THE ESSENTIAL SPECTRUM OF A SINGULAR STURM-LIOUVILLE OPERATOR

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ABSTRACT. In this paper we study the essential spectrum of the operator

$$L_A u(x) = -(A(x)u'(x))' + u(x)$$

where $A(x)$ is a positive absolutely continuous function on $(0,1)$ that resembles $x^{2\alpha}$ for some $\alpha \geq 1$. We prove that the essential spectrum of L_A coincides with the essential spectrum of the operator $L_\alpha u(x) := -(x^{2\alpha}u'(x))' + u(x)$.

1. INTRODUCTION

We consider the singular Sturm-Liouville differential operator

$$L_A u(x) := -(A(x)u'(x))' + u(x) \tag{1}$$

over the interval $(0,1)$, where $A(x) = A_\alpha(x)$ is an absolutely continuous function on $[0,1]$ such that $A(x) > 0$ for all $0 < x \leq 1$. In addition we suppose that there exist constants $c_1, c_2 > 0$ and $\alpha > 0$ such that

$$c_1 x^{2\alpha} \leq A(x) \leq c_2 x^{2\alpha}, \quad \text{for all } x \in (0,1], \text{ and} \tag{H1}$$

$$\lim_{x \to 0} x^{-2\alpha} A(x) = 1. \tag{H2}$$

Associated with (1) one can define the following operator

$$\begin{aligned} T_A : X_0^\alpha &\longrightarrow X_0^\alpha \\ f &\longmapsto T_A(f) = u, \end{aligned}$$

where u is the (unique) solution of

$$\int_0^1 A(x)u'(x)v'(x)\,\mathrm{d}x + \int_0^1 u(x)v(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x, \quad \forall\, v \in X_0^\alpha. \tag{2}$$

Here X_0^α is the space of real valued functions u in L^2 having a weak derivative satisfying $x^\alpha u' \in L^2$ such that $u(1) = 0$ (see [2, Appendix] for more details about these spaces). The fact that the operator T_A is a well defined bounded operator is a direct consequence of the

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Lax-Milgram theorem (see section 2 for some of the details). In addition, a straightforward computation tells us that if $u = T_A f$, then u is the unique weak solution of

$$\begin{cases} -(A(x)u'(x))' + u(x) = f(x) \quad \text{a.e. in } (0,1), \\ \lim_{x\to 0^+} A(x)u'(x) = 0, \\ u(1) = 0, \end{cases} \tag{3}$$

For the special case $A(x) = x^{2\alpha}$ a complete study of (3) has been developed in [2,3], exhibiting properties of existence, uniqueness, and regularity of solutions in terms of the L^2 norm of f, as well as a detailed description of the spectrum of the respective operator T_A, denoted by T_α in this particular case.

One important feature of the spectrum of T_α is that it changes from a spectrum consisting solely on isolated simple eigenvalues $\sigma(T_\alpha) = \{\lambda_i : i \in \mathbb{N}\}$ to a purely essential (continuum) spectrum when α crosses the $\alpha = 1$ barrier, namely $\sigma(T_1) = \sigma_e(T_1) = [0, \frac{4}{5}]$ (see [2, Theorem 1.17]). A study of the spectrum of T_A, and other relevant results regarding a non-linear problem, had been established by Stuart and Vuillaume in a more general setting. In the series of articles [6,8–13] the authors studied the bifurcation properties of a heavy tapered rod, and in this context with the aid of the Bernoulli-Euler bending law for beams, the differential operator $N_A u(x) = (L_A - I)u(x) = -(A(x)u'(x))'$ appears naturally (see [8, Section 1.1] for more details on how this kind of operators arise in this context). In particular, in [9] Stuart studied the spectral properties of the operator N_A under the boundary conditions

$$\lim_{x\to 0^+} A(x)u'(x) = 0 \quad \text{and} \quad u(1) = 0,$$

and also of the operator $T_A - I$ in the case $0 \le \alpha \le 1$, proving that

$$\sigma(T_A - I) = \left\{\frac{1}{\lambda} : \lambda \in \sigma(N_A)\right\} \cup \{0\}$$

and

$$\sigma_e(T_A - I) = \left\{\frac{1}{\lambda} : \lambda \in \sigma_e(N_A)\right\} \cup \{0\}.$$

Stuart also established, using a compactness argument, that for $\alpha < 1$ the spectrum of $T_A - I$ consists solely of simple eigenvalues (in particular $\sigma_e(T_A - I) = \{0\}$), but as soon as $\alpha = 1$ the essential spectrum becomes non-trivial. He also gives conditions on $A(x)$ for the existence/non-existence of eigenvalues when $\alpha = 1$.

The purpose of this work is to answer some questions raised in [2] regarding the spectrum of the operators L_A and T_A when $\alpha \ge 1$. In particular, for $\alpha = 1$ Stuart has shown that $\max \sigma_e(T_A) = \frac{4}{5}$, implying that $\sigma_e(T_A) \subseteq [0, \frac{4}{5}]$, but the question of whether $\sigma_e(T_A) = [0, \frac{4}{5}]$ remained open. For $\alpha > 1$ there is less information, as the existence/non-existence of eigenvalues in $\sigma(T_A)$ and estimates over $\sigma_e(T_A)$ or $\max \sigma_e(T_A)$ have not been discussed. The following result answers such questions.

Theorem 1. *Let $\alpha \ge 1$ and T_A be as before.*
(i) If $\alpha = 1$, then $\sigma_e(T_A) = [0, \frac{4}{5}] = \sigma_e(T_\alpha)$.
(ii) If $\alpha > 1$, then $\sigma_e(T_A) = \sigma(T_A) = [0, 1] = \sigma(T_\alpha) = \sigma_e(T_\alpha)$.

In addition, we show that the analysis of the spectrum of T_A is equivalent to the analysis of the spectrum of L_A as an unbounded operator from $D(L_A) \subset L^2(0,1)$ to $L^2(0,1)$ (the domain $D(L_A)$ will be specified later in section 5), as the following theorem shows.

Theorem 2. *Let $\alpha \ge 1$ and for $\delta \in \mathbb{R} \setminus \{0\}$ consider $\gamma = \delta^{-1}$. Then we have* *(i)* $L_A-\gamma I : D(L_A) \longrightarrow L^2$ *is an isomorphism* $\Longleftrightarrow T_A-\delta I : X_0^\alpha \longrightarrow X_0^\alpha$ *is an isomorphism.*

(ii) $L_A - \gamma I : D(L_A) \longrightarrow L^2$ *is Fredholm* $\Longleftrightarrow T_A - \delta I : X_0^\alpha \longrightarrow X_0^\alpha$ *is Fredholm.*

(iii)

$$\sigma(T_A) = \left\{ \lambda \in \mathbb{R} : \lambda^{-1} \in \sigma(L_A) \right\} \cup \{0\} ,$$

and

$$\sigma_e(T_A) = \left\{ \lambda \in \mathbb{R} : \lambda^{-1} \in \sigma_e(L_A) \right\} \cup \{0\} .$$

(iv) $T_A = L_A^{-1}\Big|_{X_0^\alpha}$.

Remark 1. It is important to mention that there are several notions of essential spectra that can be defined, however for the particular case of a self-adjoint operator on a Hilbert space, most of these notions coincide (see for instance [4, Theorem IX.1.6]). As we will see later, both operators T_A and L_A are indeed self-adjoint.

The rest of this paper is organized as follows. We establish the notation and the definitions used throughout this work in section 2. In section 2 we prove a proposition that is key in the proof of Theorem 1. Then we separate the proof of Theorem 1 into the cases $\alpha = 1$ and $\alpha > 1$ in sections 3 and 4 respectively. Finally, in section 5 we establish the connection between T_A and L_A and prove Theorem 2.

2. Preliminaries

For $\alpha > 0$, recall the definition of the real vector spaces $X^\alpha = X^\alpha(0,1)$ given in [2]

$$X^\alpha = \left\{ u \in H^1_{loc}(0,1] \, : u \in L^2(0,1), \; x^\alpha u' \in L^2(0,1) \right\} ,$$

where $H^1_{loc}(0,1]$ is the set of function belonging to the Sobolev space $H^1(K)$ for all K compact subset of $(0,1]$. Additionally, since functions in X^α are continuous away from the origin, the subset

$$X_0^\alpha = \{ u \in X^\alpha \, : u(1) = 0 \} ,$$

is a well defined closed subspace of X^α. In [2] we established that X_0^α is a Hilbert space for the inner product

$$\langle u, v \rangle_{X^\alpha} = \langle u, v \rangle + \langle x^\alpha u', x^\alpha v' \rangle ,$$

where throughout this work

$$\langle u, v \rangle = \int_0^1 u(x) v(x) \, \mathrm{d}x$$

will denote the usual inner product in $L^2 = L^2(0,1)$. Because of the Riesz representation theorem we know the existence of an operator $T_\alpha : X_0^\alpha \longrightarrow X_0^\alpha$ defined by the identity

$$\langle T_\alpha f, v \rangle_{X^\alpha} = \langle f, v \rangle \quad \forall v \in X_0^\alpha. \tag{4}$$

If we let $u = T_\alpha f$, we have shown that u is in fact the unique weak solution to the singular Sturm-Liouville equation (see [2,3])

$$\begin{cases} -(x^{2\alpha} u'(x))' + u(x) = f \quad \text{a.e. in } (0,1), \\ u(1) = 0, \\ \lim\limits_{x \to 0} x^{2\alpha} u'(x) = 0. \end{cases}$$

The following estimate also follows from [2]

$$\| T_\alpha f \|_{X^\alpha} \leq \| f \|_{L^2} \leq \| f \|_{X^\alpha} ,$$

and it asserts that the boundedness of T_α only requires f to be an L^2 function, that is, the operator T_α could be extended to $L^2 \supset X_0^\alpha$ as a bounded operator.

As we mentioned in the introduction, we will consider a function $A : [0,1] \longrightarrow [0,\infty)$ satisfying (H1) and (H2). Define the bilinear symmetric function $a : X_0^\alpha \times X_0^\alpha \longrightarrow \mathbb{R}$ by

$$a(u,v) = \int_0^1 A(x)u'(x)v'(x)\,\mathrm{d}x + \int_0^1 u(x)v(x)\,\mathrm{d}x,$$

and thanks to (H1) it is easy to find constants $K_1, K_2 > 0$ such that

$$|a(u,v)| \le K_1 \|u\|_{X^\alpha} \|v\|_{X^\alpha} \quad \text{and} \quad |a(u,u)| \ge K_2 \|u\|_{X^\alpha}^2, \tag{5}$$

thus proving that a is a bounded, coercive bilinear function over X_0^α, therefore the Lax-Milgram theorem ([1, Corollary 5.8]) tells us that there exists a unique bounded linear operator $T_A : X_0^\alpha \longrightarrow X_0^\alpha$ defined by the equation

$$a(T_A f, v) = \langle f, v \rangle \quad \forall v \in X_0^\alpha. \tag{6}$$

It is significant to observe that (5) and the symmetry of a tell us that $a(\cdot,\cdot)$ defines an inner product over X_0^α which gives an equivalent topology on X_0^α. We will use both inner products, $a(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle_{X^\alpha}$, accordingly.

Observe that for each $f \in L^2$ and if we call $u = T_A f$, then it is straightforward to see that u is the unique weak solution in X_0^α to the equation

$$\begin{cases} -(A(x)u'(x))' + u(x) = f & \text{a.e. in } (0,1), \\ u(1) = 0, & \\ \lim_{x\to 0} A(x)u'(x) = 0, & \end{cases}$$

moreover, by the definition of T_A we obtain immediately that T_A is self adjoint with respect to the inner product $a(\cdot,\cdot)$, and that

$$a(T_A u, u) = \langle u, u \rangle > 0 \quad \text{for all } u \in X_0^\alpha \setminus \{0\}$$

showing that T_A is a positive operator.

In this framework, the spectrum of the operator T_A is the set

$$\sigma(T_A) = \{\lambda \in \mathbb{R} : T - \lambda I : X_0^\alpha \to X_0^\alpha \text{ is not an isomorphism}\},$$

and that the essential spectrum of T_A is defined as

$$\sigma_e(T_A) = \{\lambda \in \sigma(T_A) : T - \lambda I : X_0^\alpha \to X_0^\alpha \text{ is not a Fredholm operator}\},$$

and to prove Theorem 1 we will use the following technical result which will allow us to characterize the essential spectrum of the operator T_A.

Proposition 1. *For $\lambda \in \mathbb{R}$ suppose there exists a sequence $\{u_n\} \in X_0^\alpha(0,1)$ such that*

(i) $\|u_n\|_{X^\alpha} = 1$,
(ii) $u_n \underset{n\to\infty}{\longrightarrow} 0$ *in the weak topology of* X_0^α,
(iii) $\|(T_\alpha - \lambda)u_n\|_{X^\alpha} = o(1)$,
(iv) $\operatorname{supp} u_n \subseteq [0, \frac{1}{n}]$,
(v) $\langle T_\alpha u_n, u_n \rangle_{X^\alpha} = \lambda + o(1)$,

where $o(1)$ is a quantity that goes to 0 as n goes to infinity. Then λ belongs to $\sigma_e(T_A)$.

Proof. To show that $\lambda \in \sigma_e(T_A)$, it is enough to find a singular sequence for T_A and λ (see [4, Theorems IX.1.3 and IX.1.6]), that is, a sequence $u_n \in X_0^\alpha$ such that $\|u_n\|_{X^\alpha} = 1$, $u_n \to 0$ weakly in X^α, and that $\|(T_A - \lambda)u_n\|_{X^\alpha} \to 0$ as $n \to \infty$.

Using the the sequence $\{u_n\}$ given in the statement of this proposition, we only need to prove $\|(T_A - \lambda)u_n\|_{X^\alpha} \to 0$, as the other conditions are already established. Observe that by the definition of T_A and T_α we have for all $v \in X_0^\alpha$ the identity

$$a(T_A u_n, v) = \langle u_n, v\rangle = \langle T_\alpha u_n, v\rangle_{X^\alpha},$$

therefore

$$a(T_A u_n, u_n) = \lambda + o(1).$$

Also, since $\operatorname{supp} u_n \subseteq [0, \frac{1}{n}]$ we have

$$\begin{aligned}|a(u_n, v) - \langle u_n, v\rangle_{X^\alpha}| &= \left|\int_0^{\frac{1}{n}} (A(x) - x^{2\alpha})u_n'(x)v'(x)\,\mathrm{d}x\right| \\ &\le \left(\int_0^{\frac{1}{n}} \left|A(x) - x^{2\alpha}\right| \left|u_n'(x)\right|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_0^{\frac{1}{n}} \left|A(x) - x^{2\alpha}\right| \left|v'(x)\right|^2 \,\mathrm{d}x\right)^{\frac{1}{2}},\end{aligned}$$

but we are assuming (H2), therefore there exists a sequence $\delta_n \longrightarrow 0$ such that

$$\left|A(x) - x^{2\alpha}\right| \le \delta_n x^{2\alpha} \quad \text{for all } 0 < x \le \frac{1}{n}.$$

Hence

$$|a(u_n, v) - \langle u_n, v\rangle_{X^\alpha}| \le \delta_n \|u_n\|_{X^\alpha} \|v\|_{X^\alpha} = \delta_n \|v\|_{X^\alpha},$$

and as a consequence we obtain

$$\begin{aligned}|a((T_A - \lambda)u_n, v)| &= |\langle (T_\alpha - \lambda)u_n, v\rangle_{X^\alpha} - \lambda\,(a(u_n, v) - \langle u_n, v\rangle_{X^\alpha})| \\ &\le \|(T_\alpha - \lambda)u_n\|_{X^\alpha} \|v\|_{X^\alpha} + \delta_n |\lambda| \|v\|_{X^\alpha}.\end{aligned} \tag{7}$$

From (5) we deduce that the norms

$$\|u\|_{X^\alpha} = \sqrt{\langle u, u\rangle_{X^\alpha}} \quad \text{and} \quad \|u\|_a = \sqrt{a(u, u)}$$

are equivalent in X^α. From this equivalence and the dual representation of the norm $\|\cdot\|_a$ (recall that $a(u, v)$ defines an inner product over X_0^α) we obtain

$$\sqrt{K_2}\,\|\varphi\|_{X^\alpha} \le \|\varphi\|_a = \sup_{v \in X^\alpha\setminus\{0\}} \frac{|a(\varphi, v)|}{\|v\|_a} \le \frac{1}{\sqrt{K_2}} \sup_{v \in X^\alpha\setminus\{0\}} \frac{|a(\varphi, v)|}{\|v\|_{X^\alpha}},$$

for all $\varphi \in X^\alpha$. Therefore for $C = K_2^{-1}$ we have

$$\|\varphi\|_{X^\alpha} \le C \sup_{v \in X^\alpha\setminus\{0\}} \frac{|a(\varphi, v)|}{\|v\|_{X^\alpha}}. \tag{8}$$

Using (7) and (8) gives

$$\begin{aligned}\|(T_A - \lambda)u_n\|_{X^\alpha} &\le C \sup_{v \in X_0^\alpha\setminus\{0\}} \frac{|a((T_A - \lambda)u_n, v)|}{\|v\|_{X^\alpha}} \\ &\le C\,(\|(T_\alpha - \lambda)u_n\|_{X^\alpha} + \delta_n |\lambda|) \\ &= o(1),\end{aligned}$$

thus concluding the proof. ■

Remark 2. Observe that the first three conditions on the sequence $\{u_n\}$ required by Proposition 1 say that $\{u_n\}$ is a singular sequence for the pair (λ, T_α), and by [4, Theorem IX.1.3] such sequence can be found for any λ in $\sigma_e(T_\alpha) = [0, \frac{4}{5}]$. However, finding a singular sequence for (λ, T_α) satisfying *in addition* $\operatorname{supp} u_n \subseteq [0, \frac{1}{n}]$ and $\langle T_\alpha u_n, u_n\rangle_{X^\alpha} = \lambda + o(1)$ requires additional work. We will do so in sections 3 and 4 to prove Theorem 1.

3. Proof of Theorem 1: case $\alpha = 1$

For $0 < \lambda < \frac{4}{5}$, let $\mu = \frac{1}{\lambda}$ and $\gamma = \sqrt{\mu - \frac{5}{4}}$. Given $\varepsilon > 0$ define

$$\tilde{w}_\varepsilon(x) = x^{\varepsilon - \frac{1}{2}} \sin(\gamma \ln x),$$

and

$$\tilde{g}_\varepsilon(x) = -2\gamma\varepsilon x^{\varepsilon - \frac{1}{2}} \cos(\gamma \ln x).$$

It is a simple exercise to see that both $\tilde{w}_\varepsilon$ and $\tilde{g}_\varepsilon$ belong to X^α for $\alpha = 1$ and all $\varepsilon > 0$; moreover, $\tilde{w}_\varepsilon$ satisfies

$$-(x^2 \tilde{w}_\varepsilon'(x))' + (1 - \mu + \varepsilon^2)\tilde{w}_\varepsilon(x) = \tilde{g}_\varepsilon(x).$$

Consider now a smooth cut-off function $\rho : \mathbb{R} \longrightarrow \mathbb{R}_+$, satisfying

$$\rho(x) = 1 \text{ for } x \leq \frac{1}{2}, \tag{9}$$

$$\rho(x) = 0 \text{ for } x \geq 1, \tag{10}$$

$$0 \leq \rho(x) \leq 1 \quad \forall x \tag{11}$$

$$\|\rho'\|_\infty + \|\rho''\|_\infty \leq C_0 \tag{12}$$

for some constant $C_0 > 0$. For $x \in [0, 1]$ define $w_\varepsilon(x) = \tilde{w}_\varepsilon\left(\frac{x}{\varepsilon}\right)$, $g_\varepsilon(x) = \tilde{g}_\varepsilon\left(\frac{x}{\varepsilon}\right)$, $\rho_\varepsilon(x) = \rho\left(\frac{x}{\varepsilon}\right)$, and let

$$\tilde{u}_\varepsilon(x) := w_\varepsilon(x)\rho_\varepsilon(x).$$

Observe that by definition $\operatorname{supp} \tilde{u}_\varepsilon \subseteq [0, \varepsilon]$. In addition, a direct computation shows that $\tilde{u}_\varepsilon$ is a solution to the equation

$$\begin{cases} -(x^2 \tilde{u}_\varepsilon'(x))' + \left(1 - \mu + \varepsilon^2\right) \tilde{u}_\varepsilon(x) = f_\varepsilon(x) & \text{for } x \in (0, 1), \\ \tilde{u}_\varepsilon \in X_0^1, \end{cases} \tag{13}$$

where

$$f_\varepsilon(x) = g_\varepsilon(x)\rho_\varepsilon(x) - 2x^2 w_\varepsilon'(x)\rho_\varepsilon'(x) - w_\varepsilon(x)(x^2\rho_\varepsilon'(x))'.$$

In terms of the operator T_1, equation (13) can be written as

$$\left(T_1 - \frac{\lambda}{1 - \lambda\varepsilon^2}\right) \tilde{u}_\varepsilon = -\frac{\lambda}{1 - \lambda\varepsilon^2} T_1 f_\varepsilon,$$

and we have the following

Lemma 1. *For $0 < \lambda < \frac{4}{5}$ let $u_\varepsilon := \tilde{u}_\varepsilon / \|\tilde{u}_\varepsilon\|_{X^1}$. As ε goes to zero one has*

(i) $\langle T_1 u_\varepsilon, u_\varepsilon\rangle_{X^1} = \lambda + o(1)$,
(ii) $\|(T_1 - \lambda)u_\varepsilon\|_{X^1} = o(1)$,
(iii) $u_\varepsilon \longrightarrow 0$ *in the weak topology of* X_0^1,

where $o(1)$ denotes a quantity that goes to 0 with ε.

Proof. Observe that

$$T_1 u_\varepsilon = \frac{\lambda}{1-\lambda\varepsilon^2} u_\varepsilon - \frac{\lambda}{(1-\lambda\varepsilon^2)\,\|\tilde{u}_\varepsilon\|_{X^1}} T_1 f_\varepsilon,$$

therefore

$$\begin{aligned}\langle T_1 u_\varepsilon, u_\varepsilon\rangle_{X^1} &= \frac{\lambda}{1-\lambda\varepsilon^2} - \frac{\lambda}{(1-\lambda\varepsilon^2)\|\tilde{u}_\varepsilon\|_{X^1}} \langle T_1 f_\varepsilon, u_\varepsilon\rangle_{X^1} \\ &= \lambda - \frac{\lambda}{(1-\lambda\varepsilon^2)\|\tilde{u}_\varepsilon\|_{X^1}} \langle T_1 f_\varepsilon, u_\varepsilon\rangle_{X^1} + o(1).\end{aligned}$$

To prove the first part of this lemma we need to estimate the middle term and show that it goes to zero as ε goes to zero. Observe that for each $x \in (0,1]$

$$\begin{aligned}|w_\varepsilon(x)| &\le \varepsilon^{\frac{1}{2}-\varepsilon} x^{\varepsilon-\frac{1}{2}}, \\ |w_\varepsilon'(x)| &\le C\varepsilon^{\frac{1}{2}-\varepsilon} x^{\varepsilon-\frac{3}{2}},\end{aligned}$$

and that since $\rho_\varepsilon(x) = 1$ for $0 \le x \le \frac{\varepsilon}{2}$ we have

$$\begin{aligned}\|\tilde{u}_\varepsilon\|_{X^1}^2 &\ge \|w_\varepsilon \rho_\varepsilon\|_{L^2}^2 \\ &\ge \int_0^{\frac{\varepsilon}{2}} |w_\varepsilon(x)|^2 \,\mathrm{d}x \\ &= \varepsilon \int_0^{\frac{1}{2}} x^{2\varepsilon-1} \sin^2(\gamma \ln x)\,\mathrm{d}x \\ &= \frac{\varepsilon}{\gamma} \int_{\gamma \ln 2}^{\infty} e^{-\frac{2\varepsilon}{\gamma}t} \sin^2(t)\,\mathrm{d}t \\ &= \frac{e^{-2\varepsilon \ln(2)}}{4} \left[1 + \frac{\varepsilon}{\varepsilon^2+\gamma^2}\left(\gamma \sin(2\gamma \ln 2) - \varepsilon\cos(2\gamma\ln 2)\right)\right] \\ &= \frac{e^{-2\varepsilon \ln(2)}}{4}(1+o(1)) \\ &= \frac{1}{4} + o(1).\end{aligned}$$

Also, since $\|T_1 f_\varepsilon\|_{X^1} \le \|f_\varepsilon\|_{L^2}$, we only need to estimate the L^2 norm of f_ε. Recall that

$$f_\varepsilon(x) = g_\varepsilon(x)\rho_\varepsilon(x) - 2x^2 w_\varepsilon'(x)\rho_\varepsilon'(x) - w_\varepsilon(x)(x^2\rho_\varepsilon'(x))'$$

and estimate each term in L^2. Firstly, as $0 \le \rho_\varepsilon(x) \le 1$ with $\operatorname{supp} \rho_\varepsilon \subseteq [0,\varepsilon]$ we can write

$$\begin{aligned}\|g_\varepsilon \rho_\varepsilon\|_{L^2}^2 &\le \int_0^{\varepsilon} \tilde{g}_\varepsilon\left(\frac{x}{\varepsilon}\right)^2 \mathrm{d}x \\ &\le 4\gamma^2\varepsilon^3 \int_0^1 y^{2\varepsilon-1}\,\mathrm{d}y \\ &= 2\gamma^2\varepsilon^2.\end{aligned}$$

For the other terms in f_ε observe that by the boundedness of the derivatives of ρ we have for all $x \in [0,1]$ the following estimates

$$\begin{aligned}|\rho_\varepsilon'(x)| &\le \frac{C_0}{\varepsilon}, \\ |\rho_\varepsilon''(x)| &\le \frac{C_0}{\varepsilon^2},\end{aligned}$$

therefore, by observing that both ρ_ε' and ρ_ε'' are also supported on the interval $[0,\varepsilon]$ we can write

$$\begin{aligned}\left\|x^2 w_\varepsilon' \rho_\varepsilon'\right\|_{L^2}^2 &\leq C\int_0^\varepsilon \left|\varepsilon^{-\frac{1}{2}-\varepsilon} x^{\frac{1}{2}+\varepsilon}\right|^2 \mathrm{d}x \\ &= C\varepsilon^{-1-2\varepsilon}\int_0^\varepsilon x^{1+2\varepsilon}\,\mathrm{d}x \\ &\leq C\varepsilon.\end{aligned}$$

Similarly,

$$\begin{aligned}\left\|x^2 w_\varepsilon \rho_\varepsilon''\right\|_{L^2}^2 &\leq C\int_0^\varepsilon \left|\varepsilon^{-\frac{3}{2}-\varepsilon} x^{\frac{3}{2}+\varepsilon}\right|^2 \mathrm{d}x \\ &= C\varepsilon^{-3-2\varepsilon}\int_0^\varepsilon x^{3+2\varepsilon}\,\mathrm{d}x \\ &\leq C\varepsilon.\end{aligned}$$

and

$$\begin{aligned}\left\|x w_\varepsilon \rho_\varepsilon'\right\|_{L^2}^2 &\leq C\int_0^\varepsilon \left|\varepsilon^{-\frac{1}{2}-\varepsilon} x^{\frac{1}{2}+\varepsilon}\right|^2 \mathrm{d}x \\ &= C\varepsilon^{-1-2\varepsilon}\int_0^\varepsilon x^{1+2\varepsilon}\,\mathrm{d}x \\ &\leq C\varepsilon.\end{aligned}$$

Hence

$$\|T_1 f_\varepsilon\|_{X^1} \leq \|f\|_{L^2} \leq C\sqrt{\varepsilon},$$

and we deduce

$$\left|\frac{1}{\|u_\varepsilon\|_{X^1}}\left\langle T_1 f_\varepsilon, u_\varepsilon\right\rangle_{X^1}\right| \leq \frac{\|T_1 f_\varepsilon\|_{X_1}}{\|\tilde{u}_\varepsilon\|_{X^1}} \leq C\sqrt{\varepsilon} = o(1),$$

thus proving the first part of the lemma.

For the second part, observe that we have established $\langle T_1 u_\varepsilon, u_\varepsilon\rangle_{X^1} = \lambda + o(1)$, therefore we can write

$$\begin{aligned}\|(T_1-\lambda)u_\varepsilon\|_{X^1}^2 &= \|T_1 u_\varepsilon\|_{X^1}^2 + \lambda^2 - 2\lambda\left\langle T_1 u_\varepsilon, u_\varepsilon\right\rangle_{X^1} \\ &= \|T_1 u_\varepsilon\|_{X^1}^2 - \lambda^2 + o(1),\end{aligned}$$

but since $\|T_1 f_\varepsilon\|_{X^1} = o(1)$ and $\|\tilde{u}_\varepsilon\|_{X^1}^{-1} = O(1)$ we obtain

$$\begin{aligned}\|T_1 u_\varepsilon\|_{X^1}^2 &= \left\|\frac{\lambda}{1-\lambda\varepsilon^2}u_\varepsilon - \frac{\lambda}{(1-\lambda\varepsilon^2)\,\|\tilde{u}_\varepsilon\|_{X^1}}T_1 f_\varepsilon\right\|_{X^1}^2 \\ &= \frac{\lambda^2}{(1-\lambda\varepsilon^2)^2} + \frac{\lambda^2}{(1-\lambda\varepsilon^2)^2}\frac{\|T_1 f_\varepsilon\|_{X^1}^2}{\|\tilde{u}_\varepsilon\|_{X^1}^2} - \frac{2\lambda^2}{(1-\lambda\varepsilon^2)^2\,\|\tilde{u}_\varepsilon\|_{X^1}}\left\langle T_1 f_\varepsilon, u_\varepsilon\right\rangle_{X^1} \\ &= \lambda^2 + o(1),\end{aligned}$$

thus

$$\|(T_1 - \lambda u_\varepsilon\|_{X^1}^2 = o(1),$$

and the second part is done.

Finally, observe that by the definition of T_1 one has for $v \in X^1_0$

$$\begin{aligned}\langle T_1 u_\varepsilon, v\rangle_{X^1} &= \langle u_\varepsilon, v\rangle \\ &= \int_0^\varepsilon u_\varepsilon(x) v(x)\,\mathrm{d}x \\ &\le \left(\int_0^\varepsilon |v(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}},\end{aligned}$$

therefore

$$\begin{aligned}|\langle u_\varepsilon, v\rangle_{X^1}| &\le \frac{1}{\lambda}\,|\langle T_1 u_\varepsilon, v\rangle_{X^1}| + \frac{1}{\lambda}\,|\langle (T_1-\lambda)u_\varepsilon, v\rangle_{X^1}| \\ &\le \frac{1}{\lambda}\left(\int_0^\varepsilon |v(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} + \frac{1}{\lambda}\,\|(T_1-\lambda)u_\varepsilon\|_{X^1}\,\|v\|_{X^1} \\ &= \frac{1}{\lambda}\left(\int_0^\varepsilon |v(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} + o(1) \\ &\underset{\varepsilon\to 0}{\longrightarrow} 0,\end{aligned}$$

hence u_ε converges weakly to 0 in X^1_0. ■

Proof of Theorem 1 when $\alpha = 1$. On the one hand, by Proposition 1 and Lemma 1 we deduce that $(0, \frac{4}{5}) \subseteq \sigma_e(T_A)$. On the other hand, in [9] it is established that $\max \sigma_e(T_A) = \frac{4}{5}$, and since T_A is a positive operator we obtain $\sigma_e(T_A) \subseteq [0, \frac{4}{5}]$. Thus we have $(0, \frac{4}{5}) \subseteq \sigma_e(T_A) \subseteq [0, \frac{4}{5}]$, but the essential spectrum is closed, consequently we deduce that $\sigma_e(T_A) = [0, \frac{4}{5}]$ as stated in the theorem. ■

4. Proof of Theorem 1: case $\alpha > 1$

For $\alpha > 1$ and $0 < \lambda < 1$, let $\mu = \frac{1}{\lambda}$ and $\beta = \frac{\sqrt{\mu-1}}{\alpha-1}$ and consider

$$w_\varepsilon(x) = \varepsilon^{\frac{\alpha}{2}} x^{-\frac{\alpha}{2}} \sin(\beta x^{1-\alpha}).$$

A direct computation shows that w_ε is a solution of

$$-(x^{2\alpha} w_\varepsilon'(x))' + (1-\mu) w_\varepsilon(x) = g_\varepsilon(x) \quad \text{in } (0,1)$$

where

$$g_\varepsilon(x) = \frac{\alpha}{2}\left(\frac{3\alpha}{2} - 1\right) \varepsilon^{\frac{\alpha}{2}} x^{\frac{3\alpha}{2}-2} \sin(\beta x^{1-\alpha}).$$

Let ρ be a smooth cut-off function with the same properties described in (9)-(12), and let $\eta(x) = 1 - \rho(x)$. For $\varepsilon > 0$ small so that $\varepsilon^{\alpha-1} < \frac{1}{2}$, define $\zeta_\varepsilon(x) = \rho\left(\frac{x}{\varepsilon}\right) \eta\left(\frac{x}{\varepsilon^\alpha}\right)$ and let

$$\tilde{u}_\varepsilon(x) = w_\varepsilon(x)\zeta_\varepsilon(x).$$

Observe that $\operatorname{supp} \tilde{u}_\varepsilon \subseteq [\frac{\varepsilon^\alpha}{2}, \varepsilon]$ and that $\tilde{u}_\varepsilon$ is a solution to

$$\begin{cases} -(x^{2\alpha}\tilde{u}_\varepsilon')' + (1-\mu)\tilde{u}_\varepsilon = f_\varepsilon & \text{in } (0,1), \\ \tilde{u}_\varepsilon \in X^\alpha_0, \end{cases} \tag{14}$$

where

$$f_\varepsilon(x) = g_\varepsilon(x)\zeta_\varepsilon(x) - 2x^{2\alpha} w_\varepsilon'(x)\zeta_\varepsilon'(x) - w_\varepsilon(x)(x^{2\alpha}\zeta_\varepsilon'(x))'.$$

If we write (14) in terms of the operator T_α we have

$$(T_\alpha - \lambda)\tilde{u}_\varepsilon = -\lambda T_\alpha f_\varepsilon,$$

for $\lambda = \frac{1}{\mu}$.

Lemma 2. *Let* $0 < \lambda < 1$ *let* $u_\varepsilon := \tilde{u}_\varepsilon / \|\tilde{u}_\varepsilon\|_{X^\alpha}$. *As* ε *goes to zero we have*

(i) $\langle T_\alpha u_\varepsilon, u_\varepsilon \rangle_{X^\alpha} = \lambda + o(1)$.
(ii) $\|(T_\alpha - \lambda) u_\varepsilon\|_{X^\alpha} = o(1)$.
(iii) $u_\varepsilon \longrightarrow 0$ *in the weak topology of* X_0^α.

Proof. Observe that $T_\alpha \tilde{u}_\varepsilon = \lambda \tilde{u}_\varepsilon - \lambda T_\alpha f_\varepsilon$, hence

$$\langle T_\alpha u_\varepsilon, u_\varepsilon \rangle_{X^\alpha} = \lambda - \frac{\lambda}{\|\tilde{u}_\varepsilon\|_{X^\alpha}^2} \langle T_\alpha f_\varepsilon, \tilde{u}_\varepsilon \rangle_{X^\alpha} .$$

Following the same argument used in the case $\alpha = 1$, it is enough to find an appropriate upper bound for $\|f_\varepsilon\|_{L^2}$ and a lower bound for $\|\tilde{u}_\varepsilon\|_{X^\alpha}$ to show that the last term goes to zero as ε goes to zero. We begin by the lower bound on $\|\tilde{u}_\varepsilon\|_{X^\alpha}$: since $\zeta_\varepsilon \equiv 1$ on $[\varepsilon^\alpha, \frac{\varepsilon}{2}]$ we can write

$$\begin{aligned}
\|\tilde{u}_\varepsilon\|_{X^\alpha}^2 &\geq \|w_\varepsilon \zeta_\varepsilon\|_{L^2}^2 \\
&\geq \int_{\varepsilon^\alpha}^{\frac{\varepsilon}{2}} |w_\varepsilon(x)|^2 \,\mathrm{d}x \\
&= \varepsilon^\alpha \int_{\varepsilon^\alpha}^{\frac{\varepsilon}{2}} \left| x^{-\frac{\alpha}{2}} \sin\left(\beta x^{1-\alpha}\right) \right|^2 \,\mathrm{d}x \\
&= \frac{\varepsilon^\alpha}{\sqrt{\mu - 1}} \int_{\beta\left(\frac{\varepsilon}{2}\right)^{1-\alpha}}^{\beta \varepsilon^{\alpha(1-\alpha)}} \sin^2(t) \,\mathrm{d}t \\
&= \frac{\varepsilon^\alpha}{2\sqrt{\mu - 1}} \left(t - \sin(t)\cos(t)\right) \Big|_{t=\beta\left(\frac{\varepsilon}{2}\right)^{1-\alpha}}^{t=\beta\varepsilon^{\alpha(1-\alpha)}} \\
&= \frac{\varepsilon^{\alpha(2-\alpha)}}{2(\alpha - 1)} \left(1 + o(1)\right),
\end{aligned}$$

because $\alpha(2-\alpha) < 1 < \alpha$.

We now estimate $\|f_\varepsilon\|_{L^2}$. To do this, observe the following obvious estimates on g_ε, w_ε, and ζ_ε on $[0,1]$:

$$\begin{aligned}
|g_\varepsilon(x)| &\leq C \varepsilon^{\frac{\alpha}{2}} x^{\frac{3\alpha}{2} - 2} \\
|w_\varepsilon(x)| &\leq \varepsilon^{\frac{\alpha}{2}} x^{-\frac{\alpha}{2}} \\
|w_\varepsilon'(x)| &\leq C \varepsilon^{\frac{\alpha}{2}} x^{-\frac{3\alpha}{2}}.
\end{aligned}$$

Additionally, recall that ρ and η are smooth functions with uniformly bounded derivatives up to the second order, consequently

$$\begin{aligned}
|\zeta_\varepsilon'(x)| &\leq \frac{C}{\varepsilon^\alpha} \\
|\zeta_\varepsilon''(x)| &\leq \frac{C}{\varepsilon^{2\alpha}},
\end{aligned}$$

for all $x \in (0,1]$. With these estimates in mind it follows

$$\|g_\varepsilon \zeta_\varepsilon\|_{L^2}^2 \le \varepsilon^\alpha \int_0^\varepsilon x^{3\alpha-4}\,\mathrm{d}x \le C\varepsilon^{4\alpha-3},$$
$$\left\|x^{2\alpha} w_\varepsilon' \zeta_\varepsilon'\right\|_{L^2}^2 \le C\varepsilon^{-\alpha} \int_0^\varepsilon x^{\alpha}\,\mathrm{d}x \le C\varepsilon,$$
$$\left\|x^{2\alpha} w_\varepsilon \zeta_\varepsilon''\right\|_{L^2}^2 \le C\varepsilon^{-3\alpha} \int_0^\varepsilon x^{3\alpha}\,\mathrm{d}x \le C\varepsilon,$$

and

$$\left\|x^{2\alpha-1} w_\varepsilon \zeta_\varepsilon'\right\|_{L^2}^2 \le C\varepsilon^{-\alpha} \int_0^\varepsilon x^{3\alpha-2}\,\mathrm{d}x \le C\varepsilon^{2\alpha-1},$$

therefore

$$\|f_\varepsilon\|_{L^2}^2 \le C\varepsilon,$$

because $2\alpha - 1 > 1$ and $4\alpha - 3 > 1$. Using once again that $\alpha > 1$ we see that

$$\left|\frac{1}{\|\tilde{u}_\varepsilon\|_{X^\alpha}} \langle T_\alpha f_\varepsilon, u_\varepsilon\rangle_{X^\alpha}\right|^2 \le C\frac{\|f_\varepsilon\|_{L^2}^2}{\|\tilde{u}_\varepsilon\|_{L^2}^2} \le C\varepsilon^{1-\alpha(2-\alpha)} = C\varepsilon^{(\alpha-1)^2} = o(1)$$

as claimed.

For the second part of the lemma notice

$$\begin{aligned}\|(T_\alpha - \lambda)u_\varepsilon\|_{X^\alpha}^2 &= \|T_\alpha u_\varepsilon\|_{X^\alpha}^2 + \lambda^2 - 2\lambda \langle T_\alpha u_\varepsilon, u_\varepsilon\rangle_{X^\alpha} \\ &= \|T_\alpha u_\varepsilon\|_{X^\alpha}^2 - \lambda^2 + o(1),\end{aligned}$$

and since $\|T_\alpha f_\varepsilon\|_{X^\alpha} \cdot \|\tilde{u}_\varepsilon\|_{X^\alpha}^{-1} = o(1)$ we deduce that

$$\begin{aligned}\|T_\alpha u_\varepsilon\|_{X^\alpha}^2 &= \left\|\lambda u_\varepsilon - \lambda \frac{T_\alpha f_\varepsilon}{\|\tilde{u}_\varepsilon\|_{X^\alpha}}\right\|^2 \\ &= \lambda^2 + \frac{\lambda^2}{\|\tilde{u}_\varepsilon\|_{X^\alpha}^2} \|T_\alpha f_\varepsilon\|_{X^\alpha}^2 - 2\frac{\lambda^2}{\|\tilde{u}_\varepsilon\|_{X^\alpha}} \langle T_\alpha f_\varepsilon, u_\varepsilon\rangle_{X^\alpha} \\ &= \lambda^2 + o(1),\end{aligned}$$

therefore

$$\|(T_\alpha - \lambda)u_\varepsilon\|_{X^\alpha}^2 = o(1)$$

and the second part is proved. Finally, observe that since $\operatorname{supp} u_\varepsilon \subseteq [0, \varepsilon]$, we can write for $v \in X_0^\alpha$

$$\begin{aligned}\langle T_\alpha u_\varepsilon, v\rangle_{X^\alpha} &= \langle u_\varepsilon, v\rangle \\ &= \int_0^\varepsilon u_\varepsilon(x) v(x)\,\mathrm{d}x \\ &\le \|u_\varepsilon\|_{L^2} \left(\int_0^\varepsilon |v(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\ &\le \left(\int_0^\varepsilon |v(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\ &= o(1),\end{aligned}$$

consequently

$$\begin{aligned}|\langle u_\varepsilon, v\rangle_{X^\alpha}| &\leq \frac{1}{\lambda}\left|\langle T_\alpha u_\varepsilon, v\rangle_{X^\alpha}\right| + \frac{1}{\lambda}\left|\langle (T_\alpha-\lambda)u_\varepsilon, v\rangle_{X^\alpha}\right| \\ &\leq \frac{1}{\lambda}\left(\int_0^\varepsilon |v(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} + \frac{1}{\lambda}\left\|(T_\alpha-\lambda)\tilde{u}_\varepsilon\right\|_{X^\alpha}\|v\|_{X^\alpha} \\ &= \frac{1}{\lambda}\left(\int_0^\varepsilon |v(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} + o(1) \\ &\underset{\varepsilon\to 0}{\longrightarrow} 0,\end{aligned}$$

thus u_ε converges weakly to 0.

■

Proof of Theorem 1 when $\alpha > 1$. From Proposition 1 and Lemma 2 we deduce that the interval $(0,1)$ is contained in $\sigma_e(T_A)$. On the other hand, since T_A is a positive self-adjoint operator for the inner product $a(\cdot,\cdot)$, and if we recall that $\|u\|_a^2 = a(u,u)$ we can write

$$\begin{aligned}\max\sigma(T_A) &= \sup\left\{\frac{a(T_A u,u)}{\|u\|_a^2} : u\in X_0^\alpha\setminus\{0\}\right\} \\ &= \sup\left\{\frac{\langle u,u\rangle}{\|u\|_a^2} : u\in X_0^\alpha\setminus\{0\}\right\}\end{aligned}$$

but

$$\frac{\langle u,u\rangle}{\|u\|_a^2} = \frac{\int_0^1 u(x)^2\,\mathrm{d}x}{\int_0^1 A(x)u'(x)^2\,\mathrm{d}x + \int_0^1 u(x)^2\,\mathrm{d}x} \leq 1,$$

for all $u\in X_0^\alpha\setminus\{0\}$, thus $\max\sigma(T_A)\leq 1$, and as a consequence $\sigma(T_A)\subseteq[0,1]$.

Summarizing, we have shown the following chain of inclusions

$$(0,1)\subseteq\sigma_e(T_A)\subseteq\sigma(T_A)\subseteq[0,1],$$

and since both $\sigma_e(T_A)$ and $\sigma(T_A)$ are closed, the result is proved. ■

5. The differential operator L_A

For $\alpha > 0$ we have defined the differential operator

$$L_A u(x) := -(A(x)u'(x))' + u(x)$$

over the interval $(0,1)$ for A satisfying (H1) and (H2). For this kind of operator it is natural to introduce the following L^2-framework: define D as the set

$$D = \left\{u\in H^2_{loc}(0,1) \,:\, u,\ (A(x)u')'\in L^2\right\},$$

and observe that the weight $A(x)$ only introduces possible singularities near the origin, therefore it is straightforward to notice that for $u\in D$ one has, after possible modifying u on a set of measure zero, that

$$u\in C^1(0,1] \text{ and } A(x)u'\in C[0,1].$$

In order to relate L_A to T_A we will follow the work of Stuart in [9, Section 6] where the relationship between L_A and T_A has been established for $\alpha\leq 1$. To do this we need to add boundary conditions to the differential operator, and the natural ones are the homogeneous boundary conditions

$$u(1) = A(x)u'(x)\Big|_{x=0} = 0.$$

With this in mind we consider the operator $L_A : D(L_A) \subset L^2 \longrightarrow L^2$, where

$$D(L_A) = \left\{ u \in D \,:\, u(1) = A(x)u'(x)\Big|_{x=0} = 0 \right\}.$$

We begin the study of this operator by recalling a density result from [2]

Lemma 3 (Lemma A.4 in [2])**.** *For each $\alpha \geq \frac{1}{2}$, the space $C_0^\infty(0,1)$ is dense in X_0^α.*

Proposition 2. *For $\alpha \geq 1$, $D(L_A) \subset X_0^\alpha$ and the inclusion is dense in the X^α-topology.*

To prove this proposition, we need the following

Lemma 4. *Let $\alpha \geq 1$, $u \in D(L_A)$ and $v \in L^2 \cap C(0,1]$. For each positive integer n, there exists $x_n < \frac{1}{n}$ such that $|A(x_n)u'(x_n)v(x_n)| \leq \frac{1}{n}$*

Proof. Indeed, take $u \in D(L_A)$ and $v \in L^2$. Observe that we can write

$$\frac{A(x)}{x} u'(x) = \frac{1}{x} \int_0^x (A(s)u'(s))' \,\mathrm{d}s,$$

thus by Hardy's inequality

$$\left\| \frac{A(x)}{x} u'(x) \right\|_{L^2} \leq C \left\| (A(x)u'(x))' \right\|_{L^2}.$$

This estimate implies that $x^{-1}A(x)u'(x)v(x)$ belongs to $L^1(0,1)$, indeed, by Hölder's inequality

$$\begin{aligned} \left\| \frac{A(x)}{x} u'(x)v(x) \right\|_{L^1} &\leq \left\| \frac{A(x)}{x} u'(x) \right\|_{L^2} \|v\|_{L^2} \\ &\leq C \left\| (A(x)u'(x))' \right\|_{L^2} \|v\|_{L^2}. \end{aligned}$$

We can now prove the lemma by contradiction: if the statement of the lemma were false, then there would exist a number $r > 0$ such that for all $x < r$

$$|A(x)u'(x)v(x)| > r,$$

but such an inequality would contradict the fact that $x^{-1}A(x)u'(x)v(x) \in L^1(0,1)$. The lemma is now proved. ■

Proof of Proposition 2. Our first claim is that $D(L_A) \subset X_0^\alpha$. Indeed, notice that the function $(A(x)u'(x))'u(x)$ belongs to $L^1(0,1)$, therefore we can write

$$\int_0^1 (A(x)u'(x))'u(x) \,\mathrm{d}x = \lim_{n\to\infty} \int_{x_n}^1 (A(x)u'(x))'u(x) \,\mathrm{d}x,$$

where x_n is the sequence from Lemma 4 for $v = u$. Since $u \in C^1(0,1]$ with $u(1) = 0$ we can integrate by parts over the interval $(x_n, 1)$ to obtain

$$\int_{x_n}^1 (A(x)u'(x))'u(x) \,\mathrm{d}x = -\int_{x_n}^1 A(x)u'(x)^2 \,\mathrm{d}x - A(x_n)u'(x_n)u(x_n),$$

therefore

$$\begin{aligned}\int_{x_n}^1 A(x)u'(x)^2\,\mathrm{d}x &= -\int_{x_n}^1 (A(x)u'(x))'u(x)\,\mathrm{d}x - A(x_n)u'(x_n)u(x_n)\\ &= -\int_0^1 (A(x)u'(x))'u(x)\,\mathrm{d}x + o(1),\end{aligned}$$

where $o(1)$ is a quantity that goes to 0 as $n \to \infty$. The monotone convergence theorem implies that

$$\int_0^1 A(x)u'(x)^2\,\mathrm{d}x = \lim_{n\to\infty}\int_{x_n}^1 A(x)u'(x)^2\,\mathrm{d}x = -\int_0^1 (A(x)u'(x))'u(x)\,\mathrm{d}x,$$

and we conclude $u \in X_0^\alpha$. Finally, observe that $C_0^\infty(0,1)$ is contained in $D(L_A)$, therefore Lemma 3 tells us that $D(L_A)$ must be also dense in X_0^α.

■

Remark 3. Observe that in the proof of Proposition 2 we have established the following identity

$$\int_0^1 A(x)u'(x)^2\,\mathrm{d}x = -\int_0^1 (A(x)u'(x))'u(x)\,\mathrm{d}x$$

for all $u \in D(L_A)$. Moreover, the same argument tells us that

$$\int_0^1 A(x)u'(x)v'(x)\,\mathrm{d}x = -\int_0^1 (A(x)u'(x))'v(x)\,\mathrm{d}x$$

holds for all $u, v \in D(L_A)$.

The following proposition is a well-known result in Sturm-Liouville theory, but for the sake of completeness we provide its proof.

Proposition 3. *Let $\alpha \geq 1$ and L_A be as before.*

(i) $D(L_A)$ is dense in L^2.
(ii) L_A is positive and self-adjoint.
(iii) If $u \in D(L_A)$ and $v \in X_0^\alpha$, then

$$\langle L_A u, v\rangle = a(u,v)$$

Proof. The density result follows directly from the density of $C_0^\infty(0,1)$ in L^2. Observe that thanks to Remark 3 one has that if $u, v \in D(L_A)$ then

$$\begin{aligned}\langle L_A u, v\rangle &= \int_0^1 \left(-(A(x)u'(x))' + u(x)\right)v(x)\,\mathrm{d}x\\ &= \int_0^1 \left(A(x)u'(x)v'(x) + u(x)v(x)\right)\,\mathrm{d}x\\ &= a(u,v)\\ &= \int_0^1 \left(-(A(x)v'(x))' + v(x)\right)u(x)\,\mathrm{d}x\\ &= \langle u, L_A v\rangle\,.\end{aligned}$$

Recall that the adjoint operator is defined by $L_a^* : D(L_a^*) \subset L^2 \longrightarrow L^2$, where

$$D(L_A^*) = \left\{v \in L^2 \,:\, \exists f \in L^2 \text{ such that } \langle L_A u, v\rangle = \langle u, f\rangle \text{ for all } u \in D(L_A)\right\},$$

in which case $L_A^*(v) = f$. The above computation tells us that $L_A \subset L_A^*$. To prove the reverse inclusion, we only need to show that $D(L_A^*) \subset D(L_A)$. Indeed, let $v \in D(L_A^*)$, then there exists $f \in L^2$ such that

$$\langle L_A u, v\rangle = \langle u, f\rangle \quad \text{for all } u \in D(L_A). \tag{15}$$

In particular, following the argument in [9, Theorem 6.2], for each $w \in C_0^\infty(0,1)$ we can consider for $s \in (0,1)$ the function

$$U[w](s) = -\int_s^1 \frac{w(t)}{A(t)}\,\mathrm{d}t,$$

and show that $U[w] \in D(L_A) \cap C^1[0,1]$ with $L_A U[w] = -w' + U[w]$. If we use $u = U[w]$ in (15) we obtain

$$\begin{aligned} -\int_0^1 w'(s)v(s)\,\mathrm{d}s &= \int_0^1 U[w](s)(f(s)-v(s))\,\mathrm{d}s \\ &= -\int_0^1 (f(s)-v(s))\int_s^1 \frac{w(t)}{A(t)}\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_0^1 w(t)\left[\frac{1}{A(t)}\int_0^t (v(s)-f(s))\,\mathrm{d}s\right]\mathrm{d}t, \end{aligned}$$

because $\frac{1}{A(t)}\int_0^t (v(s)-f(s))\,\mathrm{d}s \in L^1_{loc}(0,1)$. The above computations says that v has a weak derivative and that

$$v'(s) = \frac{1}{A(t)}\int_0^t (v(s)-f(s))\,\mathrm{d}s \qquad \text{a.e. in } (0,1).$$

From here we deduce that $v \in C(0,1]$ with $Av'|_{s=0} = 0$ and that $(Av')' = v - f$ belongs to L^2. Therefore, to prove that $v \in D(L_A)$ we only need to show that $v(1) = 0$, to do this observe that for each $u \in D(L_A) \cap C[0,1]$ we have

$$\begin{aligned} \langle u, f\rangle &= \langle L_A u, v\rangle \\ &= \langle L_A u, v - v(1)\rangle + \langle L_A u, v(1)\rangle \\ &= \langle u, L_A(v - v(1))\rangle + v(1)\int_0^1 L_A u(s)\,\mathrm{d}s \\ &= \int_0^1 u(s)\left(-(A(s)v'(s))' + v(s) - v(1)\right)\,\mathrm{d}s - A(1)u'(1)v(1) \\ &\qquad + v(1)\int_0^1 u(s)\,\mathrm{d}s \\ &= \int_0^1 u(s)f(s)\,\mathrm{d}s - A(1)u'(1)v(1). \end{aligned}$$

Hence $A(1)u'(1)v(1) = 0$ for all $u \in D(L_A) \cap C[0,1]$, therefore $v(1) = 0$. This shows that L_A is self-adjoint. Also, from Remark 3 we deduce

$$\langle L_A u, u\rangle = a(u,u) \geq K_2 \|u\|^2_{X^\alpha} \geq K_2 \|u\|^2,$$

showing that L_A is positive. Finally, Remark 3 also tells us that for $u \in D(L_A)$ and $v \in X_0^\alpha$ we have

$$\langle L_A u, v\rangle = a(u,v).$$

■

Proposition 3 tells us that L_A is a positive self-adjoint operator, therefore there exists a unique positive square root operator (see for example [5, Theorem V.3.35]), denoted by $L_A^{1/2}$ satisfying: $D(L_A) \subset D(L_A^{1/2})$ and that $(D(L_A^{1/2}), \langle\cdot,\cdot\rangle_{L_A^{1/2}})$ is a Hilbert space, where for $u, v \in D(L_A^{1/2})$ one has

$$\langle u, v\rangle_{L_A^{1/2}} = \langle u, v\rangle + \left\langle L_A^{1/2} u, L_A^{1/2} v\right\rangle.$$

In addition, the inclusion $D(L_A) \subset D(L_A^{1/2})$ is dense. We have the following result due to Stuart [7] in the context of general self adjoint operators over real Hilbert spaces:

Proposition 4. *Let $L_A^{1/2}$ be as before, then*

(i) There exists a unique operator $B_1 : D(L_A^{1/2}) \to D(L_A^{1/2})$ such that

$$\langle L_A u, v\rangle = \langle B_1 u, v\rangle_{L_A^{1/2}}.$$

(ii) There exists a unique operator $B_2 : D(L_A^{1/2}) \to D(L_A^{1/2})$ such that

$$\langle u, v\rangle = \langle B_2 u, v\rangle_{L_A^{1/2}}.$$

(iii) $\sigma(L_A) = \left\{\mu \in \mathbb{R} : B_1 - \mu B_2 \text{ is not an isomorphism in } D(L_A^{1/2})\right\}$.

(iv) $\sigma_e(L_A) = \left\{\mu \in \mathbb{R} : B_1 - \mu B_2 \text{ is not Fredholm in } D(L_A^{1/2})\right\}$.

Remark 4. We have that the Hilbert spaces $(D(L_A^{1/2}), \langle\cdot,\cdot\rangle_{L_A^{1/2}})$ and $(X_0^\alpha, \langle\cdot,\cdot\rangle_{X_0^\alpha})$ are equivalent. Indeed, for $u, v \in D(L_a)$ we have $\|u\|^2_{L_A^{1/2}} = \langle u, u\rangle + \langle L_a u, u\rangle$, hence

$$a(u,u) = \langle L_A u, u\rangle \leq \|u\|^2_{L_A^{1/2}} = \langle u, u\rangle + \langle L_a u, u\rangle \leq 2a(u,u).$$

The conclusion follows by recalling that $D(L_A)$ is dense in both $D(L_A^{1/2})$ and X_0^α.

In addition, we have that for $u, v \in D(L_A^{1/2}) = X_0^\alpha$

$$\langle u, v\rangle_{L_A^{1/2}} = \langle u, v\rangle + a(u,v) = a((T_A + I)u, v)$$

by the definition of T_A.

We can now prove Theorem 2.

Proof. We follow the proof of [9, Theorem 6.4]. Observe that thanks to Proposition 4 we can write

$$\langle B_1 u, v\rangle_{L_A^{1/2}} = \langle L_A u, v\rangle = a(u,v)$$

for all $u \in D(L_A)$ and $v \in X_0^\alpha$, but by density we conclude that this holds for all $u, v \in X_0^\alpha$. In addition, from Remark 4 we have

$$a(u,v) = \langle B_1 u, v\rangle_{L_A^{1/2}} = a((T_A + I)B_1 u, v) \quad \text{for all } u, v \in X_0^\alpha,$$

hence $(T_A + I)B_1 = I : X_0^\alpha \longrightarrow X_0^\alpha$. On the other hand

$$a((T_A + I)B_2 u, v) = \langle B_2 u, v\rangle_{L_A^{1/2}} = \langle u, v\rangle = a(T_A u, v) \quad \text{for all } u, v \in X_0^\alpha,$$

thus $(T_A + I)B_2 = T_A : X_0^\alpha \longrightarrow X_0^\alpha$. In particular we have that for every $\lambda \in \mathbb{R} \setminus \{0\}$

$$T_A - \lambda I = -\lambda(T_A + I)(B_1 - \frac{1}{\lambda}B_2),$$

and recall that T_A is a positive operator, in particular $-1 \in \rho(T_A)$, thus $T_A + I$ is an isomorphism, and the conclusion about the spectrum follows from Proposition 4.

For the last part, observe that $0 \notin \sigma(L_A)$, indeed, observe that for each $f \in L^2$ the equation

$$a(u, v) = \langle f, v \rangle \quad \text{for all } v \in X_0^\alpha$$

has a unique solution in $u \in X_0^\alpha$. Also, since this unique solution $u = T_A f \in X_0^\alpha \subset L^2$ satisfies equation (3), we see that $(A(x)u'(x))' \in L^2$, therefore $u \in D(L_A)$. This shows that the equation $L_A u = f$ has a unique solution in $D(L_A)$, and as a consequence the inverse operator $L_A^{-1} : L^2 \longrightarrow D(L_A)$ is well defined. Finally, using Proposition 3 we see that for $u \in L^2$ and for $v \in X_0^\alpha$ we can write

$$a(L_A^{-1}u, v) = \left\langle L_A(L_A^{-1}u), v \right\rangle = \langle u, v \rangle .$$

Similarly, for $u \in X_0^\alpha$ and $v \in X_0^\alpha$ we have

$$a(T_A u, v) = \langle u, v \rangle ,$$

therefore $a(T_A u, v) = a(L_A^{-1}u, v)$ for all $u, v \in X_0^\alpha$, thus $T_A = L_A^{-1}\Big|_{X_0^\alpha}$. ■

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