INTERIOR REGULARITY OF DOUBLY WEIGHTED QUASI-LINEAR EQUATIONS

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Abstract. In this article we study the quasi-linear equation

$$\begin{cases} \operatorname{div}\mathcal{A}(x,u,\nabla u) = \mathcal{B}(x,u,\nabla u) & \text{in } \Omega, \\ u \in H^{1,p}_{loc}(\Omega; w\,\mathrm{d}x) \end{cases}$$

where $\mathcal{A}$ and $\mathcal{B}$ are functions satisfying $\mathcal{A}(x,u,\nabla u) \sim w_1(|\nabla u|^{p-2}\nabla u + |u|^{p-2}u)$ and $\mathcal{B}(x,u,\nabla u) \sim w_2(|\nabla u|^{p-2}\nabla u + |u|^{p-2}u)$ for $p > 1$, a p-admissible weight function w_1, and another weight function w_2 compatible with w_1 in a suitable sense. We establish interior regularity results of weak solutions and use those results to obtain point-wise asymptotic estimates at infinity for solutions to

$$\begin{cases} -\operatorname{div}(w_1 |\nabla u|^{p-2}\nabla u) = w_2 |u|^{q-2} u & \text{in } \Omega, \\ u \in D^{1,p,w_1}(\Omega) \end{cases}$$

for a critical exponent $q > p > 1$ in the sense of Sobolev.

1. Introduction

This article is a direct continuation of [5] where we studied qualitative and quantitative properties of weak solutions to the following equation

$$\begin{cases} -\operatorname{div}\left(w_1 |\nabla u|^{p-2}\nabla u\right) = w_2 |u|^{q-2} u & \text{in } \Omega \\ u \in D^{1,p,w_1}(\Omega), \end{cases} \tag{1}$$

for equal weights $w_1 = w_2$ and $q > p > 1$ critical for the weighted Sobolev embedding from $D^{1,p,w_1}(\Omega)$ into $L^{q,w_2}(\Omega)$. In this continuation we generalize the results obtained in [5] for the case of different weights $w_1 \neq w_2$ but satisfying suitable compatibility conditions.

The main motivation behind studying this problem comes from the results in [4] where the existence to extremals to a Sobolev inequality with monomial weights was analyzed (see also [2,3]). It is known that extremals to a weighted Sobolev inequality can be viewed as positive solutions to (1) for appropriate weights w_1, w_2, and our goal is to obtain as much information as possible regarding said extremals and, in general, of solutions to (1).

As in [5] the functions w_1, w_2 will be weight functions, meaning locally Lebesgue integrable non-negative function over $\Omega \subseteq \mathbb{R}^N$ satisfying at least the following two conditions: if we abuse the notation and we also write w as the measure induced by w, that is $w(B) = \int_B w\,\mathrm{d}x$, we require that w is a doubling measure in Ω, meaning that there exists a *doubling constant* $\gamma > 0$ such that

$$w(2B) \leq \gamma w(B) \tag{2}$$

holds for every (open) ball such that $2B \subset \Omega$, where ρB denotes the ball with the same center as B but with its radius multiplied by $\rho > 0$. The smallest possible $\gamma > 0$ for which (2) holds for every ball will be denoted by $\gamma_w > 0$ from now on. Additionally we will suppose that

$$0 < w < \infty \qquad \lambda-\text{almost everywhere} \tag{3}$$

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where λ denotes the N-dimensional Lebesgue measure. Observe that these two conditions ensure that the measure w and the Lebesgue measure λ are absolutely continuous with respect to each other.

In addition to (2) and (3) we will suppose that the weight w_1 satisfies the following local $(1,p)$ Poincaré inequality: if we write $\fint_B fw\,\mathrm{d}x = \frac{1}{w(B)}\int fw\,\mathrm{d}x$ then

(PI) *Local weighted $(1,p)$-Poincaré inequality*: There exists $\rho \geq 1$ such that if $u \in C^1(\Omega)$ then for all balls $B \subset \Omega$ of radius $l(B)$ one has

$$\fint_B |u - u_{B,w_1}|\, w_1\,\mathrm{d}x \leq C_1 l(B) \left(\fint_{\rho B} |\nabla u|^p\, w_1\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{4}$$

where

$$u_{B,w} = \fint_B uw\,\mathrm{d}x$$

is the weighted average of u over B.

As it can be seen in [7, Chapter 20], when a weight function w satisfies (2), (3) and (4) then w is *p-admissible*, that is, it also satisfies the following properties

(PII) *Uniqueness of the gradient*: If $(u_n)_{n\in\mathbb{N}} \subseteq C^1(\Omega)$ satisfy

$$\int_\Omega |u_n|^p\, w_1\,\mathrm{d}x \underset{n\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \int_\Omega |\nabla u_n - v|^p\, w_1\,\mathrm{d}x \underset{n\to\infty}{\longrightarrow} 0$$

for some $v : \Omega \to \mathbb{R}^N$, then $v = 0$.

(PIII) *Local Poincaré-Sobolev inequality*: There exist constants $C_3 > 0$ and $\chi_1 > 1$ such that for all balls $B \subset \Omega$ one has

$$\left(\fint_B |u - u_{B,w_1}|^{\chi_1 p}\, w_1\,\mathrm{d}x\right)^{\frac{1}{\chi_1 p}} \leq C_2 l(B) \left(\fint_B |\nabla u|^p\, w\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{5}$$

for bounded $u \in C^1(B)$.

(PIV) *Local Sobolev inequality*: There exist constants $C_2 > 0$ and $\chi_1 > 1$ (same as above) such that for all balls $B \subset \Omega$ one has

$$\left(\fint_B |u|^{\chi_1 p}\, w_1\,\mathrm{d}x\right)^{\frac{1}{\chi_1 p}} \leq C_2 l(B) \left(\fint_B |\nabla u|^p\, w_1\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{6}$$

for $u \in C^1_c(B)$.

Remark 1.1. *As we mentioned in* [5] *the value of χ_1 comes from a dimensional constant associated to the weight, namely, it can be seen that if w is a doubling weight then*

$$\frac{w(B_R(y))}{w(B_r(x))} \leq C\left(\frac{R}{r}\right)^{D_w}, \qquad \text{for all } 0 < r \leq R < \infty \text{ with } B_r(x) \subseteq B_R(y) \subseteq \Omega. \tag{7}$$

for $D_w = \log_2 \gamma_w$, and if we denote $D_1 := \log_2 \gamma_{w_1}$ then we can take $\chi_1 = \frac{D_1}{D_1 - p}$ in (5) *and* (6)*.*

Regarding the weight w_2, in addition to satisfy (2) and (3) (in particular w_2 also satisfies (7) for $D_2 := \log_2 \gamma_{w_2}$), we require that the following compatibility condition with the weight w_1 is met: there exists $q > p$ such that

$$\frac{r}{R}\left(\frac{w_2(B_r)}{w_2(B_R)}\right)^{\frac{1}{q}} \leq C\left(\frac{w_1(B_r)}{w_1(B_R)}\right)^{\frac{1}{p}}. \tag{8}$$

holds for all balls $B_r \subset B_R \subset \Omega$. From [6] (see also [1, Theorem 7]) we know that if $1 \leq p < q < \infty$, w_1 is p-admissible, w_2 is doubling and (8) is satisfied, then the pair of weights (w_1, w_2) satisfy the (q,p)-local Poincaré-Sobolev inequality

$$\left(\fint_{B_R} |u - u_{B,w_2}|^q\, w_2\,\mathrm{d}x\right)^{\frac{1}{q}} \leq CR\left(\fint_{B_R} |\nabla u|^p\, w_1\,\mathrm{d}x\right)^{\frac{1}{p}}, \quad \forall\, u \in C^1(B_R), \tag{9}$$

and the (q,p)-local Sobolev inequality

$$\left(\fint_{B_R} |u|^q \, w_2 \, \mathrm{d}x\right)^{\frac{1}{q}} \leq CR \left(\fint_{B_R} |\nabla u|^p \, w_1 \, \mathrm{d}x\right)^{\frac{1}{p}}, \quad \forall\, u \in C_c^1(B_R). \tag{10}$$

Remark 1.2. *As it will be useful later we write* $D = \frac{qp}{q-p}$ *and* $\chi_2 = \frac{D}{D-p} = \frac{q}{p}$*. Notice that this* D *comes from* (8) *and in general it has nothing to do with* $D_2 = \log_2 \gamma_{w_2}$*, the dimensional constant associated to the doubling weight* w_2 *mentioned before.*

In order to establish the main results of this work we recall some definitions regarding weighted spaces. For an admissible weight w we consider the weighted Lebesgue space

$$L^{p,w}(\Omega) = \{\, u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |u|^p \, w \, \mathrm{d}x < \infty \,\}$$

equipped with the norm

$$\|u\|_{p,w}^p = \int_\Omega |u|^p \, w \, \mathrm{d}x.$$

The p-admissibility of w_1 is useful to have a proper definition for weighted Sobolev spaces: for an open set $\Omega \subseteq \mathbb{R}^N$ we define the weighted Sobolev space $H^{1,p,w_1}(\Omega)$

$$H^{1,p,w_1}(\Omega) = \text{the completion of } \{\, u \in C^1(\Omega) : u, \frac{\partial u}{\partial x_i} \in L^{p,w_1}(\Omega) \text{ for all } i \,\}$$

equipped with the norm

$$\|u\|_{1,p,w_1}^p = \|u\|_{p,w_1}^p + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{p,w_1}^p.$$

As we mentioned before the goal of this work is to generalize what was done in [5], that is to obtain qualitative and quantitative properties of weak solutions to (1). To do so we first study the local regularity of weak solutions the following quasi-linear problem

$$\begin{cases} \operatorname{div} \mathcal{A}(x,u,\nabla u) = \mathcal{B}(x,u,\nabla u), & \text{in } \Omega \subseteq \mathbb{R}^N \\ \qquad u \in H^{1,p,w_1}_{loc}(\Omega), \end{cases} \tag{11}$$

where $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are functions verifying the Serrin-like conditions

$$\mathcal{A}(x,u,z) \cdot z \geq w_1(x) \left(a^{-1} |z|^p - d_1 |u|^p - g\right), \tag{H1}$$

$$|\mathcal{A}(x,u,z)| \leq w_1(x) \left(a\, |z|^{p-1} + b\, |u|^{p-1} + e\right), \tag{H2}$$

$$|\mathcal{B}(x,u,z)| \leq w_2(x) \left(c\, |z|^{p-1} + d_2\, |u|^{p-1} + f\right), \tag{H3}$$

for a constant $a > 0$ and measurable functions $b, c, d_1, d_2, e, f, g : \Omega \to \mathbb{R}^+ \cup \{\, 0 \,\}$ satisfying

$$\begin{gathered} b, e \in L^{\frac{D_1}{p-1}, w_1}(B_2), \quad c \left(\frac{w_2}{w_1}\right)^{1-\frac{1}{p}} \in L^{\frac{D_1}{1-\varepsilon}, w_2}(B_2), \\ d_1, g \in L^{\frac{D_1}{p-\varepsilon}, w_1}(B_2), \quad d_2, f \in L^{\frac{D}{p-\varepsilon}, w_2}(B_2). \end{gathered} \tag{H_ε}$$

for some $0 \leq \varepsilon < 1$.

With the above into consideration, throughout the rest of this article the functions w_1, w_2 will be a non-negative locally integrable weight functions satisfying (2), (3), w_1 will satisfy the local weighted $(1,p)$-Poincaré inequality (4) and the pair (w_1, w_2) will verify the compatibility condition (8). We will also suppose that $1 < p < \min\{\, D_1, D \,\}$.

The first result of this work shows that weak solutions to (11) are locally bounded.

Theorem 1.1. *Suppose that there exists $0<\varepsilon<1$ such that* (H_ε) *is satisfied, then there exists a constant $C>0$ depending on the norms of a,b,c,d_1,d_2 such that for any weak solution to* (11) *in B_2 we have*

$$\|u\|_{L^\infty(B_1)} \le C\left([u]_{p,B_2}+k\right),$$

where

$$k=\left[\left(\fint_{B_2}|e|^{\frac{D_1}{p-1}}\,w_1\right)^{\frac{p-1}{D_1}}+\left(\fint_{B_2}|f|^{\frac{D}{p-\varepsilon}}\,w_2\right)^{\frac{p-\varepsilon}{D}}\right]^{\frac{1}{p-1}}+\left[\left(\fint_{B_2}|g|^{\frac{D_1}{p-\varepsilon}}\,w_1\right)^{\frac{p-\varepsilon}{D_1}}\right]^{\frac{1}{p}} \tag{12}$$

and for $s>1$ and $B\subseteq\Omega$ we write

$$[u]_{s,B}=\left(\fint_B|\bar u|^s\,w_1\right)^{\frac{1}{s}}+\left(\fint_B|\bar u|^s\,w_2\right)^{\frac{1}{s}} \tag{13}$$

Remark 1.3. *We have chosen to exhibit the local regularity results only for the case $B_1\subset B_2\subset\Omega$ as the general case $B_R\subseteq B_{2R}\subseteq\Omega$ can be easily obtained by a suitable scaling argument (see* [5] *where the computations are done in detail).*

Next we consider the case $\varepsilon=0$ and we show that weak solutions are in $L^{s,w_i}(B_1)$ for every $s>p$.

Theorem 1.2. *Suppose that* (H_ε) *is satisfied for $\varepsilon=0$, then there exists a constant $C>0$ depending on the norms of a,b,c,d_1,d_2 such that for any weak solution to* (11) *in B_2 satisfies*

$$[u]_{s,B_1}\le C_s\left([u]_{p,B_2}+k\right)$$

for every $s>p$ and k as in (12).

Finally, we show that the Harnack inequality holds for non-negative weak solutions to (11).

Theorem 1.3 (Harnack)**.** *Under the same hypotheses of Theorem 1.1 with the additional assumption that u is a non-negative weak solution of $\operatorname{div}\mathcal{A}=\mathcal{B}$ in B_3 then*

$$\max_{B_1}u\le C\left(\min_{B_1}u+k\right)$$

where C and k are as in Theorem 1.1.

Finally we return to (1) and we obtain a general result regarding the behavior at infinity of solutions. To do that we will suppose that in addition to the above conditions, both weights w_1,w_2 verify global Sobolev inequalities, that is, there exists a constant $C>0$ such that

$$\left(\int_\Omega|u|^{q_1}\,w_1\,\mathrm{d}x\right)^{\frac{1}{q_1}}\le C\left(\int_\Omega|\nabla u|\,w_1\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{14}$$

for $q_1=\chi_1 p$ and

$$\left(\int_\Omega|u|^{q}\,w_2\,\mathrm{d}x\right)^{\frac{1}{q}}\le C\left(\int_\Omega|\nabla u|\,w_1\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{15}$$

for q as in (8), and all $u\in C^1_c(\Omega)$. Under these assumptions, and if we define $D^{1,p,w_1}(\Omega)$ as the closure of $C^\infty_c(\Omega)$ under the (semi) norm $\|\nabla u\|_{p,w_1}$ then $D^{1,p,w_1}(\Omega)$ embeds continuously into both $L^{q_1,w_1}(\Omega)$ and $L^{q,w_2}(\Omega)$ and we are able to prove

Theorem 1.4 (Decay)**.** *Suppose $u\in D^{1,p,w_1}(\Omega)$ is a weak solution to* (1). *Then there exists $R_0>1$, $C>0$ and $\lambda>0$ such that*

$$|u(x)|\le\frac{C}{|x|^{\frac{p}{q_1-p}+\lambda}},$$

for all $|x|>R_0$ in Ω.

Remark 1.4. *It is important to mention that this decay behavior is not optimal, but it can be used as a starting point to obtain better results. This can be done with the aid of a comparison principle a the construction of a suitable barrier function depending on the weights w_1, w_2. We refer the reader to* [5, Section 4] *where power type weights and monomial weights are considered in the case $w_1 = w_2$.*

The rest of this article is dedicated to the proofs of the above results. In Section 2 we study (11) and obtain the proofs of Theorems 1.1 to 1.3 whereas in Section 3 we turn to the proof of Theorem 1.4.

2. Local estimates

Throughout the different proofs in this section we will use the dimensional constants of the weights $D_i := D_{w_i}$ as well as the local Sobolev exponents $q_1 := \frac{D_1 p}{D_1 - p}$ and $D = \frac{qp}{q-p}$ for q given by (8). With these notations we also have

$$\chi_1 = \frac{q_1}{p} = \frac{D_1}{D_1 - p} \quad \text{and} \quad \chi_2 = \frac{q}{p} = \frac{D}{D-p}$$

Following [8] (and what we did in [5]) we define $F : [k, \infty) \to \mathbb{R}$ as

$$F(x) = F_{\alpha,k,l}(x) = \begin{cases} x^\alpha & \text{if } k \le x \le l, \\ l^{\alpha-1}\left(\alpha x - (\alpha - 1)l\right) & \text{if } x > l, \end{cases}$$

which is in $C^1([k, \infty))$ with $|F'(x)| \le \alpha l^{\alpha-1}$. We consider $\bar{x} = |x| + k$ and $G : \mathbb{R} \to \mathbb{R}$ defined as

$$G(x) = G_{\alpha,k,l}(x) = \operatorname{sign}(x)\left(F(\bar{x})\left|F'(\bar{x})\right|^{p-1} - \alpha^{p-1}k^\beta\right)$$

where $\beta = 1 + p(\alpha - 1)$. Observe that G is a piecewise smooth function which is linear if $|x| > l - k$ and that both F and G satisfy

$$\begin{aligned} |G| &\le F(\bar{x})\left|F'(\bar{x})\right|^{p-1} \\ \bar{x}F'(\bar{x}) &\le \alpha F(\bar{x}) \\ F'(\bar{x}) &\le \alpha F(\bar{x})^{1-\frac{1}{\alpha}} \end{aligned}$$

and

$$G'(x) = \begin{cases} \dfrac{\beta}{\alpha}\left|F'(\bar{x})\right|^p & \text{if } |x| < l - k, \\ \left|F'(\bar{x})\right|^p & \text{if } |x| > l - k. \end{cases}$$

Finally, observe that if $\eta \in C_c^\infty(\Omega)$ and if $u \in H^{1,p,w_1}_{loc}(\Omega)$ then $\varphi = \eta^p G(u)$ is a valid test function in

$$\int_\Omega \mathcal{A}(x, u, \nabla u)\nabla\varphi + \mathcal{B}(x, u, \nabla u)\varphi = 0$$

thanks to the results in [7, Chapter 1] regarding weighted Sobolev spaces for p-admissible weights.

We can now prove the local boundedness of weak solutions.

Proof of Theorem 1.1. By using (H1)-(H3) we can write

$$\begin{aligned} |\mathcal{A}(x, u, z)| &\le w_1\left(a\,|z|^{p-1} + \bar{b}\bar{u}^{p-1}\right), \\ \mathcal{A}(x, u, z)\cdot z &\ge w_1\left(|z|^p - \bar{d}_1\bar{u}^p\right), \\ |\mathcal{B}(x, u, z)| &\le w_2\left(c\,|z|^{p-1} + \bar{d}_2\bar{u}^{p-1}\right), \end{aligned} \tag{16}$$

where

$$\begin{aligned} \bar{b} &= b + k^{1-p}e, \\ \bar{d}_1 &= d_1 + k^{-p}g, \\ \bar{d}_2 &= d_2 + k^{1-p}f, \end{aligned}$$

and $\bar{u} = |u| + k$ for $k \geq 0$ defined as[1]

$$k = \left[\left(\fint_{B_2} |e|^{\frac{D_1}{p-1}} w_1 \right)^{\frac{p-1}{D_1}} + \left(\fint_{B_2} |f|^{\frac{D}{p-\varepsilon}} w_2 \right)^{\frac{p-\varepsilon}{D}} \right]^{\frac{1}{p-1}} + \left[\left(\fint_{B_2} |g|^{\frac{D_1}{p-\varepsilon}} w_1 \right)^{\frac{p-\varepsilon}{D_1}} \right]^{\frac{1}{p}}.$$

Observe that (H_ε) implies that

$$\fint_{B_2} \left|\bar{b}\right|^{\frac{D_1}{p-1}} w_1 \leq C, \qquad \fint_{B_2} \left|\bar{d}_1\right|^{\frac{D_1}{p-\varepsilon}} w_1 \leq C, \qquad \fint_{B_2} \left|\bar{d}_2\right|^{\frac{D}{p-\varepsilon}} w_2 \leq C, \tag{17}$$

for some constant $C > 0$ depending on the respective local norms of b, d_1, d_2, e, f, g.

For a local weak solution u and arbitrary $\eta \in C_c^\infty(B_2)$ we use $\varphi = \eta^p G(u)$ and with the aid of (16) one can obtain the estimate

$$\begin{aligned} \mathcal{A} \cdot \nabla\varphi + \mathcal{B}\varphi &= \eta^p G'(u)\mathcal{A} \cdot \nabla u + p\eta^{p-1} G(u)\mathcal{A} \cdot \nabla\eta + \eta^p G(u)\mathcal{B} \\ &\geq \eta^p G'(u) w_1 \left(|\nabla u|^p - \bar{d}_1 \bar{u}^p \right) - p\eta^{p-1} \left|\nabla\eta G(u)\right| w_1 \left(a \left|\nabla u\right|^{p-1} + \bar{b}\bar{u}^{p-1} \right) \\ &\quad - \eta^p \left|G(u)\right| w_2 \left(c \left|\nabla u\right|^{p-1} + \bar{d}_2 \bar{u}^{p-1} \right) \end{aligned}$$

so that if $v = F(\bar{u})$ one reaches

$$\begin{aligned} \mathcal{A} \cdot \nabla\varphi + \mathcal{B}\varphi \geq{}& |\eta\nabla v|^p w_1 - pa \left|v\nabla\eta\right| \left|\eta\nabla v\right|^{p-1} w_1 - p\alpha^{p-1}\bar{b} \left|v\nabla\eta\right| \left|\eta v\right|^{p-1} w_1 \\ & - \beta\alpha^{p-1}\bar{d}_1 \left|\eta v\right|^p w_1 - c\eta v \left|\eta\nabla v\right|^{p-1} w_2 - \alpha^{p-1}\bar{d}_2 \left|\eta v\right|^p w_2 \end{aligned} \tag{18}$$

We integrate over B_2 and divide by $w_1(B_2)$ to obtain

$$\begin{aligned} \fint_{B_2} |\eta\nabla v|^p w_1 \leq{}& pa \fint_{B_2} |v\nabla\eta| \left|\eta\nabla v\right|^{p-1} w_1 + p\alpha^{p-1} \fint_{B_2} \bar{b} \left|v\nabla\eta\right| \left|v\eta\right|^{p-1} w_1 \\ & + \beta\alpha^{p-1} \fint_{B_2} \bar{d}_1 \left|v\eta\right|^p w_1 + \frac{1}{w_1(B_2)} \int_{B_2} cv\eta \left|\eta\nabla v\right|^{p-1} w_2 + \frac{\alpha^{p-1}}{w_1(B_2)} \int_{B_2} \bar{d}_2 \left|v\eta\right|^p w_2, \end{aligned}$$

but since $w_2(B_2) = Cw_1(B_2)$ for $C = C(x_0, w_1, w_2) = \frac{w_2(B_2)}{w_1(B_2)}$ we can write

$$\begin{aligned} \fint_{B_2} |\eta\nabla v|^p w_1 \leq{}& pa \fint_{B_2} |v\nabla\eta| \left|\eta\nabla v\right|^{p-1} w_1 + p\alpha^{p-1} \fint_{B_2} \bar{b} \left|v\nabla\eta\right| \left|v\eta\right|^{p-1} w_1 \\ & + \beta\alpha^{p-1} \fint_{B_2} \bar{d}_1 \left|v\eta\right|^p w_1 + C \fint_{B_2} cv\eta \left|\eta\nabla v\right|^{p-1} w_2 + C\alpha^{p-1} \fint_{B_2} \bar{d}_2 \left|v\eta\right|^p w_2, \end{aligned} \tag{19}$$

and each term on the right hand side can be estimated using (6), (10), and (17) as follows:

$$\fint_{B_2} |v\nabla\eta| \left|\eta\nabla v\right|^{p-1} w_1 \leq \left(\fint_{B_2} |v\nabla\eta|^p w_1 \right)^{\frac{1}{p}} \left(\fint_{B_2} |\eta\nabla v|^p w_1 \right)^{1-\frac{1}{p}}, \tag{20}$$

if D_1 the dimensional constant associated to the weight w_1 then

$$\begin{aligned} \fint_{B_2} \bar{b} \left|v\nabla\eta\right| \left|v\eta\right|^{p-1} w_1 &\leq \left(\fint_{B_2} \bar{b}^{\frac{D_1}{p-1}} w_1 \right)^{\frac{p-1}{D_1}} \left(\fint_{B_2} |v\nabla\eta|^p w_1 \right)^{\frac{1}{p}} \left(\fint_{B_2} |v\eta|^{\chi_1 p} w_1 \right)^{\frac{p-1}{\chi_1 p}} \\ &\leq C \left(\fint_{B_2} |v\nabla\eta|^p w_1 \right)^{\frac{1}{p}} \left(\fint_{B_2} |\nabla(v\eta)|^p w_1 \right)^{1-\frac{1}{p}}, \end{aligned} \tag{21}$$

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[1]If $e = f = g = 0$ we can take any $k > 0$ and at the very end we can pass to the limit $k \to 0^+$.

and

$$
\begin{aligned}
\fint_{B_2} \bar{d}_1 |v\eta|^p w_1 &= \fint_{B_2} \bar{d}_1 |v\eta|^\varepsilon |v\eta|^{p-\varepsilon} w_1 \\
&\le \left(\fint_{B_2} \bar{d}_1^{\frac{D_1}{p-\varepsilon}} w_1 \right)^{\frac{p-\varepsilon}{D_1}} \left(\fint_{B_2} |v\eta|^p w_1 \right)^{\frac{\varepsilon}{p}} \left(\fint_{B_2} |v\eta|^{\chi_1 p} w_1 \right)^{\frac{p-\varepsilon}{\chi_1 p}} \\
&\le C \left(\fint_{B_2} |v\eta|^p w_1 \right)^{\frac{\varepsilon}{p}} \left(\fint_{B_2} |\nabla(v\eta)|^p w_1 \right)^{1-\frac{\varepsilon}{p}},
\end{aligned} \tag{22}
$$

whereas for $D = \frac{pq}{q-p}$ and $\bar{c} = c\left(\frac{w_2}{w_1}\right)^{1-\frac{1}{p}}$ we have

$$
\begin{aligned}
\fint_{B_2} cv\eta |\eta\nabla v|^{p-1} w_2 &= \fint_{B_2} \bar{c} w_2^{\frac{1-\varepsilon}{D}} |v\eta|^\varepsilon w_2^{\frac{\varepsilon}{p}} |v\eta|^{1-\varepsilon} w_2^{\frac{1-\varepsilon}{q}} |\eta\nabla v|^{p-1} w_1^{1-\frac{1}{p}} \\
&\le \left(\fint_{B_2} |\bar{c}|^{\frac{D}{1-\varepsilon}} w_2 \right)^{\frac{1-\varepsilon}{D}} \\
&\quad \times \left(\fint_{B_2} |v\eta|^p w_2 \right)^{\frac{\varepsilon}{p}} \left(\fint_{B_2} |v\eta|^q w_2 \right)^{\frac{1-\varepsilon}{q}} \left(\fint_{B_2} |\eta\nabla v|^p w_1 \right)^{1-\frac{1}{p}} \\
&\le C \left(\fint_{B_2} |v\eta|^p w_2 \right)^{\frac{\varepsilon}{p}} \left(\fint_{B_2} |\nabla(v\eta)|^p w_1 \right)^{\frac{1-\varepsilon}{p}} \left(\fint_{B_2} |\eta\nabla v|^p w_1 \right)^{1-\frac{1}{p}},
\end{aligned} \tag{23}
$$

and

$$
\begin{aligned}
\fint_{B_2} \bar{d}_2 |v\eta|^p w_2 &= \fint_{B_2} \bar{d}_2 |v\eta|^\varepsilon |v\eta|^{p-\varepsilon} w_2 \\
&\le \left(\fint_{B_2} \bar{d}_2^{\frac{D}{p-\varepsilon}} w_2 \right)^{\frac{p-\varepsilon}{D}} \left(\fint_{B_2} |v\eta|^p w_2 \right)^{\frac{\varepsilon}{p}} \left(\fint_{B_2} |v\eta|^q w_2 \right)^{\frac{p-\varepsilon}{q}} \\
&\le C \left(\fint_{B_2} |v\eta|^p w_2 \right)^{\frac{\varepsilon}{p}} \left(\fint_{B_2} |\nabla(v\eta)|^p w_1 \right)^{1-\frac{\varepsilon}{p}}.
\end{aligned} \tag{24}
$$

Therefore (19), (20), (21), (22), (23) and (24) give

$$
\begin{aligned}
\fint_{B_2} |\eta\nabla v|^p w_1 &\le pa \left(\fint_{B_2} |v\nabla\eta|^p w_1 \right)^{\frac{1}{p}} \left(\fint_{B_2} |\eta\nabla v|^p w_1 \right)^{1-\frac{1}{p}} \\
&\quad + Cp\alpha^{p-1} \left[\left(\fint_{B_2} |v\nabla\eta|^p w_1 \right) + \left(\int_{B_2} |v\nabla\eta|^p w_1 \right)^{\frac{1}{p}} \left(\fint_{B_2} |\eta\nabla v|^p w_1 \right)^{1-\frac{1}{p}} \right] \\
&\quad + C\beta\alpha^{p-1} \left(\fint_{B_2} |v\eta|^p w_1 \right)^{\frac{\varepsilon}{p}} \left[\left(\fint_{B_2} |v\nabla\eta|^p w_1 \right)^{1-\frac{\varepsilon}{p}} + \left(\fint_{B_2} |\eta\nabla v|^p w_1 \right)^{1-\frac{\varepsilon}{p}} \right] \\
&\quad + C \left(\fint_{B_2} |v\eta|^p w_2 \right)^{\frac{\varepsilon}{p}} \\
&\qquad \times \left[\left(\fint_{B_2} |\eta\nabla v|^p w_1 \right)^{1-\frac{1}{p}} \left(\int_{B_2} |v\nabla\eta|^p w_1 \right)^{\frac{1-\varepsilon}{p}} + \left(\fint_{B_2} |\eta\nabla v|^p w_1 \right)^{1-\frac{\varepsilon}{p}} \right] \\
&\quad + C\alpha^{p-1} \left(\fint_{B_2} |v\eta|^p w_2 \right)^{\frac{\varepsilon}{p}} \left[\left(\fint_{B_2} |v\nabla\eta|^p w_1 \right)^{1-\frac{\varepsilon}{p}} + \left(\fint_{B_2} |\eta\nabla v|^p w_1 \right)^{1-\frac{\varepsilon}{p}} \right].
\end{aligned} \tag{25}
$$

If one considers

$$z = \frac{\left(\fint_{B_2} |\eta \nabla v|^p \, w_1\right)^{\frac{1}{p}}}{\left(\fint_{B_2} |v \nabla \eta|^p \, w_1\right)^{\frac{1}{p}}}$$

and

$$\zeta = \frac{\left(\fint_{B_2} |\eta v|^p \, w_1\right)^{\frac{1}{p}} + \left(\fint_{B_2} |\eta v|^p \, w_2\right)^{\frac{1}{p}}}{\left(\fint_{B_2} |v \nabla \eta|^p \, w_1\right)^{\frac{1}{p}}}$$

then, because $\alpha \geq 1$, (25) becomes

$$z^p \leq C \left(z^{p-1} + \alpha^{p-1}(1 + z^{p-1}) + \zeta^{\varepsilon}(z^{p-1} + z^{p-\varepsilon}) + (1+\beta)\alpha^{p-1}\zeta^{\varepsilon}(1 + z^{p-\varepsilon}) \right)$$

for some constant $C > 0$ depending on $a, b, c, d, e, f, g, w_1, w_2$ and p. With the aid of [8, Lemma 2] we obtain

$$z \leq C\alpha^{\frac{p}{\varepsilon}}(1 + \zeta)$$

which gives

$$\left(\fint_{B_2} |\eta \nabla v|^p \, w_1\right)^{\frac{1}{p}} \leq C\alpha^{\frac{p}{\varepsilon}} \left(\left(\fint_{B_2} |v \nabla \eta|^p \, w_1\right)^{\frac{1}{p}} + \left(\fint_{B_2} |\eta v|^p \, w_1\right)^{\frac{1}{p}} + \left(\fint_{B_2} |\eta v|^p \, w_2\right)^{\frac{1}{p}} \right). \tag{26}$$

Now, by (6) and (10), that is the local Sobolev inequalities for the pair (w_1, w_1) and the pair (w_1, w_2) respectively we obtain

$$\left(\fint_{B_2} |\eta v|^{\chi_i p} \, w_i\right)^{\frac{1}{\chi_i p}} \leq C\alpha^{\frac{p}{\varepsilon}} \left(\left(\fint_{B_2} |v \nabla \eta|^p \, w_1\right)^{\frac{1}{p}} + \left(\fint_{B_2} |\eta v|^p \, w_1\right)^{\frac{1}{p}} + \left(\fint_{B_2} |\eta v|^p \, w_2\right)^{\frac{1}{p}} \right), \tag{27}$$

where we recall that $\chi_1 = \frac{D_1}{D_1 - p}$ and $\chi_2 = \frac{q}{p} = \frac{D}{D-p}$.

To continue we consider a sequence of cut-off functions as follows: we take $\eta_n \in C_c^\infty(B_{h_n})$ such that $\eta_n \equiv 1$ in $B_{h_{n+1}}$ and $|\nabla \eta_n| \leq C2^n$ where $h_n = 1 + 2^{-n}$. If one recalls that both weights are doubling so that $w_i(B_{h_n}) \leq \gamma_{w_i} w_i(B_{h_{n+1}})$ we deduce from (27) that (after passing to the limit $l \to \infty$)

$$\left(\fint_{B_{h_{n+1}}} |\bar{u}|^{\alpha\chi_1 p} \, w_1\right)^{\frac{1}{\chi_1 p}} + \left(\fint_{B_{h_{n+1}}} |\bar{u}|^{\alpha\chi_2 p} \, w_2\right)^{\frac{1}{\chi_2 p}} \leq C2^n\alpha^{\frac{p}{\varepsilon}} \left[\left(\fint_{B_{h_n}} |\bar{u}|^{\alpha p} \, w_1\right)^{\frac{1}{p}} + \left(\fint_{B_{h_n}} |\bar{u}|^{\alpha p} \, w_2\right)^{\frac{1}{p}} \right], \tag{28}$$

which is valid for all $\alpha \geq 1$. Recall the definition of $[u]_{s,B}$ given by (13), that is,

$$[u]_{s,B} = \left(\fint_B |\bar{u}|^s \, w_1\right)^{\frac{1}{s}} + \left(\fint_B |\bar{u}|^s \, w_2\right)^{\frac{1}{s}}$$

and observe that if $\chi = \min\{\chi_1, \chi_2\}$ then

$$\left(\fint_{B_{h_{n+1}}} |\bar{u}|^{\chi^{n+1} p} \, w_i\right)^{\frac{1}{\chi^{n+1} p}} \leq \left(\fint_{B_{h_{n+1}}} |\bar{u}|^{\chi^n \chi_i p} \, w_i\right)^{\frac{1}{\chi^n \chi_i p}},$$

for $i = 1, 2$. Therefore, if we select $\alpha_n = \chi^n > 1$ in (28) we are led to

$$[\bar{u}]_{s_{n+1}, B_{h_{n+1}}} \leq C^{\chi^{-n}} 2^{n\chi^{-n}} \chi^{\frac{p}{\varepsilon} n \chi^{-n}} [\bar{u}]_{s_n, B_{h_n}}$$

where $s_n = p\chi^n$. And because $\chi > 1$ then $\sum_{k=0}^{\infty} k\chi^{-k}$ and $\sum_{k=0}^{\infty} \chi^{-k}$ are convergent series so we can iterate the above inequality to obtain

$$[\bar{u}]_{s_n, B_{h_n}} \leq C[\bar{u}]_{p, B_2},$$

for some constant C independent of n. After passing to the limit $n \to \infty$ we obtain

$$\|u\|_{L^\infty(B_1)} \le C\left[\left(\fint_{B_2} |u|^p\, w_1\right)^{\frac{1}{p}} + \left(\fint_{B_2} |u|^p\, w_2\right)^{\frac{1}{p}} + k\right],$$

and the result follows. ■

Proof of Theorem 1.2. Thanks to the interpolation inequality in L^{s,w_i}, it is enough to find a sequence $s_n \underset{n\to\infty}{\longrightarrow} +\infty$ for which one has

$$[\bar{u}]_{s_n,B_1} \le C_n [\bar{u}]_{p,B_2},$$

where $\bar{u} = |u| + k$. As in the proof of Theorem 1.1, by using the test function $\varphi = \eta^p G(u)$ we reach to the inequality

$$\fint_{B_2} |\eta\nabla v|^p\, w_1 \le ap \fint_{B_2} |v\nabla\eta|\, |\eta\nabla v|^{p-1}\, w_1 + p\alpha^{p-1}\fint_{B_2} \bar{b}\, |v\nabla\eta|\, |v\eta|^{p-1}\, w_1 + \beta\alpha^{p-1}\fint_{B_2} \bar{d}_1\, |v\eta|^p\, w_1$$
$$+ \fint_{B_2} cv\eta\, |\eta\nabla v|^{p-1}\, w_2 + \alpha^{p-1}\fint_{B_2} \bar{d}_2\, |v\eta|^p\, w_2,$$

but because $\varepsilon = 0$ we cannot repeat (20)-(22). Instead we firstly estimate the term involving $\bar{b}$ as follows

$$\fint_{B_2} \bar{b}\, |v\nabla\eta|\, |v\eta|^{p-1}\, w_1 \le \left(\fint_{B_2} \bar{b}^{\frac{D_1}{p-1}} w_1\right)^{\frac{p-1}{D_1}} \left(\fint_{B_2} |v\nabla\eta|^p\, w_1\right)^{\frac{1}{p}} \left(\fint_{B_2} |v\eta|^{\chi_1 p}\, w_1\right)^{\frac{p-1}{\chi_1 p}}$$
$$\le C\left(\fint_{B_2} |v\nabla\eta|^p\, w_1\right)^{\frac{1}{p}} \left[\left(\fint_{B_2} |v\nabla\eta|^p\, w_1\right)^{1-\frac{1}{p}} + \left(\fint_{B_2} |\eta\nabla v|^p\, w_1\right)^{1-\frac{1}{p}}\right].$$

For the terms involving c and $\bar{d}$ we consider $\bar{c} = c\left(\frac{w_2}{w_1}\right)^{1-\frac{1}{p}}$ and for each $M > 0$ we define the set $\mathcal{C}_M = \{\, \bar{c} \le M \,\}$ and proceed as follows

$$\fint_{B_2} cv\eta\, |\eta\nabla v|^{p-1}\, w_2 = \frac{1}{w_2(B_2)}\left[\int_{B_2\cap\mathcal{C}_M} cv\eta\, |\eta\nabla v|^{p-1}\, w_2\right.$$
$$\left. + \int_{B_2\cap\mathcal{C}_M^c} \bar{c} w_2^{\frac{1}{D}}\, |v\eta|\, w_2^{\frac{1}{q}}\, |\eta\nabla v|^{p-1}\, w_1^{1-\frac{1}{p}}\right]$$
$$\le M\left(\fint_{B_2} |v\eta|^p\, w_2\right)^{\frac{1}{p}} \left(\fint_{B_2} |\eta\nabla v|^p\, w_1\right)^{1-\frac{1}{p}}$$
$$+ \left(\frac{1}{w_2(B_2)}\int_{B_2\cap\mathcal{C}_M^c} |\bar{c}|^D\, w_2\right)^{\frac{1}{D}} \left(\fint_{B_2} |v\eta|^q\, w_2\right)^{\frac{1}{q}} \left(\fint_{B_2} |\eta\nabla v|^p\, w_1\right)^{1-\frac{1}{p}}$$
$$\le M\left(\fint_{B_2} |v\eta|^p\, w_2\right)^{\frac{1}{p}} \left(\fint_{B_2} |\eta\nabla v|^p\, w_1\right)^{1-\frac{1}{p}}$$
$$+ C\left(\fint_{B_2} |v\nabla\eta|^p\, w_1\right)^{\frac{1}{p}} \left(\fint_{B_2} |\eta\nabla v|^p\, w_1\right)^{1-\frac{1}{p}}$$
$$+ C\left(\frac{1}{w_2(B_2)}\int_{B_2\cap\mathcal{C}_M^c} |\bar{c}|^D\, w_2\right)^{\frac{1}{D}} \left(\fint_{B_2} |\eta\nabla v|^p\, w_1\right).$$

Similarly

$$\begin{aligned}\fint_{B_2} \bar{d}_1\,|v\eta|^p\,w_1 &= \frac{1}{w_1(B_2)}\left[\int_{B_2\cap\{\,\bar{d}_1\le M\,\}} \bar{d}\,|v\eta|^p\,w_1 + \int_{B_2\cap\{\,\bar{d}_1>M\,\}} \bar{d}_1\,|v\eta|^p\,w_1\right]\\
&\le M\fint_{B_2}|v\eta|^p\,w_1 + \left(\frac{1}{w_1(B_2)}\int_{B_2\cap\{\,\bar{d}_1>M\,\}} \bar{d}_1^{\frac{D_1}{p}}\,w_1\right)^{\frac{p}{D_1}}\left(\fint_{B_2}|v\eta|^{\chi_1 p}\,w_1\right)^{\frac{1}{\chi_1 p}}\\
&\le M\fint_{B_2}|v\eta|^p\,w_1 + C\left(\fint_{B_2}|v\nabla\eta|^p\,w_1\right)\\
&\quad + \left(\frac{1}{w_1(B_2)}\int_{B_2\cap\{\,\bar{d}_1>M\,\}} \bar{d}_1^{\frac{D_1}{p}}\,w_1\right)^{\frac{p}{D_1}}\left(\fint_{B_2}|\eta\nabla v|^p\,w_1\right),\end{aligned}$$

and

$$\begin{aligned}\fint_{B_2} \bar{d}_2\,|v\eta|^p\,w_2 &= \left[\frac{1}{w_2(B_2)}\int_{B_2\cap\{\,\bar{d}_2\le M\,\}} \bar{d}_2\,|v\eta|^p\,w_2 + \int_{B_2\cap\{\,\bar{d}_2>M\,\}} \bar{d}_2\,|v\eta|^p\,w_2\right]\\
&\le M\fint_{B_2}|v\eta|^p\,w_2 + \left(\frac{1}{w_2(B_2)}\int_{B_2\cap\{\,\bar{d}_2>M\,\}} \bar{d}_2^{\frac{D}{p}}\,w_2\right)^{\frac{p}{D}}\left(\fint_{B_2}|v\eta|^{q}\,w_2\right)^{\frac{1}{q}}\\
&\le M\fint_{B_2}|v\eta|^p\,w_2 + C\left(\fint_{B_2}|v\nabla\eta|^p\,w_1\right)\\
&\quad + \left(\frac{1}{w_2(B_2)}\int_{B_2\cap\{\,\bar{d}_2>M\,\}} \bar{d}_2^{\frac{D}{p}}\,w_2\right)^{\frac{p}{D}}\left(\fint_{B_2}|\eta\nabla v|^p\,w_1\right).\end{aligned}$$

Because $\bar{c}\in L^{D,w_2}$, $\bar{d}_1\in L^{\frac{D_1}{p},w_1}$ and $\bar{d}_2\in L^{\frac{D}{p},w_2}$ then for any $\delta>0$ we can find $M>0$ such that

$$\left(\frac{1}{w_2(B_2)}\int_{B_2\cap\mathcal{C}_M}|\bar{c}|^D\,w_2\right)^{\frac{1}{D}} + \left(\frac{1}{w_1(B_2)}\int_{B_2\cap\{\,\bar{d}_1>M\,\}} \bar{d}_1^{\frac{D_1}{p}}\,w_1\right)^{\frac{p}{D_1}} + \left(\frac{1}{w_2(B_2)}\int_{B_2\cap\{\,\bar{d}_2>M\,\}} \bar{d}_2^{\frac{D}{p}}\,w_2\right)^{\frac{p}{D}} \le \delta,$$

therefore for any $\alpha\ge 1$ we can find $\delta>0$ sufficiently small and a constant $C_\alpha>0$ such that

$$\begin{aligned}\fint_{B_2}|\eta\nabla v|^p\,w_1 &\le C_\alpha\left[\left(\fint_{B_2}|v\nabla\eta|^p\,w_1\right)^{\frac{1}{p}} + \left(\fint_{B_2}|v\eta|^p\,w_2\right)^{\frac{1}{p}}\right]\left(\fint_{B_2}|\eta\nabla v|^p\,w_1\right)^{1-\frac{1}{p}}\\
&\quad + C_\alpha\left(\fint_{B_2}|v\nabla\eta|^p\,w_1\right) + C_\alpha\left(\fint_{B_2}|v\eta|^p\,w_1\right) + C_\alpha\left(\fint_{B_2}|v\eta|^p\,w_2\right).\end{aligned}$$

The above inequality allows us to we use [8, Lemma 2] once again and obtain an inequality analogous to (26), namely

$$\left(\fint_{B_2}|\eta\nabla v|^p\,w_1\right)^{\frac{1}{p}} \le C_\alpha\left[\left(\fint_{B_2}|v\nabla\eta|^p\,w_1\right)^{\frac{1}{p}} + \left(\fint_{B_2}|\eta v|^p\,w_1\right)^{\frac{1}{p}} + \left(\fint_{B_2}|\eta v|^p\,w_2\right)^{\frac{1}{p}}\right] \tag{29}$$

the main difference being that the constant C_α is no longer explicit. Nonetheless we can continue the argument from the proof of Theorem 1.1 by choosing appropriate cut-off functions η to reach

$$[\bar{u}]_{s_{n+1},B_{h_{n+1}}} \le C_n[\bar{u}]_{s_n,B_{h_n}},$$

where $s_n = p\chi^n$, $h_n = 1 + 2^{-n}$ and $[u]_{s,B}$ is defined in (13). Observe that while we do not obtain a uniform estimate for C_n we can still iterate the above to conclude that

$$[\bar{u}]_{s_n,B_1} \le C_n[\bar{u}]_{p,B_2}$$

and the result is proved. ■

Proof of Theorem 1.3. Theorem 1.1 says that u is bounded on any compact subset of B_3 hence for any $\beta \in \mathbb{R}$ and any $\delta > 0$ the function $\varphi = \eta^p \bar{u}^\beta$ is a valid test function provided $\bar{u} = u + k + \delta$ and $\eta \in C_c^\infty(B_3)$. Here k is defined exactly as in Theorem 1.1.

For $\beta = 1-p$ and $v = \log \bar{u}$ this is similar to what we did in [5], the main difference is the appearance of the weight w_2. We obtain

$$(p-1)\int_{B_3} |\eta\nabla v|^p\, w_1 \le pa\int_{B_3} |\nabla\eta|\, |\eta\nabla v|^{p-1}\, w_1 + p\int_{B_3} \bar{b}\eta^{p-1}\, |\nabla\eta|\, w_1 + \int_{B_3} c\eta\, |\eta\nabla v|^{p-1}\, w_2 + (p-1)\int_{B_3} \bar{d}_1\eta^p w_1 + \int_{B_3} \bar{d}_2\eta^p w_2, \tag{30}$$

for any $\eta \in C_c^\infty(B_3)$. To continue denote by $z = \left(\int_{B_3} |\eta\nabla v|^p\, w_1\right)^{\frac{1}{p}}$ and with the aid of Hölder's inequality (30) becomes

$$z^p \le C_1 z^{p-1} + C_2,$$

where for $\bar{c} = c\left(\frac{w_2}{w_1}\right)^{1-\frac{1}{p}}$ we have

$$C_1 = \frac{pa}{p-1}\left(\int_{B_3} |\nabla\eta|^p\, w_1\right)^{\frac{1}{p}} + \frac{1}{p-1}\left(\int_{B_3} |\bar{c}\eta|^p\, w_2\right)^{\frac{1}{p}}, \tag{31}$$

$$C_2 = \frac{p}{p-1}\int_{B_3} \bar{b}\eta^{p-1}\, |\nabla\eta|\, w_1 + \int_{B_3} \bar{d}_1\eta^p w_1 + \frac{1}{p-1}\int_{B_3} \bar{d}_2\eta^p w_2, \tag{32}$$

which thanks to Young's inequality imply

$$z^p \le C(C_1^p + C_2),$$

for some constant C. To continue we estimate C_1 and C_2 using appropriate η. For any $0 < h < 2$ such that $B_h \subset B_2$ (not necessarily concentric) we have that $B_{\frac{3h}{2}} \subset B_3$ and we consider $\eta \in C_c^\infty(B_{\frac{3h}{2}})$ such that $\eta \equiv 1$ in B_h, $0 \le \eta \le 1$ and $|\nabla\eta| \le Ch^{-1}$.

We use such η in (31)-(32) and we get the following estimates using Hölder inequality and the properties of η

$$\int_{B_3} |\nabla\eta|^p\, w_1 \le \frac{C}{h^p} w_1(B_{\frac{3h}{2}}),$$

$$\int_{B_3} \bar{b}\eta^{p-1}\, |\nabla\eta|\, w_1 \le \frac{C}{h} w_1(B_{\frac{3h}{2}})^{1-\frac{p-1}{D_1}}\left(\int_{B_3} \left|\bar{b}\right|^{\frac{D_1}{p-1}}\, w_1\right)^{\frac{p-1}{D_1}},$$

$$\int_{B_3} |\bar{c}\eta|^p\, w_2 \le C w_2(B_{\frac{3h}{2}})^{1-\frac{(1-\varepsilon)p}{D}}\left(\int_{B_3} |\bar{c}|^{\frac{D}{1-\varepsilon}}\, w_2\right)^{\frac{(1-\varepsilon)p}{D}},$$

$$\int_{B_3} \bar{d}_1\eta^p w_1 \le C w_1(B_{\frac{3h}{2}})^{1-\frac{p-\varepsilon}{D_1}}\left(\int_{B_3} \left|\bar{d}_1\right|^{\frac{D_1}{p-\varepsilon}}\, w\right)^{\frac{p-\varepsilon}{D_1}},$$

$$\int_{B_3} \bar{d}_2\eta^p w_2 \le C w_2(B_{\frac{3h}{2}})^{1-\frac{p-\varepsilon}{D}}\left(\int_{B_3} \left|\bar{d}_2\right|^{\frac{D}{p-\varepsilon}}\, w\right)^{\frac{p-\varepsilon}{D}}.$$

Therefore one obtains

$$\begin{aligned}
h^p \fint_{B_h} |\nabla v|^p\, w_1 &\le \frac{h^p}{w_1(B_h)} \int_{B_3} |\eta \nabla v|^p\, w_1 \\
&\le \frac{Ch^p}{w_1(B_h)} \left(C_1^p + C_2\right) \\
&\le C \left(\frac{w_1(B_{\frac{3h}{2}})}{w_1(B_h)} + h^{p-1} \frac{w_1(B_{\frac{3h}{2}})^{1-\frac{p-1}{D_1}}}{w_1(B_h)} + h^p \frac{w_1(B_{\frac{3h}{2}})^{1-\frac{p-\varepsilon}{D_1}}}{w_1(B_h)} \right. \\
&\qquad \left. + h^p \frac{w_2(B_{\frac{3h}{2}})^{1-\frac{(1-\varepsilon)p}{D}}}{w_1(B_h)} + h^p \frac{w_2(B_{\frac{3h}{2}})^{1-\frac{p-\varepsilon}{D}}}{w_1(B_h)} \right),
\end{aligned}$$

where C depends on $\int_{B_3} \left|\bar{b}\right|^{\frac{D_1}{p-1}} w_1$, $\int_{B_3} |\bar{c}|^{\frac{D}{1-\varepsilon}} w_2$, $\int_{B_3} \left|\bar{d}_1\right|^{\frac{D_1}{p-\varepsilon}} w_1$, and $\int_{B_3} \left|\bar{d}_2\right|^{\frac{D}{p-\varepsilon}} w$. We claim that the right hand side of the above inequality is bounded independently of $0 < h \le 2$, indeed because w_1 is doubling we have

$$\frac{w_1(B_{\frac{3h}{2}})}{w_1(B_h)} \le C,$$

and also because $B_{\frac{3h}{2}} \subset B_3$ we deduce from (7) that $Ch^{D_1} w_1(B_3) \le w_1(B_{\frac{3h}{2}})$, hence

$$h^{p-1} \frac{w_1(B_{\frac{3h}{2}})^{1-\frac{p-1}{D_1}}}{w_1(B_h)} \le \frac{\gamma_{w_1} h^{p-1}}{w_1(B_{\frac{3h}{2}})^{\frac{p-1}{D_1}}} \le C,$$

also

$$h^p \frac{w_1(B_{\frac{3h}{2}})^{1-\frac{p-\varepsilon}{D_1}}}{w_1(B_h)} \le \frac{\gamma_{w_1} h^p}{w_1(B_{\frac{3h}{2}})^{\frac{p-\varepsilon}{D_1}}} \le Ch^\varepsilon.$$

From (8) we deduce

$$\begin{aligned}
h^p \frac{w_2(B_{\frac{3h}{2}})^{1-\frac{(1-\varepsilon)p}{D}}}{w_1(B_h)} &= h^p \frac{w_2(B_{\frac{3h}{2}})^{\frac{p}{q}+\varepsilon\left(1-\frac{p}{q}\right)}}{w_1(B_h)} \\
&= h^p \left(\frac{w_2(B_{\frac{3h}{2}})^{\frac{1}{q}}}{w_1(B_h)^{\frac{1}{p}}} \right)^p w_2(B_{\frac{3h}{2}})^{\varepsilon\left(1-\frac{p}{q}\right)} \\
&\le \gamma_{w_2}^{\frac{p}{q}} h^p \left(\frac{w_2(B_h)^{\frac{1}{q}}}{w_1(B_h)^{\frac{1}{p}}} \right)^p w_2(B_3)^{\varepsilon\left(1-\frac{p}{q}\right)} \\
&\le \gamma_{w_2}^{\frac{p}{q}} h^p \left(C \left(\frac{3}{h}\right) \frac{w_2(B_3)^{\frac{1}{q}}}{w_1(B_3)^{\frac{1}{p}}} \right)^p w_2(B_3)^{\varepsilon\left(1-\frac{p}{q}\right)} \\
&\le C
\end{aligned}$$

and similarly

$$h^p \frac{w_2(B_{\frac{3h}{2}})^{1-\frac{p-\varepsilon}{D}}}{w_1(B_h)} = h^p \left(\frac{w_2(B_{\frac{3h}{2}})^{\frac{1}{q}}}{w_1(B_h)^{\frac{1}{p}}} \right)^p w_2(B_{\frac{3h}{2}})^{\frac{\varepsilon}{p}\left(1-\frac{p}{q}\right)} \le C.$$

Hence for any $\varepsilon \ge 0$ each term on the right hand side is bounded independently of $0 < h \le 2$.

Finally, the local Poincaré-Sobolev inequalities (5) and (9) tell us that

$$\begin{aligned}\fint_{B_h} |v - v_{B_h}|\, w_i &\le \left(\fint_{B_h} |v - v_{B_h}|^{q_i}\, w_i\right)^{\frac{1}{q_i}} \\ &\le Ch\left(\fint_{B_h} |\nabla v|^p\, w_1\right)^{\frac{1}{p}} \\ &\le C,\end{aligned}$$

for any ball $B_h \subseteq B_2$ and both $i = 1, 2$. We conclude that

$$\fint_{B_h} |v - v_{B_h}|\, w_i \le C \tag{33}$$

where $C > 0$ is a constant not depending on h, in other words, $v \in \mathrm{BMO}(B_2, w_i\,\mathrm{d}x)$. If we denote by $\|v\|_{BMO(B_2,w_i)}$ as the least possible $C > 0$ in (33) then the John-Nirenberg lemma for doubling measures [7, Appendix II] tells us that there exist constants $p_{0,i}, C > 0$ such that

$$\fint_B e^{p_{0,i}|v - v_B|} w_i \le C$$

for all balls $B \subseteq B_2$. In particular this gives

$$\left(\fint_{B_2} e^{p_{0,i}v} w_i\right) \cdot \left(\fint_{B_2} e^{-p_{0,i}v} w_i\right) \le C^2,$$

and because $v = \log \bar{u}$ we have obtained

$$\fint_{B_2} \bar{u}^{p_{0,i}} w_i \le C\left(\fint_{B_2} \bar{u}^{-p_{0,i}} w_i\right)^{-1}.$$

Denote by $p_0 = \min\{\, p_{0,1}, p_{0,2}\,\}$ and observe that

$$\fint_{B_2} \bar{u}^{p_0} w_i \le C\left(\fint_{B_2} \bar{u}^{-p_0} w_i\right)^{-1}.$$

holds for both $i = 1, 2$ because $p_0 \le p_{0,i}$ and Hölder inequality. Therefore if we denote by $\Psi(p, h) = \left(\fint_{B_h} \bar{u}^p w_1\right)^{\frac{1}{p}} + \left(\fint_{B_h} \bar{u}^p w_2\right)^{\frac{1}{p}}$ then the above implies

$$\Psi(p_0, 2) \le C\Psi(-p_0, 2). \tag{34}$$

The rest of the proof consists in using $\varphi = \eta^p \bar{u}^\beta$ for $\beta \ne 1 - p$, 0 as test function and $v = \bar{u}^\alpha$ for α given by $p\beta = p + \alpha - 1$. This gives

$$\begin{aligned}|\alpha|^p\left(\mathcal{A}\cdot\nabla\varphi + \mathcal{B}\varphi\right) \ge{}& w_1\left(\beta\,|\eta\nabla v|^p - \beta\,|\alpha|^p\,\bar{d}_1\,|\eta v|^p\right) \\ &- w_1\Big(ap\,|\alpha|\,|\nabla\eta v|\,|\eta\nabla v|^{p-1} + p\,|\alpha|^p\,\bar{b}\,|\eta v|^{p-1}\,|\nabla\eta v|\Big) \\ &- w_2\Big(|\alpha|\,c\,|\eta v|\,|\eta\nabla v|^{p-1} + |\alpha|^p\,\bar{d}_2\,|\eta v|^{p-1}\Big)\end{aligned}$$

which after integrating over B_3 becomes

$$\begin{aligned}0 \ge{}& \fint_{B_3}\left(\beta\,|\eta\nabla v|^p - \beta\,|\alpha|^p\,\bar{d}_1\,|\eta v|^p\right) w_1 \\ &- \fint_{B_3}\Big(ap\,|\alpha|\,|\nabla\eta v|\,|\eta\nabla v|^{p-1} + p\,|\alpha|^p\,\bar{b}\,|\eta v|^{p-1}\,|\nabla\eta v|\Big)\, w_1 \\ &- C\fint_{B_3}\Big(c\,|\alpha|\,|\eta v|\,|\eta\nabla v|^{p-1} + |\alpha|^p\,\bar{d}_2\,|\eta v|^{p-1}\Big)\, w_2\end{aligned}$$

where $C = \frac{w_2(B_3)}{w_1(B_3)}$. Depending on β we have

- If $\beta > 0$ then we have

$$\beta \fint_{B_3} |\eta \nabla v|^p w_1 \leq ap\,|\alpha| \fint_{B_3} |\nabla \eta v|\,|\eta\nabla v|^{p-1} w_1 + p\,|\alpha|^p \fint_{B_3} \bar{b}\,|\eta v|^{p-1}\,|\nabla \eta v|\, w_1$$
$$+ \beta\,|\alpha|^p \fint_{B_3} \bar{d}_1\,|\eta v|^p w_1 + C\,|\alpha| \fint_{B_3} c\,|\eta v|\,|\eta \nabla v|^{p-1} w_2$$
$$+ C\,|\alpha|^p \fint_{B_3} \bar{d}_2\,|\eta v|^{p-1} w_2$$

and if we proceed as in the proof of Theorem 1.1 to estimate each integral on the right hand side we obtain

$$\left(\fint_{B_3} |\eta\nabla v|^{\chi_i p} w_i\right)^{\frac{1}{\chi_i p}} \leq C\alpha^{\frac{p}{\varepsilon}}(1+\beta^{-1})^{\frac{1}{\varepsilon}} \left[\left(\fint_{B_3} |\eta v|^p w_1\right)^{\frac{1}{p}} + \left(\fint_{B_3} |\eta v|^p w_2\right)^{\frac{1}{p}} + \left(\fint_{B_3} |\nabla \eta v|^p w_1\right)^{\frac{1}{p}}\right].$$

If $\eta \in C_c^\infty(B_h)$ is such that $\eta \equiv 1$ in $B_{h'}$ for $1 \leq h' < h \leq 2$ with $|\nabla \eta| \leq C(h-h')^{-1}$ then

$$\left(\fint_{B_{h'}} |v|^{\chi_i p} w_i\right)^{\frac{1}{\chi_i p}} \leq C\left(\frac{w_i(B_3)}{w_i(B_{h'})}\right)^{\frac{1}{\chi_i p}} \frac{\alpha^{\frac{p}{\varepsilon}}(1+\beta^{-1})^{\frac{1}{\varepsilon}}}{h-h'}$$
$$\times\left[\left(\frac{w_1(B_h)}{w_1(B_3)}\right)^{\frac{1}{p}} \fint_{B_h} |v|^p w_1 + \left(\frac{w_2(B_h)}{w_2(B_3)}\right)^{\frac{1}{p}} \fint_{B_h} |v|^p w_2\right]^{\frac{1}{p}},$$

but since $1 \leq h' < h \leq 2$ we have

$$\frac{w_i(B_3)}{w_i(B_{h'})} \leq \frac{w_i(B_{4h'})}{w_i(B_{h'})} \leq \gamma_{w_i}^2 \quad \text{and} \quad \frac{w_i(B_h)}{w_i(B_3)} \leq 1$$

hence for $\chi = \min\{\chi_1, \chi_2\}$ we have

$$\Psi(\chi p, h') \leq C\frac{\alpha^{\frac{p}{\varepsilon}}(1+\beta^{-1})^{\frac{1}{\varepsilon}}}{h-h'}\Psi(p,h). \tag{35}$$

- Similarly, for $1-p < \beta < 0$ one has

$$\Psi(\chi p, h') \leq C\frac{(1-\beta^{-1})^{\frac{1}{\varepsilon}}}{h-h'}\Psi(p,h). \tag{36}$$

- If $\beta < 1-p$ then one obtains

$$\Psi(\chi p', h') \leq C\frac{(1+|\alpha|)^{\frac{p}{\varepsilon}}}{h-h'}\Psi(p,h). \tag{37}$$

If we observe that $\Psi(s,r) \underset{s\to\infty}{\longrightarrow} 2\max_{B_r} \bar{u}$ and $\Psi(s,r) \underset{s\to-\infty}{\longrightarrow} 2\min_{B_r} \bar{u}$ then we can repeat the iterative argument from the proof of [8, Theorem 5] to deduce that (35) and (36) imply

$$\max_{B_1} \bar{u} \leq C\Psi(p_0', 2)$$

for some $p_0' \leq p_0$ chosen appropriately, whereas (37) will give

$$\min_{B_1} \bar{u} \geq C^{-1}\Psi(-p_0, 2).$$

Finally we can use (34) to obtain a constant $C > 0$ depending on the structural parameters such that

$$\max_{B_1} \bar{u} \leq C \min_{B_1} \bar{u}$$

and because $\bar{u} = u + k + \delta$ we conclude by letting $\delta \to 0^+$. ∎

3. Behavior at infinity

In this section we obtain a decay estimate for weak solutions to the equation

$$\begin{cases} -\operatorname{div}(w_1 |\nabla u|^{p-2} \nabla u) = w_2 |u|^{q-2} u & \text{in } \Omega \\ u \in D^{1,p,w_1}(\Omega) \end{cases} \tag{38}$$

where the set $\Omega \subseteq \mathbb{R}^N$ (bounded or not) is such that there exists a constant $C > 0$ for which the global weighted Sobolev inequalities (14) and (15) hold. With the aid of the results regarding the equation $\operatorname{div}\mathcal{A} = \mathcal{B}$ we are able to prove that that weak solutions to (38) are locally bounded.

Lemma 3.1. *Let $u \in D^{1,p,w}(\Omega)$ be a weak solution of*

$$-\operatorname{div}(w_1 |\nabla u|^{p-2} \nabla u) = w_2 |u|^{q-2} u \qquad \text{in } \Omega.$$

Then for every $R > 0$ such that $B_{4R}(x_0) \subseteq \Omega$ then there exists $C_R > 0$ such that

$$\|u\|_{L^\infty(B_R(x_0))} \le C_R [u]_{p,B_{4R}(x_0)}.$$

Proof. Observe that equation (38) can be written in the from $\operatorname{div}\mathcal{A} = \mathcal{B}$ for $a = 1$, $b = c = d_1 = e = f = g = 0$ and $d_2 = -|u|^{q-p}$. We first use Theorem 1.2 because from that result we know that if $d_2 \in L^{\frac{D}{p},w_2}$ then for every $s \ge 1$ and $R > 0$ the weak solution u satisfies

$$\left(\fint_{B_{2R}(x_0)} |u|^s w_1\right)^{\frac{1}{s}} + \left(\fint_{B_{2R}(x_0)} |u|^s w_2\right)^{\frac{1}{s}} \le C_{R,s} \left[\left(\fint_{B_{4R}(x_0)} |u|^p w_1\right)^{\frac{1}{p}} + \left(\fint_{B_{4R}(x_0)} |u|^p w_2\right)^{\frac{1}{p}}\right],$$

and $C_{R,s}$ depends on s and on $\left(\fint_{B_{4R}(x_0)} |d_2|^{\frac{D}{p}} w_2\right)^{\frac{p}{D}}$. But because $u \in D^{1,p,w_1}(\Omega)$ and the weights w_1, w_2 verify (8) then the local Sobolev inequality (10) holds and we have that $u \in L^{q,w_2}(\Omega)$, hence $d \in L^{\frac{D}{p},w_2}(B_{4R}(x_0)) \Leftrightarrow q = \frac{Dp}{D-p}$. In particular, this shows that $u \in L^{s,w_2}(B_{2R}(x_0))$ for every s and as a consequence $d_2 = -|u|^{q-p} \in L^{\frac{D}{p-\varepsilon},w_2}(B_{2R}(x_0))$ for every $0 < \varepsilon < p$. Therefore we can now use Theorem 1.1 to conclude that

$$\|u\|_{L^\infty(B_R(x_0))} \le C_R [u]_{p,B_{4R}(x_0)},$$

where C_R depends on $R > 0$ and the norm of u in $D^{1,p,w_1}(\Omega)$. ■

Now we would like to estimate the decay of the L^{q_1,w_1} norm of weak solutions as one leaves the set Ω.

Lemma 3.2. *Suppose $u \in D^{1,p,w_1}(\Omega)$ is a weak solution of (38), then there exists $R_0 > 0$ and $\tau > 0$ such that if $R \ge R_0$ then*

$$\|u\|_{L^{q_1,w_1}(\Omega \setminus B_R)} \le \left(\frac{R_0}{R}\right)^\tau \|u\|_{L^{q_1,w_1}(\Omega \setminus B_{R_0})}.$$

Here B_R denotes an arbitrary ball of radius R.

Proof. Because $u \in D^{1,p,w}(\Omega)$ then for $\eta \in W^{1,\infty}(\mathbb{R}^N)$ the function $\varphi = \eta^p u$ is a valid test function in

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla\varphi w_1 = \int_\Omega |u|^{q-2} u\varphi w_2.$$

On the one hand, using Young's inequality we can find $C_p > 0$ such that

$$\begin{aligned} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla\varphi w_1 &= \int_\Omega |\eta\nabla u|^p w_1 + p\int_\Omega \eta^{p-1} |\nabla u|^{p-2} \nabla u \cdot u\nabla\eta w_1 \\ &\ge \frac{1}{2}\int_\Omega |\eta\nabla u|^p w_1 - C_p \int_\Omega |u\nabla\eta|^p w_1. \end{aligned}$$

On the other hand, since $q > p$ we can write

$$\begin{aligned}\int_\Omega |u|^{q-2}\, u\varphi w_2 &= \int_\Omega u^q \eta^p w_2 \\ &= \int_\Omega |u|^{q-p}\, |\eta u|^p\, w_2 \\ &\leq \left(\int_{\operatorname{supp}\eta} |u|^q\, w_2\right)^{1-\frac{p}{q}} \left(\int_\Omega |\eta u|^q\, w_2\right)^{\frac{p}{q}}.\end{aligned}$$

Hence

$$\begin{aligned}\int_\Omega |\nabla(\eta u)|^p\, w_1 &= \int_\Omega |\eta\nabla u + u\nabla\eta|^p\, w_1 \\ &\leq 2^{p-1}\int_\Omega |\eta\nabla u|^p\, w_1 + 2^{p-1}\int_\Omega |u\nabla\eta|^p\, w_1 \\ &\leq 2^{p-1}\left(2\int_\Omega |\nabla u|^{p-2}\,\nabla u\nabla\varphi w_1 + C_p\int_\Omega |u\nabla\eta|^p\, w_1\right) + 2^{p-1}\int_\Omega |u\nabla\eta|^p\, w_1 \\ &\leq C_p\int_\Omega |u\nabla\eta|^p\, w_1 + 2^p\left(\int_{\operatorname{supp}\eta} |u|^q\, w_2\right)^{1-\frac{p}{q}} \left(\int_\Omega |\eta u|^q\, w_2\right)^{\frac{p}{q}},\end{aligned}$$

and the global Sobolev inequality (15) tells us that there exists a constant $C_{p,w_1,w_2} > 0$ such that

$$\int_\Omega |\nabla(\eta u)|^p\, w_1 \leq C_p\int_\Omega |u\nabla\eta|^p\, w_1 + C_{p,w_1,w_2}\left(\int_{\operatorname{supp}\eta} |u|^q\, w_2\right)^{1-\frac{p}{q}} \left(\int_\Omega |\nabla(\eta u)|^p\, w_1\right). \tag{39}$$

We now choose η. First of all, because $\|u\|_{q,w_2}$ is finite for any given $\varepsilon > 0$ we can find $R_0 = R_0(\varepsilon) > 0$ such that if $R \geq R_0$ then

$$\int_{\Omega\setminus B_R} |u|^q\, w_2 \leq \varepsilon.$$

With this in mind we choose $R_0 > 0$ such that

$$C_{p,w_1,w_2}\left(\int_{\Omega\setminus B_{R_0}} |u|^q\, w_2\right)^{1-\frac{p}{q}} \leq \frac{1}{2},$$

and we suppose that $R \geq R_0$ from now on. We consider $\eta \in W^{1,\infty}(\mathbb{R}^N)$, such that $0 \leq \eta \leq 1$, $\eta(x) = 0$ for $x \in B_R$, $\eta(x) = 1$ for $x \notin B_{2R}$, and $|\nabla\eta| \leq CR^{-1}$. If we use such η in (39) we obtain a constant $C > 0$ independent of R such that

$$\int_\Omega |\nabla(\eta u)|^p\, w_1 \leq C_p\int_\Omega |u\nabla\eta|^p\, w_1$$

which after using (14) gives

$$\left(\int_\Omega |\eta u|^{q_1}\, w_1\right)^{\frac{1}{q_1}} \leq C\left(\int_\Omega |u\nabla\eta|^p\, w_1\right)^{\frac{1}{p}}. \tag{40}$$

By the choice of η we also have

$$\begin{aligned}
\int_\Omega |u\nabla\eta|^p\, w_1 &\le CR^{-p}\int_{\Omega\cap B_{2R}\setminus B_R} |u|^p\, w_1 \\
&\le CR^{-p}\left(w_1(\Omega\cap B_{2R})\right)^{1-\frac{1}{\chi_1}}\left(\int_{\Omega\cap B_{2R}\setminus B_R}|u|^{q_1}\, w_1\right)^{\frac{1}{\chi_1}} \\
&\le CR^{-p}\left(w_1(\Omega\cap B_{R_0})\left(\frac{2R}{R_0}\right)^{D_1}\right)^{1-\frac{1}{\chi_1}}\left(\int_{\Omega\cap B_{2R}\setminus B_R}|u|^{q_1}\, w_1\right)^{\frac{1}{\chi_1}} \\
&= C\left(\frac{w_1(\Omega\cap B_{R_0})}{R_0^{D_1}}\right)^{1-\frac{1}{\chi_1}} R^{D_1(1-\frac{1}{\chi_1})-p}\left(\int_{\Omega\cap B_{2R}\setminus B_R}|u|^{q_1}\, w_1\right)^{\frac{1}{\chi_1}} \\
&\le CR^{D_1(1-\frac{1}{\chi_1})-p}\left(\int_{\Omega\cap B_{2R}\setminus B_R}|u|^{q_1}\, w_1\right)^{\frac{1}{\chi_1}} \\
&= C\left(\int_{\Omega\cap B_{2R}\setminus B_R}|u|^{q_1}\, w_1\right)^{\frac{1}{\chi_1}} \qquad (41)
\end{aligned}$$

where we have used (7) and the fact that $\frac{1}{q_1} = \frac{1}{D_1} - \frac{1}{p}$. From (40) and (41) we obtain

$$\int_\Omega |\eta u|^{q_1}\, w_1 \le C\int_{\Omega\cap B_{2R}\setminus B_R}|u|^{q_1}\, w_1,$$

for some constant $C>0$ depending on p, q_1, R_0 but independent of R. To continue, observe that since $\eta\equiv 1$ on B_{2R}^c we can write

$$\begin{aligned}
\int_{\Omega\setminus B_{2R}}|u|^{q_1}\, w_1 &\le \int_\Omega |\eta u|^{q_1}\, w_1 \\
&\le C\int_{\Omega\cap B_{2R}\setminus B_R}|u|^{q_1}\, w_1 \\
&= C\int_{\Omega\setminus B_R}|u|^{q_1}\, w_1 - C\int_{\Omega\setminus B_{2R}}|u|^{q_1}\, w_1,
\end{aligned}$$

thus, if $\theta = \frac{C}{C+1}\in(0,1)$ then we obtain

$$\int_{\Omega\setminus B_{2R}}|u|^{q_1}\, w_1 \le \theta\int_{\Omega\setminus B_R}|u|^{q_1}\, w_1.$$

Then just as in [5] one can find $\tau>0$ such that

$$\int_{\Omega\setminus B_R}|u|^{q_1}\, w_1 \le \left(\frac{R_0}{R}\right)^\tau\int_{\Omega\setminus B_{R_0}}|u|^{q_1}\, w_1$$

for $\tau = -q_1\log_2\theta > 0$. ■

Lemma 3.3. *Suppose that $u\in D^{1,p,w_1}(\Omega)$ is a weak solution of*

$$-\operatorname{div}(w_1\,|\nabla u|^{p-2}\,\nabla u) = w_2\,|u|^{q-2}\,u \qquad \text{in } \Omega. \tag{42}$$

Then for each $s>\max\{q_1,q\}$ there exists $R_0>0$ (depending on s) such that if $R\ge R_0$ then there exists $C=C(p,q_1,q,w_1,w_2;s)>0$ for which

$$\|u\|_{L^{s,w_i}(\Omega\setminus B_{2R})} \le \frac{C}{R^{\frac{p}{q_1-p}-o_s(1)}}\,\|u\|_{L^{q_1,w_1}(\Omega\setminus B_R)},$$

for both $i=1,2$, where $o_s(1)$ is a quantity that goes to 0 as $s\to\infty$.

Proof. Firstly notice that thanks to the $L^{s,w}$ interpolation inequality it is enough to exhibit a sequence $s_n \underset{n\to\infty}{\longrightarrow} +\infty$ for which one has

$$\|u\|_{L^{s_n,w_i}(\Omega\setminus B_{2R})} \leq \frac{C}{R^{\frac{p}{q_1-p}-o_n(1)}} \|u\|_{L^{q_1,w_1}(\Omega\setminus B_R)}.$$

Observe that in the context of (11) we can view (42) as div$\mathcal{A} = \mathcal{B}$ where $a = 1$, $b = c = d_1 = e = f = g = 0$ and $d_2 = \bar{d}_2 = -|u|^{q-p}$. The assumption $u \in D^{1,p,w_1}(\Omega)$ tells us that $\varphi = \eta^p G(u)$ is valid test function and we can follow the notation of the proof Theorem 1.1, in fact, since $e = f = g = 0$ we can further suppose that $k > 0$ is arbitrary in the definition of both F and G. Starting with (18) we now integrate over Ω to obtain

$$\int_\Omega |\eta\nabla v|^p w_1 \leq p\int_\Omega |v\nabla\eta| |\eta\nabla v|^{p-1} w_1 + (\alpha-1)\alpha^{p-1}\int_\Omega d_2 |v\eta|^p w_2,$$

where $v = F(\bar{u})$. From the above we obtain

$$\int_\Omega |\nabla(\eta v)|^p w_1 \leq C_\alpha\left(\int_\Omega |v\nabla\eta|^p w_1 + \int_\Omega |u|^{q-p} |v\eta|^p w_2\right),$$

and with the help of (15) we can write

$$\begin{aligned}\int_\Omega |u|^{q-p} |v\eta|^p w_2 &\leq \left(\int_{\operatorname{supp}\eta} |u|^q w_2\right)^{1-\frac{p}{q}} \left(\int_\Omega |v\eta|^q w_2\right)^{\frac{p}{q}} \\ &\leq C_{p,w_1,w_2}\left(\int_{\operatorname{supp}\eta} |u|^q w_2\right)^{1-\frac{p}{q}} \left(\int_\Omega |\nabla(v\eta)|^p w_1\right),\end{aligned}$$

therefore we have

$$\int_\Omega |\nabla(\eta v)|^p w_1 \leq C_\alpha\int_\Omega |v\nabla\eta|^p w_1 + C_{p,\alpha,w_1,w_2}\left(\int_{\operatorname{supp}\eta} |u|^q w_2\right)^{1-\frac{p}{q}} \left(\int_\Omega |\nabla(v\eta)|^p w_1\right).$$

We now select η. Because $u \in D^{1,p,w_1}(\Omega)$ and that (15) holds then we know that $u \in L^{q,w_2}(\Omega)$, therefore for any given $\nu > 0$ we can find $R_0 = R_0(\nu) > 0$ such that

$$\int_{\Omega\setminus B_R} |u|^q w_2 \leq \nu, \qquad \forall\, R \geq R_0.$$

With this in mind we choose $R_0 = R_0(\alpha) > 0$ such that

$$C_{p,\alpha,w_1,w_2}\left(\int_{\Omega\setminus B_R} |u|^q w_2\right)^{1-\frac{p}{q}} \leq \frac{1}{2},$$

and we suppose that $R \geq R_0$ to obtain that if $\operatorname{supp}\eta \subset B_R^c$ then

$$\int_\Omega |\nabla(\eta v)|^p w_1 \leq C_\alpha\int_\Omega |v\nabla\eta|^p w_1,$$

and using (14), (15) and passing to the limits $l \to +\infty$, $k \to 0^+$ give

$$\left(\int_\Omega |\eta u^\alpha|^{q_1} w_1\right)^{\frac{1}{q_1}} \leq C_\alpha\left(\int_\Omega |u^\alpha\nabla\eta|^p w_1\right)^{\frac{1}{p}}, \tag{43}$$

$$\left(\int_\Omega |\eta u^\alpha|^{q} w_2\right)^{\frac{1}{q}} \leq C_\alpha\left(\int_\Omega |u^\alpha\nabla\eta|^p w_1\right)^{\frac{1}{p}}. \tag{44}$$

We now select η: for $n \geq 0$ we consider $R_n = R(2-2^{-n})$ and a smooth function η such that $0 \leq \eta \leq 1$, $\eta(x) = 0$ for $|x| \leq R_n$, $\eta(x) = 1$ for $|x| \geq R_{n+1}$ and satisfies $|\nabla\eta| \leq \frac{C2^n}{R}$,

$$\operatorname{supp}\eta \subseteq \Omega\setminus B_{R_n}$$
$$\operatorname{supp}\nabla\eta \subseteq \Omega\cap B_{R_n}\setminus B_{R_{n+1}}.$$

Therefore if for $n \geq 1$ we take $\alpha_n = \left(\frac{q_1}{p}\right)^n$ in (43) then we obtain

$$\left(\int_{\Omega\setminus B_{R_{n+1}}} |u|^{\frac{q_1^{n+1}}{p^n}} w_1\right)^{\frac{p^n}{q_1^{n+1}}} \leq \left(\frac{C_n}{R}\right)^{\frac{p^n}{q_1^n}} \left(\int_{\Omega\setminus B_{R_n}} |u|^{\frac{q_1^n}{p^{n-1}}} w_1\right)^{\frac{p^{n-1}}{q_1^n}},$$

or equivalently, if $s_n = \frac{q_1^n}{p^{n-1}}$ and $\mathcal{U}_n = \|u\|_{L^{s_n,w_1}(\Omega\setminus B_{R_n})}$,

$$\mathcal{U}_{n+1} \leq \frac{\tilde{C}_n}{R^{\frac{p^n}{q_1^n}}}\mathcal{U}_n,$$

for $\tilde{C}_n = C_n^{\left(\frac{p}{q_1}\right)^n}$, which after iterating gives

$$\mathcal{U}_n \leq \left(\frac{\prod_{i=1}^{n-1}\tilde{C}_i}{R^{\sum_{i=1}^{n-1}\left(\frac{p}{q_1}\right)^i}}\right)\mathcal{U}_1,$$

and since

$$\sum_{i=1}^{n-1}\left(\frac{p}{q_1}\right)^i = \frac{p}{q_1-p} - \frac{q_1}{q_1-p}\left(\frac{p}{q_1}\right)^n = \frac{p}{q_1-p} - o_n(1),$$

because $q_1 > p$ we obtain that for any $s > q_1$

$$\|u\|_{L^{s,w_1}(\Omega\setminus B_{2R})} \leq \frac{C_s}{R^{\frac{p}{q_1-p}-o_s(1)}}\|u\|_{L^{q_1,w_1}(\Omega\setminus B_R)},$$

because $\mathcal{U}_1 \leq \|u\|_{L^{q_1,w_1}(\Omega\setminus B_R)}$, $\mathcal{U}_n \geq \|u\|_{L^{s_n,w_1}(B_{2R})}$.

With the same choice of η and α in (44) we have

$$\begin{aligned}\left(\int_{\Omega\setminus B_{R_{n+1}}} |u|^{\frac{q_1^n q}{p^n}} w_2\right)^{\frac{p^n}{q_1^n q}} &\leq \left(\frac{C_n}{R}\right)^{\frac{p^n}{q_1^n}} \left(\int_{\Omega\setminus B_{R_n}} |u|^{\frac{q_1^n}{p^{n-1}}} w_1\right)^{\frac{p^{n-1}}{q_1^n}}\\ &= \left(\frac{C_n}{R}\right)^{\frac{p^n}{q_1^n}}\mathcal{U}_n\\ &\leq \left(\frac{\prod_{i=1}^{n}\tilde{C}_i}{R^{\sum_{i=1}^{n}\left(\frac{p}{q_1}\right)^i}}\right)\mathcal{U}_1,\end{aligned}$$

and just as before we deduce that

$$\|u\|_{L^{s,w_2}(\Omega\setminus B_{2R})} \leq \frac{C_s}{R^{\frac{p}{q_1-p}-o_s(1)}}\|u\|_{L^{q_1,w_1}(\Omega\setminus B_R)}$$

for $s > q$. ∎

Now we are in position to prove Theorem 1.4:

Proof of Theorem 1.4. Consider the value of $R_0 > 0$ given in Lemma 3.2, and suppose that $x \in \Omega\setminus B_{2R_0}$. Fix $0 < r < \frac{R_0}{4}$ so that $B_r(x) \subseteq \Omega$ and use Lemma 3.1 to obtain

$$|u(x)| \leq \|u\|_{L^\infty(B_r(x))} \leq C_r[u]_{p,B_{2r}} \leq C_r\left[\left(\int_{B_{2r}} |u|^s w_1\right)^{\frac{1}{s}} + \left(\int_{B_{2r}} |u|^s w_2\right)^{\frac{1}{s}}\right],$$

for any $s > p$. If we consider $R = \frac{|x|}{4}$, then by geometric considerations we deduce that $B_{2r}(x) \subseteq \Omega\setminus B_{2R}$ hence

$$\left(\int_{B_{2r}} |u|^s w_i\right)^{\frac{1}{s}} \leq \left(\int_{\Omega\setminus B_{2R}} |u|^s w_i\right)^{\frac{1}{s}}.$$

Now we fix s large enough so that $o_s(1) \le \frac{\tau}{2}$ in Lemma 3.3, where $\tau > 0$ is taken from Lemma 3.2, by doing that we obtain

$$\begin{aligned}\|u\|_{L^{s,w_2}(\Omega\setminus B_{2R})} + \|u\|_{L^{s,w_1}(\Omega\setminus B_{2R})} &\le \frac{C}{R^{\frac{p}{q_1-p}-o_s(1)}} \|u\|_{L^{q_1,w_1}(\Omega\setminus B_R)} \\ &\le \frac{C}{R^{\frac{p}{q_1-p}-\frac{\tau}{2}}} \|u\|_{L^{q_1,w_1}(\Omega\setminus B_R)} \\ &\le \frac{C}{R^{\frac{p}{q_1-p}-\frac{\tau}{2}}} \left(\frac{R_0}{R}\right)^{\tau} \|u\|_{L^{q_1,w_1}(\Omega\setminus B_{R_0})},\end{aligned}$$

therefore, by putting all together we obtain

$$|u(x)| \le \frac{CR_0^\tau}{R^{\frac{p}{q_1-p}+\frac{\tau}{2}}} \|u\|_{L^{q_1,w}(\Omega\setminus B_{R_0})} = \frac{C}{|x|^{\frac{p}{q_1-p}+\lambda}},$$

for some constant $C > 0$ independent of $|x| \ge 2R_0$, and the result is proved for $\tilde{R} = 2R_0$. ■

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