,1) &\longrightarrow L^2(0,1) \\ f &\longmapsto T_D f = u_D, \end{aligned} \tag{15}$$

where u_D is characterized by (12). We also define, for any $\alpha > 0$, the following "Neumann-Canonical" operator T_α,

$$\begin{aligned} T_\alpha : L^2(0,1) &\longrightarrow L^2(0,1) \\ f &\longmapsto T_\alpha f = \begin{cases} u_N \text{ if } 0 < \alpha < \dfrac{1}{2}, \\ u_C \text{ if } \alpha \geq \dfrac{1}{2}, \end{cases} \end{aligned} \tag{16}$$

where u_N and u_C are characterized by (13) and (14) respectively. By Theorem A.3 in the Appendix A, we know that T_D is a compact operator for any $0 < \alpha < \frac{1}{2}$ while T_α is compact if and only if $\alpha < 1$.

In what follows, for given $\nu \in \mathbb{R}$, the function $J_\nu\colon (0,\infty) \longrightarrow \mathbb{R}$ denotes the *Bessel function of the first kind of parameter* ν. We use the positive increasing sequence $\{j_{\nu k}\}_{k=1}^\infty$ to denote all the positive zeros of the function J_ν (see e.g. [11] for a comprehensive treatment of Bessel functions). The results about the spectrum of the operators T_D and T_α read as:

Theorem 1.16 (Spectrum of the Dirichlet Operator)**.** *For $0 < \alpha < \frac{1}{2}$, define $\nu_0 = \frac{\frac{1}{2}-\alpha}{1-\alpha}$, and let $\mu_{\nu_0 k} = 1 + (1-\alpha)^2 j_{\nu_0 k}^2$. Then*

$$\sigma(T_D) = \{0\} \cup \left\{ \lambda_{\nu_0 k} := \frac{1}{\mu_{\nu_0 k}} \right\}_{k=1}^\infty .$$

For any $k \in \mathbb{N}$, the functions defined by

$$u_{\nu_0 k}(x) := x^{\frac{1}{2}-\alpha} J_{\nu_0}(j_{\nu_0 k} x^{1-\alpha})$$

is the eigenfunction of T_D corresponding to the eigenvalue $\lambda_{\nu_0 k}$. Moreover, for fixed $0 < \alpha < \frac{1}{2}$ and k sufficiently large, we have

$$\mu_{\nu_0 k} = 1 + (1-\alpha)^2 \left[\left(\frac{\pi}{2}\left(\nu_0 - \frac{1}{2}\right) + \pi k \right)^2 - \left(\nu_0^2 - \frac{1}{4}\right) \right] + O\left(\frac{1}{k}\right). \tag{17}$$

Theorem 1.17 (Spectrum of the "Neumann-Canonical" Operator)**.** *Assume $\alpha > 0$ and let T_α be the operator defined above.*

(i) *For $0 < \alpha < 1$, define $\nu = \frac{\alpha - \frac{1}{2}}{1-\alpha}$, and let $\mu_{\nu k} = 1 + (1-\alpha)^2 j_{\nu k}^2$. Then*

$$\sigma(T_\alpha) = \{0\} \cup \left\{ \lambda_{\nu k} := \frac{1}{\mu_{\nu k}} \right\}_{k=1}^\infty .$$

For any $k \in \mathbb{N}$, the functions defined by

$$u_{\nu k}(x) := x^{\frac{1}{2}-\alpha} J_\nu(j_{\nu k} x^{1-\alpha})$$

is the eigenfunction of T_α corresponding to the eigenvalue $\lambda_{\nu k}$. Moreover, for fixed $0 < \alpha < 1$ and k sufficiently large, we have

$$\mu_{\nu k} = 1 + (1-\alpha)^2 \left[\left(\frac{\pi}{2} \left(\nu - \frac{1}{2} \right) + \pi k \right)^2 - \left(\nu^2 - \frac{1}{4} \right) \right] + O\left(\frac{1}{k} \right). \tag{18}$$

(ii) *For $\alpha = 1$, the operator T_1 has no eigenvalues, and the spectrum is exactly $\sigma(T_1) = \left[0, \frac{4}{5}\right]$.*
(iii) *For $\alpha > 1$, the operator T_α has no eigenvalues, and the spectrum is exactly $\sigma(T_\alpha) = [0, 1]$.*

Recall that the discrete spectrum of an operator T is defined as

$$\sigma_d(T) = \{\lambda \in \sigma(T) : T - \lambda I \text{ is a Fredholm operator}\},$$

and the essential spectrum is defined as

$$\sigma_e(T) = \sigma(T) \backslash \sigma_d(T).$$

We have the following corollary about the essential spectrum.

Corollary 1.18 (Essential Spectrum of the "Neumann-Canonical" Operator)**.** *Assume that $\alpha > 0$ and let T_α be the operator defined above.*

(i) *For $0 < \alpha < 1$, $\sigma_e(T_\alpha) = \{0\}$.*
(ii) *For $\alpha = 1$, $\sigma_e(T_1) = \left[0, \frac{4}{5}\right]$.*
(iii) *For $\alpha > 1$, $\sigma_e(T_\alpha) = [0, 1]$.*

Remark 20. This corollary follows immediately from the fact (see e.g. Theorem IX.1.6 of [5]) that, for any self-adjoint operator T on a Hilbert space, $\sigma_d(T)$ consists of the isolated eigenvalues with finite multiplicity. In fact, for Corollary 1.18 to hold, it suffices to prove that $\sigma_d(T) \subset EV(T)$, where $EV(T)$ is the set of all the eigenvalues. We present in Section 4.1.2 a simple proof of this inclusion.

As the reader can see in Theorem 1.17, when $\alpha < 1$ the spectrum of the operator T_α is a discrete set and when $\alpha = 1$ the spectrum of T_1 becomes a closed interval, so a natural question is whether $\sigma(T_\alpha)$ converges to $\sigma(T_1)$ as $\alpha \to 1^-$ in some sense. The answer is positive as the reader can check in the following

Theorem 1.19. *Let $\alpha \leq 1$. For the spectrum $\sigma(T_\alpha)$, we have*

(i) *$\sigma(T_\alpha) \subset \sigma(T_1)$ for all $\frac{2}{3} < \alpha < 1$.*
(ii) *For every $\lambda \in \sigma(T_1)$, there exists a sequence $\alpha_m \to 1^-$ and a sequence of eigenvalues $\lambda_m \in \sigma(T_{\alpha_m})$ such that $\lambda_m \to \lambda$ as $m \to \infty$.*

Remark 21. Notice that in particular $\sigma(T_\alpha) \to \sigma(T_1)$ in the Hausdorff metric sense, that is

$$d_H(\sigma(T_\alpha), \sigma(T_1)) \to 0, \text{ as } \alpha \to 1^-,$$

where $d_H(X, Y) = \max\left\{\sup_{x \in X} \inf_{y \in Y} |x - y|, \sup_{y \in Y} \inf_{x \in X} |x - y|\right\}$ is the Hausdorff metric (see e.g. Chapter 7 of [7]).

Remark 22. When $\alpha \leq 1$, the spectrum of T_α has been investigated by C. Stuart [9]. In fact, he considered the more general differential operator $Nu = -(A(x)u')'$ under the conditions $u(1) = 0$ and $\lim_{x \to 0^+} A(x)u'(x) = 0$, with

$$A \in C^0([0, 1]);\ A(x) > 0, \forall x \in (0, 1] \text{ and } \lim_{x \to 0^+} \frac{A(x)}{x^{2\alpha}} = 1. \tag{19}$$

Notice that if $A(x) = x^{2\alpha}$, we have the equality $T_\alpha = (N+I)^{-1}$, where the inverse is taken in the space $L^2(0,1)$. When $\alpha < 1$, C. Stuart proves that $\sigma(N)$ consists of isolated eigenvalues; this is deduced from a compactness argument. When $\alpha = 1$, C. Stuart proves that $\max \sigma_e\left((N+I)^{-1}\right) = \frac{4}{5}$. On the other hand, C. Stuart has constructed an elegant example of function A satisfying (19) with $\alpha = 1$ such that $(N+I)^{-1}$ admits an eigenvalue in the interval $(\frac{4}{5}, 1]$. Moreover, G. Vuillaume (in his thesis [10] under C. Stuart) used a variant of this example to get an arbitrary number of eigenvalues in the interval $(\frac{4}{5}, 1]$. However, we still have an

Open Problem 1. If A satisfies (19) for $\alpha = 1$, is it true that $\sigma_e\left((N+I)^{-1}\right) = [0, \frac{4}{5}]$?

Similarly, when $\alpha > 1$, one can still consider the differential operator $Nu = -(A(x)u')'$ under the conditions $u(1) = 0$ and $\lim_{x\to 0^+} A(x)u'(x) = 0$, where A satisfies (19), and the operator $(N+I)^{-1}$, where the inverse is taken in the space $L^2(0,1)$, is still well-defined. By the same argument as in the case $A(x) = x^{2\alpha}$ (Theorem 1.17 (iii)) we know that $\sigma\left((N+I)^{-1}\right) \subset [0,1]$. However, we still have

Open Problem 2. Assume that A satisfies (19) for $\alpha > 1$.

(i) Is it true that $\sigma\left((N+I)^{-1}\right) = [0,1]$?
(ii) Is it true that $\max \sigma_e\left((N+I)^{-1}\right) = 1$, or more precisely $\sigma_e\left((N+I)^{-1}\right) = [0,1]$?

The rest of the paper is organized as the following. We begin by proving the uniqueness results in Section 2. We then prove the existence and regularity results in Section 3. The analysis of the spectrum of the operators T_D and T_α is performed in Section 4. Finally we present in Appendix A some properties about weighted Sobolev spaces used throughout this work.

2. Proofs of all the Uniqueness Results

In this section we will provide the proofs of the uniqueness results stated in the Introduction.

Proof of Theorem 1.2. Since $u \in C^0(0,1]$ with $\lim_{x\to 0^+} u(x) = 0$, we have that $u \in C^0[0,1]$. Notice that, for any $0 < x < 1$, we can write $x^{2\alpha}u'(x) = u'(1) - \int_x^1 u(s)ds$, which implies that $x^{2\alpha}u' \in C[0,1]$. Then we can multiply the equation (5) by u and integrate by parts over $[\epsilon, 1]$, and with the help of the boundary condition we obtain

$$\int_\epsilon^1 x^{2\alpha}u'(x)^2dx + \int_\epsilon^1 u(x)^2dx = x^{2\alpha}u'(x)u(x)|_\epsilon^1 \to 0, \text{ as } \epsilon \to 0^+.$$

Therefore, $u = 0$. □

Proof of Theorem 1.5. We first claim that $u \in C^0[0,1]$. Since $u \in C^1(0,1]$ and $\lim_{x\to 0^+} x^{2\alpha}u'(x) = 0$, there exists $C > 0$ such that $-Cx^{-2\alpha} \le u'(x) \le Cx^{-2\alpha}$, which implies that $-Cx^{1-2\alpha} \le u(x) \le Cx^{1-2\alpha}$, hence $u \in L^\infty(0,1)$ because $0 < \alpha < \frac{1}{2}$. Write $u'(x) = \frac{1}{x^{2\alpha}} \int_0^x u(s)ds$ and deduce that $u' \in L^\infty(0,1)$, thus $u \in W^{1,\infty}(0,1)$. In particular $u \in C^0[0,1]$.

Then we can multiply the equation (8) by u and integrate by parts over $[\epsilon, 1]$, and with the help of the boundary condition we obtain

$$\int_\epsilon^1 x^{2\alpha}u'(x)^2dx + \int_\epsilon^1 u(x)^2dx = x^{2\alpha}u'(x)u(x)|_\epsilon^1 \to 0, \text{ as } \epsilon \to 0^+.$$

Therefore, $u \equiv 0$. □

Proof of (i) of Theorem 1.8 and (i) of Theorem 1.12. As in the proof of Theorem 1.5, it is enough to show that $u \in C^0[0,1]$. As before, the boundary condition implies that $u(x) \sim x^{1-2\alpha}$, which gives $u \in L^{\frac{1}{\alpha}}(0,1)$. To prove that $u \in C^0[0,1]$, we first write $x^{2\alpha-1}u'(x) = \frac{1}{x}\int_0^x u(s)ds$. Let $p_0 := \frac{1}{\alpha} > 1$. Since $u \in L^{p_0}(0,1)$, one can apply Hardy's inequality and obtain $\left\|x^{2\alpha-1}u'\right\|_{L^{p_0}} \leq C\left\|u\right\|_{L^{p_0}}$. Since $u(1) = 0$, this implies that $u \in X^{2\alpha-1,p_0}_{\cdot 0}(0,1)$. By Theorem A.2, we have two alternatives

- $u \in L^q(0,1)$ for all $q < \infty$ when $\alpha \leq \frac{2}{3}$ or
- $u \in L^{p_1}(0,1)$ where $p_1 := \frac{1}{3\alpha-2} > p_0$ when $\frac{2}{3} < \alpha < 1$.

If the first case happens and $u \in L^q(0,1)$ for all $q < \infty$, then we apply Hardy's inequality and obtain $u \in X^{2\alpha-1,q}_{\cdot 0}(0,1)$ for all $q < \infty$, which embeds into $C^0[0,1]$ for q large enough. If the second alternative occurs and we apply Hardy's inequality once more, we conclude that $u \in X^{2\alpha-1,p_1}_{\cdot 0}(0,1)$. Therefore, either $u \in L^q(0,1)$ for all $q < \infty$ when $\alpha \leq \frac{4}{5}$ or $u \in L^{p_2}(0,1)$ where $p_2 = \frac{1}{5\alpha-4}$ when $\frac{4}{5} < \alpha < 1$. By repeating this argument finitely many times we can conclude that $u \in C^0[0,1]$. □

Proof of (ii) of Theorem 1.8. Let $\alpha = \frac{1}{2}$ and suppose that $u \in H^2_{loc}(0,1]$ satisfies

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 \text{ on } (0,1], \\ u(1) = 0, \\ \lim\limits_{x\to 0^+} \dfrac{u(x)}{1-\ln(x)} = 0. \end{cases}$$

Notice that $u \in C(0,1]$ together with $\lim_{x\to 0^+}(1-\ln x)^{-1}u(x) = 0$ and the integrability of $\ln x$, gives $u \in L^1(0,1)$. Define $w(x) = u(x)(1-\ln x)^{-1}$. It is enough to show that $w = 0$. Notice that w solves

$$\begin{cases} (x(1-\ln x)w'(x))' = (1-\ln x)w(x) + w'(x) \text{ on } (0,1), \\ w(1) = 0, \\ w(0) = 0. \end{cases} \tag{20}$$

We integrate equation (20) to obtain

$$x(1-\ln x)w'(x) = w'(1) - \int_x^1 (1-\ln s)w(s)dx = u'(1) - \int_x^1 u(s)ds.$$

Since $u \in L^1(0,1)$, the above computation shows that $x(1-\ln x)w'(x) \in C[0,1]$. Now we multiply (20) by w and we integrate by parts over $[\epsilon, 1]$ to obtain

$$\int_\epsilon^1 x(1-\ln x)w'(x)^2dx + \int_\epsilon^1 (1-\ln x)w^2(x)dx = x(1-\ln x)w'(x)w(x)|_\epsilon^1 - \frac{1}{2}w^2(x)|_\epsilon^1 \to 0,$$

as $\epsilon \to 0^+$, proving that $w = 0$. □

At this point we would like to mention that the proof of (iii) of Theorem 1.8 and (iii) of Theorem 1.12 will be postponed to Proposition 3.4 of Section 3.2.

Proof of (iv) of Theorem 1.8 and (ii) of Theorem 1.12. Let $\frac{1}{2} < \alpha < 1$ and suppose that $u \in H^2_{loc}(0,1]$ satisfies

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 \text{ on } (0,1], \\ u(1) = 0, \\ \lim_{x\to 0^+} x^{2\alpha-1}u(x) = 0. \end{cases}$$

Notice that $u \in C(0,1]$ together with $\lim_{x\to 0^+} x^{2\alpha-1}u(x) = 0$ and the integrability of $x^{1-2\alpha}$ for $\alpha < 1$, gives $u \in L^1(0,1)$. Define $w(x) = x^{2\alpha-1}u(x)$. We will show that $w = 0$. Notice that w satisfies

$$\begin{cases} -(xw'(x))' + (2\alpha-1)w'(x) + x^{1-2\alpha}w(x) = 0 \text{ on } (0,1], \\ w(1) = 0, \\ w(0) = 0. \end{cases} \tag{21}$$

Integrate (21) to obtain

$$xw'(x) = w'(1) - \int_x s^{1-2\alpha}w(s)ds = u'(1) - \int_x^1 u(s)ds,$$

from which we conclude $xw'(x) \in C[0,1]$. Finally, multiply (21) by w and integrate by parts over $[\epsilon,1]$ to obtain

$$\int_\epsilon^1 xw'(x)^2dx + \int_\epsilon^1 x^{1-2\alpha}w(x)^2dx = xw'(x)w(x)|_\epsilon^1 - \left(\alpha - \frac{1}{2}\right)w^2(\epsilon).$$

Letting $\epsilon \to 0^+$ and we conclude that $w = 0$. □

Proof of Theorem 1.15. Assume that (i) holds. Suppose that $u \in H^2_{loc}(0,1]$ satisfies

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 \text{ on } (0,1], \\ u(1) = 0, \\ \lim_{x\to 0^+} x^{\frac{3+\sqrt{5}}{2}}u'(x) = 0. \end{cases}$$

Let $v(x) = x^{\frac{1+\sqrt{5}}{2}}u(x)$. Then $v \in H^2_{loc}(0,1]$ and it satisfies

$$\begin{cases} -(xv'(x))' + \sqrt{5}v'(x) = 0 \text{ on } (0,1], \\ v(1) = 0, \\ \lim_{x\to 0^+} \left(xv'(x) - \frac{1+\sqrt{5}}{2}v(x)\right) = 0, \end{cases} \tag{22}$$

from which we obtain that $xv' - \frac{1+\sqrt{5}}{2}v \in C[0,1]$ and $xv' - \sqrt{5}v \in H^1(0,1)$. Therefore $v \in C[0,1]$. Multiply (22) by v and integrate over $[\epsilon,1]$ to obtain

$$\int_\epsilon^1 xv'(x)^2dx + \frac{1}{2}v^2(\epsilon) = \left(xv'(x) - \frac{1+\sqrt{5}}{2}v(x)\right)v(x)|_\epsilon^1 \to 0, \text{ as } \epsilon \to 0^+.$$

Therefore v is constant and thus $v(x) \equiv v(1) = 0$.

Assume that (ii) holds. Suppose that $u \in H^2_{loc}(0,1]$ satisfies

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 \text{ on } (0,1], \\ u(1) = 0, \\ \lim_{x\to 0^+} x^{\frac{1+\sqrt{5}}{2}} u(x) = 0. \end{cases}$$

Let $w(x) = x^{\frac{1+\sqrt{5}}{2}} u(x)$. Then $w \in H^2_{loc}(0,1]$ and it satisfies

$$\begin{cases} -(xw'(x))' + \sqrt{5}w'(x) = 0 \text{ on } (0,1], \\ w(1) = 0, \\ w(0) = 0. \end{cases} \tag{23}$$

Therefore $xw' + \sqrt{5}w \in H^1(0,1)$, $w \in C[0,1]$, and $xw' \in C[0,1]$. Multiply (23) by w and integrate over $[\epsilon, 1]$ to obtain

$$\int_\epsilon^1 xw'(x)^2 dx = xw'(x)w(x)|_\epsilon^1 - \frac{\sqrt{5}}{2} w^2(x)|_\epsilon^1 \to 0, \text{ as } \epsilon \to 0^+.$$

Therefore w is constant, so $w(x) \equiv w(1) = 0$.

Assume that (iii) holds. Suppose that $u \in H^2_{loc}(0,1]$ satisfies

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 \text{ on } (0,1], \\ u(1) = 0, \\ \lim_{x\to 0^+} x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u'(x) = 0. \end{cases}$$

Define $g(x) = e^{\frac{x^{1-\alpha}}{1-\alpha}} u(x)$. Then $g \in H^2_{loc}(0,1]$ and it satisfies

$$\begin{cases} -(x^{2\alpha}g'(x))' + (x^\alpha g(x))' + x^\alpha g'(x) = 0 \text{ on } (0,1], \\ g(1) = 0, \\ \lim_{x\to 0^+} \left(x^{\frac{3\alpha}{2}} g'(x) - x^{\frac{\alpha}{2}} g(x) \right) = 0. \end{cases}$$

Multiply the above by g and integrate over $[\epsilon, 1]$ to obtain

$$\begin{aligned} \int_\epsilon^1 x^{2\alpha} g'(x)^2 dx &= x^{2\alpha} g'(x) g(x)|_\epsilon^1 - x^\alpha g^2(x)|_\epsilon^1 \\ &= \left(x^{\frac{3\alpha}{2}} g'(x) - x^{\frac{\alpha}{2}} g(x) \right) x^{\frac{\alpha}{2}} g(x)|_\epsilon^1. \end{aligned} \tag{24}$$

We now study the function $h(x) := x^{\frac{\alpha}{2}} g(x)$. We have

$$\begin{aligned} h(x) &= -\int_x^1 h'(s) ds \\ &= -\int_x^1 \left(\frac{\alpha}{2} s^{\frac{\alpha}{2}-1} g(s) + s^{\frac{\alpha}{2}} g'(s) \right) ds \\ &= \frac{\alpha}{2} \int_x^1 s^{\frac{3\alpha}{2}-1} g'(s) ds - \left(x^{\frac{3\alpha}{2}} g'(x) - x^{\frac{\alpha}{2}} g(x) \right) \\ &= -\frac{\alpha}{2} \left(\frac{3\alpha}{2} - 1 \right) \int_x^1 s^{\frac{3\alpha}{2}-2} g(s) ds - \frac{\alpha}{2} x^{\alpha-1} h(x) - \left(x^{\frac{3\alpha}{2}} g'(x) - x^{\frac{\alpha}{2}} g(x) \right). \end{aligned}$$

Hence we can write

$$h(x)=\left[1+\frac{\alpha}{2}x^{\alpha-1}\right]^{-1}\left[-\frac{\alpha}{2}\left(\frac{3\alpha}{2}-1\right)\int_x^1 s^{\frac{3\alpha}{2}-2}g(s)ds-\left(x^{\frac{3\alpha}{2}}g'(x)-x^{\frac{\alpha}{2}}g(x)\right)\right].$$

We claim that there exists a sequence $\epsilon_n\to 0$ so that

$$\lim_{n\to\infty}\left|\int_{\epsilon_n}^1 s^{\frac{3\alpha}{2}-2}g(s)ds\right|<\infty.$$

Otherwise, assume that $\lim_{\epsilon\to 0^+}\int_\epsilon^1 s^{\frac{3\alpha}{2}-2}g(s)ds=\pm\infty$. Then

$$\lim_{x\to 0^+}x^{\frac{\alpha}{2}}e^{\frac{x^{1-\alpha}}{1-\alpha}}u(x)=\lim_{x\to 0^+}h(x)=\pm\infty.$$

This forces $\lim_{x\to 0^+}u(x)=\pm\infty$, so L'Hopital's rule applies to u and one obtains that

$$\lim_{x\to 0^+}x^{\frac{\alpha}{2}}e^{\frac{x^{1-\alpha}}{1-\alpha}}u(x)=\lim_{x\to 0^+}\frac{x^{\frac{3\alpha}{2}}e^{\frac{x^{1-\alpha}}{1-\alpha}}u'(x)}{-\frac{\alpha}{2}x^{\alpha-1}-1}=0,$$

which is a contradiction. Therefore $\lim_{\epsilon_n\to 0^+}h(\epsilon_n)$ exists for some sequence $\epsilon_n\to 0$. Finally, use that sequence $\epsilon_n\to 0^+$ in (24) to obtain that $\int_0^1 x^{2\alpha}g'(x)^2dx=0$, which gives g is constant, that is $g(x)\equiv g(1)=0$.

Assume that (iv) holds. Suppose that $u\in H^2_{loc}(0,1]$ satisfies

$$\begin{cases}-(x^{2\alpha}u'(x))'+u(x)=0 \text{ on } (0,1],\\ u(1)=0,\\ \lim_{x\to 0^+}x^{\frac{\alpha}{2}}e^{\frac{x^{1-\alpha}}{1-\alpha}}u(x)=0.\end{cases}$$

Let $p(x)=e^{\frac{x^{1-\alpha}}{1-\alpha}}u(x)$, then w satisfies

$$\begin{cases}-(x^{2\alpha}p'(x))'+(x^{\alpha}p(x))'+x^{\alpha}p'(x)=0 \text{ on } (0,1],\\ p(1)=0,\\ \lim_{x\to 0^+}x^{\frac{\alpha}{2}}p(x)=0.\end{cases}\tag{25}$$

We claim that $\lim_{x\to 0^+}x^{\frac{3\alpha}{2}}p'(x)$ exists, thus implying that $x^{\frac{3\alpha}{2}}p'(x)$ belongs to $C[0,1]$. Define $q(x)=x^{\frac{3\alpha}{2}}p'(x)$, then using (25) we obtain that, for $0<x<1$,

$$q'(x)=-\frac{\alpha}{2}x^{\frac{3\alpha}{2}-1}p'(x)+\alpha x^{\frac{\alpha}{2}-1}p(x)+2x^{\frac{\alpha}{2}}p'(x).$$

A direct computation shows that, for $0<x<1$,

$$\int_x^1 q'(s)ds=\frac{\alpha}{2}\left(\frac{3\alpha}{2}-1\right)\int_x^1 x^{\frac{3\alpha}{2}-2}p(s)ds+\frac{\alpha}{2}x^{\alpha-1}x^{\frac{\alpha}{2}}p(x)-2x^{\frac{\alpha}{2}}p(x).$$

Since $x^{\frac{\alpha}{2}}p(x)\in C[0,1]$, we obtain that $x^{\frac{3\alpha}{2}-2}p(x)\in L^1(0,1)$ which implies that $x^{\frac{3\alpha}{2}}p'(x)=q(x)=-\int_x^1 q'(s)ds$ is continuous and that the $\lim_{x\to 0^+}q(x)$ exists. We now multiply (25) by $p(x)$ and integrate by parts to obtain

$$\int_0^1 x^{2\alpha}p'(x)^2=x^{\frac{3\alpha}{2}}p'(x)x^{\frac{\alpha}{2}}p(x)|_0^1=0.$$

Thus proving that $p(x)$ is constant, i.e. $p(x)\equiv p(1)=0$.

Finally assume that (v) holds. Define $k(x) = x^{2\alpha}u'(x)$. Notice that since $u \in L^1(0,1) \cap H^2_{loc}(0,1]$, from the equation we obtain that $k(x) = u'(1) - \int_x^1 u(s)ds$, so $k(x) \in C^0[0,1]$. We claim that $k(0) = 0$. Otherwise, near the origin $u'(x) \sim \frac{1}{x^{2\alpha}}$ and $u(x) \sim \frac{1}{x^{2\alpha-1}}$, which contradicts $u \in L^1(0,1)$. Therefore, $\lim_{x\to 0^+} x^{2\alpha}u'(x) = 0$. We are now in the case where (i) or (iii) applies, so we can conclude that $u = 0$. □

3. Proofs of all the Existence and the Regularity Results

Our proof of the existence results will mostly use functional analysis tools. We take the weighted Sobolev space X^α defined in (9) and its subspaces X^α_{00} and X^α_0 defined by (11) and (10). As we can see from the Appendix A, X^α equipped with the inner product given by

$$(u,v)_\alpha = \int_0^1 \left(x^{2\alpha}u'(x)v'(x) + u(x)v(x)\right)dx,$$

is a Hilbert space. X^α_{00} and X^α_0 are well defined closed subspaces. We define two notions of weak solutions as follows: given $0 < \alpha < \frac{1}{2}$ and $f \in L^2(0,1)$ we say u is a *weak solution of the first type* of (1) if $u \in X^\alpha_{00}$ satisfies

$$\int_0^1 x^{2\alpha}u'(x)v'(x)dx + \int_0^1 u(x)v(x)dx = \int_0^1 f(x)v(x)dx, \text{ for all } v \in X^\alpha_{00}; \tag{26}$$

and given $\alpha > 0$ and $f \in L^2(0,1)$ we say that u is a *weak solution of the second type* of (1) if $u \in X^\alpha_0$ satisfies

$$\int_0^1 x^{2\alpha}u'(x)v'(x)dx + \int_0^1 u(x)v(x)dx = \int_0^1 f(x)v(x)dx, \text{ for all } v \in X^\alpha_0. \tag{27}$$

The existence of both solutions are guaranteed by Riesz Theorem. Actually, (26) is equivalent to (12), while (27) is equivalent to (13) or (14) (see e.g. Theorem 5.6 of [1]). As we will see later, the weak solution of the first type is exactly the solution u_D mentioned in the Introduction, whereas the weak solution of the second type corresponds to either u_N when $0 < \alpha < \frac{1}{2}$ or u_C when $\alpha \geq \frac{1}{2}$.

3.1. **The Dirichlet Problem.**

Proof of Theorem 1.1. We will actually prove that the solution of (26) is the solution we are looking for in Theorem 1.1. Notice that by taking $v \in C_0^\infty(0,1)$ in (26) we obtain that $w(x) := x^{2\alpha}u'(x) \in H^1(0,1)$ with $(x^{2\alpha}u'(x))' = u(x) - f(x)$ and $\|w'\|_{L^2} \leq 2\|f\|_{L^2}$. Also since $u \in X^\alpha_{00}$ we have that $u(0) = u(1) = 0$.

Now we write

$$u(x) = \int_0^x u'(s)ds = -\frac{1}{1-2\alpha}\int_0^x \left(s^{2\alpha}u'(s)\right)' s^{1-2\alpha}ds + \frac{xu'(x)}{1-2\alpha},$$

where we have used that $\lim_{s\to 0^+} su'(s) = \lim_{s\to 0^+} s^{2\alpha}u'(s)\cdot s^{1-2\alpha} = 0$ for all $\alpha < \frac{1}{2}$. It implies that

$$x^{2\alpha-1}u(x) = \frac{x^{2\alpha}u'(x)}{1-2\alpha} + \frac{x^{2\alpha-1}}{2\alpha-1}\int_0^x \left(s^{2\alpha}u'(s)\right)' s^{1-2\alpha}ds,$$

and

$$\left(x^{2\alpha-1}u(x)\right)' = x^{2\alpha-2}\int_0^x \left(s^{2\alpha}u'(s)\right)' s^{1-2\alpha}ds.$$

From here, since $\alpha < \frac{1}{2}$, we obtain

$$\left|\left(x^{2\alpha-1}u(x)\right)'\right| \leq \frac{1}{x}\int_0^x \left(s^{2\alpha}u'(s)\right)' ds,$$

so Hardy's inequality gives

$$\left\|\left(x^{2\alpha-1}u\right)'\right\|_{L^2} \leq 2\left\|\left(x^{2\alpha}u'\right)'\right\|_{L^2} \leq 4\left\|f\right\|_{L^2}.$$

Therefore, $\left\|x^{2\alpha-1}u\right\|_{H^1} \leq C\left\|f\right\|_{L^2}$, where C is a constant depending only on α. Combining this result and the fact that $x^{2\alpha}u' \in H^1(0,1)$, we conclude that $x^{2\alpha}u \in H^2(0,1)$.

Also notice that $u \in C^{0,1-2\alpha}[0,1]$ is a direct consequence of $x^{2\alpha-1}u \in C[0,1]\cap C^1(0,1]$. The proof is finished. □

Proof of Remark 1. Take $f \in C_0^\infty(0,1)$. We know that $u(x) = A\phi_1(x) + B\phi_2(x) + F(x)$ where $\phi_1(x)$ and $\phi_2(x)$ are two linearly independent solutions of the equation $-(x^{2\alpha}u'(x))' + u(x) = 0$ and

$$F(x) = \phi_1(x)\int_0^x f(s)\phi_2(s)ds - \phi_2(x)\int_0^x f(s)\phi_1(s)ds.$$

Moreover, one can see that $\phi_i(x) = x^{\frac{1}{2}-\alpha}f_i\left(\frac{x^{1-\alpha}}{1-\alpha}\right)$ where $f_i(z)$'s are two linearly independent solutions of the Bessel equation

$$z^2\phi''(z) + z\phi'(z) - \left(z^2 + \left(\frac{\frac{1}{2}-\alpha}{1-\alpha}\right)^2\right)\phi(z) = 0.$$

By the properties of the Bessel function (see e.g. Chapter III of [11]), we know that near the origin,

$$\phi_1(x) = a_1x^{1-2\alpha} + a_2x^{3-4\alpha} + a_3x^{5-6\alpha} + \cdots, \text{ for } 0 < \alpha < \frac{1}{2},$$

and

$$\phi_2(x) = b_1 + b_2x^{2-2\alpha} + b_3x^{4-4\alpha} + b_4x^{6-6\alpha} + \cdots, \text{ for } 0 < \alpha < 1.$$

Also,

$$\phi_1(0) = 0,\ \phi_2(0) \neq 0,\ \phi_1(1) \neq 0, \quad \text{for } 0 < \alpha < \frac{1}{2},$$

$$\lim_{x\to 0^+}|\phi_1(x)| = \infty,\ \lim_{x\to 0^+}\phi_2(x) = b_1, \quad \text{for } \alpha \geq \frac{1}{2},$$

and

$$\lim_{x\to 0^+}x^{2\alpha}\phi_1'(x) \neq 0,\ \lim_{x\to 0^+}x^{2\alpha}\phi_2'(x) = 0,\ \phi_2(1) \neq 0, \text{ for } 0 < \alpha < 1.$$

Notice that $F(x) \equiv 0$ near the origin. Therefore, when imposing the boundary conditions $u(0) = u(1) = 0$, we obtain $u(x) = A\phi_1(x) + F(x)$ with $A = -\frac{F(1)}{\phi_1(1)}$. Take f such that

$$F(1) = \int_0^1 f(s)[\phi_2(s)\phi_1(1) - \phi_1(s)\phi_2(1)]ds \neq 0.$$

Then $u(x) \sim \phi_1(x)$ near the origin and we get the desired power series expansion. □

Proof of Remark 3. From the proof of Theorem 1.1, we conclude that $w \in C^0[0,1]$ with $\left\|w\right\|_\infty \leq 2\left\|f\right\|_{L^2}$. From here we have

$$|u'(x)| = \left|w(x)x^{-2\alpha}\right| \leq \left\|w\right\|_\infty x^{-2\alpha}.$$

Thus, for $1 \leq p < \frac{1}{2\alpha}$,

$$\left\|u'\right\|_{L^p} \leq \left\|w\right\|_\infty\left\|x^{-2\alpha}\right\|_{L^p(0,1)} \leq C(\alpha,p)\left\|f\right\|_2.$$

□

Proof of Remark 5. If we take $f(x) := -(x^{2\alpha}u'(x))' + u(x)$, where $u(x) = x^{1-2\alpha}(x-1)$, we will see that $u \notin C^{0,\beta}[0,1]$, $\forall \beta > 1-2\alpha$. When $u(x) = x^{\frac{7}{4}-2\alpha}(x-1)$, we will see that $x^{2\alpha-1}u \notin H^2(0,1)$, $x^{2\alpha}u' \notin H^2(0,1)$, and $x^{2\alpha}u \notin H^3(0,1)$. □

Proof of Remark 6. From [3] we know that the function g exists and $x^{2\alpha}g'(x) \in L^\infty(0,1)$. Therefore, integration by parts gives

$$\int_0^1 f(x)g(x)dx = \int_0^1 -(x^{2\alpha}u'(x))'g(x) + u(x)g(x)dx = \lim_{x\to 0^+} x^{2\alpha}u'(x).$$

And the L'Hopital's rule immediately implies that

$$\lim_{x\to 0^+} x^{2\alpha-1}u(x) = \lim_{x\to 0^+} \frac{1}{1-2\alpha}x^{2\alpha}u'(x) = \frac{1}{1-2\alpha}\int_0^1 f(x)g(x)dx.$$

□

Before we prove Theorem 1.3, we need the following lemma.

Lemma 3.1. *Let $0 < \alpha < \frac{1}{2}$ and $k_0 \in \mathbb{N}$. Assume $u \in W^{k_0+1,p}_{loc}(0,1)$ for some $p \geq 1$. If $\lim_{x\to 0^+} u(x) = 0$ and $\lim_{x\to 0^+} x^{k-2\alpha}\frac{d^{k-1}}{dx^{k-1}}\left(s^{2\alpha}u'(s)\right) = 0$ for all $1 \leq k \leq k_0$, then for $0 < x < 1$*

$$\frac{d^k}{dx^k}\left(x^{2\alpha-1}u(x)\right) = x^{2\alpha-k-1}\int_0^x s^{k-2\alpha}\frac{d^k}{ds^k}\left(s^{2\alpha}u'(s)\right)ds, \quad \textit{for all } 1 \leq k \leq k_0.$$

Moreover

$$\left\|\frac{d^k}{dx^k}\left(x^{2\alpha-1}u\right)\right\|_{L^p} \leq C\left\|\frac{d^k}{dx^k}\left(x^{2\alpha}u'\right)\right\|_{L^p},$$

where C is a constant depending only on p, α and k.

Proof. When $k_0 = 1$ we can write

$$\begin{aligned}\left(x^{2\alpha-1}u(x)\right)' &= \left(x^{2\alpha-1}\int_0^x s^{2\alpha}u'(s)\left(\frac{s^{1-2\alpha}}{1-2\alpha}\right)' ds\right)' \\ &= \left(\frac{x^{2\alpha-1}}{2\alpha-1}\int_0^x \left(s^{2\alpha}u'(s)\right)' s^{1-2\alpha}ds + \frac{x^{2\alpha}u'(x)}{1-2\alpha}\right)' \\ &= x^{2\alpha-2}\int_0^x \left(s^{2\alpha}u'(s)\right)' s^{1-2\alpha}ds.\end{aligned}$$

The rest of the proof is a straightforward induction argument. We omit the details. The norm bound is obtained by Fubini's Theorem when $p = 1$ and by Hardy's inequality when $p > 1$. □

Proof of Theorem 1.3. Notice that $\lim_{x\to 0^+} x^{2-2\alpha}\left(s^{2\alpha}u'(s)\right)'=0$ since both u and f are continuous. With the aid of Lemma 3.1 for $k_0 = 2$ we can write

$$\left(x^{2\alpha-1}u(x)\right)'' = x^{2\alpha-3}\int_0^x s^{2-2\alpha}\left(s^{2\alpha}u'\right)'' ds = x^{2\alpha-3}\int_0^x s^{2-2\alpha}\left(u(s) - f(s)\right)' ds.$$

The result is obtained by using the estimate in Lemma 3.1. □

Proof of Remark 8. We use the same notation as in the proof of Remark 1. We know that $u(x) = A\phi_1(x) + B\phi_2(x) + F(x)$ where $\phi_1(x)$ and $\phi_2(x)$ are two linearly independent solutions of the equation $-(x^{2\alpha}u'(x))' + u(x) = 0$ and

$$F(x) = 1, \text{ if } f \equiv 1,$$

or

$$F(x) = \phi_1(x)\int_0^x f(s)\phi_2(s)ds - \phi_2(x)\int_0^x f(s)\phi_1(s)ds, \text{ if } f \in C_0^\infty(0,1).$$

In either case we have $F \in C[0,1]$. We also know that

$$\lim_{x\to 0^+} |\phi_1(x)| = \infty, \ \lim_{x\to 0^+} \phi_2(x) = b_1, \ \text{ for } \alpha \geq \frac{1}{2}.$$

Therefore, if one wants a continuous function at the origin, one must have $A = 0$. Then $u(x) = B\phi_2(x)+F(x)$. We see now that the conditions $u(1) = 0$ and $\lim_{x\to 0^+} u(x) = 0$ are incompatible. □

3.2. **The Neumann Problem and the "Canonical" Problem.**

Proof of Theorems 1.4, 1.7, 1.11. For $0 < \alpha < 1$, let $u \in X_0^\alpha$ solving

$$\int_0^1 x^{2\alpha}u'(x)v'(x)dx + \int_0^1 u(x)v(x)dx = \int_0^1 f(x)v(x)dx, \text{ for all } v \in X_0^\alpha.$$

First notice that

$$\|u\|_{L^2} + \|x^\alpha u'\|_{L^2} \leq \|f\|_{L^2}.$$

Also, if we take $v \in C_0^\infty(0,1)$, then $x^{2\alpha}u' \in H^1(0,1)$ with $(x^{2\alpha}u'(x))' = u(x) - f(x)$.

We now proceed to prove that $w(x) := x^{2\alpha}u'(x)$ vanishes at $x = 0$. Take $v \in C^2[0,1]$ with $v(1) = 0$ as a test function and integrate by parts to obtain

$$0 = \int_0^1 \left(-(x^{2\alpha}u'(x))' + u(x) - f(x)\right)v(x)dx = \lim_{x\to 0^+} x^{2\alpha}u'(x)v(x).$$

The claim is obtained by taking any such v with $v(0) = 1$.

The above shows that $w(x) := x^{2\alpha}u'(x) \in H^1(0,1)$ with $w(0) = 0$. Then, notice that for any function $w \in H^1(0,1)$ with $w(0) = 0$ one can write

$$|w(x)| = \left|\int_0^x w'(x)dx\right| \leq x^{\frac{1}{2}}\left(\int_0^x w'(x)^2dx\right)^{\frac{1}{2}},$$

thus

$$\lim_{x\to 0^+} x^{2\alpha-\frac{1}{2}}u'(x) = 0.$$

Also, Hardy's inequality implies that $\frac{w}{x} \in L^2(0,1)$ with $\left\|\frac{w}{x}\right\|_{L^2} \leq 2\|w'\|_{L^2}$. Now recall that $w'(x) = (x^{2\alpha}u'(x))' = u(x) - f(x)$, so $\|w'\|_{L^2} \leq \|u\|_{L^2} + \|f\|_{L^2} \leq 2\|f\|_{L^2}$. Hence we have the estimate $\left\|x^{2\alpha-1}u'\right\|_{L^2} \leq 4\|f\|_{L^2}$.

In order to prove $\left\|x^{2\alpha}u''\right\|_{L^2} \leq C\|f\|_{L^2}$, one only need to apply the above estimates and notice that $x^{2\alpha}u''(x) = (x^{2\alpha}u'(x))' - 2\alpha x^{2\alpha-1}u'(x)$.

By Theorem A.2, property (i) of Theorems 1.4, 1.7, 1.11 is a direct consequence of the fact that $u \in X_0^{2\alpha-1}$.

Finally we establish the property (ii) of Theorem 1.11. For $\alpha = \frac{3}{4}$, first notice that

$$\int_0^1 \frac{u^2(x)}{x(1-\ln x)}dx \le -\int_0^1 x\left(\frac{2u(x)u'(x)}{x(1-\ln x)} - \frac{u^2(x)}{x^2(1-\ln x)} + \frac{u^2(x)}{x^2(1-\ln x)^2}\right)dx$$
$$= -2\int_0^1 \frac{u(x)u'(x)}{1-\ln x}dx + \int_0^1 \frac{u^2(x)}{x(1-\ln x)}dx - \int_0^1 \frac{u^2(x)}{x(1-\ln x)^2}dx,$$

thus

$$\int_0^1 \frac{u^2(x)}{x(1-\ln x)^2}dx \le 2\left|\int_0^1 \frac{u(x)}{x^{\frac{1}{2}}(1-\ln x)}x^{\frac{1}{2}}u'(x)dx\right|. \tag{28}$$

Now Holder's inequality gives $(1-\ln x)^{-1}x^{-\frac{1}{2}}u(x) \in L^2(0,1)$. Therefore

$$\left((1-\ln x)^{-1}u^2(x)\right)' = (1-\ln x)^{-2}x^{-1}u^2(x) + 2(1-\ln x)^{-1}x^{-\frac{1}{2}}u(x)x^{\frac{1}{2}}u'(x) \in L^1(0,1),$$

so $\lim_{x\to 0^+}(1-\ln x)^{-\frac{1}{2}}u(x)$ exists. If the limit is non-zero, then near the origin $(1-\ln x)^{-1}x^{-\frac{1}{2}}u(x) \sim (1-\ln x)^{\frac{1}{2}}x^{-\frac{1}{2}} \notin L^2(0,1)$, which is a contradiction. For $\frac{3}{4} < \alpha < 1$, notice that

$$x^{4\alpha-3}u^2(x) = -\int_x^1 \left(t^{4\alpha-3}u^2(t)\right)'dt = -(4\alpha-3)\int_x^1 t^{4\alpha-4}u^2(t)dt - 2\int_x^1 t^{4\alpha-3}u'(t)u(t)dt.$$

Since we know $x^{2\alpha-1}u' \in L^2(0,1)$, Theorem A.1 implies that $x^{2\alpha-2}u \in L^2(0,1)$, hence $\lim_{x\to 0^+} x^{2\alpha-\frac{3}{2}}u(x)$ exists. If the limit is non-zero, then near the origin $u(x) \sim x^{\frac{3}{2}-2\alpha} \notin L^{\frac{2}{4\alpha-3}}(0,1)$, which is a contradiction. □

Proof of Remark 10 for all $0 < \alpha < 1$. First notice that $x^{2\alpha-\frac{1}{2}}u'(x) = \frac{1}{\sqrt{x}}\int_0^x (u(s)-f(s))ds$. Therefore, $\left|x^{2\alpha-\frac{1}{2}}u'(x)\right| \le 2\|f\|_{L^2}$. i.e. $K(x) \le 2$.

On the other hand, for fixed $0 < x \le \frac{1}{2}$, define

$$f(t) = \begin{cases} x^{-\frac{1}{2}} & \text{if } 0 < t \le x \\ 0 & \text{if } x < t < 1. \end{cases}$$

Then $\|f\|_{L^2} = 1$. Consider first the case when $\frac{3}{4} < \alpha < 1$. From Theorem 1.11 we obtain that $u \in X_0^{2\alpha-1}$, which embeds into L^{p_0} for $p_0 = \frac{2}{4\alpha-3} > 2$. Thus one obtains that $\left|\frac{1}{\sqrt{x}}\int_0^x u(s)ds\right| \le x^{\frac{1}{2}-\frac{1}{p_0}}$. Then

$$K_\alpha(x) \ge \left|\frac{1}{\sqrt{x}}\int_0^x (u(s)-f(s))ds\right| \ge 1 - x^{\frac{1}{2}-\frac{1}{p_0}} \ge 1 - \left(\frac{1}{2}\right)^{\frac{1}{2}-\frac{1}{p_0}}.$$

Therefore $K_\alpha(x) \ge \delta_\alpha$ for $\delta_\alpha := 1 - \left(\frac{1}{2}\right)^{\frac{1}{2}-\frac{1}{p_0}}$. Notice that when $0 < \alpha \le \frac{3}{4}$, then $u \in L^p$ for all $p > 1$, so the above argument remains valid. The proof is now finished. □

Proof of Remark 11 for all $\alpha < \frac{3}{4}$. To prove (7), first notice that, from [3], the function h exists and $x^{\frac{1}{2}}h \in L^\infty(0,1)$. Therefore, integration by parts gives

$$\int_0^1 f(x)h(x)dx = \int_0^1 (-(x^{2\alpha}u'(x))'h(x) + u(x)h(x))dx = \lim_{x\to 0^+} u(x).$$

□

In order to prove the further regularity results we need the following

Lemma 3.2. *Let $\alpha > 0$ be a real number and $k_0 \geq 0$ be an integer. Assume $u \in W^{k_0+2,p}_{loc}(0,1)$ for some $p \geq 1$, and $\lim_{x\to 0^+} x^k \frac{d^k}{dx^k}\left(x^{2\alpha}u'(x)\right) = 0$ for all $0 \leq k \leq k_0$. Then for $0 < x < 1$*

$$\frac{d^k}{dx^k}\left(x^{2\alpha-1}u'(x)\right) = \frac{1}{x^{k+1}}\int_0^x s^k \frac{d^{k+1}}{ds^{k+1}}\left(s^{2\alpha}u'(s)\right) ds, \quad \text{for all } 0 \leq k \leq k_0.$$

Moreover

$$\left\| \frac{d^k}{dx^k}\left(x^{2\alpha-1}u'\right)\right\|_{L^p} \leq C \left\| \frac{d^{k+1}}{dx^{k+1}}\left(x^{2\alpha}u'\right)\right\|_{L^p},$$

where C is a constant depending only on p, α and k.

Proof. If $k_0 = 0$ then the statement is obvious. When $k_0 = 1$, the condition $x\left(x^{2\alpha}u'(x)\right)' \to 0$ gives

$$\begin{aligned}
\left(x^{2\alpha-1}u'(x)\right)' &= \left(\frac{1}{x}\int_0^x \left(s^{2\alpha}u'(s)\right)' ds\right)' \\
&= \left(-\frac{1}{x}\int_0^x s\left(s^{2\alpha}u'(s)\right)'' ds + \left(x^{2\alpha}u'(x)\right)'\right)' \\
&= \frac{1}{x^2}\int_0^x s\left(s^{2\alpha}u'(s)\right)'' ds.
\end{aligned}$$

The rest of the proof is a straightforward induction argument. We omit the details. The norm bound is obtained by Fubini's Theorem when $p = 1$ and by Hardy's inequality when $p > 1$. ☐

Proof of Theorem 1.6. Assume that $f \in W^{1,\frac{1}{2\alpha}}(0,1)$. First notice that for $1 \leq p < \frac{1}{2\alpha}$ we have $u' \in L^p$ since $x^{2\alpha}u' \in H^1(0,1)$. Also notice that $x(x^{2\alpha}u'(x))' = x(u-f) \to 0$ since both u and f are continuous. We use Lemma 3.2 for $k_0 = 1$ to conclude

$$\left\|(x^{2\alpha-1}u')'\right\|_{L^p} \leq C\left\|(x^{2\alpha}u')''\right\|_{L^p} = C\left\|(u-f)'\right\|_{L^p} \leq C\left\|f\right\|_{W^{1,p}},$$

where C is a constant only depending on p and α. Recall that $x^{2\alpha}u'' = u - 2\alpha x^{2\alpha-1}u' - f \in W^{1,p}(0,1)$. It implies

$$|u''(x)| = \left|x^{2\alpha}u''\right| x^{-2\alpha} \leq C\left\|f\right\|_{W^{1,p}} x^{-2\alpha},$$

where C is a constant only depending on p and α. The above inequality gives that $u \in W^{2,p}(0,1)$ for all $1 \leq p < \frac{1}{2\alpha}$, with the corresponding estimate.

Assume now $f \in W^{2,\frac{1}{2\alpha}}(0,1)$. We first notice that $x^2\left(x^{2\alpha}u'(x)\right)'' = x^2(u-f)' = x^{2\alpha}u'(x)x^{2-2\alpha} - x^2f'(x) \to 0$ as $x \to 0^+$ since $f \in C^1[0,1]$. This allows us to apply Lemma 3.2 and obtain

$$\left(x^{2\alpha-1}u'(x)\right)'' = \frac{1}{x^3}\int_0^x s^2\left(s^{2\alpha}u'(s)\right)''' ds = \frac{1}{x^3}\int_0^x s^2\left(u(s)-f(s)\right)'' ds.$$

Lemma 3.2 also gives the desired estimate. ☐

Proof of Remark 12, 15, 18. It is enough to prove the following claim: there exists $f \in C_0^\infty(0,1)$ such that the solution u can be expanded near the origin as

$$u(x) = b_1 + b_2x^{2-2\alpha} + b_3x^{4-4\alpha} + b_4x^{6-6\alpha} + \cdots \tag{29}$$

where $b_1 \neq 0$, $b_2 \neq 0$.

We use the same notation as the proof of Remark 1. Take $f \in C_0^\infty(0,1)$. We know that $u(x) = A\phi_1(x) + B\phi_2(x) + F(x)$ where $\phi_1(x)$ and $\phi_2(x)$ are two linear independent solutions of the equation $-(x^{2\alpha}u'(x))' + u(x) = 0$ and

$$F(x) = \phi_1(x)\int_0^x f(s)\phi_2(s)ds - \phi_2(x)\int_0^x f(s)\phi_1(s)ds.$$

Moreover,

$$\lim_{x\to 0^+} x^{2\alpha}\phi_1'(x) \neq 0,\ \lim_{x\to 0^+} x^{2\alpha}\phi_2'(x) = 0,\ \phi_2(1) \neq 0,\ \text{for } 0<\alpha<1.$$

Notice that $F(x) \equiv 0$ near the origin. Therefore, the boundary conditions $\lim_{x\to 0^+} x^{2\alpha}u'(x) = u(1) = 0$ imply that we have $u(x) = B\phi_2(x) + F(x)$ with $B = -\frac{F(1)}{\phi_2(1)}$. Take f such that

$$F(1) = \int_0^1 f(s)[\phi_2(s)\phi_1(1) - \phi_1(s)\phi_2(1)]ds \neq 0.$$

Then $u(x) \sim \phi_2(x)$ near the origin and we get the desired power series expansion. □

Proof of Theorem 1.9. When $k=0$ we have already established that $u \in X^0 = H^1(0,1)$. Also, we have that $xu'' \in L^2$, so $(xu)'' = (u + xu')' = 2u' + xu''$, that is $xu \in H^2(0,1)$.

When $k=1$, notice that $x\left(xu'(x)\right)' = x\left(u - f\right) \to 0$ since both f and u are in $H^1(0,1)$. we use Lemma 3.2 to write

$$u''(x) = \frac{1}{x^2}\int_0^x s\left(su'(s)\right)'' ds = \frac{1}{x^2}\int_0^x s\left(u(s) - f(s)\right)' ds.$$

We conclude that $u'' \in L^2(0,1)$ using Lemma 3.2. The rest of the proof is a straightforward induction argument using Lemma 3.2. We omit the details. □

Lemma 3.3. *Suppose $0<\alpha<1$ and let $f \in L^\infty(0,1)$. If u is the solution of (27), then $u \in C^0[0,1]$ and $x^{2\alpha-1}u' \in L^\infty(0,1)$ with*

$$\|u\|_{L^\infty} + \left\|x^{2\alpha-1}u'\right\|_{L^\infty} \leq C\,\|f\|_{L^\infty},$$

where C is a constant depending only on α.

Proof. To prove $x^{2\alpha-1}u' \in L^\infty(0,1)$, it is enough to show that $u \in L^\infty(0,1)$ with $\|u\|_{L^\infty} \leq C\,\|f\|_{L^\infty}$. Indeed, if this is the case, by (27) we obtain that $x^{2\alpha}u' \in W^{1,\infty}(0,1)$ with $\lim_{x\to 0^+} x^{2\alpha}u'(x) = 0$. By Hardy's inequality, we obtain that $\left\|x^{2\alpha-1}u'\right\|_{L^\infty} \leq C_\alpha\,\|f\|_{L^\infty}$.

Now we proceed to prove that $u \in C^0[0,1]$. First notice that if $\alpha < \frac{3}{4}$ then $u \in C^0[0,1]$ by Theorem 1.7. So we only need to study what happens when $\frac{3}{4} \leq \alpha < 1$.

Suppose $\frac{3}{4} \leq \alpha < 1$. Since $u \in X^{2\alpha-1}$ we can use Theorem A.2 to say that $u \in L^{p_0}(0,1)$ for $p_0 = \frac{2}{4\alpha-3}$, so $g := f - u \in L^{p_0}(0,1)$. From (27) we obtain that $\left(x^{2\alpha}u'(x)\right)' = g(x)$, therefore $x^{2\alpha}u' \in W^{1,p_0}(0,1)$. Now since $p_0 > 1$ and $\lim_{x\to 0^+} x^{2\alpha}u'(x) = 0$, we are allowed to use Hardy's inequality and obtain that $x^{2\alpha-1}u' \in L^{p_0}(0,1)$. Using Theorem A.2 once more gives that either $u \in C^0[0,1]$ if $\alpha < \frac{7}{8}$, in which case we are done, or $u \in L^{p_1}(0,1)$ for $p_1 := \frac{2}{8\alpha-7}$ if $\frac{7}{8} \leq \alpha < 1$. If we are in the latter case, we repeat the argument. This process stops in finite time since $\alpha < 1$, thus proving that $u \in C^0[0,1]$. □

Proof of Theorem 1.10, 1.13. We begin by recalling from Lemma 3.3 that if $f \in L^\infty(0,1)$ then $x^{2\alpha-1}u' \in L^\infty(0,1)$, so $|u'(x)| \le \left\|x^{2\alpha-1}u'(x)\right\|_{L^\infty} x^{1-2\alpha}$. This readily implies $u \in W^{1,p}(0,1)$. Now just as in the proof of Theorem 1.6 we can use Lemma 3.2 and write

$$(x^{2\alpha-1}u'(x))' = \frac{1}{x^2}\int_0^x s(s^{2\alpha}u'(s))''ds = \frac{1}{x^2}\int_0^x s(u(s)-f(s))'ds.$$

Notice that $|xu'(x)| \le \left\|x^{2\alpha-1}u'\right\|_{L^\infty} x^{2-2\alpha}$. From here we obtain

$$\left|(x^{2\alpha-1}u'(x))'\right| \le C\left(\left\|x^{2\alpha-1}u'\right\|_{L^\infty} x^{1-2\alpha} + \|f'\|_{L^p}\right).$$

The conclusion then follows by integration. ☐

Proof of Remark 16. First notice that, from the proof of (ii) of Theorem 1.11, when $\alpha = \frac{3}{4}$,

$$\left|(1-\ln x)^{-\frac{1}{2}}\, u(x)\right| \le C\left\|x^{\frac{1}{2}}u'(x)\right\|_{L^2} \le C\|f\|_{L^2},$$

and when $\frac{3}{4} < \alpha < 1$,

$$\left|x^{2\alpha-\frac{3}{2}}u(x)\right| \le C_\alpha \|x^\alpha u'(x)\|_{L^2} \le C_\alpha \|f\|_{L^2}.$$

That is, $\widetilde{K}_\alpha(x) \le C_\alpha$.

On the other hand, we can write

$$\begin{aligned} u(x) =& \int_x^1 \frac{1}{t^{2\alpha}}\int_0^t (u(s)-f(s))dsdt \\ =& \frac{1}{1-2\alpha}\left(\frac{1}{x^{2\alpha-1}}\int_0^x f(t)dt + \int_x^1 \frac{f(t)}{t^{2\alpha-1}}dt\right) \\ &+ \frac{1}{1-2\alpha}\left(\int_0^1 (u(t)-f(t))dt - \frac{1}{x^{2\alpha-1}}\int_0^x u(t)dt - \int_x^1 \frac{u(t)}{t^{2\alpha-1}}dt\right). \end{aligned}$$

When $\alpha = \frac{3}{4}$, for fixed $0 < x \le \frac{1}{2}$, take

$$f(t) = \begin{cases} 0 & \text{if } 0 < t \le x \\ t^{-\frac{1}{2}}(-\ln x)^{-\frac{1}{2}} & \text{if } x < t < 1. \end{cases}$$

Then $\|f\|_{L^2} = 1$. Since $u \in L^p(0,1)$ for all $p < \infty$, we can say that, there exists $M_\alpha > 0$ independent of x such that

$$\left|\int_0^1 (u(t)-f(t))dt - \frac{1}{x^{2\alpha-1}}\int_0^x u(t)dt - \int_x^1 \frac{u(t)}{t^{2\alpha-1}}dt\right| \le M_\alpha.$$

Then

$$\widetilde{K}_\alpha(x) \ge \frac{1}{2\alpha-1}\left(\frac{(-\ln x)^{\frac{1}{2}}}{(1-\ln x)^{\frac{1}{2}}} - \frac{M_\alpha}{(1-\ln x)^{\frac{1}{2}}}\right).$$

When $\frac{3}{4} < \alpha < 1$, for fixed $0 < x \le \frac{1}{2}$, take

$$f(t) = \begin{cases} x^{-\frac{1}{2}} & \text{if } 0 < t \le x \\ 0 & \text{if } x < t < 1. \end{cases}$$

Then $\|f\|_{L^2} = 1$. Since $u \in L^{p_0}(0,1)$ for $p_0 = \frac{2}{4\alpha-3} > 2$, we can say that, there exists $M_\alpha > 0$ and $\gamma_\alpha > 0$ such that

$$\left|x^{2\alpha-\frac{3}{2}}\int_0^1 (u(t)-f(t))dt - \frac{1}{\sqrt{x}}\int_0^x u(t)dt - x^{2\alpha-\frac{3}{2}}\int_x^1 \frac{u(t)}{t^{2\alpha-1}}dt\right| \le M_\alpha x^{\gamma_\alpha}.$$

Then

$$\widetilde{K}_\alpha(x) \geq \frac{1}{2\alpha-1}\left(1 - M_\alpha x^{\gamma_\alpha}\right).$$

Now, for $\frac{3}{4} \leq \alpha < 1$, take $\epsilon_\alpha > 0$ such that $\widetilde{K}_\alpha(x) \geq \frac{1}{4}$ for all $0 < x < \epsilon_\alpha$. If $\epsilon_\alpha < x \leq \frac{1}{2}$, we take $f(t) = -2(3-2\alpha)t + 3(4-2\alpha)t^2 + t^{3-2\alpha} - t^{4-2\alpha}$, hence $u(t) = t^{3-2\alpha} - t^{4-2\alpha}$. Notice that $0 < \|f\|_{L^2} \leq 10$, so we obtain

$$\widetilde{K}_\alpha(x) \geq \frac{x^{\frac{3}{2}} - x^{\frac{5}{2}}}{10} \geq \frac{\epsilon_\alpha^{\frac{3}{2}} - \epsilon_\alpha^{\frac{5}{2}}}{10} > 0,$$

for all $\epsilon_\alpha \leq x \leq \frac{1}{2}$. The result follows when we take $\delta_\alpha := \min\left\{\frac{1}{4}, \frac{\epsilon_\alpha^{\frac{3}{2}} - \epsilon_\alpha^{\frac{5}{2}}}{10}\right\}$. □

Proof of Theorem 1.14. Let u be the solution of (27). By definition of u, we have that $u \in L^2(0,1)$ and $x^\alpha u' \in L^2(0,1)$. As in the proof of Theorem 1.4, we have that u satisfies (1), $w(x) = x^{2\alpha}u'(x) \in H^1(0,1)$, $w(0) = 0$ and for any function v in X_0^α,

$$\lim_{x\to 0^+} x^{2\alpha}u'(x)v(x) = 0.$$

Take $v(x) = x^\alpha u'(x) - u'(1)$. Since $\alpha \geq 1$, we have

$$x^\alpha (x^\alpha u'(x))' = w'(x) - \alpha x^{\alpha-1} x^\alpha u'(x) \in L^2(0,1),$$

which means that $v \in X_0^\alpha$. Thus we obtain

$$\lim_{x\to 0^+} x^{3\alpha} {u'}^2(x) = 0.$$

To prove that $\lim_{x\to 0^+} x^{\frac{\alpha}{2}}u(x) = 0$, we first claim that $\lim_{x\to 0^+} x^{\frac{\alpha}{2}}u(x)$ exists. To do this, we write $x^\alpha u^2(x) = -\int_x^1 (s^\alpha u^2(s))'ds$. Notice that

$$(x^\alpha u^2(x))' = \alpha x^{\alpha-1}u^2(x) + 2x^\alpha u'(x)u(x) \in L^1(0,1).$$

Therefore

$$\lim_{x\to 0^+} x^\alpha u^2(x) = -\int_0^1 (s^\alpha u^2(s))'ds.$$

Now, we can conclude that $\lim_{x\to 0^+} x^{\frac{\alpha}{2}}u(x) = 0$. Otherwise, $u(x) \sim \frac{1}{x^{\frac{\alpha}{2}}} \notin L^2(0,1)$. □

Before we finish this section, we present a proposition which will be used when dealing with the spectral analysis of the operator T_α. Also, this proposition gives the postponed proof of (iii) of Theorem 1.8 and (iii) of Theorem 1.12.

Proposition 3.4. *Given $\frac{1}{2} \leq \alpha \leq 1$ and $f \in L^2(0,1)$, suppose that $u \in H^2_{loc}(0,1]$ solves*

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = f(x) \text{ in } (0,1), \\ u(1) = 0, \\ u \in L^{\frac{1}{2\alpha-1}}(0,1). \end{cases} \tag{30}$$

Then u is the weak solution obtained from (27).

Proof. We claim that $x^\alpha u' \in L^2(0,1)$. To do this, define $w(x) = x^{2\alpha}u'(x)$. Then $w \in H^1(0,1)$. If $w(0) \neq 0$, then without loss of generality one can assume that there exists $\delta > 0$ such that $0 < M_1 \leq w(x) \leq M_2$ for all $x \in [0,\delta]$. Therefore,

$$\int_x^\delta \frac{M_1}{t^{2\alpha}} dt \leq \int_x^\delta u'(t)dt \leq \int_x^\delta \frac{M_2}{t^{2\alpha}} dt, \ \forall x \in (0,\delta].$$

It implies that

$$M_1(\ln\delta - \ln x) \leq u(\delta) - u(x) \leq M_2(\ln\delta - \ln x), \ \forall x \in (0,\delta],$$

when $\alpha = \frac{1}{2}$, and

$$\frac{M_1}{2\alpha - 1}\left(\frac{1}{x^{2\alpha-1}} - \frac{1}{\delta^{2\alpha-1}}\right) \leq u(\delta) - u(x) \leq \frac{M_2}{2\alpha - 1}\left(\frac{1}{x^{2\alpha-1}} - \frac{1}{\delta^{2\alpha-1}}\right), \ \forall x \in (0,\delta],$$

when $\alpha > \frac{1}{2}$. In either situation, we reach a contradiction with $u \in L^{\frac{1}{2\alpha-1}}(0,1)$. Therefore, $w(0) = 0$, so Hardy's inequality gives

$$\|x^\alpha u'\|_2^2 = \int_0^1 \frac{w^2(x)}{x^{2\alpha}} \leq \int_0^1 \frac{w^2(x)}{x^2} < \infty.$$

Since $w \in H^1(0,1)$ satisfies $w(0) = 0$, we conclude that, in the same way as in the proof of Theorem 1.7, that $\lim_{x\to 0^+} x^{-\frac{1}{2}}w(x) = 0$. Now, integrate (30) against any test function $v \in X_0^\alpha$ on the interval $[\epsilon, 1]$ and obtain

$$\int_\epsilon^1 x^{2\alpha}u'(x)v'(x)dx + \epsilon^{2\alpha}u'(\epsilon)v(\epsilon) + \int_\epsilon^1 u(x)v(x)dx = \int_\epsilon^1 f(x)v(x)dx.$$

Since $\frac{1}{2} \leq \alpha \leq 1$, we write

$$\epsilon^{2\alpha}u'(\epsilon)v(\epsilon) = \left[\epsilon^{2\alpha-\frac{1}{2}}w(\epsilon)\right]\left[\epsilon^{\frac{1}{2}}v(\epsilon)\right].$$

The estimate (47) tells us that $\left|x^{\frac{1}{2}}v(x)\right| \leq C_\alpha \|v\|_\alpha$, so we can send $\epsilon \to 0^+$ and obtain (27) as desired. ☐

4. Analysis of the Spectrum

4.1. The Operator T_α.

In this section we study the spectrum of the operator T_α. We divide this section into three parts. In subsection 4.1.1 we study the eigenvalue problem of T_α for all $\alpha > 0$. In subsection 4.1.2 we explore the rest of the spectrum of T_α for the non-compact case $\alpha \geq 1$. Finally, in subsection 4.1.3, we give the proof of Theorem 1.19.

4.1.1. *The Eigenvalue Problem for all* $\alpha > 0$.

In this subsection, we focus on finding the eigenvalues and eigenfunctions of T_α. That is, we seek $(u,\lambda) \in L^2(0,1) \times \mathbb{R}$ such that $u \neq 0$ and $T_\alpha u = \lambda u$. By definition of T_α in Section 1.6, we have $\lambda \neq 0$ and the pair (u,λ) satisfies

$$\int_0^1 x^{2\alpha}u'(x)v'(x)dx + \int_0^1 u(x)v(x)dx = \frac{1}{\lambda}\int_0^1 u(x)v(x)dx, \ \forall v \in X_0^\alpha. \tag{31}$$

From here we see right away that if $\lambda > 1$ or $\lambda < 0$, then Lax-Milgram Theorem applies and equation (31) has only the trivial solution. Also, a direct computation shows that $u \equiv 0$ is the only solution when $\lambda = 1$. This implies that all the eigenvalues belong to the interval $(0,1)$. So we will analyze (31) only for $0 < \lambda < 1$.

As the existence and uniqueness results show, it amounts to study the following ODE for $\mu := \frac{1}{\lambda} > 1$,

$$-(x^{2\alpha}u'(x))' + u(x) = \mu u(x) \quad \text{on } (0,1), \tag{32}$$

under certain boundary behaviors. To solve (32), we will use Bessel's equation

$$y^2 f''(y) + y f'(y) + (y^2 - \nu^2) f(y) = 0 \quad \text{on } (0,\infty). \tag{33}$$

Indeed, we have the following

Lemma 4.1. *For $\alpha \neq 1$ and any $\beta > 0$, let f_ν be any solution of (33) with parameter $\nu^2 = \left(\frac{\alpha-\frac{1}{2}}{\alpha-1}\right)^2$ and define $u(x) = x^{\frac{1}{2}-\alpha} f_\nu(\beta x^{1-\alpha})$. Then u solves*

$$-(x^{2\alpha}u'(x))' = \beta^2(\alpha-1)^2 u(x).$$

Proof. Notice that by definition $u'(x) = (\frac{1}{2}-\alpha)x^{-\frac{1}{2}-\alpha} f_\nu(\beta x^{1-\alpha}) + \beta(1-\alpha)x^{\frac{1}{2}-2\alpha} f_\nu'(\beta x^{1-\alpha})$, and thus $x^{2\alpha}u'(x) = (\frac{1}{2}-\alpha)x^{-\frac{1}{2}+\alpha} f_\nu(\beta x^{1-\alpha}) + \beta(1-\alpha)x^{\frac{1}{2}} f_\nu'(\beta x^{1-\alpha})$. A direct computation shows that

$$(x^{2\alpha}u'(x))' = -\left(\alpha-\frac{1}{2}\right)^2 x^{\alpha-\frac{3}{2}} f_\nu(\beta x^{1-\alpha}) + \beta(\alpha-1)^2 x^{-\frac{1}{2}} f_\nu'(\beta x^{1-\alpha}) + \beta^2(\alpha-1)^2 x^{\frac{1}{2}-\alpha} f_\nu''(\beta x^{1-\alpha}).$$

Using (33) evaluated at $y = \beta x^{1-\alpha}$ gives

$$\beta^2 x^{2(1-\alpha)} f_\nu''(\beta x^{1-\alpha}) + \beta x^{1-\alpha} f_\nu'(\beta x^{1-\alpha}) = (\nu^2 - \beta^2 x^{2(1-\alpha)}) f_\nu(\beta x^{1-\alpha}). \tag{34}$$

Multiply (34) by $(\alpha-1)^2 x^{\alpha-\frac{3}{2}}$ and obtain

$$\beta^2(\alpha-1)^2 x^{\frac{1}{2}-\alpha} f_\nu''(\beta x^{1-\alpha}) + \beta(\alpha-1)^2 x^{-\frac{1}{2}} f_\nu'(\beta x^{1-\alpha}) = (\nu^2(\alpha-1)^2 x^{\alpha-\frac{3}{2}} - \beta^2(\alpha-1)^2 x^{\frac{1}{2}-\alpha}) f_\nu(\beta x^{1-\alpha}).$$

Thus we obtain, by our choice of ν,

$$\begin{aligned}
(x^{2\alpha}u'(x))' &= -\left(\alpha-\frac{1}{2}\right)^2 x^{\alpha-\frac{3}{2}} f_\nu(\beta x^{1-\alpha}) + (\nu^2(\alpha-1)^2 x^{\alpha-\frac{3}{2}} - \beta^2(\alpha-1)^2 x^{\frac{1}{2}-\alpha}) f_\nu(\beta x^{1-\alpha}) \\
&= \left(-\left(\alpha-\frac{1}{2}\right)^2 + \nu^2(\alpha-1)^2\right) x^{\alpha-\frac{3}{2}} f_\nu(\beta x^{1-\alpha}) - \beta^2(\alpha-1)^2 x^{\frac{1}{2}-\alpha} f_\nu(\beta x^{1-\alpha}) \\
&= -\beta^2(\alpha-1)^2 x^{\frac{1}{2}-\alpha} f_\nu(\beta x^{1-\alpha}) \\
&= -\beta^2(\alpha-1)^2 u(x).
\end{aligned}$$

The proof is now completed. □

We will need a few known facts about Bessel functions, which we summarize in the following Lemmas (for the proofs see e.g. Chapter III of [11]).

Lemma 4.2. *For non-integer ν, the general solution to equation (33) can be written as*

$$f_\nu(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x). \tag{35}$$

The function $J_\nu(x)$ is called the Bessel function of the first kind of order ν. This function has the following power series expansion

$$J_\nu(x) = \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)}\left(\frac{x}{2}\right)^{2m+\nu}.$$

A similar expression can be obtained for $J_\nu'(x)$ by differentiating $J_\nu(x)$.

Lemma 4.3. *For non-negative integer ν, the general solution to equation* (33) *can be written as*

$$f_\nu(x) = C_1 J_\nu(x) + C_2 Y_\nu(x). \tag{36}$$

The function $J_\nu(x)$ is the same as the one from Lemma 4.2, and the function $Y_\nu(x)$ is called the Bessel function of second kind which satisfies the following asymptotics: for $0 < x << 1$,

$$Y_\nu(x) \sim \begin{cases} \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + \gamma\right] & \text{if } \nu = 0, \\ -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu & \text{if } \nu > 0, \end{cases}$$

where $\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{k} - \ln(n)\right)$ is Euler's constant.

Remark 23. We have been using the notation $f(x) \sim g(x)$. This notation means that there exists constants $c_1, c_2 > 0$ such that

$$c_1\,|g(x)| \le |f(x)| \le c_2\,|g(x)|\,.$$

Remark 24. Suppose that $\alpha \neq 1$, and let $\beta = \frac{\sqrt{\mu-1}}{|\alpha-1|}$. Then Lemma 4.1, 4.2 and 4.3 guarantee that the general solution of (32) is given by

$$u(x) = \begin{cases} C_1 x^{\frac{1}{2}-\alpha} J_\nu(\beta x^{1-\alpha}) + C_2 x^{\frac{1}{2}-\alpha} J_{-\nu}(\beta x^{1-\alpha}) & \text{if } \nu \text{ is not an integer}, \\ C_1 x^{\frac{1}{2}-\alpha} J_\nu(\beta x^{1-\alpha}) + C_2 x^{\frac{1}{2}-\alpha} Y_\nu(\beta x^{1-\alpha}) & \text{if } \nu \text{ is an non-negative integer}. \end{cases} \tag{37}$$

Now the problem has been reduced to select the eigenfunctions from the above family.

We first study the eigenvalue problem for the compact case $0 < \alpha < 1$.

Proof of (i) of Theorem 1.17. We first consider the case when $0 < \alpha < \frac{1}{2}$. In this case notice that $\nu = \frac{\alpha-\frac{1}{2}}{1-\alpha}$ is negative and non-integer. From theorems 1.4 and 1.5, and equations (31), (32) and (37), we have that the eigenfunction is of the form

$$u(x) = C_1 x^{\frac{1}{2}-\alpha} J_\nu(\beta x^{1-\alpha}) + C_2 x^{\frac{1}{2}-\alpha} J_{-\nu}(\beta x^{1-\alpha})$$

with $\beta = \frac{\sqrt{\mu-1}}{|\alpha-1|}$, $\lim_{x\to 0^+} x^{2\alpha}u'(x) = 0$ and $u(1) = 0$. Then Lemma 4.2 gives that $x^{2\alpha}u'(x) \sim C_2 \frac{\beta^{-\nu}\left(\frac{1}{2}-\alpha\right)}{2^{-\nu}\Gamma(-\nu+1)}$. so the boundary condition $\lim_{x\to 0^+} x^{2\alpha}u'(x) = 0$ forces C_2 to vanish. Therefore $u(x) = C_1 x^{\frac{1}{2}-\alpha} J_\nu(\beta x^{1-\alpha})$. Now, the condition $u(1) = 0$ forces β to satisfy $J_\nu(\beta) = 0$, that is β must be a positive root of the the Bessel function J_ν, for $\nu = \frac{\alpha-\frac{1}{2}}{1-\alpha}$.

Therefore, we conclude that if we let $j_{\nu k}$ be the k-th positive root of $J_\nu(x)$, then

$$u_{\nu k}(x) = x^{\frac{1}{2}-\alpha} J_\nu(j_{\nu k} x^{1-\alpha}),\ k = 1, 2, \cdots$$

are the eigenfunctions and the corresponding eigenvalues are given by

$$\lambda_{\nu k} = \frac{1}{1 + (1-\alpha)^2 j_{\nu k}^2},\ k = 1, 2, \cdots.$$

Next, we investigate the case when $\frac{1}{2} \le \alpha < 1$. In this case, $\nu = \frac{\alpha-\frac{1}{2}}{1-\alpha}$ is non-negative and could be integer or non-integer. Using Lemma 4.2 and 4.3, we obtain the asymptotics of the general solution near the origin,

$$u(x) \sim \begin{cases} \frac{C_1\beta^\nu}{\Gamma(\nu+1)2^\nu} + \frac{C_2 2^\nu}{\beta^\nu\Gamma(1-\nu)} x^{1-2\alpha} & \text{if } \alpha > \frac{1}{2}, \text{ and } \nu \text{ is not an integer}, \\ \frac{C_1\beta^\nu}{\Gamma(\nu+1)2^\nu} - \frac{2^\nu\Gamma(\nu)C_2}{\beta^\nu\pi} x^{1-2\alpha} & \text{if } \alpha > \frac{1}{2}, \text{ and } \nu \text{ is an integer}, \\ \frac{C_1\beta^\nu}{\Gamma(\nu+1)2^\nu} + \frac{2C_2}{\pi}\left[\ln(\beta\sqrt{x}) + \gamma\right] & \text{if } \alpha = \frac{1}{2}. \end{cases}$$

Now Proposition 3.4 says that it is enough to impose $u \in L^{\frac{1}{2\alpha-1}}(0,1)$ which forces $C_2 = 0$ and $u(x) = C_1 x^{\frac{1}{2}-\alpha} J_\nu(\beta x^{1-\alpha})$. Moreover, the condition $u(1) = 0$ forces β to satisfy $J_\nu(\beta) = 0$, that is β must be a positive root of the Bessel function J_ν, for $\nu = \frac{\alpha-\frac{1}{2}}{1-\alpha}$.

As before we conclude that

$$u_{\nu k}(x) = x^{\frac{1}{2}-\alpha} J_\nu(j_{\nu k} x^{1-\alpha}),\ k = 1, 2, \cdots$$

are the eigenfunctions and the corresponding eigenvalues are given by

$$\lambda_{\nu k} = \frac{1}{1 + (1-\alpha)^2 j_{\nu k}^2},\ k = 1, 2, \cdots.$$

Finally, the asymptotic behavior of $j_{\nu k}$ as $k \to \infty$ is well understood (see e.g. Chapter XV of [11]). We have

$$j_{\nu k} = k\pi + \frac{\pi}{2}\left(\nu - \frac{1}{2}\right) - \frac{4\nu^2 - 1}{8\left(k\pi + \frac{\pi}{2}\left(\nu - \frac{1}{2}\right)\right)} + O\left(\frac{1}{k^3}\right). \tag{38}$$

Using (38), we obtain that

$$\mu_{\nu k} = 1 + (1-\alpha)^2 \left[\left(\frac{\pi}{2}\left(\nu - \frac{1}{2}\right) + \pi k\right)^2 - \left(\nu^2 - \frac{1}{4}\right)\right] + O\left(\frac{1}{k}\right).$$

□

Next we consider the case $\alpha = 1$. In this case, the equation (37) is not the general solution for (32). However, as the reader can easily verify, the general solution for (32) when $\alpha = 1$ is given by

$$u(x) = \begin{cases} C_1 x^{-\frac{1}{2}+\sqrt{\frac{5}{4}-\mu}} + C_2 x^{-\frac{1}{2}-\sqrt{\frac{5}{4}-\mu}} & \text{for } \mu < \frac{5}{4}, \\ C_1 x^{-\frac{1}{2}} + C_2 x^{-\frac{1}{2}} \ln x & \text{for } \mu = \frac{5}{4}, \\ C_1 x^{-\frac{1}{2}} \cos\left(\sqrt{\mu - \frac{5}{4}} \ln x\right) + C_2 x^{-\frac{1}{2}} \sin\left(\sqrt{\mu - \frac{5}{4}} \ln x\right) & \text{for } \mu > \frac{5}{4}. \end{cases} \tag{39}$$

With equation (39) in our hands, we can prove the following:

Proposition 4.4. *If $\alpha = 1$, then T_α has no eigenvalues.*

Proof. For the general solution given by (39), we impose $u(1) = 0$, and obtain that any non-trivial solution has the form:

$$u(x) = \begin{cases} C x^{-\frac{1}{2}+\sqrt{\frac{5}{4}-\mu}} \left(1 - x^{-2\sqrt{\frac{5}{4}-\mu}}\right) & \text{for } \mu < \frac{5}{4}, \\ C x^{-\frac{1}{2}} \ln x & \text{for } \mu = \frac{5}{4}, \\ C x^{-\frac{1}{2}} \sin\left(\sqrt{\mu - \frac{5}{4}} \ln x\right) & \text{for } \mu > \frac{5}{4}, \end{cases}$$

for some $C \neq 0$. From here we see right away that if $\mu \geq \frac{5}{4}$ then $u \notin L^2(0,1)$. And when $\mu < \frac{5}{4}$, we obtain that

$$\int_0^1 u^2(x)dx = C^2 \int_0^1 x^{-1+2\sqrt{\frac{5}{4}-\mu}} \left(1 - x^{-2\sqrt{\frac{5}{4}-\mu}}\right)^2 dx.$$

Let $y = x^{2\sqrt{\frac{5}{4}-\mu}}$, so this integral becomes

$$\int_0^1 u^2(x)dx = C^2 \int_0^1 \left(1 - \frac{1}{y}\right)^2 dy \geq \frac{C^2}{4} \int_0^{\frac{1}{2}} \frac{1}{y^2} dy = +\infty.$$

This says that when $\alpha = 1$, there are no eigenvalues and eigenfunctions. □

Finally we investigate the case $\alpha > 1$. To investigate the eigenvalue problem in this case, we need the following fact about the Bessel's equation.

Lemma 4.5. *Assume that $f_\nu(t)$ is a non-trivial solution of Bessel's equation*

$$t^2 f_\nu''(t) + t f_\nu'(t) + (t^2 - \nu^2) f_\nu(t) = 0. \tag{40}$$

Then $\int_s^\infty t f_\nu^2(t)dt = \infty$, $\forall s > 0, \forall \nu > 0$.

Proof. We first define the function $g_\nu(t) = f_\nu(bt)$, for some $b \neq 1$. Then $g_\nu(t)$ satisfies the ODE

$$t^2 g_\nu''(t) + t g_\nu'(t) + (b^2 t^2 - \nu^2) g_\nu(t) = 0. \tag{41}$$

From equation (40) and (41), we have

$$t^2(f_\nu''(t)g_\nu(t) - f_\nu(t)g_\nu''(t)) + t(f_\nu'(t)g_\nu(t) - f_\nu(t)g_\nu'(t)) + t^2(1-b^2)f_\nu(t)g_\nu(t) = 0,$$

or

$$t(f_\nu''(t)g_\nu(t) - f_\nu(t)g_\nu''(t)) + (f_\nu'(t)g_\nu(t) - f_\nu(t)g_\nu'(t)) + t(1-b^2)f_\nu(t)g_\nu(t) = 0,$$

i.e.

$$\frac{d}{dt}\left[t(f_\nu'(t)g_\nu(t) - f_\nu(t)g_\nu'(t))\right] + t(1-b^2)f_\nu(t)g_\nu(t) = 0.$$

Integrating the above equation we obtain

$$\begin{aligned}\int_s^N t f_\nu(t) g_\nu(t) dt &= \frac{N(f_\nu'(N)g_\nu(N) - f_\nu(N)g_\nu'(N))}{b^2-1} - \frac{s(f_\nu'(s)g_\nu(s) - f_\nu(s)g_\nu'(s))}{b^2-1} \\ &= \frac{N f_\nu'(N) f_\nu(bN) - bN f_\nu(N) f_\nu'(bN)}{b^2-1} - \frac{s f_\nu'(s) f_\nu(bs) - bs f_\nu(s) f_\nu'(bs)}{b^2-1} \\ &\triangleq A - B.\end{aligned}$$

We then pass the limit as $b \to 1$. Notice that

$$\begin{aligned}\lim_{b\to1} A &= \lim_{b\to1} \frac{N f_\nu'(N) f_\nu(bN) - bN f_\nu(N) f_\nu'(bN)}{b^2-1} \\ &= \lim_{b\to1} \frac{N^2 f_\nu'(N) f_\nu'(bN) - N f_\nu(N) f_\nu'(bN) - bN^2 f_\nu(N) f_\nu''(bN)}{2b} \\ &= \frac{N^2 f_\nu'(N) f_\nu'(N) - N f_\nu(N) f_\nu'(N) - N^2 f_\nu(N) f_\nu''(N)}{2} \\ &= \frac{1}{2}\left(N^2 f_\nu'^2(N) + N^2 f_\nu^2(N) - \nu^2 f_\nu^2(N)\right),\end{aligned}$$

and

$$\begin{aligned}\lim_{b\to1} B &= \lim_{b\to1} \frac{s f_\nu'(s) f_\nu(bs) - bs f_\nu(s) f_\nu'(bs)}{b^2-1} \\ &= \frac{1}{2}\left(s^2 f_\nu'^2(s) + s^2 f_\nu^2(s) - \nu^2 f_\nu^2(s)\right).\end{aligned}$$

Therefore

$$\int_s^N t f_\nu^2(t) dt = \frac{1}{2}\left(N^2 f_\nu'^2(N) + N^2 f_\nu^2(N) - \nu^2 f_\nu^2(N)\right) - \frac{1}{2}\left(s^2 f_\nu'^2(s) + s^2 f_\nu^2(s) - \nu^2 f_\nu^2(s)\right).$$

Sending $N \to \infty$, we deduce from the asymptotic behavior of the Bessel's function that $\int_s^\infty t f_\nu^2(t)dt = \infty$. □

Proposition 4.6. *If $\alpha > 1$, then T_α has no eigenvalues.*

Proof. We argue by contradiction. Suppose $\lambda = \frac{1}{\mu}$ is an eigenvalue and $u \in L^2(0,1)$ is the corresponding eigenfunction, then $\mu > 1$ and the pair (u, λ) satisfies (32). Lemma 4.1 says that $u(x) = x^{\frac{1}{2}-\alpha} f_\nu(\beta x^{1-\alpha})$ where $\beta = \frac{\sqrt{\mu-1}}{\alpha-1}$ and $f_\nu(t)$ is a non-trivial solution of

$$t^2 f_\nu''(t) + t f_\nu'(t) + (t^2 - \nu^2) f_\nu(t) = 0.$$

Applying the change of variable $\beta x^{1-\alpha} = t$ and Lemma 4.5 gives

$$\begin{aligned}\int_0^1 u^2(x)dx &= \int_0^1 x^{1-2\alpha} f_\nu^2(\beta x^{1-\alpha})dx \\ &= \frac{1}{\beta(\alpha-1)} \int_\beta^\infty \left(\frac{t}{\beta}\right)^{\frac{1-2\alpha}{1-\alpha} + \frac{1}{1-\alpha} - 1} f_\nu^2(t)dt \\ &= \frac{1}{\beta^2(\alpha-1)} \int_\beta^\infty t f_\nu^2(t)dt = \infty,\end{aligned}$$

which is a contradiction. □

4.1.2. ***The Rest of the Spectrum for the Case*** $\alpha \geq 1$.

We have found the eigenvalues of T_α for all $\alpha > 0$. Next we study the rest of the spectrum for the non-compact case $\alpha \geq 1$. It amounts to study the surjectivity of the operator $T_\alpha - \lambda I$ in $L^2(0,1)$, that is, given $f \in L^2(0,1)$, we want determine whether there exists $h \in L^2(0,1)$ such that $(T - \lambda)h = f$. Since $\|T_\alpha\| \leq 1$, T_α is a positive operator, and T_α is not surjective, we can assume that $0 < \lambda \leq 1$. By letting $u = \lambda h + f$, the existence of the function $h \in L^2(0,1)$ is equivalent to the existence of the function $u \in L^2(0,1)$ satisfying

$$T_\alpha \left(\frac{u - f}{\lambda}\right) = u.$$

By the definition of T_α in Section 1.6, the above equation can be written as

$$\int_0^1 \left(x^{2\alpha} u'(x)v'(x) + \left(1 - \frac{1}{\lambda}\right) u(x)v(x)\right) dx = -\frac{1}{\lambda} \int_0^1 f(x)v(x)dx, \ \forall v \in X_0^\alpha. \tag{42}$$

Since we proved that there are no eigenvalues when $\alpha \geq 1$, a real number λ is in the spectrum of the operator T_α if and only if there exists a function $f \in L^2(0,1)$ such that (42) is not solvable. To study the solvability of (42) we introduce the following bilinear form,

$$a_\alpha(u,v) \triangleq \int_0^1 x^{2\alpha} u'(x)v'(x)dx + \left(1 - \frac{1}{\lambda}\right) \int_0^1 u(x)v(x)dx, \tag{43}$$

and we first study the coercivity of $a_1(u,v)$.

Lemma 4.7. *If* $\lambda > \frac{4}{5}$*, then* $a_1(u,v)$ *is coercive in* X_0^1.

Proof. We use Theorem A.1 and obtain

$$\begin{aligned}
a_1(u,u) &= \int_0^1 (xu'(x))^2dx - \left(\frac{1}{\lambda}-1\right)\int_0^1 u^2(x)dx \\
&\geq \int_0^1 (xu'(x))^2dx - 4\left(\frac{1}{\lambda}-1\right)\int_0^1 (xu'(x))^2 \\
&= \left(1-4\left(\frac{1}{\lambda}-1\right)\right)\int_0^1 (xu'(x))^2dx \\
&\geq \frac{1}{5}\left(1-4\left(\frac{1}{\lambda}-1\right)\right)\|u\|^2_{X_0^1}.
\end{aligned}$$

Thus if $\lambda > \frac{4}{5}$, this bilinear form is coercive. □

Now we can prove the next

Proposition 4.8. *For $\alpha = 1$, the spectrum of the operator T_1 is exactly $\sigma(T_1) = \left[0, \frac{4}{5}\right]$.*

Proof. The coercivity of $a_1(u,v)$ gives immediately that $\sigma(T_1) \subset \left[0, \frac{4}{5}\right]$. To prove the reverse inclusion, we first claim that $(T_1 - \lambda)u = -\lambda$ is not solvable when $0 < \lambda \leq \frac{4}{5}$. Otherwise, by equation (42), there would exist $\mu = \frac{1}{\lambda}$ and $u \in L^2(0,1)$ such that

$$\begin{cases} -(x^2u'(x))' + (1-\mu)u(x) = 1, \\ \qquad\qquad\qquad\qquad u(1) = 0. \end{cases} \tag{44}$$

Equation (44) can be solved explicitly as

$$u(x) = \begin{cases} x^{-\frac{1}{2}}\left[C - \left(C + \frac{1}{1-\mu}\right)\ln x\right] + \frac{1}{1-\mu} & \text{for } \mu = \frac{5}{4}, \\ C_\mu x^{-\frac{1}{2}} \sin\left(A_\mu + \sqrt{\mu - \frac{5}{4}}\ln x\right) + \frac{1}{1-\mu} & \text{for } \mu > \frac{5}{4}, \end{cases}$$

where $C_\mu = \frac{C^2 + \frac{1}{(1-\mu)^2}}{\sqrt{\mu - \frac{5}{4}}}$, $\sin A_\mu = \frac{C}{C^2 + \frac{1}{(1-\mu)^2}}$ and C could be any real number. So we have that

$$\left\| u(x) - \frac{1}{1-\mu}\right\|^2_{L^2(0,1)} = \begin{cases} \int_{-\infty}^0 \left(C - \left(C + \frac{1}{1-\mu}\right)y\right)^2 dy & \text{for } \mu = \frac{5}{4}, \\ C_\mu \int_{-\infty}^0 \sin^2(A_\mu + y)\,dy & \text{for } \mu > \frac{5}{4}. \end{cases}$$

Notice that the right hand side above is $+\infty$ independently of C, thus proving that $u \notin L^2(0,1)$. Therefore $(T_1 - \lambda)h = -\lambda$ is not solvable in $L^2(0,1)$ for $0 < \lambda \leq \frac{4}{5}$. Also $0 \in \sigma(T_1)$, because T_1 is not surjective. This gives $\left[0, \frac{4}{5}\right] \subset \sigma(T_1)$ as claimed. □

Proposition 4.9. *For $\alpha > 1$, the spectrum of the operator T_α is exactly $\sigma(T_\alpha) = [0,1]$.*

Proof. As we already know, $\sigma(T_\alpha) \subset [0,1]$. So let us prove the converse. We first claim that the equation $(T_\alpha - \lambda)u = -\lambda$ is not solvable for $0 < \lambda < 1$. As before, this amounts to solve

$$-(x^{2\alpha}u'(x))' + (1-\mu)u(x) = 1,$$

where $\mu = \frac{1}{\lambda}$. Lemma 4.1 implies that $u(x) = x^{\frac{1}{2}-\alpha} f_\nu(\beta x^{1-\alpha}) + 1$ where $\beta = \frac{\sqrt{\mu-1}}{\alpha-1}$ and $f_\nu(t)$ is a non-trivial solution of

$$t^2 f_\nu''(t) + t f_\nu'(t) + (t^2 - \nu^2) f_\nu(t) = 0.$$

By Lemma 4.5 we conclude that $\|u\|_2 = \infty$. So $(T_\alpha - \lambda)h = -\lambda$ is not solvable when $\lambda \in (0,1)$.

When $\lambda = 1$, take $f(x) = -\lambda x^{\epsilon-\frac{1}{2}}$, where $\epsilon > 0$ is to be determined, and try to solve $(T_\alpha - I)u = f$, which is equivalent to solve

$$\begin{cases} -(x^{2\alpha}u'(x))' = x^{\epsilon-\frac{1}{2}}, \\ u(1) = 0. \end{cases}$$

The general solution of this ODE is given by

$$u(x) = \frac{1}{(\frac{1}{2}+\epsilon)(\frac{3}{2}+\epsilon-2\alpha)} x^{\frac{3}{2}+\epsilon-2\alpha} + Cx^{-2\alpha+1} - C - \frac{1}{(\frac{1}{2}+\epsilon)(\frac{3}{2}+\epsilon-2\alpha)}.$$

We choose $0 < \epsilon < 2\alpha - 2$ so that $\frac{3}{2} + \epsilon - 2\alpha < -\frac{1}{2}$. Therefore, $\|u\|_2 = \infty$ independently of C, thus $(T_\alpha - I)u = f$ is not solvable. Hence $(0, 1] \subset \sigma(T_\alpha)$. Also $0 \in \sigma(T_\alpha)$; thus the result is proved. ☐

Proof of Corollary 1.18. To prove (i), it is enough to notice that when $0 < \alpha < 1$ the operator T_α is compact and $R(T_\alpha)$ is not closed.

To prove (ii) and (iii), by the definition of essential spectrum and the fact that T_α has no eigenvalue when $\alpha \geq 1$, it is enough to show that $\sigma_d(T_\alpha) \subset EV(T_\alpha)$, where $EV(T_\alpha)$ is the set of the eigenvalues. Actually, for $\lambda \in \sigma_d(T_\alpha)$, we claim that $\dim N(T_\alpha - \lambda I) \neq 0$. Suppose the contrary, then $\dim N(T_\alpha - \lambda I) = 0$, and one obtains that

$$R(T_\alpha - \lambda I)^\perp = N(T_\alpha^* - \lambda I) = N(T_\alpha - \lambda I) = \{0\}.$$

Since $T_\alpha - \lambda I$ is Fredholm, it means that $R(T_\alpha - \lambda I)$ is closed and therefore $R(T_\alpha - \lambda I) = L^2(0, 1)$. That leads to the bijectivity of $T_\alpha - \lambda I$, which contradicts with $\lambda \in \sigma_d(T_\alpha)$. ☐

4.1.3. ***The proof of Theorem 1.19***.

Proof. To prove (i), it is equivalent to prove that $\mu_{\nu k} \geq \frac{5}{4}$ for all $k = 1, 2, \dots$ and $\nu > \frac{1}{2}$. Indeed, since $\nu > \frac{1}{2}$, we have the following inequality (see [6]) for all $k = 1, 2, \dots$

$$j_{\nu k} > \nu + \frac{k\pi}{2} - \frac{1}{2} \geq \nu + \frac{\pi - 1}{2},$$

so

$$(1-\alpha)j_{\nu k} = \frac{1}{2(\nu+1)} j_{\nu k} \geq \frac{1}{2} + \frac{\pi - 3}{4(\nu+1)} \geq \frac{1}{2}.$$

Thus $\mu_{\nu k} = 1 + (1-\alpha)^2 j_{\nu k}^2 \geq \frac{5}{4}$.

To prove (ii), from [6] we obtain that for fixed $x > 0$, we have

$$\lim_{\nu\to\infty} \frac{j_{\nu,\nu x}}{\nu} = i(x), \tag{45}$$

where $i(x) := \sec\theta$ and θ is the unique solution in $\left(0, \frac{\pi}{2}\right)$ of $\tan\theta - \theta = \pi x$. Using this fact, and the definition of ν, we can write

$$\mu_{\nu k} = 1 + (1-\alpha)^2 j_{\nu k}^2 = 1 + \left(\alpha - \frac{1}{2}\right)^2 \left(\frac{j_{\nu k}}{\nu}\right)^2.$$

Define $\nu_k = \frac{k}{x}$ (or equivalently, $\alpha_k = 1 - \frac{1}{2\left(\frac{k}{x}+1\right)}$), then (45) implies that

$$\mu_m := \mu_{\nu_m m} = 1 + \left(\alpha_m - \frac{1}{2}\right)^2 i^2(x)\left(1 + o(1)\right),$$

where $o(1)$ is a quantity that goes to 0 as $m \to \infty$. So for fixed $x > 0$ we find that (notice that $m \to \infty$ implies $\nu_m \to \infty$, which necessarily implies that $\alpha_m \to 1^-$)

$$\lambda_m := \frac{1}{\mu_m} \to \frac{1}{1 + \frac{1}{4} i^2(x)} =: \lambda(x).$$

It is clear from the definition of $i(x)$, that $i(x)$ is injective and that $i((0, +\infty)) = (1, +\infty)$, which gives that $\lambda(x)$ is injective and $\lambda((0, +\infty)) = \left(0, \frac{4}{5}\right)$. So we only need to take care of the endpoints, that is 0 and $\frac{4}{5}$. Firstly, consider $j_{\nu 1}$, the first root of $J_\nu(x)$. It is known that (see e.g. Chapter XV of [11])

$$j_{\nu 1} = \nu + O(\nu^{\frac{1}{3}}) \text{ as } \nu \to \infty.$$

Consider $\mu_m = \mu_{m1} = 1 + \left(\alpha_m - \frac{1}{2}\right)^2 (1 + o(1))$, where $\alpha_m = 1 - \frac{1}{2(m+1)}$, and $o(1)$ goes to 0 as $m \to \infty$. This implies that

$$\lambda_m \to \frac{4}{5} \text{ as } \alpha_m \to 1^-.$$

To conclude the proof of (ii), recall that T_α is compact for all $\alpha < 1$ so $0 \in \sigma(T_\alpha)$. □

Proof of Remark 21. Notice that part (i) in Theorem 1.19 gives $\sup_{x \in \sigma(T_\alpha)} \inf_{y \in \sigma(T_1)} |x - y| = 0$ for all $\frac{2}{3} < \alpha < 1$. Therefore, it is enough to prove

$$\lim_{\alpha \to 1^-} \sup_{x \in \sigma(T_1)} \inf_{y \in \sigma(T_\alpha)} |x - y| = 0.$$

Indeed, the compactness of $\sigma(T_1)$ implies that, for any $\epsilon > 0$, there exists $\{x_i\}_{i=1}^n \in \sigma(T_1)$ such that

$$\sup_{x \in \sigma(T_1)} \inf_{y \in \sigma(T_\alpha)} |x - y| \le \max_{i=1,\ldots,n} d(x_i, \sigma(T_\alpha)) + \frac{\epsilon}{2}.$$

Then part (ii) in Theorem 1.19 gives the existence of $\alpha_\epsilon < 1$ such that $d(x_i, \sigma(T_\alpha)) \le \frac{\epsilon}{2}$ for all $\alpha_\epsilon < \alpha < 1$ and all $i = 1, \ldots, n$. □

4.2. The operator T_D.

Proof of Theorem 1.16. In order to find all the eigenvalues and eigenfunctions, we need the nontrivial solutions of

$$\begin{cases} -(x^{2\alpha} u'(x))' + u(x) = \mu u(x) \text{ on } (0, 1), \\ u(0) = u(1) = 0. \end{cases}$$

Let $\nu_0 = \frac{\frac{1}{2} - \alpha}{1 - \alpha}$, which is positive and never an integer. Equation (37) gives us its general solution

$$u(x) = C_1 x^{\frac{1}{2} - \alpha} J_{\nu_0}(\beta x^{1-\alpha}) + C_2 x^{\frac{1}{2} - \alpha} J_{-\nu_0}(\beta x^{1-\alpha}),$$

where $\beta = \frac{\sqrt{\mu - 1}}{|\alpha - 1|}$. The asymptotic of J_{ν_0} when $0 < x << 1$ yields

$$u(x) \sim \frac{C_1 k^{\nu_0}}{\Gamma(\nu_0 + 1) 2^{\nu_0}} x^{1 - 2\alpha} + \frac{C_2 2^{\nu_0}}{k^{\nu_0} \Gamma(1 - \nu_0)},$$

so imposing $u(0) = 0$ forces $C_2 = 0$. i.e. $u(x) = C_1 x^{\frac{1}{2} - \alpha} J_{\nu_0}(\beta x^{1-\alpha})$. Then $u(1) = 0$ forces β to satisfy $J_{\nu_0}(\beta) = 0$, that is β must be a positive root of the Bessel function J_{ν_0}, for $\nu_0 = \frac{\frac{1}{2} - \alpha}{1 - \alpha}$.

Therefore, we conclude that

$$u_{\nu_0 k}(x) = x^{\frac{1}{2} - \alpha} J_{\nu_0}(j_{\nu_0 k} x^{1-\alpha}), \ k = 1, 2, \cdots$$

are the eigenfunctions and the corresponding eigenvalues are given by

$$\lambda_{\nu_0 k} = \frac{1}{1+(1-\alpha)^2 j_{\nu_0 k}^2}, \; k = 1, 2, \cdots .$$

The behavior of $\mu_{\nu_0 k}$ is then obtained from the asymptotic of $j_{\nu_0 k}$ just as we did in the study of the operators T_α. We omit the details. □

Appendix A

For $\alpha > 0$ and $1 \leq p \leq \infty$ define

$$X^{\alpha,p}(0,1) = \left\{ u \in W^{1,p}_{loc}(0,1) \; : \; u \in L^p(0,1), x^\alpha u' \in L^p(0,1) \right\}.$$

Notice that the functions in $X^{\alpha,p}(0,1)$ are continuous away from 0. It makes sense to define the following subspace

$$X^{\alpha,p}_{\cdot 0}(0,1) = \{ u \in X^{\alpha,p}(0,1) \; : \; u(1) = 0 \}.$$

When $p = 2$, we simplify the notation and write $X^\alpha := X^{\alpha,2}(0,1)$ and $X^\alpha_0 := X^{\alpha,2}_{\cdot 0}(0,1)$. The space $X^{\alpha,p}(0,1)$ is equipped with the norm

$$\|u\|_{\alpha,p} = \|u\|_{L^p(0,1)} + \|x^\alpha u'\|_{L^p(0,1)},$$

or sometimes, if $1 < p < \infty$, with the equivalent norm

$$\left(\|u\|^p_{L^p(0,1)} + \|x^\alpha u'\|^p_{L^p(0,1)} \right)^{\frac{1}{p}}.$$

The space X^α is equipped with the scalar product

$$(u,v)_\alpha = \int_0^1 \left(x^{2\alpha} u'(x) v'(x) + u(x) v(x) \right) dx,$$

and with the associated norm

$$\|u\|_\alpha = \left(\|u\|^2_{L^2(0,1)} + \|x^\alpha u'\|^2_{L^2(0,1)} \right)^{\frac{1}{2}}.$$

One can easily check that, for $\alpha > 0$ and $1 \leq p \leq \infty$, the space $X^{\alpha,p}(0,1)$ is a Banach space and $X^{\alpha,p}_{\cdot 0}(0,1)$ is a closed subspace. When $1 < p < \infty$ the space is reflexive. Moreover, the space X^α is a Hilbert space.

Weighted Sobolev spaces have been studied in more generality (see e.g. [8]). However, since our situation is more specific, we briefly discuss some properties which are relevant for our study.

Theorem A.1. *For $1 \leq p \leq \infty$, let β be any real number such that $\beta + \frac{1}{p} > 0$. Assume that $u \in W^{1,p}_{loc}(0,1]$ and $u(1) = 0$. Then*

$$\left\| x^\beta u \right\|_{L^p} \leq C_{p,\beta} \left\| x^{\beta+1} u' \right\|_{L^p}, \tag{46}$$

where $C_{p,\beta} = \frac{p}{1+p\beta}$ for $1 \leq p < \infty$ and $C_{\infty,\beta} = \frac{1}{\beta}$. In particular, for $1 \leq p < \infty$ and $0 < \alpha \leq 1$, $|u|_{\alpha,p} := \|x^\alpha u'\|_{L^p}$ defines an equivalent norm for $X^{\alpha,p}_{\cdot 0}(0,1)$.

Proof. We first assume $1 \le p < \infty$ and write

$$\begin{aligned}\int_\epsilon^1 x^{p\beta} |u(x)|^p \, dx &= -\int_\epsilon^1 x \left(x^{p\beta} |u(x)|^p\right)' dx - \epsilon^{p\beta+1} |u(\epsilon)|^p \\ &\le -\int_\epsilon^1 x \left(x^{p\beta} |u(x)|^p\right)' dx \\ &= -p\beta \int_\epsilon^1 x^{p\beta} |u(x)|^p \, dx - p \int_\epsilon^1 x^{p\beta+1} |u(x)|^{p-2} u(x)u'(x)dx.\end{aligned}$$

Applying Holder's inequality, we obtain

$$(1+p\beta) \int_\epsilon^1 x^{p\beta} |u(x)|^p \, dx \le p \int_\epsilon^1 x^{p\beta} |u(x)|^p x^{\beta+1} |u'(x)| \, dx \le p \left\|x^\beta u\right\|_{L^p}^{p-1} \left\|x^{\beta+1} u'\right\|_{L^p}.$$

Then equation (46) is derived for $1 \le p < \infty$ and $C_{p,\beta} = \frac{p}{1+p\beta}$. When $p = \infty$, it is understood that $\frac{1}{p} = 0$ and $\beta > 0$, so we pass the limit for $p \to \infty$ in equation (46) and obtain

$$\left\|x^\beta u\right\|_{L^\infty} \le \frac{1}{\beta} \left\|x^{\beta+1} u'\right\|_{L^\infty}.$$

□

Theorem A.2. *For $0 < \alpha \le 1$, $1 \le p \le \infty$, the space $X^{\alpha,p}(0,1)$ is continuously embedded into*

(i) $C^{0,1-\frac{1}{p}-\alpha}[0,1]$ *if* $0 < \alpha < 1 - \frac{1}{p}$ *and* $p \ne 1$,
(ii) $L^q(0,1)$ *for all* $q < \infty$ *if* $\alpha = 1 - \frac{1}{p}$,
(iii) $L^{\frac{p}{p\alpha - p + 1}}(0,1)$ *if* $1 - \frac{1}{p} < \alpha \le 1$ *and* $p \ne \infty$.

Proof. For all $0 < x < y < 1$, we write $|u(y) - u(x)| \le \int_x^y |s^\alpha u'(s)| \, s^{-\alpha} ds$. Applying Holder's inequality, we obtain

$$|u(y) - u(x)| \le C_{\alpha,p} \left\|s^\alpha u'\right\|_{L^p} \begin{cases} x^{-\alpha} & \text{if } p = 1 \\ \left|y^{1-\frac{\alpha p}{p-1}} - x^{1-\frac{\alpha p}{p-1}}\right|^{\frac{p-1}{p}} & \text{if } 1 < p < \infty \text{ and } \alpha \ne 1 - \frac{1}{p} \\ \left|\ln y - \ln x\right|^{\frac{p-1}{p}} & \text{if } 1 < p < \infty \text{ and } \alpha = 1 - \frac{1}{p} \\ \left|y^{1-\alpha} - x^{1-\alpha}\right| & \text{if } p = \infty \text{ and } \alpha \ne 1 \\ \left|\ln y - \ln x\right| & \text{if } p = \infty \text{ and } \alpha = 1. \end{cases} \tag{47}$$

Then assertions (i) and (ii) of Theorem A.2 follow directly from equation (47).

Next, we prove the assertion (iii) with $u \in X^{\alpha,p}_{\cdot 0}(0,1)$. That is, for $1 \le p < \infty$, $1 - \frac{1}{p} < \alpha \le 1$ and $u \in W^{1,p}_{loc}(0,1]$ with $u(1) = 0$, we claim

$$\|u\|_{L^{\frac{p}{p\alpha - p+1}}} \le \frac{p\alpha}{p\alpha - p + 1} \left(\frac{1}{\alpha}\right)^\alpha 2^{1-\alpha} \left\|x^\alpha u'\right\|_{L^p}. \tag{48}$$

If $\alpha = 1$, estimate (48) is a special case of (46). We now prove (48) for $p = 1$ and $0 < \alpha < 1$. Notice that, from equation (46),

$$\begin{aligned}\left\|x^\alpha u\right\|_{L^\infty} &\le \left\|(x^\alpha u)'\right\|_{L^1} \\ &\le \alpha \left\|x^{\alpha-1} u\right\|_{L^1} + \left\|x^\alpha u'\right\|_{L^1} \\ &\le 2 \left\|x^\alpha u'\right\|_{L^1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^1 |u(x)|^{\frac{1}{\alpha}}\, dx &= -\frac{1}{\alpha}\int_0^1 x\,|u(x)|^{\frac{1}{\alpha}-2}\, u(x)u'(x)dx - \lim_{x\to 0^+} x\,|u(x)|^{\frac{1}{\alpha}} \\ &\le \frac{1}{\alpha}\,\|x^\alpha u'\|_{L^1}\,\left\|x^{1-\alpha}\,|u(x)|^{\frac{1}{\alpha}-1}\right\|_{L^\infty} \\ &\le \frac{1}{\alpha}2^{\frac{1-\alpha}{\alpha}}\,\|x^\alpha u'\|_{L^1}^{\frac{1}{\alpha}}\,.\end{aligned}$$

That is

$$\|u\|_{L^{\frac{1}{\alpha}}} \le \left(\frac{1}{\alpha}\right)^{\alpha} 2^{1-\alpha}\,\|x^\alpha u'\|_{L^1}\,. \tag{49}$$

Finally we assume $1 < p < \infty$ and $1 - \frac{1}{p} < \alpha < 1$, we proceed as in the proof of the Sobolev-Gagliardo-Nirenberg inequality. That is, applying the inequality (49) to $u(x) = |v(x)|^\gamma$, for some $\gamma > 1$ to be chosen, it gives

$$\left(\int_0^1 |v(x)|^{\frac{\gamma}{\alpha}}\, dx\right)^{\alpha} \le \gamma\left(\frac{1}{\alpha}\right)^{\alpha} 2^{1-\alpha}\int_0^1 |v(x)|^{\gamma-1}\,|v'(x)|\,x^\alpha dx.$$

Using Holder inequality yields

$$\left(\int_0^1 |v(x)|^{\frac{\gamma}{\alpha}}\, dx\right)^{\alpha} \le \gamma\left(\frac{1}{\alpha}\right)^{\alpha} 2^{1-\alpha}\,\|x^\alpha v'\|_{L^p}\left(\int_0^1 |v(x)|^{\frac{p(\gamma-1)}{p-1}}\right)^{1-\frac{1}{p}}.$$

Let $\frac{\gamma}{\alpha} = \frac{p(\gamma-1)}{p-1}$. That is $\gamma = \frac{p\alpha}{p\alpha-p+1} > 1$ and the above inequality gives the desired result.

Finally, the assertion (iii) in the general case follows immediately from (48), because $\|u\|_{L^p} \le \|u - u(1)\|_{L^p} + |u(1)|$, while $u - u(1) \in X^{\alpha,p}_{\cdot 0}(0,1)$ and $|u(1)| \le (2^{p\alpha}+1)\,\|u\|_{\alpha,p}$. □

We would like to point out that, by the assertion (i) in Theorem A.2, we can define, for $1 < p \le \infty$ and $0 < \alpha < 1 - \frac{1}{p}$,

$$X^{\alpha,p}_{00}(0,1) = \{u \in X^{\alpha,p}(0,1)\ :\ u(0) = u(1) = 0\}\,.$$

Remark 25. Notice that the inequalities (46) and (48) are particular cases of the inequalities proved by Caffarelli, Kohn and Nirenberg. For further reading on this topic we refer to their paper [2].

Theorem A.3. *Let $1 \le p \le \infty$. Then $X^{\alpha,p}(0,1)$ is compactly embedded into $L^p(0,1)$ for all $\alpha < 1$. On the other hand, the embedding is not compact when $\alpha \ge 1$.*

Proof. We first prove that, for $1 \le p < \infty$ and $0 < \alpha < 1$, the space $X^{\alpha,p}_{\cdot 0}(0,1)$ is compactly embedded into $L^p(0,1)$. Let $\mathcal{F}$ be the unit ball in $X^{\alpha,p}_{\cdot 0}(0,1)$. It suffices to prove that $\mathcal{F}$ is totally bounded in $L^p(0,1)$. Notice that, by equation (47), $\forall\epsilon > 0$, there exists a positive integer m, such that

$$\|u\|_{L^p(0,\frac{2}{m})} < \epsilon,\ \forall u \in \mathcal{F}.$$

Define $\phi(x) \in C^\infty(\mathbb{R})$ with $0 \le \phi \le 1$ such that

$$\phi(x) = \begin{cases} 0 & \text{if } x \le 1 \\ 1 & \text{if } x \ge 2, \end{cases}$$

and take $\phi_m(x) = \phi(mx)$. Now $\phi_m\mathcal{F}$ is bounded in $W^{1,p}(0,1)$, and therefore is totally bounded in $L^p(0,1)$. Hence we may cover $\phi_m\mathcal{F}$ by a finite number of balls of radius ϵ in $L^p(0,1)$, say

$$\phi_m\mathcal{F} \subset \bigcup_i B(g_i,\epsilon),\ g_i \in L^p(0,1).$$

We claim that $\bigcup_i B(g_i, 3\epsilon)$ covers $\mathcal{F}$. Indeed, given $u \in \mathcal{F}$ there exists some i such that

$$\|\phi_m u - g_i\|_{L^p(0,1)} < \epsilon.$$

Therefore,

$$\begin{aligned}\|u - g_i\|_{L^p(0,1)} &\le \|\phi_m u - g_i\|_{L^p(0,1)} + \|u - \phi_m u\|_{L^p(0,1)} \\ &< \epsilon + 2\|u\|_{L^p(0,\frac{2}{m})} \\ &\le 3\epsilon.\end{aligned}$$

Hence we conclude that $\mathcal{F}$ is totally bounded in $L^p(0,1)$.

To prove the compact embedding for $X^{\alpha,p}(0,1)$ with $1 \le p < \infty$ and $0 < \alpha < 1$, notice that for any sequence $\{v_n\} \subset X^{\alpha,p}(0,1)$ with $\|v_n\|_{\alpha,p} \le 1$. One can define $u_n(x) = v_n(x) - v_n(1) \in X^{\alpha,p}_{\cdot 0}(0,1)$. Then

$$\|u_n\|_{\alpha,p} = \|x^\alpha u_n'\|_{L^p} = \|x^\alpha v_n'\|_{L^p} \le 1.$$

What we just proved shows that there exists $u \in L^p(0,1)$ such that, up to a subsequence, $u_n \to u$ in L^p. Notice in addition that $|v_n(1)| \le (2^{p\alpha} + 1)\|v\|_{\alpha,p} \le 2^{p\alpha} + 1$, thus there exists $M \in \mathbb{R}$ such that, after maybe extracting a further subsequence, $v_n(1) \to M$. Then it is clear that $v_n(x) \to u(x) + M$ in L^p.

We now prove the embedding is not compact when $1 \le p < \infty$ and $\alpha \ge 1$. To do so, define the sequence of functions

$$v_n(x) = \left(\frac{1}{nx(1-\ln x)^{1+\frac{1}{n}}}\right)^{\frac{1}{p}},$$

and

$$u_n(x) = v_n(x) - \left(\frac{1}{n}\right)^{\frac{1}{p}}, \ \forall n \ge 2.$$

Clearly $\|v_n\|_{L^p(0,1)} = 1$ and $1 - \left(\frac{1}{2}\right)^{\frac{1}{p}} \le \|u_n\|_{L^p(0,1)} \le 2$. Also $\|xu_n'\|_{L^p(0,1)} \le \frac{6}{p}$. It means that $\{u_n(x)\}_{n=2}^{\infty}$ is a bounded sequence in $X^{\alpha,p}_{\cdot 0}(0,1)$ for $\alpha \ge 1$. However, it has no convergent subsequence in $L^p(0,1)$ since $u_n \to 0$ a.e. and $\|u_n\|_{L^p(0,1)}$ is uniformly bounded below.

If $p = \infty$ and $0 < \alpha < 1$, take $u \in X^{\alpha,\infty}(0,1)$ and equation (47) implies that

$$|u(x) - u(y)| \le C_\alpha \|x^\alpha u'\|_{L^\infty} |x - y|^{1-\alpha}.$$

Therefore, the embedding is compact by the Ascoli-Arzela theorem. To prove that the embedding is not compact for $p = \infty$ and $\alpha \ge 1$, define the sequence of functions

$$\phi_n(x) = \begin{cases} -\frac{\ln x}{\ln n} & \text{if } \frac{1}{n} \le x \le 1 \\ 1 & \text{if } 0 \le x < \frac{1}{n}. \end{cases}$$

We can see that ϕ_n is a bounded sequence in $X^{\alpha,\infty}(0,1)$ for $\alpha \ge 1$. However it has no convergent subsequence in $L^\infty(0,1)$ since $\phi_n \to 0$ a.e but $\|\phi_n\|_{L^\infty} = 1$. ☐

We conclude the Appendix with the following density result, which is not used in the paper but is of independent interest.

Theorem A.4. *Assume* $1 \le p < \infty$.

(i) *If* $p \ne 1$ *and* $0 < \alpha < 1 - \frac{1}{p}$, *we have that* $C^\infty([0,1])$ *is dense in* $X^{\alpha,p}(0,1)$ *and that* $C_0^\infty(0,1)$ *is dense in* $X^{\alpha,p}_{00}(0,1)$.

(ii) *If* $\alpha > 0$ *and* $\alpha \geq 1 - \frac{1}{p}$*, we have that* $C_0^\infty(0,1]$ *is dense in* $X^{\alpha,p}(0,1)$.

Proof. For any $1 \leq p < \infty$, $\alpha > 0$ and $u \in X^{\alpha,p}(0,1)$, we first claim that there exists a sequence $\{\epsilon_n > 0\}$ with $\lim_{n\to\infty} \epsilon_n = 0$ such that:

- either $|u(\epsilon_n)| \leq C$ uniformly in n, or
- $|u(\epsilon_n)| \leq |u(x)|$ for all n and $0 < x < \epsilon_n$.

Indeed, if $|u(x)|$ is unbounded along every sequence converging to 0, we would have $\lim_{x\to 0^+} |u(x)| = +\infty$, in which case we can define $\epsilon_n > 0$ to be such that $|u(\epsilon_n)| = \min_{0<x\leq \frac{1}{n}} |u(x)|$, thus completing the argument. In the rest of this proof, for any $u \in X^{\alpha,p}(0,1)$, sequence $\{\epsilon_n\}$ is chosen to have the above property.

We first prove (i). Assume $1 < p < \infty$ and $0 < \alpha < 1 - \frac{1}{p}$. To prove that $C^\infty([0,1])$ is dense in $X^{\alpha,p}(0,1)$, it suffices to show that $W^{1,p}(0,1)$ is dense in $X^{\alpha,p}(0,1)$. Take $u \in X^{\alpha,p}(0,1)$. Define

$$u_n(x) = \begin{cases} u(\epsilon_n) & \text{if } 0 < x \leq \epsilon_n \\ u(x) & \text{if } \epsilon_n < x \leq 1. \end{cases}$$

Then one can easily check that $u_n \in W^{1,p}(0,1)$ and that $u_n \to u$ in $X^{\alpha,p}(0,1)$ by the dominated convergence theorem. To prove that $C_0^\infty(0,1)$ is dense in $X_{00}^{\alpha,p}(0,1)$, it suffices to show that $W_0^{1,p}(0,1)$ is dense in $X_{00}^{\alpha,p}(0,1)$, to do so, we adapt a technique by H. Brezis (see the proof of Theorem 8.12 of [1], page 218): Take $G \in C^1(\mathbb{R})$ such that $|G(t)| \leq |t|$ and

$$G(t) = \begin{cases} 0 & \text{if } |t| \leq 1 \\ t & \text{if } |t| > 2. \end{cases}$$

For $u \in X_{00}^{\alpha,p}(0,1)$, define $u_n = \frac{1}{n} G(nu)$. Then one can easily check that $u_n \in C_0(0,1) \cap X^{\alpha,p}(0,1) \subset W_0^{1,p}(0,1)$ and that $u_n \to u$ in $X^{\alpha,p}(0,1)$ by the dominated convergence theorem.

To prove the assertion (ii), we notice that it is enough to prove that $C_0^\infty(0,1)$ is dense in $X_{\cdot 0}^{\alpha,p}(0,1)$. Indeed, for any $u \in X^{\alpha,p}(0,1)$, define $\phi(x) \in C_0^\infty(0,1]$ such that $|\phi(x)| \leq 1$ with

$$\phi(x) = \begin{cases} 1 & \text{if } \frac{2}{3} \leq x \leq 1 \\ 0 & \text{if } 0 \leq x \leq \frac{1}{3}. \end{cases}$$

Define $v(x) := u(x) - \phi(x)u(1)$, then $v \in X_{\cdot 0}^{\alpha,p}(0,1)$. If we can approximate v by $v_n \in C_0^\infty(0,1)$, then $u_n(x) = v_n(x) + \phi(x)u(1)$ belongs to $C_0^\infty(0,1]$ and it approximates u in $X_{\cdot 0}^{\alpha,p}(0,1)$. So let $\alpha > 1 - \frac{1}{p}$ and $1 \leq p < \infty$, to prove that $C_0^\infty(0,1)$ is dense in $X_{\cdot 0}^{\alpha,p}(0,1)$, it suffices to show that $W_0^{1,p}(0,1)$ is dense in $X_{\cdot 0}^{\alpha,p}(0,1)$. To do so, for fixed $u \in X_{\cdot 0}^{\alpha,p}(0,1)$, define

$$u_n(x) = \begin{cases} \frac{u(\epsilon_n)}{\epsilon_n} x & \text{if } 0 \leq x \leq \epsilon_n \\ u(x) & \text{if } \epsilon_n < x \leq 1. \end{cases}$$

Then $u_n \in W_0^{1,p}(0,1)$ and on the interval $(0, \epsilon_n)$ we have either $|u_n(x)| \leq |u(x)|$ and $|u_n'(x)| \leq \frac{|u(x)|}{x}$, or $|u_n(x)| \leq C$ and $|u_n'(x)| \leq \frac{C}{x}$ where C is independent of n. In both cases, since $\alpha > 1 - \frac{1}{p}$ and $x^{\alpha-1}u(x) \in L^p$ by Theorem A.1, one can conclude that $u_n \to u$ in $X^{\alpha,p}(0,1)$ by the dominated convergence theorem.

For $\alpha = 1 - \frac{1}{p}$ and $1 < p < \infty$, again, it suffices to prove that $W_0^{1,p}(0,1)$ is dense in $X_{\cdot 0}^{\alpha,p}(0,1)$. For fixed $u \in X_{\cdot 0}^{\alpha,p}(0,1)$, define

$$u_n(x) = \begin{cases} \frac{u(\epsilon_n)(1-\ln\epsilon_n)}{1-\ln x} & \text{if } 0 \le x \le \epsilon_n \\ u(x) & \text{if } \epsilon_n < x \le 1. \end{cases}$$

One can easily check that $u_n \in C[0,1] \cap X^{\alpha,p}(0,1)$ and $u_n(0) = u_n(1) = 0$. On the interval $(0, \epsilon_n)$, we have either $|u_n(x)| \le |u(x)|$ and $|u_n'(x)| \le \frac{|u(x)|}{x(1-\ln x)}$, or $|u_n| \le C$ and $|u_n'(x)| \le \frac{C}{x(1-\ln x)}$ where C is independent of n. Notice that by using the same trick used in estimate (28), one can show that $x^{-\frac{1}{p}}(1-\ln x)^{-1}u \in L^p(0,1)$ for any $u \in X_{\cdot 0}^{1-\frac{1}{p},p}(0,1)$ with $1 < p < \infty$. Therefore, one can conclude that $u_n \to u$ in $X^{\alpha,p}(0,1)$.

The above shows that that $\{u \in C[0,1] \cap X^{\alpha,p}(0,1) \;:\; u(0) = u(1) = 0\}$ is dense in $X_{\cdot 0}^{\alpha,p}(0,1)$. Finally, notice that by using the same argument used to prove (i), we obtain that $W_0^{1,p}(0,1)$ is dense in $\{u \in C[0,1] \cap X^{\alpha,p}(0,1) \;:\; u(0) = u(1) = 0\}$, thus concluding the proof. □

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