# ASYMPTOTICS OF THE MOMENTS OF QUADRATIC ASYMPTOTICALLY SYMMETRIC TIME NON-LOCAL BIRTH-DEATH PROCESSES

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Abstract. In this paper, we study the moments of semi-Markovian versions of classical birth-death processes, focusing on the so-called *Quadratic Asymptotically Symmetric (QAS) birth-death processes*. By means of Tauberian theorems, we provide a complete description of their asymptotic behavior. Our results show a dichotomous pattern: when the birth rate dominates the death rate, the moments grow exponentially, while if the death rate exceeds the birth rate, the moments decay slowly. This contrasts with classical birth-death processes, where moment growth and decay are always exponential.

## 1. Introduction

The theory of birth-death processes is a basic framework for studying how the size of a population changes as time evolves. Such processes are continuous-time Markov chains that count the number of *particles in a system* as time elapses. More precisely, in a population with $n$ individuals, each individual could give birth to another at a rate $b_n \geq 0$ or could die at a rate $d_n \geq 0$.

These processes have proven to be versatile and have been successfully applied in various fields, such as demography, queuing theory, and epidemiology (see, e.g., [9, 14, 28] and the references therein). However, the Markovian nature of these processes limits their applicability in analyzing phenomena that exhibit *long memory* or are influenced by *environmental conditions with random fluctuations*. To address these limitations, several researchers have proposed generalizations and extensions of birth-death processes, see, e.g., [4, 6, 7, 10, 19, 21, 22, 23, 24, 30], and the references therein.

Since the probability of birth or death events depends not only on the current state but also on the system's history, the processes discussed above are known as *nonlocal birth-death processes*. A key contribution in this field comes from Ascione, Leonenko, and Pirozzi [4], who studied a specific class of nonlocal birth-death processes as discrete approximations of *Pearson diffusions*. Specifically, they have considered a solvable birth-death process $\mathcal{N}(t)$ and a subordinator $\sigma_\Phi$ associated with a Bernstein function $\Phi$, and they analyze th features of a nonlocal birth-death process induced by $\mathcal{N}$ and $\Phi$ as a compound process of the form:

$$\mathcal{N}_\Phi(t) := \mathcal{N}(E_\Phi(t)), \quad t \geq 0,$$

where $E_\Phi$ denotes the inverse subordinator of $\sigma_\Phi$, which is independent of $\mathcal{N}$. Among other results, they have proved that these processes admit an invariant measure, which also serves as the limit measure for any starting distribution. They also provide the correlation structure of the stochastic processes in terms of the potential measure of the involved subordinator and the eigenfunctions of the nonlocal in-time derivatives.

The primary goal of this paper is to deepen the understanding of the properties of this type of nonlocal processes $\mathcal{N}_\Phi(t)$. To this end, we will consider two main assumptions. First, we examine birth-death processes with *Quadratic asymptotically Symmetric (QAS)* transition rates, that is, the birth and death rates are given by

$$b_n = \beta n^2 + \delta_b n + \gamma \quad \text{and} \quad d_n = \beta n^2 + \delta_d n, \quad n \in \mathbb{N},$$

where $\beta$, $\delta_b$, $\delta_d$, and $\gamma$ are non-negative constants. Second, we consider a specific class of Bernstein functions, defined by $\Phi(\lambda) = \lambda \widehat{h}(\lambda)$ for $\lambda > 0$, where $h$ is a function of type $(\mathcal{PC})'$ and $\widehat{h}$ represents the

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Laplace transform of $h$. The condition $(\mathcal{PC})'$ means that $h \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ is a nonnegative, non-increasing function, and there exists $\ell \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ such that $\ell \notin L_1(\mathbb{R}_+)$ and $h * \ell = 1$ on $(0, \infty)$.

While at first glance, this choice of Bernstein function may appear restrictive, it in fact encompasses a remarkably broad class of nonlocal birth-death processes, including the fractional case, as shown in Section 4 below. On the other hand, although birth-death processes with quadratic transition rates have been studied in the local setting (see, e.g., [17, 28]), their nonlocal counterparts remain largely unexplored. To the best of our knowledge, a systematic investigation of these processes within the nonlocal framework has yet to be undertaken. This gap in the literature motivates our study and underscores the relevance of our approach.

## 2. Main Results

In order to present our main results, we need to introduce some notation and definitions that we use throughout the text.

The Laplace transform of a function $f : [0, \infty) \to \mathbb{R}$ defined on the half-line will be denoted by

$$\mathscr{L}(f; \lambda) = \widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt,$$

whenever the last integral is convergent. For readability, we will use the symbol $\mathscr{L}$ when the function $f$ has a lengthy expression, such as functions involving convolutions.

**Definition 2.1.** We say that a non-negative function $h \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ is of type $(\mathcal{PC})$ if there is a non-negative non-increasing $\ell \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ such that $h * \ell = 1$ on the interval $(0, \infty)$. In those cases where $\ell \notin L_1(\mathbb{R}_+)$, we will say that $h$ is of type $(\mathcal{PC})'$. In order to emphasize the existence of the function $\ell$, throughout the text we write $(h, \ell) \in (\mathcal{PC})'$.

We emphasize that the condition $(\mathcal{PC})$ has been successfully applied in the study of subdiffusion processes, as demonstrated in works such as [2, 15, 26], among others. Given that $(\mathcal{PC})' \subset (\mathcal{PC})$, this condition establishes a robust framework for analyzing this class of nonlocal processes.

**Definition 2.2.** A $C^\infty$-function $f \colon (0, \infty) \to \mathbb{R}$ is called completely monotonic if $(-1)^n f^{(n)}(\lambda) \geq 0$ for all $n \in \mathbb{N}_0$ and $\lambda > 0$. Further, a $C^\infty$-function $g \colon (0, \infty) \to \mathbb{R}$ is called Bernstein function if $g(\lambda) \geq 0$ for all $\lambda > 0$, and $g'$ is completely monotonic. The class of completely monotonic functions and Bernstein functions will be denoted by $(\mathcal{CM})$ and $(\mathcal{BF})$, respectively.

A detailed collection of the most important properties of the classes $(\mathcal{CM})$ and $(\mathcal{BF})$ can be found in [29]. This reference provides a thorough analysis of their analytic characteristics, key theorems, and applications. In particular, it discusses the interplay between these function classes and probability theory.

**Definition 2.3.** Consider a birth-death process $\mathcal{N}(t)$ and a subordinator $\sigma_\Phi$ associated with a Bernstein function $\Phi$, with inverse subordinator $E_\Phi$ (independent of $\mathcal{N}$). The *time non-local birth-death process* induced by $\mathcal{N}$ and $\Phi$ is a stochastic process of the form

$$\mathcal{N}_\Phi(t) := \mathcal{N}(E_\Phi(t)), \quad t \geq 0.$$

For $n \in \mathbb{N}$, the $n$-th moment $M_n^\Phi(t)$ of the process $\mathcal{N}_\Phi(t)$ is defined by

$$M_n^\Phi(t) = \sum_{k=0}^{\infty} k^n P_k^\Phi(t), \quad t \geq 0,$$

where $P_k^\Phi(t)$ is the transition probability of $\mathcal{N}_\Phi(t)$, given by

$$P_k^\Phi(t) = \Pr\{\mathcal{N}_\Phi(t) = k \mid \mathcal{N}_\Phi(0) = N\}, \quad \text{for} \quad t > 0,$$

and

$$P_k^\Phi(0) = \begin{cases} 1, & k = N, \\ 0, & k \neq N. \end{cases}$$

For further insights into these processes, we refer the reader to [4, Section 2], where the authors study such processes as discrete approximations of the so-called *Pearson diffusion*.

**Theorem 2.1.** *Let $\Phi$ be a Bernstein function given by $\Phi(\lambda) = \lambda\widehat{h}(\lambda)$ for some $(h, \ell) \in (\mathcal{PC})'$. Consider a classical birth-death process $\mathcal{N}(t)$ with birth rate $b_k = \beta k^2 + \delta_b k + \gamma$ and death rate $d_k = \beta k^2 + \delta_d k$, where $\beta$, $\delta_b$, $\delta_d$, and $\gamma$ are non-negative constants. If the moments of $\mathcal{N}(t)$ are finite, then the moments of $\mathcal{N}_\Phi(t)$ are also finite. Moreover, the following expressions hold:*

$$M_0^\Phi(t) = 1, \qquad M_1^\Phi(t) = N s_{c_1}(t) + \gamma(1 * r_{c_1})(t), \quad t \geq 0, \tag{2.1}$$

*and*

$$M_2^\Phi(t) = N^2 s_{c_2}(t) + (2\gamma + \delta_b + \delta_d)(r_{c_2} * M_1^\Phi)(t) + \gamma(1 * r_{c_2})(t), \quad t \geq 0, \tag{2.2}$$

*where*

$$c_1 = \delta_b - \delta_d, \quad \text{and} \quad c_2 = 2(\beta + \delta_b - \delta_d).$$

*Additionally, for $n \geq 3$, we have the following recursive formula:*

$$M_n^\Phi(t) = N^n s_{c_n}(t) + \sum_{j=0}^{n-1} \kappa_{n,j}(r_{c_n} * M_j^\Phi)(t), \quad t \geq 0, \tag{2.3}$$

*where*

$$c_n = (\delta_d - \delta_b)n - \beta n(n-1), \quad \text{and} \quad \kappa_{n,j} = \begin{cases} 2\beta\binom{n}{j+2} + (\delta_b - \delta_d)\binom{n}{j+1} + \gamma\binom{n}{j}, & j \quad \text{even}, \\ \gamma\binom{n}{j} + (\delta_b + \delta_d)\binom{n}{j+1}, & j \quad \text{odd}. \end{cases}$$

*Here, for $n \in \mathbb{N}$, the functions $s_{c_n}$ and $r_{c_n}$ correspond to the scalar resolvent functions associated with the function $\ell$ defined in Appendix A below.*

Notice that the Bernstein functions that we are considering constitute a subclass of *special* Bernstein functions, in the sense that the conjugate of $\Phi$ is still a Bernstein function. Indeed, if $\Phi(\lambda) = \lambda\,\widehat{h}(\lambda)$, then the conjugate is given by

$$\Phi^\star(\lambda) = \frac{\lambda}{\Phi(\lambda)} = \frac{1}{\widehat{h}(\lambda)},$$

that is clearly a Bernstein function since $\widehat{h}$ is completely monotone. Nevertheless, for functions of this form, the identification of $h$ with the tail of the Lévy measure is immediate. Furthermore, the co-Sonine kernel $\ell$ is the potential measure of the involved subordinator, i.e.

$$\ell(t) = \frac{d}{dt}\mathbb{E}[E_\Phi(t)].$$

To guarantee that $(h, \ell)$ satisfy $(\mathcal{PC})'$, one just needs to ensure that

$$\lim_{\lambda\to+\infty} \frac{\Phi(\lambda)}{\lambda} = \lim_{\lambda\to+\infty} \widehat{h}(\lambda) = 0, \qquad \lim_{\lambda\to 0} \Phi(\lambda) = 0, \qquad \lim_{t\to 0} h(t) = +\infty, \qquad -\int_0^{+\infty} t\, dh(t) = +\infty.$$

For further details, check the relative section in [29].

**Remark 2.1.** It is important to note that $\kappa_{n,j}$ is not defined for $j = 0$. However, we can extend the definition, by setting $\kappa_{n,0} := c_n = (\delta_d - \delta_b)n - \beta n(n-1)$. To emphasize the significance of this term, we will refer to it as $c_n$.

**Remark 2.2.** Theorem 2.1 establishes that all the moments of the process $\mathcal{N}_\Phi(t)$ are well defined, provided that $\mathcal{N}(t)$ satisfies the same property. This criterion is notably broad, imposing minimal restrictions. Indeed, [4, Section 2] provides a comprehensive set of results demonstrating conditions under which all moments of $\mathcal{N}(t)$ are guaranteed to be finite.

In order to analyze the asymptotic behavior of $M_n^\Phi(t)$ as $t \to \infty$, we must to introduce an additional concept.

**Definition 2.4.** Let $L\colon (0,\infty) \to (0,\infty)$ and $\varrho \in \mathbb{R}$. We say that $L$ is a regularly function at infinity of index $\varrho$, if for all $x > 0$ we have that

$$\lim_{t\to\infty} \frac{L(tx)}{L(t)} = x^\varrho.$$

We denote this class of functions by $\mathcal{RV}_\varrho^\infty$. In the case that $\varrho = 0$, these functions are known as slowly varying at infinity functions and they are denoted by $\mathcal{SV}^\infty$.

**Theorem 2.2.** *Let $\Phi$ be a Bernstein function given by $\Phi(\lambda) = \lambda\widehat{h}(\lambda)$ for some $(h, \ell) \in (\mathcal{PC})'$. Consider a classical birth-death process $\mathcal{N}(t)$ with birth rates $b_k = \beta k^2 + \delta_b k + \gamma$ and death rates $d_k = \beta k^2 + \delta_d k$, where $\beta$, $\delta_b, \delta_d$ and $\gamma$ are non-negative constants. Assume that $M_1^\Phi(t)$ is finite. The following assertions hold.*

(i) *If $\delta_b > \delta_d$ then*

$$M_1^\Phi(t) \sim K_1 \cdot \exp(\omega_1 t), \quad \textit{as} \quad t \to \infty,$$

*where $\omega_1 > 0$ is the unique solution of the equation $\lambda\widehat{h}(\lambda) = \delta_b - \delta_d$ and*

$$K_1 = \frac{\big(N(\delta_b - \delta_d) + \gamma\big)}{\omega_1(\widehat{h}(\omega_1) + \omega_1\dot{\widehat{h}}(\omega_1))}.$$

*where $\dot{\widehat{h}}$ corresponds to the derivative of the Laplace transform of $h$.*

(ii) *If $\delta_d > \delta_b$ and the mapping $t \mapsto \widehat{\ell}(t^{-1})$ is regularly varying of index $\varrho < 1$, then*

$$M_1^\Phi(t) \sim \frac{N}{1 + (\delta_d - \delta_b)\widehat{\ell}(t^{-1})} + \frac{\gamma}{\delta_d - \delta_b}, \quad \textit{as} \quad t \to \infty.$$

(iii) *If $\delta_d = \delta_b$ and the mapping $t \mapsto \widehat{\ell}(t^{-1})$ is regularly varying of index $-1 < \varrho$, then*

$$M_1^\Phi(t) \sim N + \gamma\widehat{\ell}(t^{-1}), \quad \textit{as} \quad t \to \infty.$$

*Here, the notation $f(t) \sim g(t)$ as $t \to \infty$ means that $\lim_{t\to\infty} f(t)/g(t) = 1$.*

**Remark 2.3.** This remark has two key aspects. On one hand, the conditions in Theorem 2.2 are broad and flexible, encompassing many standard examples of pairs $(h, \ell) \in (\mathcal{PC})'$. On the other hand, it highlights that the asymptotic behavior of $M_1^\Phi(t)$ as $t \to \infty$ can vary significantly and is not necessarily exponential, as in the classical case. Specifically, when the birth rate asymptotically exceeds the death rate, the moments exhibit exponential growth. Conversely, when the birth rate is asymptotically smaller than the death rate, the moments decay at a slow rate. In Section 4, we present several examples that illustrate the diverse range of asymptotic behaviors for these moments.

We point out that due to the recursive nature of the formula for the higher order moments $M_k^\Phi(t)$ of $\mathcal{N}_\Phi(t)$, the proof of the preceding result can be straightforwardly extended to compute the asymptotic behavior of $M_k^\Phi(t)$ for any $k \geq 2$. For the sake of brevity, we will establish this result only for the second-order moment.

**Theorem 2.3.** *Let $\Phi$ be a Bernstein function given by $\Phi(\lambda) = \lambda\widehat{h}(\lambda)$ for some $(h, \ell) \in (\mathcal{PC})'$. Consider a classical birth-death process $\mathcal{N}(t)$ with birth rates $b_k = \beta k^2 + \delta_b k + \gamma$ and death rates $d_k = \beta k^2 + \delta_d k$, where $\beta$, $\delta_b, \delta_d$ and $\gamma$ are non-negative constants. Assume that $M_2^\Phi(t)$ is finite. The following assertions hold.*

(i) *If $\delta_d < \delta_b + \beta$, then*

$$M_2^\Phi(t) \sim K_2 \cdot \exp(\omega_2 t), \quad \textit{as} \quad t \to \infty,$$

*where $\omega_2 > 0$ is the unique solution of the equation $\lambda\widehat{h}(\lambda) = 2(\delta_b - \delta_d + \beta)$ and*

$$K_2 = \frac{1}{\widehat{h}(\omega_2) + \omega_2\dot{\widehat{h}}(\omega_2))}\left[N^2\widehat{h}(\omega_2) + \frac{c_2 + N\kappa_{2,1}}{\omega_2} + \frac{\kappa_{2,1}(\gamma - c_1)}{\omega_2(2\beta + \delta_b - \delta_d)}\right].$$

(ii) *If $\delta_d > \delta_b + \beta$ and the mapping $t \mapsto \widehat{\ell}(t^{-1})$ is regularly varying of index $\varrho < 1$, then*

$$M_2^\Phi(t) \sim \frac{N^2}{1 + 2(\delta_d - \delta_b - \beta)\widehat{\ell}(t^{-1})} + \frac{c_2 + \kappa_{2,1}N}{2(\delta_d - \delta_b - \beta)} + \frac{\kappa_{2,1}(\gamma - c_1)}{2(\delta_d - \delta_b)(\delta_d - \delta_b - \beta)} \quad \textit{as} \quad t \to \infty.$$

(iii) *If $\delta_d = \delta_b + \beta$ and the mapping $t \mapsto \widehat{\ell}(t^{-1})$ is regularly varying of index $-1 < \varrho$, then*

$$M_2^\Phi(t) \sim N^2 + \left(c_2 + \kappa_{2,1}N + \frac{\kappa_{2,1}(\gamma - c_1)}{c_1}\right)\widehat{\ell}(t^{-1}), \quad \textit{as} \quad t \to \infty.$$

*Here, the constants $c_1, c_2$ and $\kappa_{2,1}$ are given by $c_1 = \delta_b - \delta_d$, $c_2 = 2\beta + 2(\delta_b - \delta_d)$, and $\kappa_{2,1} = 2\gamma + (\delta_b + \delta_d)$.*

**Remark 2.4.** Note that the asymptotic behavior of $M_2^\Phi(t)$ is, up to multiplicative constants, similar the asymptotic behavior of $M_1^\Phi$, which again is markedly different to the classical case.

## 3. Proof of main results

In order to prove our main results, we need to establish several previous technical results. We begin establishing the following Lemma, whose proof is obtained from [4, Theorem 4.4 and Theorem 5.1], see also [3].

**Lemma 3.1.** *Let $\Phi$ be a Bernstein function given by $\Phi(\lambda) = \lambda\widehat{h}(\lambda)$ for some $(h, \ell) \in (\mathcal{PC})'$. Consider a birth-death process $\mathcal{N}(t)$ with birth rates $b_k = \beta k^2 + \delta_b k + \gamma$ and death rates $d_k = \beta k^2 + \delta_d k$, for some non-negative constants $\beta$, $\delta_b$, $\delta_d$ and $\gamma$. The transition probability $P_n^\Phi$ of the time non-local birth-death process $\mathcal{N}_\Phi(t)$ satisfies the following evolution equation*

$$\partial_t(h * (P_n^\Phi - P_n^\Phi(0))) = b_{n-1}P_{n-1}^\Phi(t) - (b_n + d_n)P_n^\Phi(t) + d_{n+1}P_{n+1}^\Phi(t), \quad t \geq 0. \tag{3.1}$$

**Remark 3.1.** Since $h * \ell = 1$ on $(0, \infty)$, the equation (3.1) can be rewritten as the following integral Volterra equation

$$P_n^\Phi(t) = P_n^\Phi(0) + b_{n-1}(P_{n-1}^\Phi * \ell)(t) - (b_n + d_n)(P_n^\Phi * \ell)(t) + d_{n+1}(P_{n+1}^\Phi * \ell)(t), \quad t \geq 0.$$

This framework provides several advantages in obtaining a formula for the moments of $\mathcal{N}_\Phi(t)$. For instance, in the following result we exploit the theory of Volterra equations to represent $P_n^\Phi(t)$ in terms of the probability transitions of the classic birth-death process.

**Lemma 3.2.** *Let $\Phi$ be a Bernstein function given by $\Phi(\lambda) = \lambda\widehat{h}(\lambda)$ for some $(h, \ell) \in (\mathcal{PC})'$. Consider a birth-death process $\mathcal{N}(t)$ with birth rates $b_k = \beta k^2 + \delta_b k + \gamma$ and death rates $d_k = \beta k^2 + \delta_d k$, for some non-negative constants $\beta$, $\delta_b$, $\delta_d$ and $\gamma$. Then the transition probability $P_n^\Phi$ can be represented as follows*

$$P_n^\Phi(t) = -\int_0^\infty P_n(\tau) d_\tau W_\Phi(t, \tau), \quad t \geq 0, \tag{3.2}$$

*where $P_n$ is the probability density transition of the classical birth-death process $\mathcal{N}(t)$ and $W_\Phi$ is the propagation function associated to the Bernstein function $\Phi$, defined in Appendix B below.*

***Proof.*** It follows from Remark 3.1 that the Laplace transform of $P_n^\Phi$ satisfies the following system of infinite evolution equations

$$\widehat{P_n^\Phi}(\lambda) - \frac{P_n^\Phi(0)}{\lambda} = b_{n-1}\,\widehat{\ell}(\lambda)\,\widehat{P_{n-1}^\Phi}(\lambda) - (b_n + d_n)\,\widehat{\ell}(\lambda)\,\widehat{P_n^\Phi}(\lambda) + d_{n+1}\widehat{\ell}(\lambda)\,\widehat{P_{n+1}^\Phi}(\lambda), \quad \lambda > 0. \tag{3.3}$$

On the other hand, it is well known (cf. [5, Section 8.3]) that the probability transitions of a classical birth-death process satisfy the following differential equation

$$P_n'(t) = b_{n-1}P_{n-1}(t) - (b_n + d_n)P_n(t) + d_{n+1}P_{n+1}(t), \quad t > 0,\, n \geq 1, \qquad P_n(0) = N\,\delta_{n,1}, \quad n \geq 1.$$

Therefore, the Laplace transform of $P_n$ satisfies

$$\widehat{P}_n(\sigma) - \frac{P_n(0)}{\sigma} = b_{n-1}\frac{\widehat{P}_{n-1}(\sigma)}{\sigma} - (b_n + d_n)\frac{\widehat{P}_n(\sigma)}{\sigma} + d_{n+1}\frac{\widehat{P}_{n+1}(\sigma)}{\sigma}, \quad \sigma > 0.$$

Since the preceding relation is valid for all $\sigma > 0$, and $\lambda\widehat{h}(\lambda) > 0$ for all $\lambda > 0$, we can consider $\lambda\widehat{h}(\lambda)$ instead of $\sigma$. Hence, we have that

$$\widehat{P}_n(\lambda\widehat{h}(\lambda)) - \frac{P_n(0)}{\lambda\widehat{h}(\lambda))} = b_{n-1}\frac{\widehat{P}_{n-1}(\lambda\widehat{h}(\lambda))}{\lambda\widehat{h}(\lambda)} - (b_n + d_n)\frac{\widehat{P}_n(\lambda\widehat{h}(\lambda))}{\lambda\widehat{h}(\lambda)} + d_{n+1}\frac{\widehat{P}_{n+1}(\lambda\widehat{h}(\lambda))}{\lambda\widehat{h}(\lambda)}, \quad \lambda > 0,$$

or equivalently

$$\begin{aligned}\widehat{h}(\lambda)\widehat{P}_n(\lambda\widehat{h}(\lambda)) - \frac{P_n(0)}{\lambda} =& b_{n-1}\widehat{\ell}(\lambda)\widehat{h}(\lambda)\widehat{P}_{n-1}(\lambda\widehat{h}(\lambda)) - (b_n + d_n)\widehat{\ell}(\lambda)\widehat{h}(\lambda)\widehat{P}_n(\lambda\widehat{h}(\lambda)) \\ &+ d_{n+1}\widehat{\ell}(\lambda)\widehat{h}(\lambda)\widehat{P}_{n+1}(\lambda\widehat{h}(\lambda)), \quad \lambda > 0.\end{aligned} \tag{3.4}$$

Since the solution to this equation is unique, it follows from (3.3) and (3.4) that

$$\widehat{P_n^\Phi}(\lambda) = \widehat{h}(\lambda)\widehat{P}_n(\lambda\widehat{h}(\lambda)), \quad \lambda > 0.$$

Now, for $\Phi$ and $n \in \mathbb{N}$, we define

$$B_n(t) = -\int_0^\infty P_n(\tau) d_\tau W_\Phi(t, \tau), \quad t \geq 0,\ n \in \mathbb{N}, \tag{3.5}$$

where $P_n(t)$ is the state probability function of the classical pure-birth process $\mathcal{N}(t)$, and $W_\Phi$ is the propagation function associated to the Bernstein function $\Phi$. Taking Laplace transform into the both sides of (3.5), we have that

$$\begin{aligned}\widehat{B}_n(\lambda) &= \widehat{h}(\lambda)\int_0^\infty P_n(\tau)e^{-\tau\lambda\widehat{h}(\lambda)}d\tau, \quad \lambda > 0.\\ &= \widehat{h}(\lambda)\widehat{P}_n(\lambda\widehat{h}(\lambda)), \quad \lambda > 0.\end{aligned}$$

The identity (3.2) follows directly from the uniqueness of the Laplace transform. ■

**Remark 3.2.** As we have mentioned before, Lemma 3.2 allows us to represent $P_n^\Phi(t)$ in terms of the probability density transition of the classical birth-death process $\mathcal{N}(t)$. This type of representation is known in the specialized literature as *subordination formulas*. We refer the interested reader to [1, 15, 25, 26] and the references therein to consult subordination formulas applied in another contexts.

**Corollary 3.1.** *Let $\Phi$ be a Bernstein function given by $\Phi(\lambda) = \lambda\widehat{h}(\lambda)$ for some $h \in (\mathcal{PC})'$. Consider a birth-death process $\mathcal{N}(t)$ with birth rates $b_k = \beta k^2 + \delta_b k + \gamma$ and death rates $d_k = \beta k^2 + \delta_d k$, for some non-negative constants $\beta$, $\delta_b$, $\delta_d$ and $\gamma$. The $n$-th moment of $\mathcal{N}_\Phi(t)$ can be represented as follows:*

$$M_n^\Phi(t) = -\int_0^\infty M_n(\tau)d_\tau W_\Phi(t,\tau), \quad t \geq 0, \tag{3.6}$$

*where $M_n$ is the $n$-th moment of the classical birth-death process $\mathcal{N}(t)$ and $W_\Phi$ is the propagation function associated to the Bernstein function $\Phi$, defined in Appendix B below.*

***Proof.*** Let $n \in \mathbb{N}$. By definition $M_n^\Phi(t)$ is given by

$$M_n^\Phi(t) = \sum_{k=0}^\infty k^n P_k^\Phi(t), \quad t \geq 0.$$

Hence, the formula (3.6) follows directly from Lemma 3.2 and the Dominated Convergence Theorem. ■

The formula (3.6) provides a helpful representation of the moments of $\mathcal{N}_\Phi(t)$ in terms of the moments of $\mathcal{N}(t)$. For this reason, we present a recursive representation of the moments of $\mathcal{N}(t)$.

**Lemma 3.3.** *Consider a classical birth-death process $\mathcal{N}(t)$ with birth rates $b_k = \beta k^2 + \delta_b k + \gamma$ and death rates $d_k = \beta k^2 + \delta_d k$, for some non-negative constants $\beta$, $\delta_b$, $\delta_d$ and $\gamma$. Assume that the $\mathcal{N}(t)$ admits moments of any order. Then, $M_0(t) = 1$,*

$$M_1(t) = Ne^{(\delta_b-\delta_d)t} + \gamma\int_0^t e^{(\delta_b-\delta_d)s}ds, \tag{3.7}$$

*and*

$$\begin{aligned}M_2(t) =& N^2 e^{2(\beta+\delta_b-\delta_d)t} + (2\gamma+\delta_b+\delta_d)\int_0^t e^{2(\beta+\delta_b-\delta_d)s}M_1(t-s)ds\\ &+\gamma\int_0^t e^{2(\beta+\delta_b-\delta_d)s}ds.\end{aligned} \tag{3.8}$$

*Additionally, for $n \geq 3$ we have that*

$$M_n(t) = N^n\, e^{c_n t} + \sum_{j=1}^n \kappa_{n,j}\int_0^t M_{n-j}(s)e^{c_n(t-s)}ds, \qquad n \in \mathbb{N}, \tag{3.9}$$

*where, for $n \geq 2$ and $1 \leq j \leq n-1$,*

$$c_n = (\delta_d - \delta_b)n - \beta n(n-1), \quad \text{and} \quad \kappa_{n,j} = \begin{cases} 2\beta\binom{n}{j+2} + (\delta_b-\delta_d)\binom{n}{j+1} + \gamma\binom{n}{j}, & j \quad \text{even},\\ \gamma\binom{n}{j} + (\delta_b+\delta_d)\binom{n}{j+1}, & j \quad \text{odd}.\end{cases}$$

***Proof.*** Let $n \in \mathbb{N}$. We have that

$$M_n(t) = \sum_{k=0}^\infty k^n P_k(t), \quad t \geq 0.$$

It is a well known fact that the probability transitions $P_n$ satisfy the following system of differential equations

$$\frac{d}{dt}P_n(t) = b_{n-1}P_{n-1}(t) - (b_n + d_n)P_n(t) + d_{n+1}P_{n+1}(t), \quad t \geq 0. \tag{3.10}$$

Therefore, we have that

$$\begin{aligned}\frac{d}{dt}M_n(t) =& \sum_{k=0}^{\infty}\Big((k-1)^n b_{k-1}P_{k-1}(t) - k^n b_k P_k(t)\Big) + \sum_{k=0}^{\infty}\big(k^n - (k-1)^n\big)b_{k-1}P_{k-1}(t) \\ &+ \sum_{k=0}^{\infty}\Big((k+1)^n d_{k+1}P_{k+1}(t) - k^n d_k P_k(t)\Big) + \sum_{k=0}^{\infty}\big(k^n - (k+1)^n\big)d_{k+1}P_{k+1}(t),\end{aligned}$$

which implies that

$$\begin{aligned}\frac{d}{dt}M_n(t) =& -\lim_{k\to\infty} k^n b_k P_k(t) + \lim_{k\to\infty}(k+1)^n d_{k+1}P_{k+1}(t) \\ &+ \sum_{k=0}^{\infty}\big(k^n - (k-1)^n\big)b_{k-1}P_{k-1}(t) + \sum_{k=0}^{\infty}\big(k^n - (k+1)^n\big)d_{k+1}P_{k+1}(t).\end{aligned}$$

Since $\mathcal{N}(t)$ admits moments of any order, and both birth and death rates are quadratic polynomials when $\beta \neq 0$ or linear polynomials when $\beta = 0$, it follows that $\lim_{k\to\infty} k^n P_k(t) = 0$ for all $t \geq 0$ and $n \in \mathbb{N}$. Therefore, we have that

$$\lim_{k\to\infty} k^n b_k P_k(t) = 0 \quad \text{and} \quad \lim_{k\to\infty}(k+1)^n d_{k+1}P_{k+1}(t) = 0.$$

This in turn implies that

$$\frac{d}{dt}M_n(t) = \sum_{k=0}^{\infty}\big(k^n - (k-1)^n\big)b_{k-1}P_{k-1}(t) + \sum_{k=0}^{\infty}\big(k^n - (k+1)^n\big)d_{k+1}P_{k+1}(t).$$

Since $b_{-1} = 0$ we have that

$$\begin{aligned}\frac{d}{dt}M_n(t) &= \sum_{k=1}^{\infty}\big((k^n - (k-1)^n\big)b_{k-1}P_{k-1}(t) + \sum_{k=0}^{\infty}\big((k^n - (k+1)^n\big)d_{k+1}P_{k+1}(t) \\ &= \sum_{k=0}^{\infty}\big((k+1)^n - k^n\big)b_k P_k(t) + \sum_{k=0}^{\infty}\big((k-1)^n - k^n\big)d_k P_k(t) \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{n-1}\binom{n}{j}k^j\big(b_k + (-1)^{n-j}d_k\big)P_k(t).\end{aligned} \tag{3.11}$$

The right hand side of (3.11) can be rewritten as follows

$$\begin{aligned}\frac{d}{dt}M_n(t) =& \sum_{k=0}^{\infty}\binom{n}{n-1}k^{n-1}(b_k - d_k)P_k(t) + \sum_{k=0}^{\infty}\binom{n}{n-2}k^{n-2}(b_k + d_k)P_k(t) \\ &+ \sum_{k=0}^{\infty}\sum_{j=0}^{n-3}\binom{n}{j}k^j(b_k + (-1)^{n-j}d_k)P_k(t).\end{aligned}$$

Since $b_k = \beta k^2 + \delta_b k + \gamma$ and $d_k = \beta k^2 + \delta_d k$, we have that

$$b_k + d_k = 2\beta k^2 + (\delta_b + \delta_d)k + \gamma \quad \text{and} \quad b_k - d_k = (\delta_b - \delta_d)k + \gamma.$$

Consequently

$$\sum_{k=0}^{\infty}\binom{n}{n-1}k^{n-1}(b_k - d_k)P_k(t) = n(\delta_b - \delta_d)M_n(t) + n\gamma M_{n-1}(t), \quad t \geq 0,$$

and

$$\sum_{k=0}^{\infty}\binom{n}{n-2}k^{n-2}(b_k+d_k)P_k(t) = n(n-1)\beta M_n(t)+\frac{n(n-1)}{2}(\delta_b+\delta_d)M_{n-1}(t) + \gamma\frac{n(n-1)}{2}M_{n-2}(t), \quad t\geq 0.$$

Therefore

$$\frac{d}{dt}M_n(t)+c_nM_n(t)=\sum_{j=1}^{n}\kappa_{n,j}M_{n-j}(t), \quad t\geq 0, \tag{3.12}$$

where, for $n\geq 2$ and $1\leq j\leq n-1$,

$$\kappa_{n,j}:=\begin{cases}2\beta\binom{n}{j+2}+(\delta_b-\delta_d)\binom{n}{j+1}+\gamma\binom{n}{j}, & j \ \text{even},\\ \gamma\binom{n}{j}+(\delta_b+\delta_d)\binom{n}{j+1}, & j \ \text{odd}.\end{cases}$$

A straightforward computation shows that the solution of (3.12) is given by

$$M_n(t)=M_n(0)\,e^{\kappa_{n,0}t}+\sum_{j=1}^{n}\kappa_{n,j}\int_0^t M_{n-j}(s)e^{\kappa_{n,0}(t-s)}ds, \quad t\geq 0,$$

Since $P_k(0)=\delta_{k,N}$ we have that

$$M_n(0)=\sum_{k=0}^{\infty}k^nP_k(0)=N^n,$$

which proves our claim. ∎

***Proof of Theorem 2.1.*** Let $n\in\mathbb{N}$. According to Corollary 3.1 we have that

$$M_n^{\Phi}(t)=-\int_0^{\infty}M_n(\tau)d_{\tau}W_{\Phi}(t,\tau), \quad t\geq 0.$$

Hence, it follows from Corollary 3.1 that the Laplace transform of $M_n^{\Phi}$ is given by

$$\widehat{M_n^{\Phi}}(\lambda)=\int_0^{\infty}M_n(\tau)\widehat{h}(\lambda)e^{-\tau\lambda\widehat{h}(\lambda)}=\widehat{h}(\lambda)\widehat{M}_n(\lambda\widehat{h}(\lambda)), \quad \lambda\geq 0.$$

On the other hand, it follows from Lemma 3.3 that the Laplace transform of $M_n$ is given by the following recursive expression

$$\widehat{M}_n(z)=\frac{N^n}{z+c_n}+\sum_{j=|}^{n}\frac{\kappa_{n,j}}{z+c_n}\,\widehat{M}_j(z), \quad z\geq 0.$$

Replacing $z$ by $\lambda\widehat{h}(\lambda)$ and multiplying the preceding expression by $\widehat{h}(\lambda)$, we obtain that

$$\widehat{h}(\lambda)\,\widehat{M}_n(\lambda\widehat{h}(\lambda))=N^n\,\frac{\widehat{h}(\lambda)}{\lambda\widehat{h}(\lambda)+c_n}+\sum_{j=1}^{n}\frac{\kappa_{n,j}}{\lambda\widehat{h}(\lambda)+c_n}\widehat{h}(\lambda)\widehat{M}_j(\lambda\widehat{h}(\lambda)), \quad \lambda\geq 0.$$

It follows from (A.5) and (A.6) that

$$\widehat{s}_{c_n}(\lambda)=\frac{\widehat{h}(\lambda)}{\lambda\widehat{h}(\lambda)+c_n} \quad \text{and} \quad \widehat{r}_{c_n}(\lambda)=\frac{1}{\lambda\widehat{h}(\lambda)+c_n}, \quad \lambda>0.$$

Hence, by the uniqueness of the Laplace transform, we have that

$$M_n^{\Phi}(t)=N^n\,s_{c_n}(t)+\sum_{j=1}^{n}\kappa_{n,j}\big(r_{c_n}*M_j^{\Phi}\big)(t), \quad t\geq 0, \tag{3.13}$$

and the proof is complete. ∎

***Proof of Theorem 2.2.*** Since $M_0^\Phi(t) = 1$, it follows from Theorem 2.1 that

$$M_1^\Phi(t) = N\, s(t, \delta_d - \delta_b) + \gamma \int_0^t r(\sigma, \delta_d - \delta_b) d\sigma, \quad t \geq 0.$$

Initially, we consider the case $\delta_b > \delta_d$. In such a case, by Remark A.2 we have that the function $t \mapsto s(t, \delta_d - \delta_b)$ is non-decreasing. Moreover, since the function $\ell$ is non-negative, we have that the mapping $t \mapsto r(t, \delta_d - \delta_b)$ in non-negative as well, which implies that $M_1^\Phi$ is a non-decreasing function. On the other hand, it follows from (A.5) and (A.6) that the Laplace transform of $M_1^\Phi$ is given by

$$\widehat{M_1^\Phi}(\lambda) = \frac{1}{\lambda}\left(\frac{N\lambda \widehat{h}(\lambda) + \gamma}{\lambda \widehat{h}(\lambda) + \delta_d - \delta_b}\right), \quad \lambda > \omega_1,$$

where $\omega_1 > 0$ is the unique solution of the equation

$$\lambda \widehat{h}(\lambda) = \delta_b - \delta_d. \tag{3.14}$$

We recall that the condition $(h, \ell) \in (\mathcal{PC})'$ implies that the mapping $\lambda \mapsto \lambda \widehat{h}(\lambda)$ is an unbounded Bernstein function such that $\lim_{\lambda \to 0} \lambda \widehat{h}(\lambda) = 0$, which guarantees the existence and uniqueness of the solution to the equation (3.14). Therefore, it follows from Wiener-Ikehara Theorem C.2 that

$$M_1^\Phi(t) \sim K_1 \cdot \exp(\omega_1 t), \quad \text{as} \quad t \to \infty,$$

where $K_1$ is the residue of $\lambda \mapsto \frac{1}{\lambda}\left(\frac{N\lambda \widehat{h}(\lambda)+\gamma}{\lambda \widehat{h}(\lambda)+\delta_d-\delta_b}\right)$ at $\lambda = \omega_1$. In other words,

$$K_1 = \lim_{\lambda \to \omega_1} \frac{\lambda - \omega_1}{\lambda} \left(\frac{N\lambda \widehat{h}(\lambda) + \gamma}{\lambda \widehat{h}(\lambda) + \delta_d - \delta_b}\right).$$

Using L'Hôpital rules, we have that

$$K_1 = \frac{\left(N\,(\delta_b - \delta_d) + \gamma\right)}{\omega_1 (\widehat{h}(\omega_1) + \omega_1 \dot{\widehat{h}}(\omega_1))},$$

where the $\dot{\widehat{h}}$ means the derivative of $\widehat{h}$, and the proof of the first assertion is complete.

Assume now that $\delta_d > \delta_b$. In this case, the mapping $t \mapsto s(t, \delta_d - \delta_b)$ is nonincreasing and the mapping $t \mapsto \int_0^t r(\sigma, \delta_d - \delta_b)(\sigma) d\sigma$ is increasing. Thus, $M_1^\Phi$ is the sum of a nonincreasing function and a nondecreasing one. Therefore we can apply the Karamata-Feller Theorem separately in each summand, and obtain the asymptotic behavior of $M_1^\Phi$. To this end, we rewrite $\widehat{M_1^\Phi}$ as follows

$$\widehat{M_1^\Phi}(\lambda) = \frac{1}{\lambda^{1-\varrho}} L_1\left(\frac{1}{\lambda}\right) + \frac{1}{\lambda} L_2\left(\frac{1}{\lambda}\right), \quad \lambda > 0,$$

where $L_1, L_2 \colon (0, \infty) \to (0, \infty)$ are given by

$$L_1(t) = \frac{N t^\varrho}{1 + (\delta_d - \delta_b)\widehat{\ell}(t^{-1})}, \quad \text{and} \quad L_2(t) = \frac{\gamma \widehat{\ell}(t^{-1})}{1 + (\delta_d - \delta_b)\widehat{\ell}(t^{-1})}.$$

Let us assume that the mapping $t \mapsto \widehat{\ell}(t^{-1})$ is regularly varying of index $\varrho < 1$. In such a case, we have that both $L_1$ and $L_2$ are slowly varying at infinity functions. Thus, according Karamata-Feller Theorem C.1 we have that

$$M_1^\Phi(t) \sim \frac{N}{1 + (\delta_d - \delta_b)\widehat{\ell}(t^{-1})} + \frac{\gamma \widehat{\ell}(t^{-1})}{1 + (\delta_d - \delta_b)\widehat{\ell}(t^{-1})}, \quad \text{as} \quad t \to \infty.$$

Since $\ell \notin L_1(\mathbb{R}_+)$, we have that $\frac{\gamma \widehat{\ell}(t^{-1})}{1+(\delta_d-\delta_b)\widehat{\ell}(t^{-1})} \sim \frac{\gamma}{\delta_d - \delta_b}$ as $t \to \infty$, which in turn implies that

$$M_1^\Phi(t) \sim \frac{N}{1 + (\delta_d - \delta_b)\widehat{\ell}(t^{-1})} + \frac{\gamma}{\delta_d - \delta_b}, \quad \text{as} \quad t \to \infty.$$

In the case $\delta_d = \delta_b$, we have that

$$\widehat{M_1^\Phi}(\lambda) = \frac{N}{\lambda} + \frac{\gamma \widehat{\ell}(\lambda)}{\lambda}, \quad \lambda > 0.$$

We can rewrite the preceding expression as follows

$$\widehat{M_1^\Phi}(\lambda) = \frac{N}{\lambda} + \frac{1}{\lambda^{1+\varrho}} L\left(\frac{1}{\lambda}\right), \quad \lambda > 0.$$

where $L\colon (0,\infty) \to (0,\infty)$ is given by

$$L(t) = \widehat{\ell}(t^{-1})t^{-\varrho}, \quad t > 0.$$

Since the mapping $t \mapsto \widehat{\ell}(t^{-1})$ is regularly varying of index $\varrho > -1$, we have that $L$ is a slowly varying function, and by Karamata-Feller Theorem we have

$$M_1^\Phi(t) \sim N + \gamma\widehat{\ell}(t^{-1}), \quad \text{as} \quad t \to \infty,$$

which finishes the proof.

***Proof of Theorem 2.3.*** Recall that $c_1 = \delta_d - \delta_b$ and $c_2 = 2(\delta_d - \delta_b) - 2\beta = 2c_1 - 2\beta$. According to Theorem 2.1 we have that

$$M_2^\Phi(t) = N^2 s_{c_2}(t) + (2\gamma + \delta_b + \delta_d)(r_{c_2} * M_1^\Phi)(t) + \gamma(1 * r_{c_2})(t), \quad t \geq 0,$$

which, by the expression of $M_1^\Phi$, is equivalent to

$$\begin{aligned} M_2^\Phi(t) =& N^2\, s_{c_2}(t) + \gamma(1 * r_{c_2})(t) + (2\gamma + \delta_b + \delta_d)N\,(r_{c_2} * s_{c_1})(t) \\ &+ \gamma(2\gamma + \delta_b + \delta_d)(1 * r_{c_1} * r_{c_2})(t). \end{aligned}$$

It follows from (A.3) that

$$s_{c_1}(t) = 1 - c_1(1 * r_{c_1})(t).$$

Therefore, $M_2^\Phi(t)$ is given by

$$\begin{aligned} M_2^\Phi(t) =& N^2\, s_{c_2}(t) + \gamma(1 * r_{c_2})(t) + (2\gamma + \delta_b + \delta_d)N\,(1 * r_{c_2})(t) \\ &+ \gamma(2\gamma + \delta_b + \delta_d)(1 * r_{c_1} * r_{c_2})(t) - (2\gamma + \delta_b + \delta_d)c_1(1 * r_{c_1} * r_{c_2})(t). \end{aligned} \tag{3.15}$$

The preceding expression shows that $M_2^\Phi$ is a finite sum of monotone functions. Thus, in order to determine the asymptotic behavior of $M_2^\Phi$ we will analyze the asymptotic behavior of each term of (3.15).

Initially, we first consider $\delta_d < \delta_b + \beta$. Since $c_1 = \delta_d - \delta_b$, and $c_2 = 2c_1 - 2\beta$ a straightforward calculation shows that $c_2 < c_1$ provided that $c_1 < 0$, which precludes the case $c_1 < c_2 < 0$. Hence, we initially consider the case $c_2 < 0$.

By (A.5) and (A.6) we have that

$$\widehat{s}_{c_2}(\lambda) = \frac{1}{\lambda}\frac{\lambda\widehat{h}(\lambda)}{\lambda\widehat{h}(\lambda) + c_2}, \quad \text{and} \quad \widehat{1 * r_{c_2}}(\lambda) = \frac{1}{\lambda}\frac{1}{\lambda\widehat{h}(\lambda) + c_2}, \quad \lambda > \omega_2,$$

where $\omega_2$ is the unique solution to the equation

$$\lambda\widehat{h}(\lambda) + c_2 = 0.$$

Since $c_2 < 0$, both $s_{c_2}$ and $1 * r_{c_2}$ are non-decreasing functions. Thus, Wiener-Ikehara Theorem implies that

$$s_{c_2}(t) \sim R_2 \cdot e^{\omega_2 t}, \quad \text{and} \quad (1 * r_{c_2})(t) \sim R_3 \cdot e^{\omega_2 t}, \qquad t \to \infty,$$

where

$$R_2 = \widehat{h}(\omega_2) \lim_{\lambda \to \omega_2} \left(\frac{\lambda - \omega_2}{\lambda\widehat{h}(\lambda) + 2(\delta_d - \delta_b - \beta)}\right), \quad \text{and} \quad R_3 = \frac{1}{\omega_2} \lim_{\lambda \to \omega_2} \left(\frac{\lambda - \omega_2}{\lambda\widehat{h}(\lambda) + 2(\delta_d - \delta_b - \beta)}\right).$$

On the other hand, by the non-negativeness of both $r_{c_1}$ and $r_{c_2}$, we have that the function $1 * r_{c_1} * r_{c_2}$ is an increasing function, whose Laplace transform $\mathscr{L}$ is given by

$$\mathscr{L}(1 * r_{c_2} * r_{c_1}; \lambda) = \frac{1}{\lambda}\frac{1}{\lambda\widehat{h}(\lambda) + c_2}\frac{1}{\lambda\widehat{h}(\lambda) + c_1}, \quad \lambda > \omega_2,$$

where $\omega_2 > 0$ has been defined above. Since $c_2 < c_1$, we point out that $\lambda\widehat{h}(\lambda) + c_1 \neq 0$ in the complex half-plane $\Omega = \{z \in \mathbb{C} : Re(z) > \omega_2\}$, which implies that the function $\lambda \mapsto \mathscr{L}(1 * r_{c_1} * r_{c_2}; \lambda)$ is analytic in this complex sector and it has a simple pole in $z = \omega_2$ with residue $R_1$ given by

$$R_1 = \frac{1}{\omega_2(2\beta + \delta_b - \delta_d)} \lim_{\lambda \to \omega_2} \left( \frac{\lambda - \omega_2}{\lambda\widehat{h}(\lambda) + 2(\delta_d - \delta_b + \beta)} \right).$$

Therefore, we have that

$$M_2^\Phi(t) \sim K_2 e^{\omega_2 t}, \quad t \to \infty,$$

where

$$K_2 = N^2 R_2 + \gamma R_3 + N(2\gamma + \delta_b + \delta_d)R_3 + \gamma(2\gamma + \delta_b + \delta_d)R_1 - (2\gamma + \delta_b + \delta_d)c_1 R_1,$$

and the constants $R_1, R_2$ and $R_3$ are given above. Indeed, using L'Hôpital rules they can be computed explicitly to obtain

$$R_1 = \frac{1}{\omega_2(2\beta + \delta_b - \delta_d)(\widehat{h}(\omega_2) + \omega_2\dot{\widehat{h}}(\omega_2))}, \; R_2 = \frac{\widehat{h}(\omega_2)}{(\widehat{h}(\omega_2) + \omega_2\dot{\widehat{h}}(\omega_2))}, \text{ and } \quad R_3 = \frac{1}{\omega_2(\widehat{h}(\omega_2) + \omega_2\dot{\widehat{h}}(\omega_2))},$$

which proves our assertion.

Assume now that $\delta_d > \delta_b + \beta$ and the mapping $t \mapsto \widehat{\ell}(t^{-1})$ is regularly varying of index $\varrho < 1$. We can rewrite $\widehat{M_2^\Phi}(\lambda)$ as follows

$$\widehat{M_2^\Phi}(\lambda) = \frac{1}{\lambda^{1-\varrho}} L_1\left(\frac{1}{\lambda}\right) + \frac{1}{\lambda} L_2\left(\frac{1}{\lambda}\right), \quad \lambda > 0,$$

where $L_1, L_2\colon (0,\infty) \to (0,\infty)$ are given by

$$L_1(t) = \frac{N^2 t^\varrho}{1 + 2(\delta_d - \delta_b - \beta)\widehat{\ell}(t^{-1})}$$

and

$$L_2(t) = \frac{\widehat{\ell}(t^{-1})}{1 + 2(\delta_d - \delta_b - \beta)\widehat{\ell}(t^{-1})} \left[ \gamma + N(2\gamma + \delta_b + \delta_d) + (\gamma - c_1)\frac{(2\gamma + \delta_b + \delta_d)\widehat{\ell}(t^{-1})}{(1 + (\delta_d - \delta_b)\widehat{\ell}(t^{-1}))} \right].$$

Since $\beta > 0$ we have that $\delta_d > \delta_b$. Due to the mapping $t \mapsto \widehat{\ell}(t^{-1})$ is regularly varying of index $\varrho < 1$, we have that both $L_1$ and $L_2$ are slowly varying functions. Since $\ell \notin L_1(\mathbb{R}_+)$, and according to the Karamata-Feller Theorem C.1 we have that

$$M_2^\Phi(t) \sim \frac{N^2}{1 + 2(\delta_d - \delta_b - \beta)\widehat{\ell}(t^{-1})} + \frac{\kappa_{2,0} + \kappa_{2,1}N}{2(\delta_d - \delta_b - \beta)} + \frac{\gamma\kappa_{2,1} - \kappa_{2,1}c_1}{2(\delta_d - \delta_b)(\delta_d - \delta_b - \beta)} \quad \text{as} \quad t \to \infty.$$

Assume now that $\delta_d = \delta_b + \beta$. Since $\beta > 0$ we have that $\delta_d > \delta_b$, and

$$\widehat{M_2^\Phi}(\lambda) = \frac{N^2}{\lambda} + \frac{(1 + N)(2\gamma + \delta_b + \delta_d)\widehat{\ell}(\lambda)}{\lambda} + \frac{(\gamma - c_1)(2\gamma + \delta_b + \delta_d)(\widehat{\ell}(\lambda))^2}{\lambda(1 + c_1\widehat{\ell}(\lambda)),}$$

for any $\lambda > 0$. Proceeding in a similar way to the proof of statement (iii) in Theorem 2.2, we may use Karamata-Feller Theorem C.1 to obtain

$$M_2^\Phi(t) \sim N^2 + \left( \gamma + (2\gamma + \delta_b + \delta_d)N + \frac{(\gamma - c_1)(2\gamma + \delta_b + \delta_d) - (2\gamma + \delta_b + \delta_d)}{c_1} \right) \widehat{\ell}(t^{-1}), \quad \text{as} \quad t \to \infty,$$

which finishes the proof. ∎

**Remark 3.3.** Note that the formula in equation (3.13) provides a recursive method for calculating the moments of the process $\mathcal{N}_\Phi(t)$. As a result, this means that the proofs of Theorem 2.2 and Theorem 2.3 can be adjusted to describe the asymptotic behavior of moments of any order, not just the first and second moments. However, the resulting expressions are quite lengthy. For the sake of brevity of the text, we have omitted them in this work.

## 4. Examples

In this section, we demonstrate how Theorem 2.2 and Theorem 2.3 can be applied to various examples of function pairs $(h, \ell) \in (\mathcal{PC})'$.

**Example** 4.1. Let $\alpha \in (0,1)$ and consider the pair of functions $(h, \ell) = (g_{1-\alpha}, g_\alpha)$, where $g_\beta$ with $\beta > 0$ is the standard notation for the function

$$g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0. \tag{4.1}$$

In this case, the term $\partial_t(h*\cdot)$ becomes the time-fractional derivative $\partial_t^\alpha$ in the sense of Riemann- Liouville of order $\alpha \in (0,1)$, and the equation (3.1) is an example of the so-called time-fractional evolution equation, which have been successfully applied in the context of diffusion processes whose mean square displacement grows like a multiple of $t^\alpha$, see e.g. [11, 20].

Since $\alpha \in (0,1)$, it is clear that $(h, \ell) \in (\mathcal{PC})'$. In order to agree with the standard notation of the literature, we refer the corresponding process $\mathcal{N}_\Phi(t)$ as $\mathcal{N}_\alpha(t)$. For the same reason, we denote $M_1^\alpha(t)$ its first moment. We have that $\lambda\widehat{h}(\lambda) = \lambda^\alpha$ for $\lambda > 0$. Therefore, it follows from the assertion $(i)$ of Theorem 2.2 that in the case $\delta_b > \delta_d$ the asymptotic behavior of $M_1^\Phi$ is given by

$$M_1^\alpha(t) \sim \frac{N(\delta_b - \delta_d) + \gamma}{\alpha(\delta_b - \delta_d)} \exp\left((\delta_b - \delta_d)^{\frac{1}{\alpha}} t\right), \quad \text{as} \quad t \to \infty.$$

On the other hand, we have that $\widehat{\ell}(\lambda) = \lambda^{-\alpha}$ for $\lambda > 0$, which implies that the mapping $t \mapsto \widehat{\ell}(t^{-1})$ is regularly varying of index $\alpha \in (0,1)$. Hence, by assertion $(ii)$ of Theorem 2.2 we have that

$$M_1^\alpha(t) \sim N + \gamma t^\alpha, \quad \text{as} \quad t \to \infty,$$

when $\delta_b = \delta_d$. Analogously, assertion $(iii)$ of Theorem 2.2 implies that

$$M_1^\alpha(t) \sim \frac{N}{1 + (\delta_d - \delta_b)t^\alpha} + \frac{\gamma}{\delta_d - \delta_b}, \quad \text{as} \quad t \to \infty.$$

when $\delta_d > \delta_b$. ■

**Remark 4.1.** Example 4.1 illustrates that when $\delta_b > \delta_d$, the behavior of $M_1^\alpha(t)$ closely resembles the results obtained by Orsingher and Polito [21] for the so-called *fractional Yule-Furry process*. This characteristic feature of the process under investigation demonstrates its suitability for modeling explosively expanding populations.

We point out that a key insight for identifying nonlocal birth-death $\mathcal{N}_\Phi(t)$ processes where the first moment grows exponentially when $\delta_b > \delta_d$ is to find pairs $(h, \ell) \in (\mathcal{PC})'$ such that the equation

$$\lambda\widehat{h}(\lambda) = \delta_b - \delta_d \tag{4.2}$$

can be solved explicitly. Although this task can generally be quite challenging, in the following examples, we present several cases where the equation (4.2) can be solved effectively.

**Example** 4.2. Consider the pair of functions $(h_1, \ell_1)$ given by

$$h_1(t) = \int_0^1 g_\alpha(t) d\alpha, \quad \text{and} \quad \ell_1(t) = \int_0^\infty \frac{e^{-st}}{1+s} ds, \quad t > 0.$$

In this case, the integrodifferential operator $\partial_t(h_1*\cdot)$ is an example of the so-called operators of distributed order and the equation (3.1) can be viewed as an example of the so-called *ultraslow evolution equations*. Such equations appear in the study of diffusion processes with a logarithmic growth of the mean square displacement, see e.g. [15, 16, 18, 31]. For instance, it has been proved in [31] that $h_1 * \ell_1 = 1$, and

$$\widehat{\ell}_1(\lambda) = \frac{\log(\lambda)}{\lambda - 1}, \qquad \lambda > 0.$$

Therefore $\lim_{\lambda \to 0^+} \widehat{\ell}_1(\lambda) = +\infty$, which implies that $\ell_1 \notin L_1(\mathbb{R}_+)$, and consequently $(h_1, \ell_1) \in (\mathcal{PC})'$. By a direct computation, we have that

$$\lambda\widehat{h}_1(\lambda) = \frac{\lambda - 1}{\log(\lambda)}, \quad \lambda > 0.$$

Hence, when $\delta_b - \delta_d > 0$, the unique solution $\omega_1 > 0$ of the equation $\lambda \widehat{h}_1(\lambda) = \delta_b - \delta_d$ is given by

$$\omega_1 = \begin{cases} -(\delta_b - \delta_d)\, W_{-1}\Big( - \frac{1}{\delta_b - \delta_d} \exp\big(\frac{-1}{\delta_b - \delta_d}\big)\Big), & (\delta_b - \delta_d) \geq 1, \\ -(\delta_b - \delta_d)\, W_0\Big( - \frac{1}{\delta_b - \delta_d} \exp\big(\frac{-1}{\delta_b - \delta_d}\big)\Big), & 0 < (\delta_b - \delta_d) \leq 1, \end{cases}$$

where $W_0$ and $W_{-1}$ denote the two real branches of the so-called *Lambert product function*. Therefore, Theorem 2.2 implies that in the case $\delta_b > \delta_d$ we have that

$$M_1^{\Phi}(t) \sim K \cdot \exp(\omega_1 t), \quad \text{as} \quad t \to \infty,$$

where $\omega_1 > 0$ has been given above, and

$$K_1 = (N\,(\delta_b - \delta_d) + \gamma) \frac{\log(\omega_1)}{\omega_1 + (\delta_d - \delta_b)}.$$

On the other hand, we have that $\widehat{\ell}_1(t^{-1}) = \frac{t \cdot \log(t)}{t-1}$ for $t > 0$. Therefore, the function $t \mapsto \widehat{\ell}_1(t^{-1})$ is regularly varying of index $\varrho = 0$. Hence, by assertion $(ii)$ of Theorem 2.2 we have that

$$M_1^{\Phi}(t) \sim N + \gamma \log(t), \quad \text{as} \quad t \to \infty,$$

when $\delta_b = \delta_d$. Analogously, by the assertion $(iii)$ of Theorem 2.2 we have that

$$M_1^{\Phi}(t) \sim \frac{N}{1 + (\delta_d - \delta_b)\log(t)} + \frac{\gamma}{\delta_d - \delta_b}, \quad \text{as} \quad t \to \infty.$$

when $\delta_d > \delta_b$. ■

**Example** 4.3. Let $\epsilon \in (0,1)$. In [2, Example 3.10], the authors have proved the existence of a pair of functions $(h_\epsilon, \ell_\epsilon) \in (\mathcal{PC})$ such that

$$\widehat{h}_\epsilon(\lambda) = \frac{1}{\lambda}\left(\frac{\lambda - 1}{\log(\lambda)}\right)^{\epsilon}, \quad \text{and} \quad \widehat{\ell}_\epsilon(\lambda) = \left(\frac{\log(\lambda)}{\lambda - 1}\right)^{\epsilon}$$

We note that $\lim_{\lambda \to 0^+} \widehat{\ell}_\epsilon(\lambda) = \infty$, which implies that for all $\epsilon \in (0,1)$ the pair $(h_\epsilon, \ell_\epsilon)$ belongs to $(\mathcal{PC})'$. The corresponding Bernstein function will be denoted by $\Phi_\epsilon$. Assume that $\delta_b > \delta_d$. In this case, the equation (4.2) takes the form

$$\left(\frac{\lambda - 1}{\log(\lambda)}\right)^{\epsilon} = \delta_b - \delta_d.$$

Analogously to Example (4.2) this equation can solved by means of the Product Lambert Function. Indeed, it follows from the assertion $(i)$ of Theorem 2.2 that when $\delta_b > \delta_d$ we have that

$$M_1^{\Phi}(t) \sim K_1 \cdot \exp\big(\omega_\epsilon t\big), \quad \text{as} \quad t \to \infty,$$

where

$$\omega_\epsilon = \begin{cases} -(\delta_b - \delta_d)^{\frac{1}{\epsilon}}\, W_{-1}\left( - \frac{1}{(\delta_b - \delta_d)^{\frac{1}{\epsilon}}} \exp\left(\frac{-1}{(\delta_b - \delta_d)^{\frac{1}{\epsilon}}}\right)\right), & (\delta_b - \delta_d) \geq 1, \\ -(\delta_b - \delta_d)^{\frac{1}{\epsilon}}\, W_0\left( - \frac{1}{(\delta_b - \delta_d)^{\frac{1}{\epsilon}}} \exp\left(\frac{-1}{(\delta_b - \delta_d)^{\frac{1}{\epsilon}}}\right)\right), & 0 < (\delta_b - \delta_d) \leq 1, \end{cases}$$

and

$$K_1 = (N\,(\delta_b - \delta_d) + \gamma) \frac{(\omega_\epsilon - 1)\log(\omega_\epsilon)}{\epsilon(\delta_d - \delta_b)(\omega_\epsilon - 1 - \omega_\epsilon \log(\omega_\epsilon))}.$$

On the other hand, we have that $\widehat{\ell}_\epsilon(t^{-1}) = \left(\frac{t \cdot \log(t)}{t-1}\right)^{\epsilon}$ for $t > 0$. Therefore, the function $t \mapsto \widehat{\ell}_\epsilon(t^{-1})$ is regularly varying of index $\varrho = 0$. According to Theorem 2.2 we have that

$$M_1^{\Phi_\epsilon}(t) \sim N + \gamma\big(\log(t)\big)^{\epsilon}, \quad \text{as} \quad t \to \infty,$$

when $\delta_b = \delta_d$. Analogously, by the assertion $(iii)$ of Theorem 2.2 we have that

$$M_1^{\Phi_\epsilon}(t) \sim \frac{N}{1 + (\delta_d - \delta_b)\big(\log(t)\big)^{\epsilon}} + \frac{\gamma}{\delta_d - \delta_b}, \quad \text{as} \quad t \to \infty.$$

when $\delta_d > \delta_b$. ■

In order to present the following example, we first recall the result established in [2, Lemma 3.9].

**Lemma 4.1.** *Let $\phi, \varphi \in (\mathcal{CM})$. Then, there exists a unique $\psi \in (\mathcal{CM})$ such that*

$$\widehat{\psi}(\lambda) = \widehat{\phi}(\lambda)\widehat{\varphi}(\widehat{\phi}(\lambda)), \quad \lambda > 0.$$

This result provides a method for constructing infinitely many pairs $(k, \ell) \in (\mathcal{PC})'$. Consequently, it enables the definition of infinitely many nonlocal birth-death processes $\mathcal{N}_\Phi(t)$, each exhibiting interesting properties, as we will illustrate through the following example.

**Proposition 4.1.** *There exists a pair of functions $(h_2, \ell_2) \in (\mathcal{PC})'$ such that*

$$\lambda\widehat{h}_2(\lambda) = \frac{\widehat{\ell}_1(\lambda) - 1}{\widehat{\ell}_1(\lambda)\log(\widehat{\ell}_1(\lambda))}, \quad \lambda > 0, \tag{4.3}$$

*where $(h_1, \ell_1) \in (\mathcal{PC})'$ is defined in Example 4.2.*

***Proof.*** Let $\delta \in (0,1)$, and consider $\phi = \ell_1$ and $\varphi = g_{1-\delta}$. It well known that both $\phi$ and $\varphi$ are completely monotonic. According to Lemma 4.1 there exists $f_\delta \in (\mathcal{CM})$ such that

$$\widehat{f}_\delta(\lambda) = \left(\widehat{\ell}_1(\lambda)\right)^\delta, \quad \lambda > 0.$$

Since $f_\delta \in (\mathcal{CM})$ it follows from [13, Theorem 5.4] that there is $k_\delta \in (\mathcal{CM})$ such that

$$\widehat{k}_\delta(\lambda) = \frac{1}{\lambda}\left(\widehat{\ell}_1(\lambda)\right)^{-\delta}, \quad \lambda > 0.$$

Consider the function $h_2\colon (0,\infty) \to (0,\infty)$ defined by

$$h_2(t) = \int_0^1 k_\delta(t)d\delta, \quad t > 0.$$

Since the class of functions $(\mathcal{CM})$ is closed under sums and pointwise limits, we conclude that $h_2 \in (\mathcal{CM})$. Therefore, applying again [13, Theorem 5.4] there exists $\ell_2 \in (\mathcal{CM})$ such that $h_2 * \ell_2 = 1$. Moreover, we have that

$$\widehat{h}_2(\lambda) = \int_0^1 \frac{1}{\lambda}\left(\widehat{\ell}_1(\lambda)\right)^{-\delta}d\delta = \frac{\widehat{\ell}_1(\lambda) - 1}{\lambda\widehat{\ell}_1(\lambda)\log(\widehat{\ell}_1(\lambda))}, \quad \lambda > 0,$$

which implies that $\lambda\widehat{h}_2(\lambda) = \frac{\widehat{\ell}_1(\lambda)-1}{\widehat{\ell}_1(\lambda)\log(\widehat{\ell}_1(\lambda))}$, and the proof is complete. ∎

**Example** 4.4. Let $(h_2, \ell_2) \in (\mathcal{PC})'$ be the pair of functions defined in Proposition 4.1. Consider the Bernstein function $\Phi_2\colon (0,\infty) \to (0,\infty)$ given by

$$\Phi_2(\lambda) = \lambda\widehat{h}_2(\lambda), \quad \lambda > 0,$$

and the process $\mathcal{N}_{\Phi_2}(t)$ induced by a time non-local birth-death process $\mathcal{N}$ and the function $\Phi_2$. As mentioned above, in order to describe the asymptotic behavior of $M_1^{\Phi_2}(t)$ when $\delta_b > \delta_d$, we need solve the equation (4.2). In this case, this is equivalent to find $\mu_2 > 0$ such that

$$\frac{\widehat{\ell}_1(\mu_2) - 1}{\widehat{\ell}_1(\mu_2)\log(\widehat{\ell}_1(\mu_2))} = \delta_b - \delta_d.$$

Analogously to the examples (4.2) and (4.3), this equation can be solved using the properties of Lambert product function. More concretely, it can be solved by an iterative procedure. Indeed, if we make the change of variable $\widehat{\ell}_1(\mu_2) = \mu_1$ we have that

$$\frac{\mu_1 - 1}{\mu_1\log(\mu_1)} = \delta_b - \delta_d,$$

which implies that

$$\mu_1 = \begin{cases} \frac{-1}{(\delta_b-\delta_d)W_{-1}\left(-\frac{e^{-1/(\delta_b-\delta_d)}}{(\delta_b-\delta_d)}\right)}, & \delta_b - \delta_d \geq 1, \\ -\frac{(\delta_b-\delta_d)}{W_0((\delta_b-\delta_d)e^{-(\delta_b-\delta_d)})}, & 0 < \delta_b - \delta_d \leq 1, \end{cases}$$

where $W_0$ and $W_{-1}$ are the real branches of the Lambert product function. Since $\mu_1 = \widehat{\ell}_1(\mu_2) = \frac{\log(\mu_2)}{\mu_2 - 1}$, by using again the properties of the Lambert function, it follows that

$$\mu_2 = \begin{cases} \frac{-W_0(-\mu_1 e^{-\mu_1})}{\mu_1}, & \mu_1 \geq 1, \\ \frac{-W_{-1}(-\mu_1 e^{-\mu_1})}{\mu_1}, & 0 < \mu_1 \leq 1, \end{cases}$$

where $\mu_1$ has been defined above. Therefore, according to Theorem (2.2) we have that

$$M_1^{\Phi}(t) \sim K \exp\left(\mu_2 t\right), \quad t \to \infty,$$

where $K$ is given by

$$K = \frac{\left(N\left(\delta_b - \delta_d\right) + \gamma\right)}{\mu_2(\widehat{h}(\mu_2) + \mu_2 \dot{\widehat{h}}(\mu_2))}.$$

On the other hand, we have that

$$\widehat{\ell}_2(t^{-1}) = \left( \frac{t\log(t)\log\left(\frac{t\log(t)}{t-1}\right)}{t\log(t) - t + 1} \right), \quad t > 0.$$

By a straighforward computation, we conclude that the function $t \mapsto \widehat{\ell}_2(t^{-1})$ is a regularly varying function at infinity of index $\varrho = 0$.

$$M_1^{\Phi_2}(t) \sim N + \gamma \log(\log(t)), \quad \text{as} \quad t \to \infty,$$

when $\delta_b = \delta_d$. Analogously, by the assertion $(iii)$ of Theorem 2.2 we have that

$$M_1^{\Phi_2}(t) \sim \frac{N}{1 + (\delta_d - \delta_b)\log(\log(t))} + \frac{\gamma}{\delta_d - \delta_b}, \quad \text{as} \quad t \to \infty.$$

when $\delta_d > \delta_b$.

**Remark 4.2.** The process to define the pair $(h_2, \ell_2)$ can be slightly modified to define recursively a family of pairs $(h_n, \ell_n) \in (\mathcal{PC})'$ such that

$$\lambda \widehat{h}_{n+1}(\lambda) = \frac{\widehat{\ell}_n(\lambda) - 1}{\widehat{\ell}_n(\lambda)\log(\widehat{\ell}_n(\lambda))}, \quad \lambda > 0, \tag{4.4}$$

where $(h_1, \ell_1) \in (\mathcal{PC})'$ has been defined in Example 4.2. This allows us to consider a process $\mathcal{N}_{\Phi_n}(t)$ induced by a time non-local birth-death process $\mathcal{N}$ and the Bernstein function $\Phi_n(\lambda) = \lambda \widehat{h}_n(\lambda)$. Following the same procedure described in Example 4.4, the corresponding equation (4.2) can be solved by a recursive manner. For the sake of the brevity of the text, we omit the computations.

Although in the previously developed examples, the equation (4.2) has been difficult to solve, there are instances where this is not the case. In such a context we present the following example.

**Example** 4.5. Let $(h_1, \ell_1)$ the pair of functions in $(\mathcal{PC})'$ defined in Example 4.2. Consider the function $h_e \colon (0, \infty) \to (0, \infty)$ given by

$$h_e(t) = e^{-t}\ell_1(t), \quad t > 0.$$

We recall that the multiplication of two completely monotonic functions is completely monotonic, which implies that $h_e \in (\mathcal{CM})$. Moreover, it follows from [13, Theroem 5.4] that there exists $\ell_e \in (\mathcal{CM})$ such that $h_e * \ell_e = 1$ on $\mathbb{R}_+$. Since $\widehat{\ell}_1(\lambda) = \frac{\log(\lambda)}{\lambda - 1}$ we have that

$$\widehat{h}_e(\lambda) = \frac{\log(\lambda + 1)}{\lambda}, \quad \text{and} \quad \widehat{\ell}_e(\lambda) = \frac{1}{\log(\lambda + 1)}, \quad \lambda > 0.$$

Clearly we have that $\lim_{\lambda \to 0^+} \widehat{\ell}_e(\lambda) = +\infty$, which implies that $\ell_e \notin L_1(\mathbb{R}_+)$, and the pair $(h_e, \ell_e) \in (\mathcal{PC})'$. Consider the process $\mathcal{N}_{\Phi_e}$ where $\Phi_e(\lambda) = \lambda \widehat{h}_e(\lambda)$. In this case the equation (4.2) takes the form

$$\log(\lambda + 1) = \delta_b - \delta_d.$$

Therefore, when $\delta_b > \delta_d$ the unique solution $\omega_e > 0$ of the equation $\lambda \widehat{h}_e(\lambda) = \delta_b - \delta_d$ is given by

$$\omega_e = e^{\delta_b - \delta_d} - 1.$$

Therefore, by assertion $(i)$ of Theorem 2.2 when $\delta_b > \delta_d$, we have that

$$M_1^\Phi(t) \sim R_e \cdot \exp(\omega_e t), \quad \text{as} \quad t \to \infty,$$

where $\omega_e > 0$ has been given above, and

$$R_e = \frac{\big(N\,(\delta_b - \delta_d) + \gamma\big)e^{\delta_b - \delta_d}}{e^{\delta_b - \delta_d} - 1}.$$

## 5. Appendix

A. **Scalar Volterra equations.** By scalar Volterra equation we mean any scalar integral equation of the form

$$v(t) + \mu(\ell * v)(t) = f(t), \quad t \geq 0,$$

where $v$ is the unknown function, $\mu \in \mathbb{C}$, and the functions $f\colon \mathbb{R}_+ \to \mathbb{C}$ and $\ell\colon \mathbb{R}_+ \to \mathbb{R}$ are given. Over the years, this type of equations has been widely studied, see, e.g. [8, 13, 27]. In particular, the following two scalar Volterra equations have played an crucial role in the treatment of many nonlocal in time evolution equations.

**Definition A.1.** Let $\mu \in \mathbb{C}$ and $\ell \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$. The ***scalar resolvent function*** $s_\mu\colon \mathbb{R}_+ \to \mathbb{C}$ is defined as the unique solution of the Volterra equation

$$s_\mu(t) + \mu(s_\mu * \ell)(t) = 1, \quad t \geq 0. \tag{A.1}$$

The ***integrated scalar resolvent function*** $r_\mu$ is defined as the unique solution of the equation

$$r_\mu(t) + \mu(r_\mu * \ell)(t) = \ell(t), \quad t > 0. \tag{A.2}$$

It is well known from [13, Chapter 2, Theorem 3.1] that if $\ell \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$, then for every $\mu \in \mathbb{C}$ there exists a unique locally integrable scalar resolvent function $r_\mu \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$. Furthermore, it follows from [13, Chapter 2, Theorem 3.5] that the solution of the scalar Volterra equation is given by

$$v(t) = f(t) - \mu(r_\mu * f)(t), \quad t > 0.$$

This shows in particular that if $\ell \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ then there exists a unique scalar resolvent function $s_\mu$, and it satisfies that

$$\mu(1 * r_\mu)(t) = 1 - s_\mu(t), \quad t > 0. \tag{A.3}$$

A direct consequence of (A.3) is that for all $\mu \in \mathbb{R}$, the function $s_\mu$ is differentiable. Further, for all $\mu > 0$ we have that

$$\int_0^\infty r_\mu(t)dt \leq \frac{1}{\mu}. \tag{A.4}$$

**Remark A.1.** Both $s_\mu$ and $r_\mu$ are nonnegative functions for all $\mu \in \mathbb{R}$. For $\mu \geq 0$, this is a consequence of the complete positivity of $\ell$, see, e.g. [8, Theorem 2.1]. If $\mu < 0$, this can be seen, e.g., by a simple fixed point argument in the space of nonnegative $L_1((0,T))$-functions with arbitrary $T > 0$ and an appropriate norm, see [32] for more details.

**Remark A.2.** In Definition A.1, we will denote the corresponding scalar resolvent functions by $s(t,\mu)$ and $r(t,\mu)$. With this notation, it follows from (A.3) that if $\mu > 0$, then the function $s(\cdot,\mu)$ is a decreasing function. Meanwhile, if $\mu < 0$ then $s(\cdot,\mu)$ is increasing.

Let $(h,\ell) \in (\mathcal{PC})$ and $\mu \in \mathbb{C}$. Consider the functions $s(\cdot,\mu)$ and $r(\cdot,\mu)$ associated to $\ell$. Then the Laplace transform of both $s(\cdot,\mu)$ and $r(\cdot,\mu)$ exist, and they are respectively given by

$$\widehat{s}(\lambda,\mu) = \frac{\widehat{h}(\lambda)}{\lambda\widehat{h}(\lambda) + \mu} = \frac{1}{\lambda(1 + \mu\widehat{\ell}(\lambda))}, \quad \lambda > \lambda_0, \tag{A.5}$$

and

$$\widehat{r}(\lambda,\mu) = \frac{1}{\lambda\widehat{h}(\lambda) + \mu} = \frac{\widehat{\ell}(\lambda)}{1 + \mu\widehat{\ell}(\lambda)}, \quad \lambda > \lambda_0, \tag{A.6}$$

for some $\lambda_0 \geq 0$.

B. **Propagation function associated to a Bernstein function.** Let $(h, \ell) \in (\mathcal{PC})$. In this section we introduce the so-called *propagation function* associated to the function $\Phi(\lambda) = \lambda \widehat{h}(\lambda)$. It follows from [27, Proposition 4.5] that the completely positivity of $\ell$ implies that $\Phi(\lambda) := \lambda \widehat{h}(\lambda)$, $\lambda > 0$, is a Bernstein function. This in turn implies that, for each fixed $\tau \geq 0$, the function $\psi_\tau : (0, \infty) \to (0, \infty)$ defined by

$$\psi_\tau(\lambda) = \frac{\exp(-\tau\Phi(\lambda))}{\lambda}, \quad \lambda > 0,$$

is completely monotonic. Therefore, it follows from Bernstein's theorem ([29, Theorem 1.4]) that for every $\tau \geq 0$ there is a unique nondecreasing function $W_\Phi(\cdot, \tau) \in BV(\mathbb{R}_+)$, normalized by $W_\Phi(0, \tau) = 0$ and left-continuous, whose Laplace transform (with respect to $t$) is given by

$$\widehat{W}_\Phi(\lambda, \tau) = \frac{\exp(-\tau\Phi(\lambda))}{\lambda}, \quad \lambda > 0. \tag{A.7}$$

The function $W_\Phi(\cdot, \cdot)$ is known as the ***propagation function*** associated to a Bernstein function $\Phi$. We refer the reader to [27, Section 4.5] for several results and applications of this function to more general frameworks. For instance, the propagation function is strongly related to the scalar resolvent function $s(t, \mu)$. Indeed, for all $\mu > 0$ and $t \geq 0$ we have that

$$s(t, \mu) = -\int_0^\infty e^{-\tau\mu} d_\tau W_\Phi(t, \tau).$$

This implies that the mapping $\mu \mapsto -s(t, \mu)$ is completely monotonic with respect to the parameter $\mu$.

**Remark B.1.** Under the assumption $\Phi(\lambda) = \lambda \hat{h}(\lambda)$ with $h$ in $(\mathcal{PC})'$ the inverse subordinator $E_\Phi$ is such that, for all $t > 0$, $E_\Phi(t)$ admits a density $f_\Phi(s; t)$. Such a density can be recognized in terms of the propagator as follows:

$$f_\Phi(s; t) ds = -d_s W_\Phi(t, s).$$

Nevertheless, $s(t, \mu)$ is the Laplace transform of the inverse subordinator $E_\Phi(t)$.

C. **Tauberian theorems.** This type of theorem is a powerful tool for determining the asymptotic behavior of certain functions, provided that some information about an integral transformation of the function, such as the Laplace transform, is known. In this work, we apply two such theorems.

**Theorem C.1.** *(Karamata-Feller) Let $L : (0, \infty) \to (0, \infty)$ be a slowly varying function and $\varrho > 0$. If $g : (0, \infty) \to \mathbb{R}$ be a monotone function whose Laplace transform $\widehat{g}(z)$ exists for all $z \in \mathbb{C}_+ := \{z \in \mathbb{C} : Re(z) > 0\}$, then*

$$\widehat{g}(z) \sim \frac{1}{z^\varrho} L\Big(\frac{1}{z}\Big), \ \text{as } z \to 0, \quad \text{if and only if} \quad g(t) \sim \frac{t^{\varrho - 1}}{\Gamma(\varrho)} L(t), \ \text{as } t \to \infty.$$

*The approach here is considered on the positive real axis and the notation $f(t) \sim h(t)$ as $t \to t_*$ means that $\lim_{t \to t_*} f(t)/h(t) = 1$.*

Karamata-Feller Theorem establishes that the asymptotic behavior of a function $g(t)$ as $t \to \infty$ can be determined, under suitable conditions, by looking at the behavior of its Laplace transform $\widehat{g}(z)$ and vice versa. See the monograph [12, Section 5, Chapter XIII] for a more general version and proofs.

**Theorem C.2.** *(Wiener-Ikehara) Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function. Assume that there exist $a, c > 0$ such that the Laplace transform $\widehat{f}$ is well-defined in the set $\{z \in \mathbb{C} : Re(z) > c\}$, and the function $F$ defined by*

$$F(z) = \widehat{f}(z) - \frac{a}{z - c},$$

*has a continuous extension to $\{z \in \mathbb{C} : Re(z) \geq c\}$. Then*

$$f(t) \sim a e^{ct}, \quad \text{as} \quad t \to \infty.$$

Wiener-Ikehara's theorem can be rephrased as follows. If the Laplace transform $\widehat{f}$ of a non-decreasing function $f$ is analytic in the complex half-plane $\{z \in \mathbb{C} : Re(z) > c\}$ having a simple pole in $z = c$ with residue $a$, then

$$f(t) \sim a \cdot \exp(ct), \quad \text{as} \quad t \to \infty.$$

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