WEIGHTED PSEUDO ANTIPERIODIC SOLUTIONS FOR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES.

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Abstract. In this paper we prove the existence of weighted pseudo antiperiodic mild solutions for fractional integro-differential equations in the form

$$D^{\alpha}(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t, u(t)),$$

where $f(\cdot, u(\cdot))$ is Stepanov weighted pseudo antiperiodic and A generates a resolvent family $\{S_{\alpha}(t)\}_{t\geq 0}$ of bounded and linear operators on a Banach space X, $a \in L^1_{\rm loc}(\mathbb{R}_+)$ and $\alpha > 0$. Also, we give a short proof to show that the vector-valued space of Stepanov-like weighted pseudo antiperiodic functions is a Banach space.

1. Introduction

Let us consider the equation

$$L(u) = f, \tag{1.1}$$

where L is a linear, possibly unbounded operator, and the forcing term f belongs to some space of vector-valued functions, say $\mathcal{M}$. It is well known that mathematical understanding of the linear Equation (1.1) is meant as a preliminary critical step for the subsequent analysis of full nonlinear models. Usually, one is interested in to find conditions on the operator L such that the solution u belongs to the same space of vector-valued functions than f. Then, fixed point arguments are used to obtain the desired solution of associated nonlinear problems.

We ask for the following problem: (Q) Can the solution u be more regular that f?. In other words, is it possible to find a subspace $\mathcal{N} \subset \mathcal{M}$ such that $u \in \mathcal{N}$?.

This problem has begun to be studied recently and there are some cases in the literature where the answer is positive. For example, in [8], Diagana, N'Guérékata and Mophou solved problem (Q) taking $\mathcal{M}$ as the space of Stepanov-like weighted pseudo almost automorphic functions, $L(u) := D_t^{\alpha}u - Au$ and $\mathcal{N}$ as the subspace of weighted pseudo almost automorphic functions. Here A is a closed and linear operator defined on a Banach space X and D^{α} denotes fractional derivative of order $\alpha > 0$.

In this paper, we are able to give an affirmative answer to (Q) taking $\mathcal{M}$ as the space of Stepanov-like weighted pseudo antiperiodic functions; $\mathcal{N}$ as the space of weighted pseudo antiperiodic functions and where the class of operators is defined by

$$L(u)(t) = D^{\alpha}u(t) - Au(t) - \int_{-\infty}^{t} a(t-s)Au(s)\,ds, \tag{1.2}$$

where A generates a resolvent family $\{S_{\alpha}(t)\}_{t\geq 0}$ on Banach space X, $a \in L^1_{\rm loc}(\mathbb{R}_+)$ and $\alpha > 0$. The class of operators (1.2) has been in studied in [28]. In that paper, the author has solved the problem of maximal regularity in several spaces of functions, i.e. starting with $f \in \mathcal{M}$ and proving that the mild solution u belongs to the same subspace $\mathcal{M}$. We remark that continuity fails in the case of Stepanov type functions and only measurability and integrability are required to work with this class of functions. Hence, it justify think in the preceding problem in the context mentioned above. To the best of our

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knowledge, the existence of weighted pseudo antiperiodic solutions to Equation (1.2) in the case when the forcing term f is Stepanov weighted pseudo antiperiodic is an untreated original problem, which constitutes one of the main motivations of this work.

The existence and uniqueness of antiperiodic solutions to evolution equations have been studied in several works. We mention here Aftabizadeh, Aizicivici and Pavel [1], [2], Al-Islam, Alsulami and Diagana [5], H.L.Chen [9], Y.Q. Chen [10], Haraux [21], Okoshi [27], and N'Guérékata and Valmorin [26].

This paper is organized as follows. In Section 2, we first present some definitions and basic results of Stepanov-like type spaces and then we give a short and direct proof to the fact that the space of Stepanov-like weighted pseudo antiperiodic functions is a Banach space (Theorem 2.15). In Section 3, we first give a composition Theorem in the space of Stepanov-like weighted pseudo antiperiodic functions, assuming a compactness condition (Theorem 3.3). Then, we give sufficient conditions in order to ensure the existence and uniqueness of weighted weighted antiperiodic mild solutions where the input data f belongs to the space of Stepanov-like weighted pseudo antiperiodic functions. We finish this paper with an illustrative example to find existence and uniqueness of mild solutions for a concrete semilinear problem is given.

2. Preliminaries

In this section, we introduce some basic definitions, notations and preliminaries facts that we will use in the paper. Particularly, we give an alternative proof to show that the space of Stepanov-like weighted pseudo antiperiodic functions is a Banach space.

Throughout the paper $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are complex Banach spaces and $B(X,Y)$ is the Banach space of bounded linear operators from X to Y; when $X = Y$ we simply write $B(X)$.

We denote by

$$BC(\mathbb{R}, X) := \{f : \mathbb{R} \to X : f \text{ is continuous}, ||f||_\infty := \sup_{t\in\mathbb{R}} ||f(t)|| < \infty\},$$

the Banach space of X-valued bounded and continuous functions on $\mathbb{R}$, with natural norm.

Given a function $g : \mathbb{R} \to X$, the *Caputo (or Weyl) fractional integral* of order $\alpha > 0$ is defined by

$$D^{-\alpha} g(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} g(s) ds, \quad t \in \mathbb{R},$$

when this integral is convergent. The *Caputo (or Weyl) fractional derivative* $D^\alpha g$ of order $\alpha > 0$ is defined by

$$D^\alpha g(t) := \frac{d^n}{dt^n} D^{-(n-\alpha)} g(t), \quad t \in \mathbb{R},$$

where $n = [\alpha] + 1$. It is known that $D^\alpha D^{-\alpha} g = g$ for any $\alpha > 0$, and $D^n = \frac{d^n}{dt^n}$ holds with $n \in \mathbb{N}$. See [25] for more details.

The Mittag-Leffler function (see e.g. [24]) is defined as follows:

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_{Ha} e^\mu \frac{\mu^{\alpha-\beta}}{\mu^\alpha - z} d\mu, \quad \alpha, \beta > 0, z \in \mathbb{C},$$

where Ha is a Hankel path, i.e. a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \le |z|^{1/\alpha}$ counterclockwise. The Laplace transform of a variant of the Mittag-Leffler function is given by:

$$\mathcal{L}(t^{\beta-1} E_{\alpha,\beta}(-\rho t^\alpha))(\lambda) = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha + \rho}, \quad \rho \in \mathbb{C}, \mathrm{Re}\lambda > |\rho|^{1/\alpha}.$$

We recall the following definition [28] (see also [29] for a general treatment on resolvent families).

Definition 2.1. *Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X, and $\alpha > 0$. Given $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, we say that A is the generator of an α-resolvent family, if there exist*

$\omega \geq 0$ and a strongly continuous function $S_\alpha : [0,\infty) \to \mathcal{B}(X)$ such that $\{\frac{\lambda^\alpha}{1+\hat{a}(\lambda)} : \mathrm{Re}\lambda > \omega\} \subset \rho(A)$ and for all $x \in X$,

$$\left(\lambda^\alpha - (1+\hat{a}(\lambda))A\right)^{-1} x = \frac{1}{1+\hat{a}(\lambda)}\left(\frac{\lambda^\alpha}{1+\hat{a}(\lambda)} - A\right)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \ \ \mathrm{Re}\lambda > \omega.$$

*In this case, $\{S_\alpha(t)\}_{t\geq 0}$ is called the α-*resolvent family *generated by A.*

Now, we recall the definitions of antiperiodic functions.

Definition 2.2. *A function $f \in C(\mathbb{R}, X)$ is said to be antiperiodic if there exists a $\omega \in \mathbb{R}\setminus\{0\}$ with the property $f(t+\omega) = -f(t)$ for all $t \in \mathbb{R}$. If there exists a least positive ω with this property, it is called the anti-period of f. The collection of those functions with the same anti-period ω is denoted by $P_{\omega ap}(\mathbb{R}, X)$.*

Remark 2.3. *Note that if $f \in P_{\omega ap}(\mathbb{R}, X)$, then $f \in P_{2\omega}(\mathbb{R}, X)$, where $P_{2\omega}(\mathbb{R}, X)$ denotes the Banach space of all 2ω-periodic functions.*

Definition 2.4. *A function $f \in C(\mathbb{R}\times X, X)$ (resp., $C(\mathbb{R}\times X\times X, X)$) is said to be antiperiodic in $t \in \mathbb{R}$ and uniformly in $u \in X$ (resp. in $(u,v) \in X\times X$) if there exists a $\omega \in \mathbb{R}\setminus\{0\}$ with the property $f(t+\omega, u) = -f(t,u)$ for all $t \in \mathbb{R}$, $u \in X$. (resp. $f(t+\omega,u,v) = -f(t,u,v)$ for all $t \in \mathbb{R}$, $(u,v) \in X\times X$). The collection of those ω-antiperiodic functions is denoted by $P_{\omega ap}(\mathbb{R}\times X, X)$ (resp., $P_{\omega ap}(\mathbb{R}\times X\times X, X)$).*

Let U be denote the set of all functions $\rho : \mathbb{R} \to (0,\infty)$ in $L^1_{loc}(\mathbb{R})$ such that $\rho(t) > 0$ for all $t \in \mathbb{R}$ a.e. For a given $r > 0$ and for each $\rho \in U$, we set

$$m(r,\rho) := \int_{-r}^{r} \rho(t)\, dt.$$

Thus the space of weights U_∞ is defined by

$$U_\infty := \{\rho \in U : \lim_{r\to\infty} m(r,\rho) = \infty\}.$$

Now, for $\rho \in U_\infty$, we define

$$PAA_0(\mathbb{R}, X) := \{f \in BC(\mathbb{R}, X) : \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \|f(t)\|\rho(t)\, dt = 0\};$$

$$\begin{aligned} PAA_0(\mathbb{R}\times Y, X) := \{f \in BC(\mathbb{R}\times Y, X) \quad &: \quad f(\cdot, y) \ \text{ is bounded for each } y \in Y \\ \text{and} \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \|f(t)\|\rho(t)\, dt \quad &= \quad 0, \ \text{ uniformly in } y \in Y\}. \end{aligned}$$

Definition 2.5 ([16])**.** *Let $\rho \in U_\infty$. A function $f \in BC(\mathbb{R}, X)$ (respectively $f \in BC(\mathbb{R}\times Y, X)$) is called weighted pseudo antiperiodic if it can be expressed as $f = g + h$ where $g \in P_{\omega ap}(\mathbb{R}, X)$ (respectively $P_{\omega ap}(\mathbb{R}\times Y, X)$) and $h \in PAA_0(\mathbb{R}, X)$ (respectively $PAA_0(\mathbb{R}\times Y, X)$). We denote by $WPP_{\omega ap}(\mathbb{R}, X)$ (respectively $WPP_{\omega ap}(\mathbb{R}\times Y, X)$) the set of all such functions.*

Definition 2.6 ([17])**.** *The Bochner transform $f^b(t,s)$ with $t \in \mathbb{R}, s \in [0,1]$ of a function $f : \mathbb{R} \to X$ is defined by*

$$f^b(t,s) := f(t+s).$$

Definition 2.7 ([17])**.** *The Bochner transform $f^b(t,s,u)$ with $t \in \mathbb{R}, s \in [0,1], u \in X$ of a function $f : \mathbb{R}\times X \to X$ is defined by*

$$f^b(t,s,u) := f(t+s,u) \quad \text{for all} \quad u \in X.$$

Definition 2.8 ([17])**.** *Let $p \in [1,\infty)$. The space $BS^p(\mathbb{R}, X)$ of all Stepanov bounded functions, with exponent p, consist of all measurable functions $f : \mathbb{R} \to X$ such that $f^b \in L^\infty(\mathbb{R}, L^p(0,1;X))$. This is a Banach space with the norm*

$$\|f\|_{S^p} := \|f^b\|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p\, d\tau\right)^{\frac{1}{p}}.$$

Definition 2.9. *A function $f \in BS^p(\mathbb{R}, X)$ is called Stepanov antiperiodic if $f^b \in P_{\omega ap}(\mathbb{R}, L^p(0,1;X))$. We denote the set of all such functions by $P_{\omega ap}S^p(\mathbb{R}, X)$.*

Remark 2.10. *We note that the preceding definition implies*

$$\sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(s+\omega) + f(s)\|^p \, ds \right)^{1/p} = 0$$

which is equivalent to say that $f(t+\omega) = -f(t)$ a.e. $t \in \mathbb{R}$; that is; $\|f(t+\omega) + f(t)\|_p = 0$. This coincide with the definition of Z. Xia in [35].

Definition 2.11. *A function $f : \mathbb{R} \times X \to Y$ with $f(\cdot, u) \in BS^p(\mathbb{R}, Y)$, for each $u \in X$, is called Stepanov antiperiodic function in $t \in \mathbb{R}$ uniformly for $u \in X$ if $f(t+\omega, u) = -f(t, u)$ a.e. $t \in \mathbb{R}$ and each $u \in X$. We denote by $P_{\omega ap}S^p(\mathbb{R} \times X, Y)$ the set of such functions.*

Now, we introduce a (natural) linear operator from $BS^p(\mathbb{R}, X)$ into $L^\infty(\mathbb{R}, L^p(0,1;X))$ which will be an important tool in order to clarity some concepts and achieve our goals.

Definition 2.12. *We define the map*

$$\begin{aligned} \mathcal{B} :&BS^p(\mathbb{R}, X) \to L^\infty(\mathbb{R}, L^p(0,1;X)) \\ &f \mapsto (\mathcal{B}f)(t)(s) = f(t+s). \end{aligned}$$

Remark 2.13. *It follows from the definitions that the operator $\mathcal{B}$ is a linear isometry between $BS^p(\mathbb{R}, X)$ and $L^\infty(\mathbb{R}, L^p(0,1;X))$. More precisely*

$$\|\mathcal{B}f\|_\infty = \|f\|_{BS^p(\mathbb{R},X)}.$$

Remark 2.14. *The definition of Stepanov-like weighted pseudo antiperiodic functions given by Z. Xia in* [35] *can be written using the preceding notation. Thus, for $\rho \in U_\infty$, we say that a function f is Stepanov-like weighted pseudo antiperiodic (or S^p-weighted pseudo antiperiodic) if and only if $f \in \mathcal{B}^{-1}(P_{\omega ap}(\mathbb{R}, L^p(0,1;X))) + \mathcal{B}^{-1}(PAA_0(\mathbb{R}, L^p(0,1;X)))$. In other words,*

$$WPP_{\omega ap}S^p(\mathbb{R}, X) = \mathcal{B}^{-1}(P_{\omega ap}(\mathbb{R}, L^p(0,1;X))) + \mathcal{B}^{-1}(PAA_0(\mathbb{R}, L^p(0,1;X))) \tag{2.1}$$

Moreover, since $\mathcal{B}$ is an isometry and $P_{\omega ap}(\mathbb{R}, L^p(0,1;X)) \cap PAA_0(\mathbb{R}, L^p(0,1;X)) = \{0\}$ then the sum is direct, that is,

$$WPP_{\omega ap}S^p(\mathbb{R}, X) = \mathcal{B}^{-1}(P_{\omega ap}(\mathbb{R}, L^p(0,1;X))) \oplus \mathcal{B}^{-1}(PAA_0(\mathbb{R}, L^p(0,1;X))).$$

Based in the definition of operator $\mathcal{B}$, we prove that $WPP_{\omega ap}S^p(\mathbb{R}, X)$ is a Banach space when endowed with their natural norm.

Theorem 2.15. *$WPP_{\omega ap}S^p(\mathbb{R}, X)$ is a Banach space with the norm*

$$\|f\|_{WPP_{\omega ap}S^p(\mathbb{R},X)} := \|g\|_{BS^p(\mathbb{R},X)} + \|h\|_{BS^p(\mathbb{R},X)}$$

where $f = g + h$ with $g \in \mathcal{B}^{-1}(P_{\omega ap}(\mathbb{R}, L^p(0,1;X)))$ and $h \in \mathcal{B}^{-1}(PAA_0(\mathbb{R}, L^p(0,1;X)))$.

Proof. Let (f_n) be a Cauchy sequence in $WPP_{\omega ap}S^p(\mathbb{R}, X)$. Then $\|f_n - f_m\|_{WPP_{\omega ap}S^p(\mathbb{R},X)} \to 0$ as $n, m \to \infty$. Let $f_n = g_n + h_n$ and $f_m = g_m + h_m$ with $g_n, g_m \in \mathcal{B}^{-1}(P_{\omega ap}(\mathbb{R}, L^p(0,1;X)))$ and $h_n, h_m \in \mathcal{B}^{-1}(PAA_0(\mathbb{R}, L^p(0,1;X)))$. If $n, m \to \infty$, then

$$\|\mathcal{B}g_n - \mathcal{B}g_m\|_{L^\infty(\mathbb{R},L^p)} = \|g_n - g_m\|_{BS^p(\mathbb{R},X)} \leq \|f_n - f_m\|_{WPP_{\omega ap}S^p(\mathbb{R},X)} \to 0$$

and

$$\|\mathcal{B}g_n - \mathcal{B}g_m\|_{L^\infty(\mathbb{R},L^p)} = \|g_n - g_m\|_{BS^p(\mathbb{R},X)} \leq \|f_n - f_m\|_{WPP_{\omega ap}S^p(\mathbb{R},X)} \to 0.$$

This implies that $(\mathcal{B}g_n)$ and $(\mathcal{B}h_n)$ are Cauchy sequences in $P_{\omega ap}(\mathbb{R}, L^p(0,1;X))$ and $PAA_0(\mathbb{R}, L^p(0,1;X))$ respectively. Since $P_{\omega ap}(\mathbb{R}, L^p(0,1;X))$ and $PAA_0(\mathbb{R}, L^p(0,1;X))$ are Banach spaces (see [26] and [18] resp.) then there exist $g \in P_{\omega ap}(\mathbb{R}, L^p(0,1;X))$ and $h \in PAA_0(\mathbb{R}, L^p(0,1;X))$ such that

$$\|\mathcal{B}g_n - g\|_{L^\infty(\mathbb{R},L^p)} \to 0, \quad \|\mathcal{B}h_n - h\|_{L^\infty(\mathbb{R},L^p)} \to 0 \quad (n \to \infty).$$

Let $f_1 := \mathcal{B}^{-1}(\{g\}) \in \mathcal{B}^{-1}(P_{\omega ap}(\mathbb{R}, L^p(0,1;X)))$ and $f_2 := \mathcal{B}^{-1}(\{h\}) \in \mathcal{B}^{-1}(PAA_0(\mathbb{R}, L^p(0,1;X)))$. Note that f_1 and f_2 are well defined because $\mathcal{B}$ is injective. Let $f := f_1 + f_2 \in WPP_{\omega ap}S^p(\mathbb{R}, X)$. Then

$$\begin{aligned}\|f_n - f\|_{WPP_{\omega ap}S^p(\mathbb{R},X)} &= \|(g_n + h_n) - (f_1 + f_2)\|_{WPP_{\omega ap}S^p(\mathbb{R},X)}\\ &= \|g_n - f_1\|_{BS^p(\mathbb{R},X)} + \|h_n - f_2\|_{BS^p(\mathbb{R},X)}\\ &= \|\mathcal{B}g_n - \mathcal{B}f_1\|_{L^\infty(\mathbb{R},L^p)} + \|\mathcal{B}h_n - \mathcal{B}f_2\|_{L^\infty(\mathbb{R},L^p)}\\ &= \|\mathcal{B}g_n - g\|_{L^\infty(\mathbb{R},L^p)} + \|\mathcal{B}h_n - h\|_{L^\infty(\mathbb{R},L^p)} \to 0 \quad (n \to \infty).\end{aligned}$$

Therefore $WPP_{\omega ap}S^p(\mathbb{R}, X)$ is a Banach space. □

Theorem 2.16. *Let $\rho \in U_\infty$ be given and let $S : \mathbb{R}_+ \to B(X)$ be strongly continuous. Suppose that there exist a function $\phi \in L^1(\mathbb{R}_+)$ such that*

(a) $\|S(t)\| \le \phi(t) \quad t \ge 0$;
(b) $\phi(t)$ is increasing;
(c) $\sum_{n=0}^{\infty} \phi(n) < \infty$.

Suppose that $f \in WPP_{\omega ap}S^p(\mathbb{R}, X)$. Then

$$(S * f)(t) := \int_{-\infty}^{t} S(t-s)f(s)\,ds \in WPP_{\omega ap}(\mathbb{R}, X).$$

Proof. See [35, Lemma 36]. □

3. Weighted pseudo antiperiodic mild solutions

In this section we consider the problem of existence and uniqueness of weighted pseudo antiperiodic mild solutions for the following class of integro-differential equations

$$D^\alpha u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t, u(t)), \tag{3.1}$$

where A generates an α-resolvent family $\{S_\alpha(t)\}_{t\ge 0}$ on a Banach space X, $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, $\alpha > 0$ and the fractional derivative is understood in the sense of Caputo. Note that Equation (3.1) has the form of Equation (1.1) with $Lu = D^\alpha u(t) - Au(t) - \int_{-\infty}^{t} a(t-s)Au(s)\,ds$.

Definition 3.1. *A function $u : \mathbb{R} \to X$ is said to be a mild solution of (3.1) if*

$$u(t) = \int_{-\infty}^{t} S_\alpha(t-s)f(s, u(s))\,ds \quad (t \in \mathbb{R})$$

where $\{S_\alpha(t)\}_{t\ge 0}$ is the α-resolvent family generated by A, whenever it exists.

Now, we present the following composition theorems.

Theorem 3.2. *Assume that $F : \mathbb{R} \times X \to X$ is a bounded function that satisfies*

(a) There exists $\omega > 0$ such that $F(t+\omega, -x) = -F(t, x)$ for a.e. $t \in \mathbb{R}$ and for all $x \in X$;
(b) There exists $L > 0$ such that $\|F(t, x) - F(t, y)\| \le L\|x - y\|$ for all $x, y \in X$ and $t \in \mathbb{R}$;
(c) $u \in P_{\omega ap}S^p(\mathbb{R}, X)$.

Then $F(\cdot, u(\cdot)) \in P_{\omega ap}S^p(\mathbb{R}, X)$.

Proof. Since $\|F(t, x) - F(t, y)\| \le L\|x - y\|$ implies $\|F(t, x) - F(t, y)\|_p \le L\|x - y\|_p$, then

$$\begin{aligned}\|F(t+\omega, u(t+\omega)) + F(t, u(t))\|_p &= \|F(t+\omega, u(t+\omega)) - F(t+\omega, -u(t))\|_p\\ &\quad + \|F(t+\omega, -u(t)) + F(t, u(t))\|_p\\ &\le L\|u(t+\omega) + u(t)\|_p + \|F(t+\omega, -u(t)) + F(t, u(t))\|_p = 0.\end{aligned}$$

Therefore $F(t+\omega, u(t+\omega)) = -F(t, u(t))$ a.e. $t \in \mathbb{R}$ and consequently $F(\cdot, u(\cdot)) \in P_{\omega ap}(\mathbb{R}, X)$. □

Our next result assume a compactness condition in order to obtain invariance under composition of functions for the space of Stepanov weighted pseudo antiperiodic functions.

Theorem 3.3. *Let $\rho \in U_\infty$, $p > 1$, $f = g + \phi \in WPP_{\omega ap}S^p(\mathbb{R} \times X, X)$ with $g \in \mathcal{B}^{-1}(P_{\omega ap}(\mathbb{R} \times X, L^p(0,1;X)))$ and $\phi \in \mathcal{B}^{-1}(PAA_0(\mathbb{R} \times X, L^p(0,1;X)))$. Assume that*

(i) There exists $\omega > 0$ such that $f(t+\omega, -x) = -f(t,x)$.
(ii) There exist constants $L_f, L_g > 0$ such that

$$\|f(t,u) - f(t,v)\| \le L_f\|u - v\|, \ \ \|g(t,u) - g(t,v)\| \le L_g\|u - v\| \quad t \in \mathbb{R},\, u, v \in X.$$

(iii) $h = \alpha+\beta \in WPP_{\omega ap}S^p(\mathbb{R}, X)$ with $\alpha \in \mathcal{B}^{-1}(P_{\omega ap}(\mathbb{R}, L^p(0,1;X)))$ and $\beta \in \mathcal{B}^{-1}(PAA_0(\mathbb{R}, L^p(0,1;X)))$ is such that the set

$$K := \overline{\{\alpha(t) : t \in \mathbb{R}\}}$$

is compact in X. Then $f(\cdot, h(\cdot)) \in WPP_{\omega ap}S^p(\mathbb{R}, X)$.

Proof. We can decompose

$$f(t, h(t)) = g(t, \alpha(t)) + f(t, h(t)) - f(t, \alpha(t)) + \phi(t, \alpha(t)).$$

Set

$$F(t) := g(t, \alpha(t)), \ \ G(t) := f(t, h(t)) - f(t, \alpha(t)), \ \ H(t) := \phi(t, \alpha(t)).$$

Since $\alpha \in P_{\omega ap}S^p(\mathbb{R}, X)$ and $g \in P_{\omega ap}S^p(\mathbb{R} \times X, X)$ then by assumptions and Theorem 3.2 we obtain that $F(t) \in \mathcal{B}^{-1}(P_{\omega ap}(\mathbb{R}, L^p(0,1;X)))$.

Next we show that $G(t) \in \mathcal{B}^{-1}(PAA_0(\mathbb{R}, L^p(0,1;X)))$. Indeed

$$\begin{aligned}\int_t^{t+1} \|G(\sigma)\|^p\, d\sigma &= \int_t^{t+1} \|f(\sigma, h(\sigma)) - f(\sigma, \alpha(\sigma))\|^p\, d\sigma \\ &\le \int_t^{t+1} L_f^p \|h(\sigma) - \alpha(\sigma)\|^p\, d\sigma \\ &= \int_t^{t+1} L_f^p \|\beta(\sigma)\|^p\, d\sigma.\end{aligned}$$

Then

$$\frac{1}{m(r,\rho)} \int_{-r}^{r} \left(\int_t^{t+1} \rho(t)\|G(\sigma)\|^p\, d\sigma \right)^{1/p} dt \le \frac{L_f}{m(r,\rho)} \int_{-r}^{r} \rho(t) \left(\int_t^{t+1} \|\beta(\sigma)\|^p\, d\sigma \right)^{1/p} dt.$$

Since $\beta(\cdot) \in \mathcal{B}^{-1}(PAA_0(\mathbb{R}, L^p(0,1;X)))$ we obtain that $G(\cdot) \in \mathcal{B}^{-1}(PAA_0(\mathbb{R}, L^p(0,1;X)))$.

Next, we prove that $H(\cdot) \in \mathcal{B}^{-1}(PAA_0(\mathbb{R}, L^p(0,1;X)))$. Since $\phi \in \mathcal{B}^{-1}(PAA_0(\mathbb{R} \times X, L^p(0,1;X)))$ then for any $\epsilon > 0$ there exist $r_0 > 0$ such that $r > r_0$ implies that

$$\frac{1}{m(r,\rho)} \int_{-r}^{r} \rho(t) \left(\int_t^{t+1} \|\phi(\sigma, u)\|^p\, d\sigma \right)^{1/p} dt < \epsilon \quad (u \in X).$$

Since K is compact, we can find finite open balls O_k $(k = 1, 2, 3, ..., n)$ with center x_k and radius less than $\frac{\epsilon}{L_f + L_g}$ such that $K \subset \cup_{k=1}^m O_k$. Set $B_k := \{t \in \mathbb{R} : \alpha(t) \in O_k\}$. Then $\mathbb{R} = \cup_{k=1}^n B_k$. Let $E_1 = B_1$, $E_k = B_k \setminus (\cup_{j=1}^{k-1} B_j)$ $(2 \le k \le m)$. Thus $E_i \cap E_j = \emptyset$ for $i \ne j$. By Minkowski inequatility, for $r > r_0$ we have

$$\begin{aligned}
&\frac{1}{m(r,\rho)}\int_{-r}^{r}\rho(t)\left(\int_t^{t+1}\|\phi(\sigma,\alpha(\sigma))\|^p\,d\sigma\right)^{1/p}dt\\
&=\frac{1}{m(r,\rho)}\sum_{k=1}^{n}\int_{E_k\cap[-r,r]}\rho(t)\left(\int_t^{t+1}\|\phi(\sigma,\alpha(\sigma))\|^p\,d\sigma\right)^{1/p}dt\\
&\le\frac{1}{m(r,\rho)}\sum_{k=1}^{n}\int_{E_k\cap[-r,r]}\rho(t)\left(\int_t^{t+1}\|\phi(\sigma,\alpha(\sigma))-\phi(\sigma,\alpha(x_k))\|^p\,d\sigma\right)^{1/p}dt\\
&+\frac{1}{m(r,\rho)}\sum_{k=1}^{n}\int_{E_k\cap[-r,r]}\rho(t)\left(\int_t^{t+1}\|\phi(\sigma,\alpha(x_k))\|^p\,d\sigma\right)^{1/p}dt\\
&\le\frac{1}{m(r,\rho)}\sum_{k=1}^{n}\int_{E_k\cap[-r,r]}\rho(t)\left(\int_t^{t+1}\|f(\sigma,\alpha(\sigma))-f(\sigma,\alpha(x_k))\|^p\,d\sigma\right)^{1/p}dt\\
&+\frac{1}{m(r,\rho)}\sum_{k=1}^{n}\int_{E_k\cap[-r,r]}\rho(t)\left(\int_t^{t+1}\|g(\sigma,\alpha(\sigma))-g(\sigma,\alpha(x_k))\|^p\,d\sigma\right)^{1/p}dt\\
&+\frac{1}{m(r,\rho)}\sum_{k=1}^{n}\int_{E_k\cap[-r,r]}\rho(t)\left(\int_t^{t+1}\|\phi(\sigma,\alpha(x_k))\|^p\,d\sigma\right)^{1/p}dt\\
&\le L_f\frac{1}{m(r,\rho)}\sum_{k=1}^{n}\int_{E_k\cap[-r,r]}\rho(t)\left(\int_0^{1}\|\alpha(\sigma+t)-x_k\|^p\,d\sigma\right)^{1/p}dt\\
&+L_g\frac{1}{m(r,\rho)}\sum_{k=1}^{n}\int_{E_k\cap[-r,r]}\rho(t)\left(\int_0^{1}\|\alpha(\sigma+t)-x_k\|^p\,d\sigma\right)^{1/p}dt\\
&+\frac{1}{m(r,\rho)}\sum_{k=1}^{n}\int_{E_k\cap[-r,r]}\rho(t)\left(\int_t^{t+1}\|\phi(\sigma,\alpha(x_k))\|^p\,d\sigma\right)^{1/p}dt\\
&<2\epsilon+\sum_{k=1}^{n}\frac{1}{m(r,\rho)}\int_{-r}^{r}\rho(t)\left(\int_t^{t+1}\|\phi(\sigma,x_k)\|^p\,d\sigma\right)^{1/p}dt.
\end{aligned}$$

Then

$$\frac{1}{m(r,\rho)}\int_{-r}^{r}\rho(t)\left(\int_t^{t+1}\|\phi(\sigma,\alpha(\sigma))\|^p\,d\sigma\right)^{1/p}dt<(n+2)\epsilon\quad(r>r_0).$$

Hence

$$\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\rho(t)\left(\int_t^{t+1}\|\phi(\sigma,\alpha(\sigma))\|^p\,d\sigma\right)^{1/p}dt=0.$$

Therefore $H(\cdot)\in\mathcal{B}^{-1}(PAA_0(\mathbb{R},L^p(0,1;X)))$. It follows that $f(\cdot,h(\cdot))\in WPP_{\omega ap}S^p(\mathbb{R},X)$. □

Now, we obtain the existence and uniqueness of weighted pseudo antiperiodic solutions with help of Theorem 2.16 and Theorem 3.2.

Theorem 3.4. *Let $\rho\in U_\infty$ and $p>1$ and $f=g+h\in WPP_{\omega ap}S^p(\mathbb{R}\times X,X)$ be given. Suppose that*

(H1) There exists $\omega>0$ such that $f(t+\omega,-x)=-f(t,x)$.

(H2) There exist constants $L_f,L_g>0$ such that

$$\|f(t,u)-f(t,v)\|\le L_f\|u-v\|,\quad\|g(t,u)-g(t,v)\|\le L_g\|u-v\|,\quad t\in\mathbb{R},\,u,v\in X.$$

(H3) The operator A generates an α-resolvent family $\{S_\alpha(t)\}_{t\ge0}$ such that $\|S_\alpha(t)\|\le\varphi_\alpha(t)$, for all $t\ge0$, where $\varphi_\alpha(\cdot)\in L^1(\mathbb{R}_+)$ is nonincreasing such that $\varphi_0:=\sum_{n=0}^{\infty}\varphi_\alpha(n)<\infty$ and $L_f<\|\varphi_\alpha\|^{-1}$.

Then the Equation (3.1) has a unique mild solution in $WPP_{\omega ap}(\mathbb{R},X)$.

Proof. Consider the operator $Q : WPP_{\omega ap}(\mathbb{R}, X) \to WPP_{\omega ap}(\mathbb{R}, X)$ defined by

$$Q(u)(t) := \int_{-\infty}^{t} S(t-s)f(s,u(s))\,ds, \quad t \in \mathbb{R}.$$

First, we show that $Q(WPP_{\omega ap}(\mathbb{R}, X)) \subset WPP_{\omega ap}(\mathbb{R}, X)$. Let $u = u_1 + u_2 \in WPP_{\omega ap}(\mathbb{R}, X)$. Then $u_1 \in P_{\omega ap}(\mathbb{R}, X)$ and hence $K := \overline{\{u_1(t) : t \in \mathbb{R}\}}$ is compact. Moreover, it is clear that $u \in WPP_{\omega ap}S^p(\mathbb{R}, X)$ and hence (iii) in Theorem 3.3 is satisfied. From $(H1)$ and $(H2)$ we have the conditions (i) and (ii) in Theorem 3.3. It follows that $f(\cdot, u(\cdot)) \in WPP_{\omega ap}S^p(\mathbb{R}, X)$. On the other hand, the hypothesis $(H3)$ and Theorem 2.16 imply that $Q(u)(t) \in WPP_{\omega ap}(\mathbb{R}, X)$. Now, if $u, v \in WPP_{\omega ap}(\mathbb{R}, X)$ we have

$$\begin{aligned}
\|Q(u)(t) - Q(v)(t)\|_\infty &= \sup_{t\in\mathbb{R}} \left\| \int_{-\infty}^{t} S(t-s)[f(s,u(s)) - f(s,v(s))]\,ds \right\| \\
&\le L_f \sup_{t\in\mathbb{R}} \int_0^\infty \|S(s)\| \|u(t-s) - v(t-s)\|\,ds \\
&\le L_f \|u - v\|_\infty \int_0^\infty \varphi_\alpha(s)\,ds.
\end{aligned}$$

This proves that Q is a contraction, so by the Banach Fixed Point Theorem we conclude that Q has unique fixed point. It follows that $Q(u) = u \in WPP_{\omega ap}(\mathbb{R}, X)$ is unique. Hence u is the unique mild solution of (3.1). □

We finish this paper with a simple application that no means generality but illustrates how our hypotheses apply.

Example 3.5. We put $A = -\varrho$ in $X = \mathbb{R}$, $a(t) = \dfrac{\varrho}{4}\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$, $\varrho > 0$, $0 < \alpha < 1$, and $f(t,u) = \cos(u)g(t) + \phi(t)\cos u$, where $g(t) = \displaystyle\sum_{k=1}^{\infty} \frac{\sin((2k+1)t)}{k^2}$, and

$$\phi(t) = \begin{cases} \cos(t), & t \in [2^n - \frac{\pi}{2}, 2^n + \frac{\pi}{2}],\ n \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

The functions $g(t,u) := \cos(u)g(t)$, $h(t,u) := \phi(t)\cos u$ verify the hypothesis in Theorem 3.4 (with $\rho(t) = 1$). Thus, we have equation (1.2) in the form

$$D^\alpha u(t) = -\varrho u(t) - \frac{\varrho^2}{4} \int_{-\infty}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds + f(t,u(t)), \quad t \in \mathbb{R}. \tag{3.2}$$

From [28, Example 4.17], it follows that A generates an α-resolvent family $\{S_\alpha(t)\}_{t\ge 0}$ such that

$$\hat{S}_\alpha(\lambda) = \frac{\lambda^\alpha}{(\lambda^\alpha + 2/\varrho)^2} = \frac{\lambda^{\alpha-\alpha/2}}{(\lambda^\alpha + 2/\varrho)} \cdot \frac{\lambda^{\alpha-\alpha/2}}{(\lambda^\alpha + 2/\varrho)}.$$

Thus, we obtain explicitly

$$S_\alpha(t) = (r * r)(t)$$

with $r(t) = t^{\frac{\alpha}{2}-1}E_{\alpha,\frac{\alpha}{2}}(-\frac{\varrho}{2}t^\alpha)$, and where $E_{\alpha,\frac{\alpha}{2}}(\cdot)$ is the Mittag-Leffler function.

Note that $f \in WPP_{\omega ap}S^p(\mathbb{R}, X)$ with weight $\rho(t) = 1$ for $t \in \mathbb{R}$. Moreover,

$$||f(t,u) - f(t,v)|| \le \left(\frac{\pi^2}{6} + 1\right) ||u - v||.$$

Then, by Theorem 3.4, we can conclude that there exists a unique mild solution $u(\cdot) \in WPP_{\omega ap}(\mathbb{R}, X)$ of Eq.(3.2) provided $\|S_\alpha\| < \frac{6}{\pi^2+6}$. We remark that given $0 < \alpha < 1$, we can choose the number $\varrho > 0$ such that $\|S_\alpha\| < \frac{6}{\pi^2+6}$ as in the proof of [28, Lemma 3.9].

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