UNIQUENESS RESULTS FOR A SINGULAR NON-LINEAR STURM-LIOUVILLE EQUATION

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Abstract. In this work we study the uniqueness of solutions to the following singular non-linear Sturm-Liouville equation

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + u^p & \text{in } (0,1), \\ u > 0 & \text{in } (0,1), \\ u(1) = 0, \end{cases}$$

where $0 < \alpha < 1$, $p > 1$ and $\lambda \in \mathbb{R}$ are parameters.

We show that when $0 < \alpha \leq \frac{1}{2}$ and $p > 1$, and when $\frac{1}{2} < \alpha < 1$ and $1 < p \leq \frac{3-2\alpha}{2\alpha-1}$ uniqueness of solutions is guaranteed to hold when one imposes some appropriate behavior at the origin.

1. Introduction

We are interested in the problem of uniqueness a function u satisfying the non-linear singular Sturm-Liouville equation

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + u^p & \text{in } (0,1), \\ u > 0 & \text{in } (0,1), \\ u(1) = 0, \end{cases} \tag{1}$$

where $0 < \alpha < 1$, $p > 1$ and $\lambda \in \mathbb{R}$. More precisely, we want to understand under what condition at the origin equation (1) has at most one solution.

In [3] we proved existence of solutions to equation (1) that belong to $C[0,1]$, and now we would like to show that those solutions are in fact unique in their respective classes. One of the solutions obtained in [3], hereafter denoted by u_D, was obtained by imposing the Dirichlet condition $\lim_{x\to 0^+} u_D(x) = 0$, and the next result shows that u_D is in fact unique in this class.

Theorem 1 (Uniqueness of the Dirichlet problem)**.** *Let $0 < \alpha < \frac{1}{2}$, $\lambda \in \mathbb{R}$ and $p > 1$, then equation*

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + u^p & \text{in } (0,1), \\ u(1) = 0, \\ \lim_{x\to 0^+} u(x) = 0, \end{cases} \tag{2}$$

has at most one positive solution.

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The other solution obtained in [3] when $0 < \alpha < \frac{1}{2}$, denoted by u_N, was obtained by imposing the Neumann-type condition $\lim_{x\to 0^+} x^{2\alpha} u_N'(x) = 0$. It was established in [3] that this solution has nicer regularity, namely $x^{2\alpha-1}u_N' \in C[0,1]$, which for $0 < \alpha < \frac{1}{2}$ implies that u_N is in fact $C^1[0,1]$ and that $u_N'(0) = 0$. This second solution is also unique, as the following theorem shows:

Theorem 2 (Uniqueness of the Neumann problem)**.** *Let $0 < \alpha < \frac{1}{2}$, $\lambda \in \mathbb{R}$ and $p > 1$, then equation*

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + u^p \quad \textit{in } (0,1), \\ u(1) = 0, \\ x^{2\alpha-1}u' \in C[0,1], \end{cases} \tag{3}$$

has at most one positive solution.

The case $\frac{1}{2} \le \alpha < 1$ is a little more delicate, as uniqueness seems to depend on the exponent $p > 1$. As seen in [3], the exponent $p = \frac{3-2\alpha}{2\alpha-1}$ plays a role in the existence question. This can be seen from the fact that the weighted Sobolev spaces X^α, introduced in [4], are embedded into $L^{q+1}(0,1)$ if and only if $1 \le q \le \frac{3-2\alpha}{2\alpha-1}$, and in this case a solution to equation (1) can be produced by minimizing a suitable energy functional. This exponent turns out to be critical also for the uniqueness.

In [4] it was proved that for $\frac{1}{2} \le \alpha < 1$ the operator $-(x^{2\alpha}u')'$ has a natural boundary condition that can be imposed at the origin, and this is what we called the "Canonical" condition $\lim_{x\to 0} x^{2\alpha}u'(x) = 0$. We proved that for $1 < p \le \frac{3-2\alpha}{2\alpha-1}$ and suitable λ, equation (1) has at least one solution under this boundary condition, which we hereafter denote by u_C. This solution has the same property as u_N, namely $x^{2\alpha-1}u_C' \in C[0,1]$, and when $1 < p \le \frac{3-2\alpha}{2\alpha-1}$, this is enough to make u_C unique, as the following theorem shows.

Theorem 3 (Uniqueness of the "Canonical" Problem)**.** *Let $\frac{1}{2} \le \alpha < 1$, $\lambda \in \mathbb{R}$ and suppose $1 < p \le \frac{3-2\alpha}{2\alpha-1}$, then equation*

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + u^p \quad \textit{in } (0,1), \\ u(1) = 0, \\ x^{2\alpha-1}u'(x) \in C[0,1], \end{cases} \tag{4}$$

has at most one positive solution.

When $\frac{1}{2} < \alpha < 1$, $p > \frac{3-2\alpha}{2\alpha-1}$ and $\lambda > 0$ is sufficiently close to the first eigenvalue of the operator $-(x^{2\alpha}u')'$ under the "Canonical" boundary condition, bifurcation theory guarantees the existence of regular solutions to equation (1) (that is, a solution satisfying $u \in C[0,1]$ and $x^{2\alpha-1}u' \in C[0,1]$), however such solutions are not necessarily unique. This phenomenon had already been noticed in the study of the equation

$$\begin{cases} -\Delta u = \lambda u + u^p & \text{in } B(0,1) \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial B(0,1), \\ u > 0 & \text{in } B(0,1), \end{cases} \tag{5}$$

for $N > 2$, $p > \frac{N+2}{N-2}$ and $\lambda > 0$ sufficiently close to the first eigenvalue of $-\Delta$. For equation (5), existence is also guaranteed by bifurcation theory, but uniqueness is known to fail, as it can be seen in [2,7]. It was pointed out in [3] that equation (1) and equation (5) are related through a change of variable, so when $\frac{1}{2} < \alpha < 1$ and $p > \frac{3-2\alpha}{2\alpha-1}$ any attempt to prove uniqueness is guaranteed to fail.

The exposition of this paper is divided as follows. In section 2 we establish some preliminary results. In section 3 we prove Theorems 2 and 3, and then prove Theorem 1 in section 4.

2. Preliminaries

The following is an important proposition which will allow us to simplify the proof of our theorems. In what follows, whenever we say "$p > 1$ is sub-critical" we will mean that:

- ⋄ $p > 1$, if $0 < \alpha \leq \frac{1}{2}$, or
- ⋄ $1 < p \leq \frac{3-2\alpha}{2\alpha-1}$, if $\frac{1}{2} < \alpha < 1$.

Proposition 2.1. *Let $0 < \alpha < 1$, $\lambda \in \mathbb{R}$ and $p > 1$ be sub-critical. Suppose equation* (1) *has two distinct solutions $u_1, u_2 \in C[0,1] \cap C^2(0,1]$, such that $u_2'(1) < u_1'(1) < 0$. Then there exists a third solution $u_3 \in C[0,1] \cap C^2(0,1]$ such that $u_3'(1) \leq u_2'(1)$ and u_1 and u_3 intersect at most once in* $(0,1)$, *i.e.*

$$\# \{x \in (0,1) : u_1(x) = u_3(x)\} \leq 1.$$

To prove this proposition we need the following

Lemma 2.1. *Let $\lambda \in \mathbb{R}$, $p > 1$, $B \leq 0$, Suppose $V \in C^1[0,\infty)$ is such that both $\|V\|_{L^\infty(0,\infty)}$ and $\|V'\|_{L^1(0,\infty)}$ are finite. Let w be the unique solution of the initial value problem*

$$\begin{cases} w'' + \lambda w + |w|^{p-1} w = V(y)w + Bw' \quad \text{in } (0,\infty), \\ w(0) = 0, \\ w'(0) = 1. \end{cases} \tag{6}$$

Then $w \in W^{2,\infty}(0,\infty)$ with

$$\|w\|_{W^{2,\infty}} \leq C(\lambda, p, \|V\|_{L^\infty}, \|V'\|_{L^1}).$$

Remark 2.1. Notice that the constant which bounds $\|w\|_{2,\infty}$ *does not* depend on the constant $B \leq 0$.

Proof of Lemma 2.1. Let

$$E(w,y) = \frac{w'(y)^2}{2} + \frac{\lambda}{2} w(y)^2 + \frac{1}{p+1} |w(y)|^{p+1}.$$

By multiplying equation (6) by w' we can easily see that

$$\frac{d}{dy} E(w,y) = \frac{1}{2} V(y) \left(w(y)^2\right)' + Bw'(y)^2.$$

Now, let $\mathcal{A} = \left\{y > 0 : \max_{s \in [0,y]} w(s)^2 = w(y)^2\right\}$. Notice that since $w'(0) = 1$, we have that $(0,\varepsilon) \subset \mathcal{A}$ for small enough $\varepsilon > 0$, so $\mathcal{A}$ is not empty. For $y \in \mathcal{A}$ we integrate the above identity over $(0,y)$ to obtain

$$
\begin{aligned}
E(w,y)-E(w,0) &= \int_0^y \left(\frac{1}{2}V(s)\left(w(s)^2\right)' + Bw'(s)^2\right)ds,\\
&\le -\frac{1}{2}\int_0^y V'(s)w(s)^2 ds + \frac{1}{2}V(y)w(y)^2,\\
&\le \frac{1}{2}\left(\|V'\|_{L^1(0,\infty)} + \|V\|_{L^\infty(0,\infty)}\right)w(y)^2,
\end{aligned}
\tag{7}
$$

from where we deduce that

$$
\begin{aligned}
\frac{w'(y)^2}{2} + \frac{1}{2}\left[\lambda - \left(\|V'\|_{L^1(0,\infty)} + \|V\|_{L^\infty(0,\infty)}\right)\right]w(y)^2 + \frac{1}{p+1}|w(y)|^{p+1} &\le E(w,0)\\
&= \frac{1}{2}.
\end{aligned}
$$

Since the level sets of the function $h(x,y)=\frac{1}{2}y^2+\frac{1}{2}Rx^2+\frac{1}{p+1}|x|^{p+1}$ are bounded for all $R\in\mathbb{R}$, we obtain that $|w(y)|\le C$ for all $y\in\mathcal{A}$, where C does not depend on y. Therefore we deduce that

$$
|w(y)| \le C = C(\lambda, p, \|V\|_{L^\infty}, \|V'\|_{L^1})
$$

for all $y\ge 0$, because if this were not true, we could find a sequence such that $w(y_n)^2\to+\infty$ and, after maybe extracting a sub-sequence, that $y_n\in\mathcal{A}$, which we have shown to be impossible.

Now that we know that w is bounded, we obtain from estimate (7) and equation (6) that w' and w'' are also bounded. □

With lemma 2.1 in our pockets, we can now prove Proposition 2.1.

Proof of Proposition 2.1. To prove this result we will follow a proof by Kabeya and Tanaka in [5, Appendix A]. Without lost of generality, we will assume that

$$
\#\{x\in(0,1): u_1(x)=u_2(x)\}\ge 2,
$$

because otherwise we can simply take $u_3\equiv u_2$.

First of all notice that if u solves $-(x^{2\alpha}u')'=\lambda u+|u|^{p-1}u$ in $(0,1)$, then if one lets $c=-\frac{2-2\alpha}{p-1}<0$ and defines $w(y)=e^{cy}u(e^{-y})$, then w solves

$$
-w''+Bw'+Aw=\lambda e^{-(2-2\alpha)y}w+|w|^{p-1}w \text{ in } (0,\infty),
$$

where $A=c(1-2\alpha-c)$ and $B=2\alpha-1+2c$. Observe that $B\le 0$ whenever $p>1$ is sub-critical. Now, for $m>0$, define $w(y,m)$ as the unique solution of the initial value problem

$$
\begin{cases}
-w''+Bw'+Aw=\lambda e^{-(2-2\alpha)y}w+|w|^{p-1}w & \text{in } (0,\infty),\\
w(0)=0, w'(0)=m.
\end{cases}
\tag{8}
$$

For $i=1,2$, let $m_i=-u_i'(1)$. Then by the uniqueness of the initial value problem one has that $w_i(y):=w(y,m_i)=e^{cy}u_i(e^{-y})$ for $i=1,2$. Define $\sigma_j(m)$ as the j^{th} intersection between $w_1(y)$ and $w(y,m)$, i.e. if one lets $\sigma_0(m)=0$, then

$$
\sigma_{j+1}(m) := \inf\{y>\sigma_j(m): w_1(y)=w(y,m)\}.
$$

We claim that

(i) For $\bar{m} > m_2$ large enough there exists $y_0 < \infty$ such that $w(y, \bar{m})$ solves

$$\begin{cases} -w'' + Bw' + Aw = \lambda e^{-(2-2\alpha)y} w + w^p & \text{in } (0, y_0), \\ w > 0 & \text{in } (0, y_0), \\ w(0) = 0, w(y_0) = 0, \end{cases}$$

and $\# \{y \in (0, y_0) : w_1(y) = w(y, \bar{m})\} = 1$.

(ii) There exists $m_3 \in (m_2, \bar{m})$ such that $\sigma_2(m) \to \infty$ as $m \to m_3^-$.

(iii) If one lets $w_3(y) := w(y, m_3)$, then w_3 solves

$$\begin{cases} -w'' + Bw' + Aw = \lambda e^{-(2-2\alpha)y} w + w^p & \text{in } (0, \infty), \\ w(0) = 0, \\ w > 0, \end{cases}$$

and $\# \{y \in (0, \infty) : w_3(y) = w_1(y)\} \leq 1$.

Let us prove the claims:

Proof of (i). To prove this claim let $\tilde{w}_m(y) = m^a w(m^b y, m)$, where $a = -\frac{2}{p-1}$ and $b = -\frac{p-1}{p+1}$, then a direct computation shows that $\tilde{w}_m$ solves

$$\begin{cases} \tilde{w}_m'' + \lambda \tilde{w}_m + |\tilde{w}_m|^{p-1} \tilde{w}_m = V_m(y) \tilde{w}_m + B m^b \tilde{w}_m' & \text{in } (0, \infty), \\ \tilde{w}_m(0) = 0, \tilde{w}_m'(0) = 1, \end{cases}$$

where $V_m(y) = A m^{2b} - \lambda \left(e^{-(2-2\alpha) m^b y} - 1 \right)$. Observe that for all $m > 1$ one has $\|V_m\|_\infty \leq |A| + 2|\lambda|$ and that $\|V_m'\|_{L^1(0,\infty)} = |\lambda|$, hence, since $B \leq 0$, we can use lemma 2.1 to say that $\tilde{w}_m$, $\tilde{w}_m'$ and $\tilde{w}_m''$ are bounded *independently* of $m > 1$. By means of Arzela-Ascoli theorem we are able to find a function $\tilde{w}_\infty \in C^1[0, \infty)$ such that $\tilde{w}_m$ converges to $\tilde{w}_\infty$ in $C^1_{loc}[0, \infty)$. Now, it is easy to see that $V_m(y) \underset{m \to \infty}{\longrightarrow} 0$ uniformly over compact sets in $[0, \infty)$, hence we must have that $\tilde{w}_\infty$ is the unique solution of

$$\begin{cases} \tilde{w}_\infty'' + \lambda \tilde{w}_\infty + |\tilde{w}_\infty|^{p-1} \tilde{w}_\infty = 0 & \text{in } (0, \infty), \\ \tilde{w}_\infty(0) = 0, \tilde{w}_\infty'(0) = 1. \end{cases}$$

Multiply the above equation by $\tilde{w}_\infty'$ and integrate over $[0, y]$ to obtain

$$\frac{1}{2} \tilde{w}_\infty'(y)^2 + \frac{\lambda}{2} \tilde{w}_\infty(y)^2 + \frac{1}{p+1} |\tilde{w}_\infty(y)|^{p+1} = \frac{1}{2},$$

hence $\tilde{w}_\infty$ is periodic and one has that for $\tilde{y}_0 := \inf \{y > 0 : \tilde{w}_\infty(y) = 0\}$ then $\tilde{w}_\infty(y) > 0$ for $y \in (0, \tilde{y}_0)$ and $\tilde{w}_\infty(\tilde{y}_0) = 0$.

Finally, since $\tilde{w}_m \to \tilde{w}_\infty$ uniformly over compact sets, we have that for m large enough the claim holds.

Proof of (ii). Let $m > m_2$ and denote $w_2(y) := w(y, m_2)$. Notice that by the uniqueness of the initial value problem at $\sigma_j(m)$ one has that $w_2'(\sigma_j(m)) \neq w'(\sigma_j(m), m)$. Hence, thanks to the implicit value theorem, one obtains that $\sigma_j(m)$ varies continuously when one varies m.

Now let $[m_2, m^*)$ be the maximal interval where both σ_1 and σ_2 are finite. We claim that if $m \in [m_2, m^*)$ then $w(x, m) > 0$ in $(0, \sigma_2(m))$. Indeed, if $w(y', m') \leq 0$ for some $m' \in (m_2, m^*)$ and some $y' \in (0, \sigma_2(m'))$, we can define

$$m_0 = \inf \left\{ m \in [m_2, m^*) : \min_{y \in (0, \sigma_2(m)]} w(y, m) \leq 0 \right\}.$$

Since for $m = m_2$ we have $w(y,m) > 0$ we obtain that $m_0 \in (m_2, m']$ and that

$$\min_{y\in(0,\sigma_2(m_0)]} w(y,m_0) = 0.$$

The above implies that there is some $\hat{y} \in (0,\infty)$ such that $w(\hat{y},m_0) = w'(\hat{y},m_0) = 0$, so by the uniqueness of the initial value problem at $\hat{y}$ one obtains $w(y,m_0) \equiv 0$, which is impossible since $0 < m_2 < m_0$.

Now, by claim (i), $w(y,\bar{m})$ hits zero for some finite y, so we must have that $m^* < \bar{m}$, so the only possibility is that $\sigma_2(m) \to \infty$ as $m \nearrow m^*$. The claim is proved with $m_3 = m^*$.

Proof of (iii). Define $w_3(y) := w(y,m_3)$. There are two cases to take into account:

- $\sigma_1(m) \underset{m\nearrow m^*}{\longrightarrow} \infty$, and
- $\sigma_1(m) \underset{m\nearrow m^*}{\longrightarrow} \sigma_1 < \infty$.

Notice that by the definition of $\sigma_1(m)$ and the fact that $m > m_1$ for all $m \in [m_2, m_3)$, we have that $w_1(y) < w(y,m)$ if $y \in (0,\sigma_1(m))$ and $w_1(y) > w(y,m)$ if $y > (\sigma_1(m),\infty)$.

If $\sigma_1(m) \underset{m\nearrow m^*}{\longrightarrow} \sigma_1 < \infty$, we obtain by passing to the limit that $w_1(y) > w_3(y)$ for all $y > \sigma_1$, hence w_3 is dominated at infinity by w_1, which decays exponentially (recall that $w_1(y) = e^{cy}u_1(e^{-y})$ for $c < 0$ and that by assumption $u_1 \in C[0,1]$). Therefore w_3 must also decay exponentially and therefore by dominated convergence we obtain that w_3 is in fact the solution we are looking for (in this case there is a unique intersection between w_1 and w_3).

On the other hand, if $\sigma_1(m) \underset{m\nearrow m^*}{\longrightarrow} \infty$, we have that that for $w_1(y) < w(y,m)$ when $y \in (0,\sigma_1(m))$, then $W(y) := w_1'(y)w(y,m) - w_1(y)w'(y,m) > 0$ in $y \in (0,\sigma_1(m))$. Indeed, notice that W satisfies

$$W'(y) + BW(y) = -w_1(y)w(y,m)\left(w_1(y)^{p-1} - w(y,m)^{p-1}\right) > 0 \quad \text{in } (0,\sigma_1(m)),$$

hence $e^{By}W$ is an increasing function, but $W(0) = 0$, so $W(y) > 0$ for all $y \in (0,\sigma_1(m))$. This implies that $\dfrac{w_1(y)}{w(y,m)}$ is monotonically decreasing in $(0,\sigma_1(m))$. So $0 < \dfrac{w(y,m)}{w_1(y,m)} < \lim_{y\to 0}\dfrac{w(y,m)}{w_1(y)} = \frac{m}{m_1}$ and we have that $w(y,m) < \dfrac{m}{m_1}w_1(y)$, therefore when we pass to the limit we obtain that

$$w_3(y) < \frac{m}{m_1}w_1(y), \quad \text{for all } y > 0.$$

The conclusion is the same as before, as the above implies that w_3 decays exponentially at infinity (in this case there is no intersection between w_1 and w_3). □

Next, we recall the Pohozaev type identity established in [3]. For each $\beta \in \mathbb{R}$, we have the "energy" functional

$$\begin{aligned}(9)\quad E_{\lambda,\beta}(u)(x) = {} & \frac{1}{2}x^{2\alpha+1+\beta}u'(x)^2 + \frac{1}{p+1}x^{\beta+1}\left|u(x)\right|^{p+1} + \frac{\lambda}{2}x^{\beta+1}u(x)^2 \\ & - \frac{1}{2}\left(\beta+1-2\alpha\right)x^{2\alpha+\beta}u'(x)u(x) + \frac{\beta}{4}\left(\beta+1-2\alpha\right)x^{2\alpha-1+\beta}u(x)^2,\end{aligned}$$

and the identity satisfied by all solutions to (1)

$$E_{\lambda,\beta}(u)(x) = \frac{1}{2}u'(1)^2 - \lambda(1-\alpha+\beta)\int_x^1 s^\beta u(s)^2 ds - \left((\beta+1)\left(\frac{1}{2}+\frac{1}{p+1}\right)-\alpha\right)\int_x^1 s^\beta |u(s)|^{p+1} - \frac{\beta}{4}\left(\beta^2-(2\alpha-1)^2\right)\int_x^1 s^{2\alpha-2+\beta}u(s)^2 ds. \tag{10}$$

As it will be seen later it is convenient to choose β in the following way

$$\beta := \frac{\alpha - \frac{1}{2} - \frac{1}{p+1}}{\frac{1}{2}+\frac{1}{p+1}}. \tag{11}$$

Before explaining the reason why we select such β, let us make an observation. Firstly, we notice that for every $0<\alpha<1$, every $\lambda \in \mathbb{R}$, every $p>1$, every solution u of equation (1) satisfying u, $x^{2\alpha-1}u' \in C[0,1]$, and for β as above, then $\beta \in (\alpha-1, 2\alpha-1)$ and

$$\lim_{x\to 0^+} E_{\lambda,\beta}(u)(x) = \begin{cases} 0 & \text{if } \beta > 1-2\alpha, \\ \frac{(1-2\alpha)^2}{2}u(0)^2 & \text{if } \beta = 1-2\alpha, \\ +\infty & \text{if } \beta < 1-2\alpha, \end{cases}$$

Indeed, since $\beta > -1$, we obtain that terms of the form $x^{1+\beta}u^q(x) = o(1)$ for all $q \geq 1$ (this follows since $u \in C[0,1]$). Also

$$x^{2\alpha+\beta}u'(x)u(x) = o(1),$$

and

$$x^{2\alpha+1+\beta}u'(x)^2 = o(1).$$

This means that the only term we need to worry about is the last one in the definition of $E_{\lambda,\beta}$, that is

$$E_{\lambda,\beta}(u)(x) = \frac{\beta}{4}(\beta+1-2\alpha)\, x^{2\alpha-1+\beta}u(x)^2 + o(1). \tag{12}$$

Now, since both u and $x^{2\alpha-1}u'$ are continuous in $[0,1]$, we have that $u \in C^{0,2-2\alpha}[0,1]$, hence

$$u(x)^2 = u(0)^2 + O(x^{2-2\alpha}),$$

so we can write

$$E_{\lambda,\beta}(u)(x) = \frac{\beta}{4}(\beta+1-2\alpha)\, x^{2\alpha-1+\beta}u(0)^2 + o(1), \tag{13}$$

from where it is easily deduced that if $\beta > 1-2\alpha$, the limit is 0; when $\beta = 1-2\alpha$, then the limit is $\frac{(1-2\alpha)^2}{2}u(0)^2$; and when $\beta < 1-2\alpha$, the limit is $+\infty$.

When $0<\alpha<\frac{1}{2}$ and u solves equation (1) with $u(0)=0$, we still have that the terms of the form $x^{1+\beta}|u(x)|^q = o(1)$, so we have

$$E_{\lambda,\beta}(u)(x) = x^{1-2\alpha+\beta}\left[\frac{1}{2}x^{4\alpha}u'(x)^2 + \frac{1}{2}(2\alpha-1-\beta)x^{4\alpha-1}u'(x)u(x) + \frac{\beta}{4}(\beta+1-2\alpha)x^{4\alpha-2}u(x)^2\right] + o(1).$$

But now $x^{2\alpha-1}u$ and $x^{2\alpha}u'$ belong to $C^1[0,1]$ (this follows from the fact that $u \in C[0,1]$ and the regularity properties of the operator $-(x^{2\alpha}u')'$ given by [4, Lemma 3.1]), thus we obtain

$$E_{\lambda,\beta}(u)(x) = x^{1-2\alpha+\beta}\left[\frac{1}{2}x^{4\alpha}u'(x)^2\Big|_0 + \frac{1}{2}(2\alpha-1-\beta)x^{4\alpha-1}u'(x)u(x)\Big|_0 + \frac{\beta}{4}(\beta+1-2\alpha)x^{4\alpha-2}u(x)^2\Big|_0\right] + o(1).$$

Notice that for all $x > 0$ small enough, one must have that $u'(x) > 0$, and since $\beta < 2\alpha - 1 < 0$ we have that every term in parenthesis is positive, so for every such u we have that

$$\lim_{x\to 0} E_\lambda(u)(x) = +\infty.$$

The main motivation behind the choice of β comes from identity (10), as for β chosen as above, we obtain that the derivative of $E_{\lambda,\beta}(u)(x)$ with respect to x is a multiple to $u(x)^2$, that is

$$\frac{d}{dx}\left(E_{\lambda,\beta}(u)(x)\right) = G(x)u(x)^2,$$

where

$$G(x) := \lambda\,(1-\alpha+\beta)\,x^\beta + \frac{\beta}{4}\left(\beta^2 - (2\alpha-1)^2\right)x^{2\alpha-2+\beta}. \tag{14}$$

This is the key ingredient that will allow us to adapt a technique by Kwong and Li [6] to prove our result. In [6], the authors proved the uniqueness of positive solutions of an equation of the form

$$\begin{cases} u''(x) + f(u(x)) + g(x)u(x) = 0 & x \in (a,b), \\ u(a) = u(b) = 0, \end{cases}$$

by defining an energy function that had the property that its derivative is a multiple of the square of the function, that is the main reason behind our choice of β.

As we will see in the proof, it is necessary to impose some hypotheses over the function G in order to obtain the uniqueness: We suppose $G \in C(0,1)$ is either identically 0 or that that there exists $c \in [0,1]$ such that

$$G(x) > 0 \text{ for all } x \in (0,c), \text{ and } G(x) < 0 \text{ for all } x \in (c,1). \tag{15}$$

Let us find out when the function G defined in (14) satisfies this hypothesis. Since we are only concerned about the case $p > 1$ sub-critical, we will only consider $\beta \le 0$. It is easy to see that when $1 - 2\alpha < \beta < 0$ (or equivalently $\frac{3-4\alpha}{2\alpha-1} < p < \frac{3-2\alpha}{2\alpha-1}$), then $G(x) \to +\infty$ as $x \to 0^+$, and that depending on λ, either $G > 0$ in $(0,1)$ or G has exactly one zero in $(0,1]$. When $\beta = 0$ (that is when $p = \frac{3-2\alpha}{2\alpha-1}$), then $G(x) = \lambda\,(1-\alpha+\beta)$, so $\operatorname{sign}(G) = \operatorname{sign}(\lambda)$.

When $\beta \le 1 - 2\alpha$ (or equivalently, $1 < p \le \frac{3-4\alpha}{2\alpha-1}$, which only occurs when $\frac{1}{2} < \alpha < \frac{2}{3}$), there are two cases to take into account. When $\beta = 1 - 2\alpha$, then $\operatorname{sign}(G) = \operatorname{sign}(\lambda)$. And when $\alpha - 1 < \beta < 1 - 2\alpha$, then $G(x) \to -\infty$ as $x \to 0$, so the only way to obtain such c is that $c = 1$ and $G \le 0$ in $(0,1]$, which is satisfied when

$$\lambda \le \frac{\beta((2\alpha-1)^2 - \beta^2)}{4(1-\alpha+\beta)}.$$

It is easy to see that

$$\lambda_{\alpha,\beta} := \frac{\beta((2\alpha-1)^2-\beta^2)}{4(1-\alpha+\beta)}$$

is always a positive number which satisfies $\lambda_{\alpha,\beta} \searrow 0$ as $p > 1$ increases to the critical exponent (that is, $p \nearrow \infty$ when $\alpha \le \frac{1}{2}$ and $p \nearrow \frac{3-2\alpha}{2\alpha-1}$ when $\frac{1}{2} < \alpha < 1$). Because of this behavior is that we will only use this approach for $\lambda \le 0$. In summary we have proved the following two lemmas.

Lemma 2.2. *Suppose* $0 < \alpha < 1$, $\lambda \le 0$ *and that* $p > 1$ *is sub-critical. Let* u *be a solution of* (1) *satisfying in addition that* $x^{2\alpha-1}u' \in C[0,1]$, *then there exist* $\beta = \beta(\alpha,p) \in \mathbb{R}$ *and* $G \in C(0,1)$ *such that for* $E_{\lambda,\beta}(u)(x)$ *defined in* (9) *we have*

$$\frac{d}{dx}\left(E_{\lambda,\beta}(u)(x)\right) = G(x)u(x)^2,$$

and G *satisfies* (15) *for some* $c \in [0,1]$. *Moreover we have the following expansion of* $E_{\lambda,\beta}$

$$E_{\lambda,\beta}(u)(x) = \frac{\beta}{4}\left(\beta+1-2\alpha\right)x^{2\alpha-1+\beta}u(0)^2 + o(1). \tag{16}$$

Lemma 2.3. *Suppose* $0 < \alpha < \frac{1}{2}$, $\lambda \le 0$ *and that* $p > 1$. *Let* u *be a solution of equation* (1) *such that* $u(0) = 0$, *then there exist* $\beta = \beta(\alpha,p) \in \mathbb{R}$ *and* $G \in C(0,1)$ *such that for* $E_{\lambda,\beta}(u)(x)$ *defined in* (9) *we have*

$$\frac{d}{dx}\left(E_{\lambda,\beta}(u)(x)\right) = G(x)u(x)^2,$$

and G *satisfies* (15) *for some* $c \in [0,1]$. *Moreover we have the following expansion of* $E_{\lambda,\beta}$

$$E_{\lambda,\beta}(u)(x) = x^{1-2\alpha+\beta}\left[\frac{1}{2}x^{4\alpha}u'(x)^2\Big|_0 + \frac{1}{2}(2\alpha-1-\beta)x^{4\alpha-1}u'(x)u(x)\Big|_0 + \frac{\beta}{4}(\beta+1-2\alpha)x^{4\alpha-2}u(x)^2\Big|_0\right] + o(1).$$

For $\lambda > 0$, we will adapt a method by Adimurthi and Yadava [1] used in the study of the uniqueness of radial solutions to the equation

$$-\mathrm{div}(|\nabla u|^{m-2}\,\nabla u) = \lambda\,|u|^{m-2}\,u + u^p.$$

The idea used in [1] resembles the technique of Kwong and Li as they both use a Pohozaev type identity to prove that a single intersection between two positive solutions cannot occur.

With the above in mind, we define the new energy functional

$$\tilde{E}_\lambda(u)(x) := \frac{1}{2}x^{2\alpha+1}u'(x)^2 + \frac{1}{p+1}x\,|u(x)|^{p+1} + \frac{\lambda}{2}xu(x)^2 + \frac{1}{p+1}x^{2\alpha}u'(x)u(x), \tag{17}$$

then a direct computation shows that for every solution u of equation (1) we have the following identity

$$\frac{d}{dx}\tilde{E}_\lambda(u)(x) = \left(\frac{1}{p+1}+\frac{1}{2}-\alpha\right)x^{2\alpha}u'(x)^2 + \lambda\left(\frac{1}{2}-\frac{1}{p+1}\right)u(x)^2, \tag{18}$$

so in the derivative of this new energy functional instead of having only a term involving $u(x)^2$, there is a second term involving $u'(x)^2$. Observe that for every

$0<\alpha<1$, $\lambda>0$, and every $p>1$ sub-critical we have that both $\frac{1}{p+1}+\frac{1}{2}-\alpha$ and $\lambda\left(\frac{1}{2}-\frac{1}{p+1}\right)$ are non-negative constants which cannot be simultaneously 0.

It is easy to see that for u solving equation (1), with the additional assumption that $x^{2\alpha-1}u'\in C[0,1]$, we can write

$$\tilde{E}_\lambda(u)(x):=\frac{1}{2}x^{2\alpha+1}u'(x)^2+\frac{1}{p+1}x^{2\alpha}u'(x)u(x)+o(1),$$

and since both u and $x^{2\alpha-1}u'$ belong to $C[0,1]$ we deduce

$$\tilde{E}_\lambda(u)(x)=\frac{1}{2}x^{4\alpha-2}u'(x)^2x^{3-2\alpha}+\frac{1}{p+1}x^{2\alpha-1}u'(x)u(x)x+o(1)=o(1).$$

In summary, we have proved

Lemma 2.4. *Suppose* $0<\alpha<1$, $\lambda>0$ *and that* $p>1$ *is sub-critical. Let* $\tilde{E}_\lambda(u)(x)$ *be defined as in* (17)*, then for every* u *solution of equation* (1) *satisfying* $x^{2\alpha-1}u'\in C[0,1]$*, there exists constants* $C_1,C_2\geq 0$ *not both simultaneously* 0 *such that for all* $0<\varepsilon<1$

$$\tilde{E}_\lambda(u)(1)-\tilde{E}_\lambda(u)(\varepsilon)=C_1\int_\varepsilon^1 x^{2\alpha}u'(x)^2+C_2\int_\varepsilon^1 u(x)^2, \tag{19}$$

and that $E_\lambda(u)(\varepsilon)=o(1)$ *as* ε *approaches* 0.

3. Proof of Theorems 2 and 3

Proof. We will argue by contradiction and assume that u_1 and u_2 are two distinct solutions of equation (1) satisfying $x^{2\alpha-1}u'\in C[0,1]$. We begin the proof with an observation: Suppose $u_1<u_2$ (respectively $u_1>u_2$) in $(a,b)\subset(0,1)$, then the function

$$w(x)=x^{2\alpha}\left(u_1'(x)u_2(x)-u_1(x)u_2'(x)\right)$$

is increasing (respectively decreasing) in (a,b). Indeed, for $x\in(a,b)$ we have

$$\begin{aligned}
w'&=(x^{2\alpha}u_1')'u_2+x^{2\alpha}u_1'u_2'-(x^{2\alpha}u_2')'u_1-x^{2\alpha}u_1'u_2'\\
&=-\left(\lambda u_1+u_1^p\right)u_2+\left(\lambda u_2+u_2^p\right)u_1\\
&=u_1u_2\left(u_2^{p-1}-u_1^{p-1}\right)\\
&>0\quad(\text{respectively }<0).
\end{aligned} \tag{20}$$

Having said that, notice that by proposition 2.1 we can assume that u_1 and u_2 intersect at most once in $(0,1)$. Let us rule out first the case of no intersection, that is we can assume that u_1 and u_2 are ordered, say $u_1<u_2$ in $(0,1)$. Multiply the equation of u_1 by u_2 and integrate by parts over $(0,1)$ to obtain

$$\int_0^1 x^{2\alpha}u_1'(x)u_2'(x)dx=\lambda\int_0^1 u_1(x)u_2(x)dx+\int_0^1 u_1(x)^pu_2(x)dx,$$

where we have used that $x^{2\alpha}u_1'(x)u_2(x)\to 0$ as $x\to 0$. The same identity holds when u_1 and u_2 are interchanged. By subtracting the two identities we obtain

$$0=\int_0^1 u_1(x)u_2(x)\left(u_2(x)^{p-1}-u_1(x)^{p-1}\right)dx>0,$$

impossible.

Finally we only need to rule out the case of a unique intersection, so suppose that there is $\sigma \in (0,1)$ such that $u_1 < u_2$ in $(0,\sigma)$ and $u_1 > u_2$ in $(\sigma,1)$. For $i=1,2$, define $r_i(x) = \dfrac{u_i'(x)}{u_i(x)}$.

We claim that r_1 and r_2 do not intersect in $(0,1)$. Suppose the contrary, then there exists $\rho \in (0,1)$ such that $r_1(\rho) = r_2(\rho)$. If $\rho \geq \sigma$, then for $x \in (\rho,1)$ we have $u_1 > u_2$, so by (20) we obtain that w is decreasing in $(\rho,1)$, but by assumption $w(\rho) = \rho^{2\alpha} u_1(\rho) u_2(\rho) \left(r_1(\rho) - r_1(\rho)\right) = 0$. On the other hand since $u_1(1) = u_2(1) = 0$, we obtain that $w(1) = 0$, impossible. Similarly, if $\rho \leq \sigma$, we obtain that w is increasing; by assumption $w(\rho) = 0$ and since $x^{2\alpha} u_i'(x) u_j(x) \to 0$ for $i,j = 1,2$, we obtain that $w(0) = 0$, also impossible. Hence r_1 never intersects r_2, but since $r_1(\sigma) > r_2(\sigma)$, we must have $r_1(x) > r_2(x)$ for all $x \in (0,1)$. From here we deduce that the function $\dfrac{u_1}{u_2}$ is increasing, indeed, notice that $\left(\dfrac{u_1(x)}{u_2(x)}\right)' = \dfrac{u_1(x)}{u_2(x)} \left(r_1(x) - r_2(x)\right) > 0$.

Now we distinguish two cases: $\lambda \leq 0$ and $\lambda > 0$.

The case $\lambda \leq 0$: From lemma 2.2 there exist $\beta \in \mathbb{R}$ and a function $G \in C(0,1)$ such that for any solution u of equation (1) satisfying $x^{2\alpha-1}u' \in C[0,1]$ we have

$$\frac{d}{dx}\left(E_{\lambda,\beta}(u)(x)\right) = G(x)u(x)^2, \tag{21}$$

and G satisfies (15) for some $c \in [0,1]$. Define

$$\gamma = \begin{cases} \dfrac{u_1(c)}{u_2(c)} & \text{if } 0 \leq c < 1, \\ \dfrac{u_1'(1)}{u_2'(1)} & \text{if } c = 1, \\ 1 & \text{if } G \equiv 0. \end{cases} \tag{22}$$

By the monotonicity of $\dfrac{u_1}{u_2}$ we deduce that

$$u_1(x) < \gamma u_2(x) \text{ for } 0 < x < c \text{ and } u_1(x) > \gamma u_2(x) \text{ for } c < x < 1.$$

Now, let $0 < \varepsilon < 1$ and integrate equation (21) over $(\varepsilon,1)$ where u is replaced by u_1, to obtain

$$\frac{1}{2}u_1'(1)^2 - E_{\lambda,\beta}(u_1)(\varepsilon) = \int_\varepsilon^1 G(x)u_1(x)^2 dx.$$

Do the same for u_2, and multiply the result by γ^2 to obtain

$$\frac{\gamma^2}{2}u_2'(1)^2 - \gamma^2 E_{\lambda,\beta}(u_2)(\varepsilon) = \gamma^2 \int_\varepsilon^1 G(x)u_2(x)^2 dx.$$

Subtracting the two identities above yields

$$\int_\varepsilon^1 G(x)\left(u_1(x)^2 - \gamma^2 u_2(x)^2\right)dx = \frac{1}{2}\left(u_1'(1)^2 - \gamma^2 u_2'(1)^2\right) - \left(E_{\lambda,\beta}(u_1)(\varepsilon) - \gamma^2 E_{\lambda,\beta}(u_2)(\varepsilon)\right).$$

Notice that by the definition of γ and (15), the integrand on the left hand side is always non-positive (it is zero if and only if $G \equiv 0$). Also notice that since

$u_1(x) > \gamma u_2(x)$ for all $c < x < 1$, we obtain that

$$\gamma \leq \lim_{x\to 1^-} \frac{u_1(x)}{u_2(x)} = \frac{u_1'(1)}{u_2'(1)},$$

hence $u_1'(1)^2 - \gamma^2 u_2'(1)^2 \geq 0$. Also with the aid of (13) we have that

$$E_{\lambda,\beta}(u_1)(\varepsilon) - \gamma^2 E_{\lambda,\beta}(u_2)(\varepsilon) = \frac{\beta}{2}\left(\beta + 1 - 2\alpha\right)\varepsilon^{2\alpha-1+\beta}\left(u_1(0)^2 - \gamma^2 u_2(0)^2\right) + o(1),$$

but since $u_1(x) < \gamma u_2(x)$ for all $0 < x < c$, we obtain that $u_1(0)^2 \leq \gamma^2 u_2(0)^2$, and since for all $p > 1$ sub-critical, $\beta(\beta+1-2\alpha) \geq 0$, we can deduce that

$$\frac{1}{2}\left(u_1'(1)^2 - \gamma^2 u_2'(1)^2\right) + o(1) \leq \int_\varepsilon^1 G(x)\left(u_1(x)^2 - \gamma^2 u_2(x)^2\right) dx,$$

which by letting ε go to 0 gives

$$0 \leq \frac{1}{2}\left(u_1'(1)^2 - \gamma^2 u_2'(1)^2\right) \leq \int_0^1 G(x)\left(u_1(x)^2 - \gamma^2 u_2(x)^2\right) dx \leq 0,$$

since the last inequality is strict when $G \not\equiv 0$ we obtain a contradiction. When $G \equiv 0$, then by definition $\gamma = 1$, and we obtain that $u_1'(1) = u_2'(1)$, so $u_1 \equiv u_2$, also a contradiction.

The case $\lambda > 0$: To handle this case we first notice that if $u > 0$ solves

$$-(x^{2\alpha}u')' = \lambda u + u^p,$$

and $\lim_{x\to 0^+} x^{2\alpha}u'(x) \leq 0$, then $u'(x) < 0$ for all $x \in (0,1)$. Indeed, since $\lambda > 0$ and $u > 0$, from the equation we obtain that $x^{2\alpha}u'$ is strictly decreasing, hence for $0 < x < 1$ we have $x^{2\alpha}u'(x) < \lim_{x\to 0^+} x^{2\alpha}u'(x) \leq 0$.

Recall that we already established that $\frac{u_1}{u_2}$ is increasing, so we have that $u_1'u_2 > u_1u_2'$, and since $u_2' < 0$ for $\lambda > 0$ we obtain that

$$\frac{u_1'(x)}{u_2'(x)} < \frac{u_1(x)}{u_2(x)} \text{ for all } 0 < x < 1.$$

Let $\tilde{\gamma} = \lim_{x\to 1^-} \frac{u_1(x)}{u_2(x)} = \frac{u_1'(1)}{u_2'(1)}$, then the above implies that $u_1(x)^2 < \tilde{\gamma}^2 u_2(x)^2$ and $u_1'(x)^2 < \tilde{\gamma}^2 u_2'(x)^2$. Now, for given $0 < \varepsilon < 1$, subtract $\tilde{\gamma}^2$ times identity (19) for u_2 from identity (19) for u_1, and with the aid of lemma 2.4 we get, after sending ε to 0,

$$\frac{1}{2}\left(u_1'(1)^2 - \tilde{\gamma}^2 u_2'(1)^2\right) = C_1 \int_0^1 x^{2\alpha}\left(u_1'(x)^2 - \tilde{\gamma}^2 u_2'(x)^2\right) dx + C_2 \int_0^1 \left(u_1(x)^2 - \tilde{\gamma} u_2(x)^2\right) dx.$$

By definition of $\tilde{\gamma}$, the left hand side is identically 0. For the right hand side notice that both integrands are negative functions, and since $C_1, C_2 \geq 0$ with one of them strictly positive, we conclude that the right hand side must be negative, impossible. □

4. Proof of Theorem 1

We divide the proof into two cases: $\lambda \leq 0$ and $\lambda > 0$

Proof of Theorem 1 when $\lambda \leq 0$. The proof is by contradiction, that is we assume that we have two distinct solutions u_1, u_2 of equation (1) satisfying $u_i(0) = 0$, $i = 1, 2$. Proposition 2.1 still applies, so we can assume that u_1 and u_2 intersect at most once in $(0, 1)$. The case of no intersection is immediately ruled out as before because we still have $x^{2\alpha} u_1'(x) u_2(x) = o(1) = x^{2\alpha} u_2'(x) u_1(x)$ when $x \to 0^+$, so we only need to take care of the case of a unique intersection. Suppose that there is $\sigma \in (0, 1)$ such that $u_1 < u_2$ in $(0, \sigma)$ and $u_1 > u_2$ in $(\sigma, 1)$. Also, a line by line copy of our previous argument allows us to show that the function $\dfrac{u_1}{u_2}$ is increasing.

We continue as in the proof of the uniqueness of Theorems 2 and 3, but instead of using lemma 2.2, we will use lemma 2.3. So after defining γ as in 22 and using lemma 2.3 in the same way as we used lemma 2.2 before, gives

$$\int_\varepsilon^1 G(x) \left(u_1(x)^2 - \gamma^2 u_2(x)^2\right) dx = \frac{1}{2}\left(u_1'(1)^2 - \gamma^2 u_2'(1)^2\right) - \left(E_{\lambda,\beta}(u_1)(\varepsilon) - \gamma^2 E_{\lambda,\beta}(u_2)(\varepsilon)\right).$$

The main difference in the argument is the expansion of $E_{\lambda,\beta}(u)(\varepsilon)$ for $\varepsilon > 0$ small, in this case from lemma 2.3 we obtain that

$$\begin{aligned} E_{\lambda,\beta}(u_1)(\varepsilon) - \gamma^2 E_{\lambda,\beta}(u_2)(\varepsilon) = \varepsilon^{1-2\alpha+\beta} \Bigg[\frac{1}{2}\left(\varepsilon^{4\alpha} u_1'(\varepsilon)^2 \Big|_0 - \gamma^2 \varepsilon^{4\alpha} u_2'(\varepsilon)^2 \Big|_0 \right) \\ + \frac{1}{2}(2\alpha - 1 - \beta)\left(\varepsilon^{4\alpha-1} u_1'(\varepsilon) u_1(\varepsilon) \Big|_0 - \gamma^2 \varepsilon^{4\alpha-1} u_2'(\varepsilon) u_2(\varepsilon) \Big|_0 \right) \\ + \frac{\beta}{4}(\beta + 1 - 2\alpha)\left(\varepsilon^{4\alpha-2} u(\varepsilon)^2 \Big|_0 - \gamma^2 \varepsilon^{4\alpha-2} u(\varepsilon)^2 \Big|_0 \right) \Bigg] + o(1), \end{aligned}$$

but $u_1(x) < \gamma u_2(x)$ for all $0 < x < c$ so by L'Hôspital's rule we have that

$$\lim_{x \to 0^+} \frac{x^{2\alpha} u_1'(x)}{x^{2\alpha} u_2'(x)} < \gamma.$$

Also, since $u_2'(x) > 0$ for $x > 0$ small, we deduce that

$$\lim_{x \to 0^+} x^{2\alpha} u_1'(x) < \gamma \lim_{x \to 0^+} x^{2\alpha} u_2'(x).$$

From these observations we obtain that

$$\varepsilon^{4\alpha} u_1'(\varepsilon)^2 \Big|_0 \leq \gamma^2 \varepsilon^{4\alpha} u_2'(\varepsilon)^2 \Big|_0,$$

$$\varepsilon^{4\alpha-1} u_1'(\varepsilon) u_1(\varepsilon) \Big|_0 \leq \gamma^2 \varepsilon^{4\alpha-1} u_2'(\varepsilon) u_2(\varepsilon) \Big|_0$$

and that

$$\varepsilon^{4\alpha-2} u(\varepsilon)^2 \Big|_0 \leq \gamma^2 \varepsilon^{4\alpha-2} u(\varepsilon)^2 \Big|_0,$$

which, since $\beta < 2\alpha - 1 < 0$, imply that

$$E_{\lambda,\beta}(u_1)(\varepsilon) - \gamma^2 E_{\lambda,\beta}(u_2)(\varepsilon) \leq o(1).$$

Therefore after sending ε to 0, we obtain

$$\frac{1}{2}\left(u_1'(1)^2 - \gamma^2 u_2'(1)^2\right) \leq \int_0^1 G(x) \left(u_1(x)^2 - \gamma^2 u_2(x)^2\right) dx,$$

and we reach the same contradiction obtained in proof of the uniqueness in Theorems 2 and 3. □

For the case $\lambda > 0$ our previous ideas do not work. Instead we will use a shooting argument together with an idea of Yadava [8] where the uniqueness of positive solutions to

$$-\Delta u = u^q \pm u^p$$

in an annulus is studied.

Recall that we are interested in the uniqueness of a solution to equation

$$\begin{cases} -(x^{2\alpha}u')' = \lambda u + u^p & \text{in } (0,1), \\ u > 0 & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where $0 < \alpha < \frac{1}{2}$, $p > 1$ and $\lambda > 0$. To simplify the exposition, we will use the following change of variables: let $v(y) = u(y^{\frac{1}{1-2\alpha}})$, then a direct computation shows that v is a solution to

$$\begin{cases} -v'' = h(y)f(v) & \text{in } (0,1), \\ v > 0 & \text{in } (0,1), \\ v(0) = v(1) = 0, \end{cases} \tag{23}$$

where $h(y) = \frac{1}{(1-2\alpha)^2} y^{\frac{2\alpha}{1-2\alpha}}$ and $f(v) = \lambda v + |v|^{p-1} v$. Following [8], we introduce some notation and some properties of solutions to the equation

$$-v'' = h(y)f(v). \tag{24}$$

Let $F(v) = \int_0^v f(s)ds = \frac{\lambda}{2}v^2 + \frac{1}{p+1}|v|^{p+1}$ and define

$$E(y) := \frac{1}{2}yv'(y)^2 + yh(y)F(v(y)) - \frac{1}{2}v'(y)v(y).$$

A direct computation shows that if v solves equation (24), then

$$E'(y) := h(y)\left(F(v(y)) + f(v(y))v(y)\right) + yh'(y)F(v(y)). \tag{25}$$

Also, for $A \in \mathbb{R}$ to be fixed, we let

$$g_A(y) := yv'(y) + Av(y). \tag{26}$$

A straightforward computation gives that g_A satisfies

$$g_A' = (1+A)v' - yh(y)f(v)$$

and

$$-g_A'' = h(y)f'(v)g + I(A,v), \tag{27}$$

where

$$I(A,v) = \left((2+A)h(y) + yh'(y)\right)f(v) - Ah(y)f'(v)v.$$

We also need to introduce the linearized equation

$$-w'' = h(y)f'(v)w. \tag{28}$$

A useful identity obtained from equations (27) and (28) is that for any $a < b$,

$$\int_a^b I(A,v(y))w(y)dy = \left[yw'v' - Aw'v - (1+A)v'w + yh(y)f(v)w\right]\Big|_a^b. \tag{29}$$

We also need the following identity satisfied by all solutions of equation (24): Let $a < y$, then

$$v^2\left(\frac{yv'(y)}{v(y)}\right)' = \left[(v'(y) - yh(y)f(v(y)))v(y) - yv'(y)^2\right]\Big|_a + yh(y)\left[2F(v(y)) - f(v(y))v(y)\right]\Big|_a^y - \int_a^y \left[h(s)\left(2F(v(s)) + f(v(s))v(s)\right) + 2sh'(s)F(v(s))\right] ds. \tag{30}$$

Now, let $v(y,m)$ be the unique solution of the initial value problem

$$\begin{cases} -v'' = h(y)f(v), \\ v(0) = 0, \\ v'(0) = m, \end{cases} \tag{31}$$

and define $r(m)$ as the first zero of $v(y,m)$, i.e. $r(m) := \inf\{y > 0 : v(y,m) = 0\}$. Notice that the uniqueness of the solution to equation (23) is guaranteed if we can prove $r(m) = 1$ has at most one solution. To do this we will show that $r(m)$ is monotone for all $m > 0$, and this is the content of the following

Proposition 4.1. *Given $m > 0$, then $\dot{r}(m) \neq 0$.*

Remark. The $\dot{r}(m)$ notation means derivative with respect to m.

The proof of this proposition requires the following

Lemma 4.1. *For given $m > 0$, let $v(y,m)$ be the unique solution of equation (24), and let $r(m)$ be as above. Then $\frac{yv'}{v} < 0$ for all $y < r(m)$.*

Proof. We have that $v(s) > 0$ for all $s < r(m)$. From identity (30) we have that for $a = 0$ and $0 < y < r(m)$

$$\begin{aligned} v^2\left(\frac{yv'}{v}\right)' &= \left[(v' - yh(y)f(v))v - yv'^2\right]\Big|_0 + yh(y)\left[2F(v) - f(v)v\right]\Big|_0^y \\ &\quad - \int_0^y \left[h(y)\left(2F(v) + f(v)v\right) + 2yh'(y)F(v)\right] \\ &= yh(y)\left[2F(v(y)) - f(v(y))v(y)\right] - \int_0^y \left[h(y)\left(2F(v) + f(v)v\right) + 2yh'(y)F(v)\right] \\ &= -\frac{p-1}{(1-2\alpha)^2(p+1)} y^{\frac{1}{1-2\alpha}} v(y)^{p+1} \\ &\quad - \frac{1}{(1-2\alpha)^2} \int_0^y s^{\frac{2\alpha}{1-2\alpha}} \left[\lambda\left(\frac{2-2\alpha}{1-2\alpha} v(s)^2 + \left(\frac{2}{(p+1)(1-2\alpha)} + 1\right) v(s)^{p+1}\right)\right] ds \\ &< 0, \end{aligned}$$

for all $p > 1$, $0 < \alpha < \frac{1}{2}$ and $\lambda > 0$. ∎

Proof of Proposition 4.1. Suppose the contrary and assume $m_0 > 0$ is such that $\dot{r}(m_0) = 0$. By the definition of $r(m)$ we have that $v(r(m), m) = 0$. Differentiate this equation with respect to m to obtain

$$w(r(m)) + v'(r(m), m)\dot{r}(m) = 0,$$

where $w(y) := w(y, m)$ is the unique solution of

$$\begin{cases} -w'' = h(y)f'(v(y,m))w, \\ w(0) = 0, \\ w'(0) = 1. \end{cases}$$

Since $\dot{r}(m_0) = 0$ we have that $w(r(m_0)) = 0$. Let y_0 be the largest zero of w that is less that $r(m_0)$, i.e. $y_0 = \sup\{y \in (0, r(m_0)) : w(y) = 0\}$. A constant multiple of w (which we denote the same) must solve

$$\begin{cases} -w'' = h(y)f'(v(y,m))w, \\ w(0) = 0, \\ w(r(m_0)) = 0, \\ w'(r(m_0)) = v'(r(m_0), m_0) < 0. \end{cases}$$

Now for $A := \frac{2-2\alpha}{(p-1)(1-2\alpha)}$, consider g_A defined in (26). We claim that g_A has exactly one zero in $(0, r(m))$ for all $m > 0$. Indeed, notice that solving $g_A(y) = 0$ is equivalent to solving

$$\frac{yv'(y)}{v(y)} = -A. \tag{32}$$

From lemma 4.1, the quantity $\dfrac{yv'(y)}{v(y)}$ is monotonically decreasing, and it satisfies $\displaystyle\lim_{y\to 0^+} \frac{yv'}{v} = 1$ and $\displaystyle\lim_{y\to r(m)^-} \frac{yv'}{v} = -\infty$. Since $-A = -\frac{2-2\alpha}{(p-1)(1-2\alpha)} < 0$, we have a unique solution to equation (32), and hence $g_A(s) = 0$ has exactly one zero. Let $s_0 \in (0, r(m_0))$ be that unique zero.

<u>Claim:</u> $y_0 < s_0$.

Notice that $\dfrac{w}{v}$ is increasing in $(y_0, r(m_0))$, indeed, let $z = w'v - v'w$, so it is enough to prove that $z(y) > 0$. Suppose that $z(\bar{y}) = 0$ for some $\bar{y} \in (y_0, r(m_0))$. Since $z(r(m_0)) = 0$ we obtain that

$$\begin{aligned} 0 &= z(r(m_0)) - z(\bar{y}) \\ &= \int_{\bar{y}}^{r(m_0)} z' \\ &= \int_{\bar{y}}^{r(m_0)} w''v - v''w \\ &= \int_{\bar{y}}^{r(m_0)} h(y)\left(f(v) - f'(v)v\right) w. \end{aligned}$$

Since $w > 0$ in $(y_0, r(m_0))$, $h(y) > 0$ and since $f(v) > f'(v)v$ for all $v > 0$ we obtain a contradiction. Hence $z(y)$ does not change sign, but since $z(y_0) = w'(y_0)v(y_0) > 0$ we obtain that $z(y) > 0$ for all $y \in (y_0, r(m_0))$.

Now since $w'(r(m_0)) < 0$, $w > 0$ in $(y_0, r(m_0))$ and the fact that $\frac{w}{v}$ is increasing we deduce that $w < v$ in $(y_0, r(m_0))$. From identity (29) we obtain that

$$\int_{y_0}^{r(m_0)} I(A, v)w = r(m_0)w'(r(m_0))^2 - g_A(y_0)v'(y_0),$$

but from the choice of A we have that, since $h(y) > 0$,

$$\begin{aligned}\int_{y_0}^{r(m_0)} I(A, v)w &= \lambda \left(\frac{2-2\alpha}{1-2\alpha}\right) \int_{y_0}^{r(m_0)} h(y)vw \\ &< \lambda \left(\frac{2-2\alpha}{1-2\alpha}\right) \int_{y_0}^{r(m_0)} h(y)v^2 \\ &< \lambda \left(\frac{2-2\alpha}{1-2\alpha}\right) \int_{0}^{r(m_0)} h(y)v^2,\end{aligned}$$

but from (25) we deduce that

$$\lambda \left(\frac{2-2\alpha}{1-2\alpha}\right) \int_{0}^{r(m_0)} h(y)v^2 < r(m_0)v'(r(m_0))^2,$$

hence

$$g_A(y_0)v'(y_0) > 0,$$

and since $v'(y_0) > 0$, we deduce that $g_A(y_0) > 0$. But

$$g'_A(0) = (1 + A)u'(0, m) = (1 + A)m > 0,$$

so $g_A(y) > 0$ if and only if $y < s_0$, hence $y_0 < s_0$.

Now, let $y_1 = \sup\{y < y_0 : v(y) = 0\}$. By definition, $v < 0$ in (y_1, y_0), but from identity (29) we obtain

$$\int_{y_1}^{y_0} I(A, v)w = [w'g_A - wg'_A]\Big|_{y_1}^{y_0} = w'(y_0)g_A(y_0),$$

so

$$0 < w'(y_0)g_A(y_0) = \lambda \left(\frac{2-2\alpha}{1-2\alpha}\right) \int_{y_1}^{y_0} h(y)vw < 0,$$

hence we conclude that $v(0) \neq 0$, a contradiction. Therefore $\dot{r}(m) \neq 0$. □

Proof of Theorem 1 when $\lambda > 0$.

From Proposition 4.1, we deduce that $r(m)$ is either monotonically increasing or monotonically decreasing, hence $r(m) = 1$ has at most one solution. This proves the theorem. □

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