DISCRETE-TIME THEOREMS FOR GLOBAL AND POINTWISE DICHOTOMIES OF COCYCLES OVER SEMIFLOWS

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Abstract. In this paper, we consider linear skew-product semiflows on bundles of Banach fibers over a locally compact metric space. Our aim is to give discrete-time theorems for the existence of global and pointwise continuous-time dichotomies with no invariant unstable manifolds. We involve here a concept of exponential dichotomy for skew-product semiflows weaker than the concept used by Sacker-Sell [29] and Magalhaes [16]; our definition (of no past exponential dichotomy) follows roughly the definition given by Chow and Leiva [4] in the sense that we allow the unstable subspace to have infinite dimension. The main improvement is that we go even more general and we do not assume *a priori* that the cocycle is invertible on the unstable space (actually we do not even assume that the unstable subspace is invariant under the cocycle). Roughly speaking, we prove that if the solution of the corresponding inhomogeneous variational difference equation belongs to any sequence space (on which the right shift is an isometry) for every inhomogenity from the same class of sequence spaces, then the continuous-time solutions of the variational homogeneous differential equation will exhibit a (no past) exponential dichotomic behavior. This approach has many advantages among which we emphasize on the facts that the above condition is very general (since the class of sequence spaces that we use includes almost all the known sequence spaces, as the classical p-summable spaces, sequence Orlicz spaces, etc.). Since we use a discrete-time technique we are not forced to require any continuity or measurability hypotheses on the trajectories of the exponentially bounded cocycle. Also, it is worth to mention that from discrete-time conditions we get informations about the continuous-time behavior of the solutions of differential variational equations.

1. Introduction

The analysis of linear skew-product (semi)flows with finite-dimensional fibers is already a classical topic in the study of the asymptotic behavior of the solutions of differential equations. The extension of the above analysis to the infinite-dimensional framework is due mainly to J. Hale (we refer the reader to [9, p. 60]).

In this paper, we consider linear skew-product semiflows on bundles of Banach fibers over a locally compact metric space. It is worth to mention that there is an increasing "applications-wise" interest for this topic, since it was pointed out the presence of continuous cocycle on attractors of dissipative PDEs, in particular the Navier–Stokes equation. New implementations of the theory in the ideal fluid dynamics raised new questions that, as far as we know, have not been explicitly answered before. One of them includes precise formulation the relationship between exponential dichotomy and existence of Mañé sequences or points.

The aim of the present paper is to give discrete-time theorems for the existence of global and pointwise continuous-time dichotomies of an exponentially bounded cocycle. Our philosophy follows roughly the classical route of pointwise construction of stable and unstable foliations (as in [3, 4, 29, 30]). It is worth to mention that the existence of exponential dichotomies is by far one

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of the most important concepts arising in the theory of dynamical systems. This topic has for instance a privileged position in the Hadamard–Perron theory of invariant manifolds for dynamical systems, and in many aspects of the theory of stability. It is also known the importance of the concept of exponential dichotomy in the bifurcation theory. However, in this setting the exponential dichotomy is better shaped by its younger sibling, the exponential trichotomy. In particular, topics such as the reduction principle and center manifold theorem, the robustness of periodic solutions and invariant manifolds as seen in the Poincaré–Melnikov scenario rely heavily on the theory of exponential trichotomies

The history of the study of exponential dichotomies of linear differential equations goes back to 30's, when O. Perron [21] shows interest in the problem of conditional stability of a system $\dot{x}(t) = A(t)x(t)$ and its connection with the existence of bounded solutions of the equation $\dot{x}(t) = A(t)x(t) + f(x,t)$, where the state space is a Banach space X and $t \mapsto A(t) : \mathbb{R} \to B(X)$ is bounded, continuous in the strong operator topology. We have to emphasize here on the milestone contributions to these problems by Massera–Schäffer [18], Daleckij–Krein [8], Levinson [13] , Coppel [7], Sacker–Sell [29] and Palmer [20]. For recent results in this direction we refer the reader to [19, 24, 25, 26].

For the case of discrete-time systems, analogous results were firstly obtained by T. Li in [12]. In his paper from 1934, it can be seen almost the same approach as in Perron's work, but in discrete-time arguments. The Perron–Li approach has been extended for discrete-time systems in the infinite-dimensional case by C.V. Coffman and J.J. Schäffer [5] and D. Henry [10]. For recent results we refer the reader to the papers of A. Ben-Artzi and I. Gohberg [1], M. Pinto [22], J.P. La Salle [31]. Also, applications of this "discrete-time theory" to the stability theory of linear continuous-time systems in infinite-dimensional spaces have been obtained firstly by K.M. Przyluski and S. Rolewicz in [27].

In a paper from 1996, S.N. Chow and H. Leiva, propose two new definitions of exponential dichotomy for skew-product semiflows (see [3]). The need for a better insight to the problem of exponential dichotomy arose from the fact that for a time dependent linear differential equation with unbounded operator $A(t)$, the solutions, generally speaking, either cannot be extended in the direction of the negative times, or can be extended, but not uniquely. For example, for parabolic partial differential equations many authors have studied these problems, including D. Henry [10], X.B. Lin [15] and J. Hale [9]. For the case of functional differential equations we can see the work done by X.B. Lin [14]. All the problems above can be analyzed in the larger setting of a linear skew-product semiflow (LSPS). It is worth to note that in [30], R.J. Sacker and G.R. Sell deal already with a concept of exponential dichotomy for skew-product semiflow but restrained by the additional requirement that the unstable subspace has finite dimension, and they give a sufficient condition for the existence of exponential dichotomy for skew-product semiflow. This concept is also used by L.T. Magalhaes in [16]. We involve here a concept of exponential dichotomy for skew-product semiflows weaker than the concept used by Sacker-Sell and Magalhaes; our definition (of no past exponential dichotomy) follows roughly the definition given by Chow and Leiva (see [4]) in the sense that we allow the unstable subspace to have infinite dimension. The main improvement is that we go even more general and we do not assume *a priori* that the cocycle is invertible on the unstable space (actually we do not even assume that the unstable subspace is invariant under the cocycle).

We claim that we arrive here to a final stage for the Perron–Li line of results from the following perspective. First, as we mentioned above, we use a more general concept of exponential dichotomy and secondly, we prove that the Perron–Li conditions are valid for a whole class of sequences spaces (class that includes all the sequence spaces considered previously). Roughly speaking, we prove that if the solution of the corresponding inhomogeneous variational difference equation belongs to any sequence space (on which the right shift is an isometry) for every inhomogenity from the same class of sequence spaces, then the continuous-time solutions of the variational homogeneous differential equation will exhibit a (no past) exponential dichotomic behavior. This approach has many advantages among which we emphasize on the facts that the above condition is very general (since the class of sequence spaces that we use includes almost all the known sequence spaces, as the classical ℓ^p spaces, sequence Orlicz spaces, etc.). Since we use a discrete-time technique we are not forced to require any continuity or measurability hypotheses on the trajectories of the exponentially bounded cocycle. Also, it is worth to mention that from discrete-time conditions we get informations about the continuous-time behavior of the solutions.

2. Sequence Schäffer spaces

Let $\mathbb{N}$ denote the set of all nonnegative integers and put $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. We denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}_+$ the set of all nonnegative real numbers and by $[t]$ the greatest integer less than or equal with $t \in \mathbb{R}$. The linear space of all real-valued sequences $s : \mathbb{N} \to \mathbb{R}$ will be denoted by $\mathcal{S}$. Throughout this paper $(X, \|\cdot\|)$ is a real or complex Banach space and we consider $\mathcal{S}(X)$ the linear space of all sequences $f : \mathbb{N} \to X$. For a X-valued sequence $f : \mathbb{N} \to X$, we shall associate the sequence $\|f\| : \mathbb{N} \to \mathbb{R}$ defining $\|f\|(n) = \|f(n)\|$ for all $n \in \mathbb{N}$. We also consider two linear operators $R, L : \mathcal{S}(X) \to \mathcal{S}(X)$ defined by

$$Rf(n) = \begin{cases} f(n-1) & , \ n \in \mathbb{N}^* \\ 0 & , \ n = 0 \end{cases} \quad , \quad Lf(n) = f(n+1)$$

known as the *right shift operator*, respectively the *left shift operator*. A simple verification gives us $LRf = f$ and $RLf(n) = f(n)$ for $n \in \mathbb{N}^*$, $RLf(0) = 0$, for all $f \in \mathcal{S}(X)$. If $A \subset \mathbb{N}$, the characteristic function of A will be denoted by χ_A and for the simplicity of notation put $\delta_k = \chi_{\{k\}}$ for each $k \in \mathbb{N}$.

Definition 2.1. *A Banach space $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ of sequences is said to be a* sequence Schäffer space *if the following conditions hold*

(s_1) $\delta_0 \in \mathcal{E}$,
(s_2) *if $f \in \mathcal{E}$, then $Rf \in \mathcal{E}$ and $\|Rf\|_{\mathcal{E}} = \|f\|_{\mathcal{E}}$,*
(s_3) *if $f \in \mathcal{S}$ and $g \in \mathcal{E}$ such that $|f| \le |g|$, then $f \in \mathcal{E}$ and $\|f\|_{\mathcal{E}} \le \|g\|_{\mathcal{E}}$.*

We call *ad hoc* the above class of sequence spaces as “sequence Schäffer spaces” as a recognition of the contribution of J.J. Schäffer that uses for the first time this class of spaces in the study of linear difference equations (see [5]).

Remark 2.2. *By (s_1) and (s_2) we have that any sequence with finite support is contained in any sequence Schäffer space, hence $\chi_{\{0,1,\ldots,n\}} \in \mathcal{E}$ for any sequence Schäffer space $\mathcal{E}$ and $n \in \mathbb{N}$. The third property is called the ideal property and will play a central role in our investigations.*

Example 2.3. *Common instances of sequence Schäffer spaces are the spaces of p-summable sequences, namely for $p \in [1, \infty)$,*

$$\ell^p = \{\, f : \mathbb{N} \to \mathbb{R} : \sum_{k=0}^{\infty} |f(k)|^p < \infty \,\} \ , \ \textit{with the norm } \|f\|_p = \left(\sum_{k=0}^{\infty} |f(k)|^p \right)^{1/p}$$

and

$$\ell^\infty = \{\, f : \mathbb{N} \to \mathbb{R} : \sup_{n \in \mathbb{N}} |f(n)| < \infty \,\} \ , \ \textit{with the norm } \|f\|_\infty = \sup_{n \in \mathbb{N}} |f(n)| \ .$$

The subspace of ℓ^∞, $\ell^\infty_0 = \{\, f \in \ell^\infty : \lim_{n\to\infty} f(n) = 0 \,\}$ (often denoted c_0) with the induced norm is another example of sequence Schäffer space.

It is easy to check that $(c, \|\cdot\|_\infty)$ (the space of all convergent sequences) is not a sequence Schäffer space.

The spaces ℓ^1, ℓ^∞, ℓ_0^∞ occupy particularly important positions in the class of sequence Schäffer spaces. For $\mathcal{E}$ a sequence Schäffer space, we shall define the sequences $\alpha_\mathcal{E}, \beta_\mathcal{E} \in \mathcal{S}$ by

$$\alpha_\mathcal{E}(n) = \inf\{\, L > 0 \,:\, \sum_{k=0}^{n} |f(k)| \leq L\|f\|_\mathcal{E}\,,\ \text{for all } f \in \mathcal{E}\,\}\ ,$$
$$\beta_\mathcal{E}(n) = \|\chi_{\{0,1,\dots,n\}}\|_\mathcal{E}\ ,$$

which are both nondecreasing and $\beta_\mathcal{E}(n) > 0$, for all $n \in \mathbb{N}$.

Example 2.4. *Other remarkable examples of sequence Schäffer spaces are the* sequence Orlicz spaces. *Let $\varphi : \mathbb{R} \to [0,\infty]$ be a left continuous, nondecreasing function and not identically* 0 *or* ∞ *on* $(0,\infty)$. *The Young function attached to φ is defined by* $\Phi(t) = \int_0^t \varphi(s)ds\ ,\ t \geq 0$. *Consider:*

$$\ell^\Phi = \{\, f \in \mathcal{S} \,:\ \textit{exists } c > 0 \textit{ such that} \sum_{k=0}^{\infty} \Phi(c^{-1}|f(k)|) < \infty\,\} \textit{ with the norm}$$

$$\|f\|_\Phi = \inf\{\, c > 0 \ :\ \sum_{k=0}^{\infty} \Phi(\,c^{-1}|f(k)|\,) \leq 1\,\} \quad \textit{(the Luxemburg norm)}\ .$$

For the Banach space $(\ell^\Phi, \|\cdot\|_\Phi)$ the conditions (s_1), (s_2) and (s_3) are verified, hence $(\ell^\Phi, \|\cdot\|_\Phi)$ is a sequence Schäffer space.

For $1 \leq p < \infty$, taking $\varphi(t) = pt^{p-1}$ we have that $(\ell^\Phi, \|\cdot\|_\Phi) \equiv (\ell^p, \|\cdot\|_p)$. Even ℓ^∞ is a sequence Orlicz space, obtained from $\varphi(t) = 0$ for $t \in [0,1]$ and $\varphi(t) = \infty$ for $t > 1$.

Remark 2.5. *By simple computations we obtain that*

$$\alpha_{\ell^p}(n) = (n+1)^{1-\frac{1}{p}} \quad , \quad \beta_{\ell^p}(n) = (n+1)^{\frac{1}{p}}$$

for $1 \leq p \leq \infty$ (with the convetion $\frac{1}{\infty} = 0$) and for the sequence Orlicz spaces,

$$\alpha_{\ell^\Phi}(n) = (n+1)\Phi^{-1}\left(\frac{1}{n+1}\right) \quad , \quad \beta_{\ell^\Phi}(n) = \frac{1}{\Phi^{-1}(\frac{1}{n+1})}\ .$$

Remark 2.6. *Let $p \in [1,\infty)$. If $(\ell^\Phi, \|\cdot\|_\Phi) = (\ell^p, \|\cdot\|_p)$, then $\lim_{t\to 0} \frac{\Phi(t)}{t^p} = 1$.*

Indeed, if $(\ell^\Phi, \|\cdot\|_\Phi) = (\ell^p, \|\cdot\|_p)$, then $\|\chi_{\{0,1,\dots,n\}}\|_\Phi = \|\chi_{\{0,1,\dots,n\}}\|_p$, for all $n \in \mathbb{N}$, which is equivalent with

$$\Phi^{-1}\left(\frac{1}{n+1}\right) = \left(\frac{1}{n+1}\right)^{\frac{1}{p}} \quad , \quad \textit{for all } n \in \mathbb{N}\ .$$

Let $x \in (0,1]$ and $m = \left[\frac{1}{x}\right] \in \mathbb{N}^*$. Using the fact that Φ^{-1} is nondecreasing we have that

$$\left(\frac{1}{m+1}\right)^{\frac{1}{p}} = \Phi^{-1}\left(\frac{1}{m+1}\right) \leq \Phi^{-1}(x) \leq \Phi^{-1}\left(\frac{1}{m}\right) = \left(\frac{1}{m}\right)^{\frac{1}{p}}$$

which implies that

$$\left[\frac{1}{([1/x]+1)x}\right]^{\frac{1}{p}} \leq \frac{\Phi^{-1}(x)}{x^{\frac{1}{p}}} \leq \left[\frac{1}{x[1/x]}\right]^{\frac{1}{p}}\ ,$$

for all $x \in (0,1]$. Hence $\lim_{x\to 0} \Phi^{-1}(x)x^{-\frac{1}{p}} = 1$ and

$$\lim_{u\to 0} \frac{\Phi(u)}{u^p} = \lim_{u\to 0} \frac{1}{\left[\frac{\Phi^{-1}(\Phi(u))}{(\Phi(u))^{1/p}}\right]^p} = 1\ .$$

Example 2.7. *Consider* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *by* $\varphi(t) = \sum_{m=1}^{\infty} \frac{\sqrt[m]{t}}{m^2}$ *. Then,*

$$\Phi(t) = \int_0^t \varphi(s)ds = \sum_{m=1}^{\infty} \frac{t^{1+\frac{1}{m}}}{m(m+1)} .$$

We claim that $\ell^\Phi \neq \ell^p$*, no matter how we choose* $p \in [1, \infty)$.

Indeed, $\lim_{t\to 0} \frac{\Phi(t)}{t} = 0$ *and* $\lim_{t\to 0} \frac{\Phi(t)}{t^p} = \infty$*, for all* $p \in (1, \infty)$ *and using the above remark, our claim follows easilly. Also,* $\ell^\Phi \neq \ell^\infty$ *since* $\chi_{\mathbb{N}} \in \ell^\infty \setminus \ell^\Phi$.

For two Banach spaces $(B_1, \|\cdot\|_1)$ and $(B_2, \|\cdot\|_2)$, we say that B_1 is continuously embedded in B_2 (and we use the notation $B_1 \hookrightarrow B_2$) if $B_1 \subset B_2$ and there exists $c > 0$ such that $\|f\|_2 \le c\|f\|_1$ for all $f \in B_1$. For the following three propositions, proofs can be retrieved from [5, Section 3].

Proposition 2.8. *If* $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ *is a sequence Schäffer space, then* $\ell^1 \hookrightarrow \mathcal{E} \hookrightarrow \ell^\infty$ *with* $\beta_{\mathcal{E}}(0)\|f\|_\infty \le \|f\|_{\mathcal{E}}$ *for all* $f \in \mathcal{E}$ *and* $\|f\|_{\mathcal{E}} \le \beta_{\mathcal{E}}(0)\|f\|_1$ *for all* $f \in \ell^1$.

Proposition 2.9. *If* $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$, $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ *are sequence Schäffer spaces, then* $\mathcal{E} \hookrightarrow \mathcal{F}$ *if and only if* $\mathcal{E} \subset \mathcal{F}$.

Proposition 2.10. *Let* $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ *be a sequence Schäffer space. The following characterizations hold:*

(i) $\alpha_{\mathcal{E}}$ *is bounded if and only if* $\mathcal{E} = \ell^1$ *and* $\|\cdot\|_1 \sim \|\cdot\|_{\mathcal{E}}$*, that is, the norms* $\|\cdot\|_1$ *and* $\|\cdot\|_{\mathcal{E}}$*, are equivalent;*
(ii) $\beta_{\mathcal{E}}$ *is bounded if and only if* $\ell_0^\infty \subset \mathcal{E}$.

For $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ a sequence Schäffer space and X a Banach space, we consider $\mathcal{E}(X) = \{ f \in \mathcal{S}(X) : \|f\| \in \mathcal{E} \}$ and $\|f\|_{\mathcal{E}(X)} = \| \|f\| \|_E$. To prove that $(\mathcal{E}(X), \|\cdot\|_{\mathcal{E}(X)})$ is a Banach space see, for example, [25, Remark 2.1] or [5, Lemma 3.8]. The following properties of this space are simple verifications.

Proposition 2.11. *The space* $(\mathcal{E}(X), \|\cdot\|_{\mathcal{E}(X)})$ *is a Banach space with the following properties:*

(i) *if* $f \in \mathcal{S}(X)$ *has finite support, then* $f \in \mathcal{E}(X)$*;*
(ii) *if* $f \in \mathcal{E}(X)$*, then* $Rf \in \mathcal{E}(X)$ *and* $\|Rf\|_{\mathcal{E}(X)} = \|f\|_{\mathcal{E}(X)}$*;*
(iii) *if* $f \in \mathcal{S}(X)$ *and* $g \in \mathcal{E}(X)$ *such that* $\|f\| \le \|g\|$*, then* $f \in \mathcal{E}(X)$ *and* $\|f\|_{\mathcal{E}(X)} \le \|g\|_{\mathcal{E}(X)}$.

To prevent any further confusion, let us fix the notation $\mathcal{B}(X)$ for the Banach algebra of bounded linear operators acting on X. Norms on both Banach spaces X and $\mathcal{B}(X)$ will be denoted by $\|\cdot\|$.

3. Cocycles over semiflows. Linear skew-product semiflows

Let X be a Banach space (called the state space) and Θ a metric space (called the base space).

Definition 3.12. *A mapping* $\sigma : \Theta \times \mathbb{R}_+ \to \Theta$ *is called a* semiflow *if*

(i) $\sigma(\theta, 0) = \theta$ *for all* $\theta \in \Theta$ *and*
(ii) $\sigma(\theta, t + s) = \sigma(\sigma(\theta, t), s)$ *for all* $\theta \in \Theta$ *and* $t, s \in \mathbb{R}_+$.

If the semiflow σ *is continuous (with respect to the product topology), then it is said to be a* continuous semiflow*. In other terms, a (continuous) semiflow is a (continuous) group action of* $(\mathbb{R}_+, +)$ *on* Θ*. By* $\mathcal{O}_\sigma(\theta) = \{ \sigma(\theta, t) : t \in \mathbb{R}_+ \}$ *we denote* the orbit of θ by the semiflow σ.

The definition of a flow (respectively, of a continuous flow) is obtained by replacing $\mathbb{R}_+$ *with* $\mathbb{R}$.

Example 3.13. *Let* $\Theta = \mathbb{R}_+$ *and* $\sigma(\theta, t) = \theta + t$*, for all* $\theta, t \in \mathbb{R}_+$*. Then,* $\{\sigma(\theta, t)\}_{\theta, t\in\mathbb{R}_+}$ *is a continuous semiflow. If* $\Theta = \mathbb{R}$ *and* σ *is defined on* $\mathbb{R} \times \mathbb{R}$ *by the same expression as above, then* σ *is an example of continuous flow.*

If one considers $\Theta = \mathbb{R}$ *and* $\sigma(\theta, t) = \theta - t$*, for all* $\theta \in \mathbb{R}$ *and* $t \in \mathbb{R}_+$*, then* $\{\sigma(\theta, t)\}_{\theta, t\in\mathbb{R}_+}$ *is also a continuous semiflow.*

Definition 3.14. *Let $\sigma : \Theta \times \mathbb{R}_+ \to \Theta$ be a semiflow. A* cocycle *over σ is an operator-valued function $\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{B}(X)$ that satisfies the following properties:*

(i) $\Phi(\theta, 0) = I$ *for all $\theta \in \Theta$ (where I denotes the identity operator);*
(ii) $\Phi(\theta, t+s) = \Phi(\sigma(\theta, t), s)\Phi(\theta, t)$ *for all $\theta \in \Theta$ and $t, s \in \mathbb{R}_+$.*

If, in addition, there exists $M, \omega > 0$ such that

(iii) $\|\Phi(\theta, t)\| \leq Me^{\omega t}$ *for all $t \in \mathbb{R}_+$,*

then Φ is said to be exponentially bounded*. If σ is a continuous semiflow and for every $x \in X$, the function $(\theta, t) \mapsto \Phi(\theta, t)x : \Theta \times \mathbb{R}_+ \to X$ is continuous (the strong continuity property), then Φ is a* strongly continuous cocycle*.*

Note that the operators in a (strongly continuous) cocycle are not assumed to be invertible. For this reason, the cocycle is parametrized by $t \in \mathbb{R}_+$, but not by $t \in \mathbb{R}$. If X is finite-dimensional, then a strongly continuous cocycle can always be extended to $t \in \mathbb{R}$ (see [2, Remark 6.2]).

Remark 3.15. *The notion of a cocycle generalizes the classic notion of a (two-parameter) evolution family, that is a family $\{U(t, s)\}_{t \geq s \geq 0} \subset \mathcal{B}(X)$, for which the following conditions hold*

(i) $U(t, t) = I$*, for all $t \geq 0$;*
(ii) $U(t, s)U(s, r) = U(t, r)$*, for all $t \geq s \geq r \geq 0$;*
(iii) $U(\cdot, s)x$ *is continuous on $[s, \infty)$, for all $s \geq 0$ and $x \in X$;*
(iv) $U(t, \cdot)x$ *is continuous on $[0, t]$, for all $t > 0$ and $x \in X$;*
(v) *there exist $M, \omega > 0$ such that $\|U(t, s)\| \leq Me^{\omega(t-s)}$, for all $t \geq s \geq 0$.*

Let $\Theta = \mathbb{R}_+$ and consider the first semiflow from Example 3.13. We define $\Phi_U(\theta, t) = U(\theta + t, \theta)$, for all $\theta, t \in \mathbb{R}_+$. Then, $\{\Phi_U(\theta, t)\}_{\theta, t \in \mathbb{R}_+}$ is a strongly continuous, exponentially bounded cocycle (over the semiflow σ considered in Example 3.13).

Cocycles commonly arise as the solutions of variational equations on Banach spaces. Let σ be a continuous semiflow on Θ and $\{A(\theta)\}_{\theta \in \Theta}$ be a family of (possibly unbounded) densely defined, closed, linear operators acting on the Banach space X. A strongly continuous cocycle Φ (over the semiflow σ) is said to solve the variational equation

$$\dot{u}(t) = A(\sigma(\theta, t))u(t) \quad , \quad (\theta, t) \in \Theta \times \mathbb{R}_+ \ , \tag{3.1}$$

if for any $\theta \in \Theta$, there exists a dense subset $Z_\theta \subset D(A(\theta))$ such that for every $u_\theta \in Z_\theta$, the function $\Phi(\theta, \cdot)u_\theta$ is differentiable on $\mathbb{R}_+$, $u(t) := \Phi(\theta, t)u_\theta \in D(A(\sigma(\theta, t)))$ for all $t \in \mathbb{R}_+$ and it verifies the above differential equation.

In general, a linear skew-product (semi)flow is a dynamical system on a vector bundle such that each transformation is linear when restricted to a fiber of the bundle. To avoid technical complications for the general case, we will define the notion of a linear skew-product (semi)flow in the setting of a trivial vector bundle. It is worth to mention that the theory is valid for general vector bundles, but the topology of nontrivial bundles plays no role in the analysis [2, Chapter 6]. For details of Banach bundles we refer the reader to [32, Chapter 4].

Definition 3.16. *The* linear skew-product semiflow (LSPS) *associated with the cocycle Φ over the semiflow σ is the dynamical system $\pi = (\Phi, \sigma)$ on the Banach bundle $E = X \times \Theta$ defined by*

$$\pi : X \times \Theta \times \mathbb{R}_+ \to X \times \Theta \quad , \quad \pi(x, \theta, t) = (\Phi(\theta, t)x, \sigma(\theta, t)) \ .$$

4. Global and pointwise (no past) exponential dichotomy. Admissibility

In this section we shall give definitions for several concepts of exponential dichotomy of a cocycle $\Phi = \{\Phi(\theta, t)\}_{(\theta, t) \in \Theta \times \mathbb{R}_+}$ over a semiflow $\sigma = \{\sigma(\theta, t)\}_{(\theta, t) \in \Theta \times \mathbb{R}_+}$. All the definitions can be stated equivalently for the associated linear skew-product semiflow $\pi = (\Phi, \sigma)$. We begin with the definition of "global" exponential dichotomy, that is dichotomy on the entire base space. Such a behavior will be referred throughout this paper as exponential dichotomy on Θ.

Definition 4.17. *The cocyle Φ over the semiflow σ has an* exponential dichotomy on Θ *if there exist a family of projectors $\{P(\theta)\}_{\theta\in\Theta}$ and the constants $N,\nu>0$ such that the following conditions hold:*

(i) $\Phi(\theta,t)P(\theta)=P(\sigma(\theta,t))\Phi(\theta,t)$, *for all* $(\theta,t)\in\Theta\times\mathbb{R}_+$;
(ii) $\Phi(\theta,t)_{|}:\operatorname{Ker}P(\theta)\to\operatorname{Ker}P(\sigma(\theta,t))$ *is an isomorphism, for all* $(\theta,t)\in\Theta\times\mathbb{R}_+$;
(iii) $\|\Phi(\theta,t)x\|\leq Ne^{-\nu t}\|x\|$, *for all* $(\theta,t)\in\Theta\times\mathbb{R}_+$, $x\in\operatorname{Im}P(\theta)$;
(iv) $\|\Phi(\theta,t)_{|}^{-1}x\|\leq Ne^{-\nu t}\|x\|$, *for all* $(\theta,t)\in\Theta\times\mathbb{R}_+$, $x\in\operatorname{Ker}P(\sigma(\theta,t))$.

Exponential dichotomy for a cocycle that commes from a variational equation as (3.1) means that X can be decomposed, at every point $\theta\in\Theta$, as a direct sum between two subspaces such that solutions starting in the first subspace (respectively, in the second one) decay exponentially in forward time (respectively, in backward time). Assuming the existence of an exponential dichotomy we practically force the solutions that starts in the second subspace to exist for negative time. However, there are situations which require to drop off this requirement. We will extend the notion of exponential dichotomy by replacing the exponential decay in negative time for the solutions starting in the second subspace with an exponential blow-up in positive time.

Definition 4.18. *The cocyle Φ over the semiflow σ has a* no past exponential dichotomy on Θ *if there exist a family of projectors $\{P(\theta)\}_{\theta\in\Theta}$ and the constants $N,\nu>0$ such that the following conditions hold:*

(i) $\|\Phi(\theta,t)x\|\leq Ne^{-\nu t}\|x\|$, *for all* $(\theta,t)\in\Theta\times\mathbb{R}_+$, $x\in\operatorname{Im}P(\theta)$;
(ii) $\|\Phi(\theta,t)x\|\geq \dfrac{1}{N}e^{\nu t}\|x\|$, *for all* $(\theta,t)\in\Theta\times\mathbb{R}_+$, $x\in\operatorname{Ker}P(\theta)$.

Remark 4.19. *It is obvious that the existence of an exponential dichotomy on Θ implies the existence of a no past exponential dichotomy on Θ. Assuming that* $\dim\operatorname{Ker}P(\theta)<\infty$ *for all $\theta\in\Theta$ and condition (i) in Definition 4.17, we get an equivalence between the above two definitions. However, for infinite-dimensional subspaces* $\operatorname{Ker}P(\theta)$*, the inequality (ii) of Definition 4.18 does not imply the inequality (iv) of Definition 4.17. The inequalities from Definition 4.18 together with condition (i) from Definition 4.17 defines the notion of hyperbolic cocycle (see* [2, Definition 6.15]*).*

The next two notions are pointwise (or, nonuniform with respect to the base space) versions of the previous ones. Definition 4.20 and Definition 4.21 describe the dichotomic behavior of the cocycle on a single orbit.

Definition 4.20. *The cocyle Φ over the semiflow σ has an* exponential dichotomy at the point $\theta_0\in\Theta$ *if there exist a family of projectors $\{P_{\theta_0}(t)\}_{t\geq0}$ and the constants $N_{\theta_0},\nu_{\theta_0}>0$ such that the following conditions hold:*

(i) $\Phi(\sigma(\theta_0,t_0),t)P_\theta(t_0)=P_{\theta_0}(t_0+t)\Phi(\sigma(\theta_0,t_0),t)$, *for all* $t_0,t\geq0$;
(ii) $\Phi(\sigma(\theta_0,t_0),t)_{|}:\operatorname{Ker}P_{\theta_0}(t_0)\to\operatorname{Ker}P_{\theta_0}(t_0+t)$ *is an isomorphism, for all* $t_0,t\geq0$;
(iii) $\|\Phi(\sigma(\theta_0,t_0),t)x\|\leq N_{\theta_0}\,e^{-\nu_{\theta_0}\,t}\,\|x\|$, *for all* $x\in\operatorname{Im}P_{\theta_0}(t_0)$ *and* $t_0,t\geq0$;
(iv) $\|\Phi(\sigma(\theta_0,t_0),t)_{|}^{-1}x\|\leq N_{\theta_0}\,e^{-\nu_{\theta_0}\,t}\,\|x\|$, *for all* $x\in\operatorname{Ker}P_{\theta_0}(t_0+t)$ *and* $t_0,t\geq0$.

Definition 4.21. *The cocyle Φ over the semiflow σ has a* no past exponential dichotomy at the point $\theta_0\in\Theta$ *if there exist a family of projectors $\{P_{\theta_0}(t)\}_{t\geq0}$ and the constants $N_{\theta_0},\nu_{\theta_0}>0$ such that the following conditions hold:*

(i) $\|\Phi(\sigma(\theta_0,t_0),t)x\|\leq N_{\theta_0}\,e^{-\nu_{\theta_0}\,t}\,\|x\|$, *for all* $x\in\operatorname{Im}P_{\theta_0}(t_0)$ *and* $t_0,t\in\mathbb{R}_+$;
(ii) $\|\Phi(\sigma(\theta_0,t_0),t)x\|\geq \dfrac{1}{N_{\theta_0}}\,e^{\nu_{\theta_0}\,t}\,\|x\|$, *for all* $x\in\operatorname{Ker}P_{\theta_0}(t_0)$ *and* $t_0,t\in\mathbb{R}_+$.

Remark 4.22. *Note that if the family $\{P_{\theta_0}(t)\}_{t\geq0}$ is of one-to-one projectors, then* $\operatorname{Im}P_{\theta_0}(t)=X$ *for all $t\geq0$. Thus the concept of (no past) exponential dichotomy in θ_0 overlaps the concept of* *exponential stability in θ_0. We say that the cocycle Φ (over the semiflow σ) is exponentially stable in θ_0 (respectively, on Θ) if there exist $N_{\theta_0}, \nu_{\theta_0} > 0$ (respectively, $N, \nu > 0$) such that*

$$\|\Phi(\theta, t)\| \leq N_{\theta_0} e^{-\nu_{\theta_0} t} \ , \ (\textit{respectively, } \|\Phi(\theta, t)\| \leq N e^{-\nu t})$$

for all $t \geq 0$ and $\theta \in \mathcal{O}_\sigma(\theta_0)$ (respectively, $\theta \in \Theta$).

Remark 4.23. *If the cocycle Φ has a (no past) exponential dichotomy at the point θ_0, for any $\theta_0 \in \Theta$, $\sup_{\theta_0 \in \Theta} N_{\theta_0} < \infty$ and $\inf_{\theta_0 \in \Theta} N_{\theta_0} > 0$, then Φ has a (no past) exponential dichotomy on Θ, but in general this is not true as the following example points out.*

Remark 4.24. *Let Φ be a cocycle over a semiflow σ, $\theta \in \Theta$. Consider $X_{1,\theta} = \{x \in X : \lim\limits_{t \to \infty} \Phi(\theta, t)x = 0\}$. If there exist $N_\theta, \nu_\theta > 0$ and a projector P_θ such that*

$$\|\Phi(\theta, t)x\| \leq N_\theta e^{-\nu_\theta t} \|x\| \ , \ \textit{for all } t \in \mathbb{R}_+ \textit{ and } x \in \operatorname{Im} P_\theta$$

$$\|\Phi(\theta, t)x\| \geq \frac{1}{N_\theta} e^{\nu_\theta t} \|x\| \ , \ \textit{for all } t \in \mathbb{R}_+ \textit{ and } x \in \operatorname{Ker} P_\theta \ ,$$

then $\operatorname{Im} P_\theta = X_{1,\theta}$. Therefore, $X_{1,\theta}$ is a closed linear subspace of X.

Indeed, if $x \in \operatorname{Im} P_\theta$, since $\|\Phi(\theta, t)x\| \leq N_\theta \, e^{-\nu_\theta \, t} \, \|x\|$ for all $t \geq 0$, it follows that $\lim_{t \to \infty} \Phi(\theta, t)x = 0$. Conversely, if $x \in X_{1,\theta}$ let $u \in \operatorname{Im} P_\theta$, $v \in \operatorname{Ker} P_\theta$ such that $x = u + v$. Then,

$$\|\Phi(\theta, t)x\| \geq \|\Phi(\theta, t)v\| - \|\Phi(\theta, t)u\| \geq \frac{1}{N_\theta} \, e^{\nu_\theta \, t} \, \|v\| - N_\theta \, e^{-\nu_\theta \, t} \, \|u\| \ ,$$

for all $t \geq 0$. If we suppose that $v \neq 0$, we are led to $\lim_{t \to \infty} \|\Phi(\theta, t)x\| = \infty$ which contradicts $x \in X_{1,\theta}$. Therefore $v = 0$ and $x = u \in \operatorname{Im} P_\theta$.

Remark 4.25. *For a cocycle Φ over a semiflow σ, let $\{P(\theta)\}_{\theta \in \Theta}$ be a family of projectors on X and for each $\theta \in \Theta$, take $Q(\theta) = I - P(\theta)$, $X_1(\theta) = \operatorname{Im} P(\theta)$ and $X_2(\theta) = \operatorname{Im} Q(\theta) = \operatorname{Ker} P(\theta)$. For $(\theta, t) \in \Theta \times \mathbb{R}_+$ we have that:*

(i) *$X_1(\,\cdot\,)$ is "$\Phi(\theta, t)$-invariant", that is $\Phi(\theta, t)X_1(\theta) \subset X_1(\sigma(\theta, t))$, if and only if $P(\sigma(\theta, t))\Phi(\theta, t)P(\theta) = \Phi(\theta, t)P(\theta)$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$;*
(ii) *$X_2(\,\cdot\,)$ is "$\Phi(\theta, t)$-invariant", that is $\Phi(\theta, t)X_2(\theta) \subset X_2(\sigma(\theta, t))$, if and only if $Q(\sigma(\theta, t))\Phi(\theta, t)Q(\theta) = \Phi(\theta, t)Q(\theta)$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$;*
(iii) *$X_1(\,\cdot\,)$ and $X_2(\,\cdot\,)$ are both "$\Phi(\theta, t)$-invariant" if and only if $P(\sigma(\theta, t))\Phi(\theta, t) = \Phi(\theta, t)P(\theta)$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$.*

Lemma 4.26. *If $h : \mathbb{N} \to \mathbb{R}_+$ is a sequence, $H > 0$, $n_0 \in \mathbb{N}^*$ and $\eta \in (0, 1)$ such that*

(i) *$h(k) \leq Hh(n)$, for all $k \in \{n, n+1, ..., n+n_0\}$, $n \in \mathbb{N}$ and*
(ii) *$h(n + n_0) \leq \eta \, h(n)$, for all $n \in \mathbb{N}$,*

then there exist $N, \nu > 0$ such that

$$h(n) \leq N e^{-\nu(n-m)} h(m) \ , \ \textit{for all } n, m \in \mathbb{N}, n \geq m \ .$$

Proof. Let $n, m \in \mathbb{N}$ such that $n \geq m$. Then, there exist $k \in \mathbb{N}$ and $r \in \{0, 1, ..., n_0 - 1\}$ such that $n - m = kn_0 + r$. Therefore, $h(n) = h(m + r + kn_0) \leq \eta h(m + r + (k-1)n_0)$ and in another $(k-1)$ steps we obtain $h(n) \leq \eta^k h(m + r)$. Applying now the first property, we have that

$$h(n) \leq H \eta^k h(m) \ .$$

Now, take $\nu := -\frac{1}{n_0} \ln \eta > 0$ and $N := He^{\nu n_0} > 0$ to obtain $h(n) \leq He^{\nu r} e^{-\nu(n-m)} \leq Ne^{-\nu(n-m)} h(m)$. □

Lemma 4.27. *If $h : \mathbb{N} \to \mathbb{R}_+$ is a sequence, $H > 0$, $n_0 > 0$ and $\eta > 1$ such that*

(i) *$h(k) \geq Hh(n)$, for all $k \in [n, n + n_0]$, $n \in \mathbb{N}$ and*
(ii) *$h(n + n_0) \geq \eta \, h(n)$, for all $n \in \mathbb{N}$,*

then there exist $N, \nu > 0$ such that

$$h(n) \geq N e^{\nu(n-m)} h(m) \ , \ \textit{for all } n, m \in \mathbb{N},\, n \geq m \ .$$

Proof. It is an analogue of the proof of Lemma 4.26. □

If $A : \Theta \to \mathcal{B}(X)$ is continuous, $f \in L^1_{loc}(\mathbb{R}_+; X)$ and σ a continuous semiflow on Θ, then the function

$$u(t) = \Phi(\theta, t)x_0 + \int_0^t \Phi(\sigma(\theta, s), t-s) f(s) ds$$

is called the *mild solution* of the inhomogeneous initial-value problem $(A, f; x)$:

$$\begin{cases} \dot{u}(t) = A(\sigma(\theta, t))u(t) + f(t) \ , \ (\theta, t) \in \Theta \times \mathbb{R}_+ \\ u(0) = x_0 \end{cases} ,$$

where the cocycle Φ solves the homogeneous differential equation $\dot{u}(t) = A(\sigma(\theta, t)u(t)$.

We set the expression of the mild solution of the inhomogeneous equation in discrete-time to give the following definition of admissibility of a pair of sequence Schäffer spaces to a cocycle in some point of the base space.

Definition 4.28. *Let $\mathcal{E}, \mathcal{F}$ be two sequence Schäffer spaces, Φ a cocycle over the semiflow σ and $\theta \in \Theta$. The pair $(\mathcal{E}, \mathcal{F})$ is said to be* θ-admissible to Φ *if for every $f \in \mathcal{E}(X)$, there exists $x \in X$ such that $p_f(\,\cdot\,; x, \theta) \in \mathcal{F}(X)$, where*

$$p_f(n; x, \theta) = \Phi(\theta, n)x + \sum_{k=0}^{n} \Phi(\sigma(\theta, k), n-k) f(k) \ ,$$

for each $n \in \mathbb{N}$.

5. Main Results

In this section, for $\pi = (\Phi, \sigma)$ a LSPS on $E = X \times \Theta$ associated to the exponentially bounded cocycle Φ over the semiflow σ (with M, ω assuring the exponential boundedness of Φ), a point $\theta \in \Theta$ and $\mathcal{E}$, $\mathcal{F}$ two sequence Schäffer spaces, we consider

$$X_{1,\mathcal{F}}(\theta) = \{\, x \in X \ : \ (\Phi(\theta, n)x)_{n \in \mathbb{N}} \in \mathcal{F}(X) \,\}$$

and for the sake of readability we take $X_1(\theta) = X_{1,\ell_0^\infty}(\theta)$ (which coincides with $X_{1,\theta}$ considered in Remark 4.24). Obviously, $X_{1,\mathcal{F}}(\theta)$, $X_1(\theta)$ are vector subspaces of X. In what follows, we make the next assumption.

Hypothesis. The vector subspace $X_{1,\mathcal{F}}(\theta)$ is closed and admits a closed complement, i.e. there exists $X_{2,\mathcal{F}}(\theta)$ a closed vector subspace of X such that $X = X_{1,\mathcal{F}}(\theta) \oplus X_{2,\mathcal{F}}(\theta)$.

We denote by $P_{\mathcal{F}}(\theta)$ the projection onto $X_{1,\mathcal{F}}(\theta)$ along $X_{2,\mathcal{F}}(\theta)$ and set $Q_{\mathcal{F}}(\theta) = I - P_{\mathcal{F}}(\theta)$ the complementary projection. In the first subsection, we will prove that in the case of a no past exponential dichotomy for Φ, $X_{1,\mathcal{F}}(\theta)$ coincides with $X_1(\theta)$.

5.1. Necessary conditions for no past exponential dichotomies.

After some preliminaries we prove in Theorem 5.33 that if the pair $(\mathcal{E}, \mathcal{F})$ is θ-admissible to the exponentially bounded cocycle Φ (over a semiflow σ), then the family of functions $\{\Phi(\theta, \cdot)x : \mathbb{R}_+ \to X\}_{x \in X}$ exhibits a dichotomic behavior. The restriction over such a pair $(\mathcal{E}, \mathcal{F})$ is that $\mathcal{E}$ and $\mathcal{F}$ do not occupy simultaneously boundary positions in the chain of sequence Scäffer spaces (in the sense of Proposition 2.8).

Proposition 5.29. *For any $t \geq 0$ we have that:*

(i) $\Phi(\theta,t)X_{1,\mathcal{F}}(\theta)\subset X_{1,\mathcal{F}}(\sigma(\theta,t))$ *and*
(ii) $\Phi(\theta,t)x\neq 0$, *for all* $x\in X_{2,\mathcal{F}}(\theta)\setminus\{0\}$.

Proof. (i) Let $x\in X_{1,\mathcal{F}}(\theta)$, $t\geq 0$ and take $y=\Phi(\theta,t)x$. Then,

$$\|\Phi(\sigma(\theta,t),n)y\|=\|\Phi(\theta,n+t)x\|\leq\|\Phi(\sigma(\theta,n),t)\|\cdot\|\Phi(\theta,n)x\|\leq Me^{\omega t}\|\Phi(\theta,n)x\|$$

for all $n\in\mathbb{N}$. Since $(\|\Phi(\theta,n)x\|)_{n\in\mathbb{N}}\in\mathcal{F}$, it follows that $y\in X_{1,\mathcal{F}}$. Thus, $X_{1,\mathcal{F}}(\,\cdot\,)$ is "$\Phi(\theta,t)$-invariant" (in the sense of Remark 4.25).

To prove (ii) assume for a contradiction that there exist $t\geq 0$ and $x\in X_{2,\mathcal{F}}(\theta)\setminus\{0\}$ such that $\Phi(\theta,t)x=0$. Then, for each $n\in\mathbb{N}$, $n\geq t$ we have that

$$\Phi(\theta,n)x=\Phi(\sigma(\theta,t),n-t)\Phi(\theta,t)x=0$$

and thus $(\Phi(\theta,n)x)_{n\in\mathbb{N}}\in\mathcal{F}(X)$. It follows that $x\in X_{1,\mathcal{F}}(\theta)$, which is impossible since $x\in X_{2,\mathcal{F}}(\theta)\setminus\{0\}$. We proved that $\Phi(\theta,t)|_{X_{2,\mathcal{F}}(\theta)}:X_{2,\mathcal{F}}(\theta)\to X_{2,\mathcal{F}}(\sigma(\theta,t))$ is one-to-one, for all $t\geq 0$. □

Proposition 5.30. *If the pair* $(\mathcal{E},\mathcal{F})$ *is* θ*-admissible to the cocycle* Φ *(over the semiflow* σ*), then for each* $f\in\mathcal{E}(X)$*, there exists a unique* $x_{f,\theta}\in X_{2,\mathcal{F}}(\theta)$ *such that* $p_f(\,\cdot\,;x_{f,\theta},\theta)\in\mathcal{F}(X)$.

Proof. Let $f\in\mathcal{E}(X)$ and $x\in X$ given by Definition 4.28. Considering $y=x-P_{\mathcal{F}}(\theta)x=Q_{\mathcal{F}}(\theta)x$ we have that $y\in X_{2,\mathcal{F}}(\theta)$ and $p_f(n;y,\theta)=p_f(n;x,\theta)-\Phi(\theta,n)P_{\mathcal{F}}(\theta)x$, for all $n\in\mathbb{N}$. Since $p_f(\,\cdot\,;x,\theta)\in\mathcal{F}(X)$ and $(\Phi(\theta,n)P_{\mathcal{F}}(\theta)x)_n\in\mathcal{F}(X)$, it follows that $p_f(\,\cdot\,;y,\theta)\in\mathcal{F}(X)$.

To prove the uniqueness of y, suppose that there exists $z\in X_{2,\mathcal{F}}(\theta)$ with the property $p_f(\,\cdot\,;z,\theta)\in\mathcal{F}(X)$. Since $p_f(n;y,\theta)-p_f(n;z,\theta)=\Phi(\theta,n)(y-z)$, we have that $y-z\in X_{1,\mathcal{F}}(\theta)\cap X_{2,\mathcal{F}}(\theta)$ and therefore $z=y$.

The unique vector $y\in X_{2,\mathcal{F}}(\theta)$ will be denoted by $x_{f,\theta}$. □

Proposition 5.31. *If the pair* $(\mathcal{E},\mathcal{F})$ *is* θ*-admissible to the cocycle* Φ *(over the semiflow* σ*), there exists a constant* $K=K(\theta)>0$ *such that*

$$\|x_{f,\theta}\|\leq K\|f\|_{\mathcal{E}(X)}\quad\textit{and}\quad\|p_f(\,\cdot\,;x_{f,\theta},\theta)\|_{\mathcal{F}(X)}\leq K\|f\|_{\mathcal{E}(X)}\ ,$$

for all $f\in\mathcal{E}(X)$.

Proof. We define the operator

$$\mathcal{U}_\theta:\mathcal{E}(X)\to X_{2,\mathcal{F}}(\theta)\times\mathcal{F}(X)\quad,\quad\mathcal{U}_\theta f=(\,x_{f,\theta}\,,\,p_f(\,\cdot\,;x_{f,\theta},\theta)\,)$$

From a simple verification it results that $\mathcal{U}_\theta$ is a linear operator. Now, we will show that it is also closed.

Let $(f_n)_{n\in\mathbb{N}}\subset\mathcal{E}(X)$ such that $\|f_n-f\|_{\mathcal{E}(X)}\xrightarrow[n\to\infty]{}0$ and $\|\mathcal{U}_\theta f_n-(y,g)\|_{X_{2,\mathcal{F}}(\theta)\times\mathcal{F}(X)}\xrightarrow[n\to\infty]{}0$, where $f\in\mathcal{E}(X)$, $y\in X_{2,\mathcal{F}}(\theta)$ and $g\in\mathcal{F}(X)$. For each $n\in\mathbb{N}$, we take $x_n=x_{f_n,\theta}\in X_{2,\mathcal{F}}(\theta)$ and $u_n=p_{f_n}(\,\cdot\,;x_n,\theta)\in\mathcal{F}(X)$. We have that $\|x_n-y\|\xrightarrow[n\to\infty]{}0$ and $\|u_n-g\|_{\mathcal{F}(X)}\xrightarrow[n\to\infty]{}0$.

On the one hand, since

$$\|u_n(k)-g(k)\|\leq\frac{1}{\beta_F(0)}\|u_n-g\|_{\mathcal{F}(X)}\ ,$$

we have that $\lim_{n\to\infty}u_n(k)=g(k)$, for all $k\in\mathbb{N}$. On the other hand,

$$\begin{aligned}\|u_n(k)-p_f(k;y,\theta)\|&=\|\Phi(\theta,k)(x_n-y)+\sum_{j=0}^{k}\Phi(\sigma(\theta,j),k-j)(f_n(j)-f(j))\|\\&\leq\|\Phi(\theta,k)\|\,\|x_n-y\|+\sum_{j=0}^{k}\|\Phi(\sigma(\theta,j),k-j)\|\,\|f_n(j)-f(j)\|\end{aligned}$$

which implies $\lim_{n\to\infty} u_n(k) = p_f(k; y, \theta)$, for all $k \in \mathbb{N}$. It follows that $p_f(\,\cdot\,; y, \theta) = g \in \mathcal{F}(X)$ and from Proposition 5.30 we have $y = x_{f,\theta}$. Therefore $\mathcal{U}_\theta f = (y; g)$.

Hence, $\mathcal{U}_\theta$ is a closed linear operator and by the Closed-Graph Theorem it is also bounded which means that there exists $K > 0$ such that

$$\|x_{f,\theta}\| + \|p_f(\,\cdot\,; x_{f,\theta}, \theta)\|_{\mathcal{F}(X)} = \|\mathcal{U}_\theta f\|_{X_{2,\mathcal{F}}(\theta)\times\mathcal{F}(X)} \le K\|f\|_{\mathcal{E}(X)}$$

and the proof is complete. □

A simple and useful evaluation that results from Proposition 2.8 and Proposition 5.31 is the following remark.

Remark 5.32. *If the pair $(\mathcal{E}, \mathcal{F})$ is θ-admissible to the cocycle Φ (over the semiflow σ), then (E, ℓ^∞) is θ-admissible to Φ and $\|p_f(n; x_{f,\theta}, \theta)\| \le \frac{K(\theta)}{\beta_F(0)}\|f\|_{\mathcal{E}(X)}$, for all $f \in \mathcal{E}(X)$ and $n \in \mathbb{N}$.*

Theorem 5.33. *Let $\mathcal{E}$, $\mathcal{F}$ be two sequence Schäffer spaces such that $\ell^1 \ne \mathcal{E}$ or $\ell_0^\infty \not\subset \mathcal{F}$, Φ an exponentially bounded cocycle over the semiflow σ and $\theta \in \Theta$. If the pair $(\mathcal{E}, \mathcal{F})$ is θ-admissible to Φ, then there exist two constants $N = N(\theta)$, $\nu = \nu(\theta) > 0$ such that*

$$\|\Phi(\theta, t)x\| \le Ne^{-\nu t}\|x\|\ ,\ \textit{for all } t \in \mathbb{R}_+ \textit{ and } x \in X_{1,\mathcal{F}}(\theta)$$
$$\|\Phi(\theta, t)x\| \ge \frac{1}{N}e^{\nu t}\|x\|\ ,\ \textit{for all } t \in \mathbb{R}_+ \textit{ and } x \in X_{2,\mathcal{F}}(\theta)\ .$$

Proof of Part I. We study the asymptotic behavior of $\Phi(\theta, \cdot\,)x$ when $x \in X_{1,\mathcal{F}}(\theta)$.

Let $x \in X_{1,\mathcal{F}}(\theta)$, $j \in \mathbb{N}$ and consider the sequence

$$f_j : \mathbb{N} \to X \quad , \quad f_j(n) = \delta_j(n)\Phi(\theta, j)x\ . \tag{5.1}$$

which is in $\mathcal{E}(X)$ with $\|f_j\|_{\mathcal{E}(X)} = \beta_{\mathcal{E}}(0)\|\Phi(\theta, j)x\|$.

Observe that if $n \ge j$, $p_{f_j}(n; 0, \theta) = \Phi(\theta, n)x$ and if $0 \le n < j$, $p_{f_j}(n; 0, \theta) = 0$. Then, $p_{f_j}(\,\cdot\,; 0, \theta) \in \mathcal{F}(X)$ and from $\|p_{f_j}(n; 0, \theta)\| \le \frac{K(\theta)}{\beta_F(0)}\|f_j\|_{\mathcal{E}(X)}$ we obtain that $\|\Phi(\theta, n)x\| \le K(\theta)\frac{\beta_{\mathcal{E}}(0)}{\beta_F(0)}\|\Phi(\theta, j)x\|$, for all $n \in \mathbb{N}$, $n \ge j$. In particular, we have that $\|\Phi(\theta, n)x\| \le K(\theta)\frac{\beta_{\mathcal{E}}(0)}{\beta_F(0)}\|x\|$, for all $n \in \mathbb{N}$. For $t \ge 0$, taking $n = [t]$ we deduce that

$$\|\Phi(\theta, t)x\| = \|\Phi(\sigma(\theta, n), t - n)\Phi(\theta, n)x\| \le Me^{\omega(t-n)}\|\Phi(\theta, n)x\| \le Me^{\omega}K(\theta)\frac{\beta_{\mathcal{E}}(0)}{\beta_F(0)}\|x\|\ .$$

Since the constants $C'_{1,\theta} = K(\theta)\beta_{\mathcal{E}}(0)\beta_F(0)^{-1}$, $C_{1,\theta} := Me^{\omega}C'_{1,\theta}$ do not depend on x we can write down

$$\|\Phi(\theta, t)x\| \le C_{1,\theta}\|x\|\ ,\ \text{for all } t \ge 0 \text{ and } x \in X_{1,\mathcal{F}}(\theta)\ . \tag{5.2}$$

For $x \in X_{1,\mathcal{F}}(\theta)$, $n, k \in \mathbb{N}$ we have that

(5.3)

$$\begin{aligned}\|\Phi(\theta, n+k)x\|\sum_{j=n}^{n+k}\delta_j(m) &= \sum_{j=n}^{n+k}\|\Phi(\theta, n+k)x\|\delta_j(m) \le C'_{1,\theta}\sum_{j=n}^{n+k}\|\Phi(\theta, j)x\|\delta_j(m)\\ &\le C'_{1,\theta}\sum_{j=n}^{\infty}\|\Phi(\theta, j)x\|\delta_j(m) = C'_{1,\theta}\,\|p_{f_n}(m; 0, \theta)\|\ ,\ \text{for all } m \in \mathbb{N}, m \ge n\ ,\end{aligned}$$

which implies that $\|\Phi(\theta, n+k)x\|\sum_{j=n}^{n+k}\delta_j \le C'_{1,\theta}\,\|p_{f_n}(\,\cdot\,; 0, \theta)\|$. Therefore

$$\|\ \|\Phi(\theta, n+k)x\|\sum_{j=n}^{n+k}\delta_j\ \|_{\mathcal{F}} \le C'_{1,\theta}\,\|\ \|p_{f_n}(\,\cdot\,; 0, \theta)\|\ \|_{\mathcal{F}}\ ,$$

or equivalently,

$$\|\Phi(\theta,n+k)x\|\,\|\chi_{\{n,n+1,\dots,n+k\}}\|_{\mathcal{F}} \le C'_{1,\theta}\|p_{f_n}(\,\cdot\,;0,\theta)\|_{\mathcal{F}(X)}\ .$$

By Proposition 5.31 we can write down

$$\|\Phi(\theta,n+k)x\| \le \frac{C'_{1,\theta}K(\theta)\beta_{\mathcal{E}}(0)}{\beta_{\mathcal{F}}(k)}\|\Phi(\theta,n)x\|\ ,\ \text{for all } k\in\mathbb{N} \text{ and } x\in X_{1,\mathcal{F}}(\theta)\ . \tag{5.4}$$

If $\ell_0^\infty \not\subset \mathcal{F}$, then β_F is not bounded and therefore it exists $n_0\in\mathbb{N}^*$ such that

$$\eta := C'_{1,\theta}K(\theta)\frac{\beta_{\mathcal{E}}(0)}{\beta_{\mathcal{F}}(n_0)} < 1 \quad \text{and} \quad \|\Phi(\theta,n+n_0)x\| \le \eta\|\Phi(\theta,n)x\|\ ,\ \text{for all } n\in\mathbb{N},\ x\in X_{1,\mathcal{F}}(\theta)\ .$$

If $\ell_0^\infty \subset \mathcal{F}$, then $E\neq\ell^1$. From Proposition 2.8, it follows that there exists $h\in\mathcal{E}\setminus\ell^1$. Consider

$$\gamma:\mathbb{N}\to\mathbb{R}\quad,\quad \gamma(k)=\sum_{j=0}^{k}|h(j)|\quad, \tag{5.5}$$

which is nondecreasing and $\lim_{k\to\infty}\gamma(k)=\infty$. For $x\in X_{1,\mathcal{F}}$ and $n,k\in\mathbb{N}$, the sequence

$$g:\mathbb{N}\to X\quad,\quad g(m)=\chi_{\{n,n+1,\dots,n+k\}}(m)\,|R^n h(m)|\,\Phi(\theta,m)x \tag{5.6}$$

has finite support, thus $g\in\mathcal{E}(X)$. Observing that $\|g(m)\|\le C'_{1,\theta}\|\Phi(\theta,n)x\|\,|R^n h(m)|$ for every $m\in\mathbb{N}$, we are led to the evaluation: $\|g\|_{\mathcal{E}(X)}\le C'_{1,\theta}\|h\|_{\mathcal{E}}\|\Phi(\theta,n)x\|$. Since, for $m<n$, $p_g(m;0,\theta)=0$ and for $m\ge n$

(5.7)

$$\begin{aligned} p_g(m;0,\theta) &= \sum_{j=0}^{m}\Phi(\sigma(\theta,j),m-j)g(j) = \sum_{j=n}^{m}\Phi(\sigma(\theta,j),m-j)\Phi(\theta,j)x|h(j-n)|\chi_{\{n,n+1,\dots,n+k\}}(j) \\ &= \left(\sum_{j=0}^{m-n}\chi_{\{n,n+1,\dots,n+k\}}(n+j)|h(j)|\right)\Phi(\theta,m)x\quad, \end{aligned}$$

we have that $\|p_g(m;0,\theta)\|\le\gamma(k)\|\Phi(\theta,n)x\|$, for all $n\in\mathbb{N}$. Since $x\in X_{1,\mathcal{F}}$, $(\Phi(\theta,n)x)_{n\in\mathbb{N}}\in\mathcal{F}(X)$, thus we have that $p_g(\,\cdot\,;0,\theta)\in\mathcal{F}(X)$. Taking $m=n+k$ in $\|p_g(m;0,\theta)\|\le\frac{K(\theta)}{\beta_{\mathcal{F}}(0)}\|g\|_{\mathcal{E}(X)}$, we obtain that

$$\gamma(k)\|\Phi(\theta,n+k)x\| \le \frac{K(\theta)C'_{1,\theta}}{\beta_{\mathcal{F}}(0)}\|h\|_{\mathcal{E}}\|\Phi(\theta,n)x\|\ . \tag{5.8}$$

Therefore, there exists $n_0\in\mathbb{N}^*$ such that

$$\eta := \frac{K(\theta)C'_{1,\theta}\|h\|_{\mathcal{E}}}{\beta_{\mathcal{F}}(0)\gamma(n_0)} < 1 \quad \text{and} \quad \|\Phi(\theta,n+n_0)x\| \le \eta\|\Phi(\theta,n)x\|\ ,\ \text{for all } n\in\mathbb{N},\ x\in X_{1,\mathcal{F}}(\theta)\ .$$

In both cases, we obtained the existence of some $n_0\in\mathbb{N}^*$ and $\eta\in(0,1)$ satisfying the condition (ii) from Lemma 4.26, while for the first condition put $H:=C'_{1,\theta}>0$. In consequence, there exist $N'_{1,\theta}$, $\nu_{1,\theta}>0$ such that

$$\|\Phi(\theta,n)x\| \le N'_{1,\theta}\,e^{-\nu_{1,\theta}\,n}\,\|x\|\ ,\ \text{for all } n\in\mathbb{N} \text{ and } x\in X_{1,\mathcal{F}}(\theta)\ .$$

To obtain the above property in the continuous case, simply take $N_{1,\theta}=M\,e^{\omega+\nu_{1,\theta}}\,N'_{1,\theta}$. Indeed, for $t\ge0$, put $n=[t]$ to obtain

$$\|\Phi(\theta,t)\| \le \|\Phi(\sigma(\theta,n),t-n)\|\,\|\Phi(\theta,n)\| \le Me^{\omega}\,N'_{1,\theta}\,e^{-\nu_{1,\theta}\,n}\,\|x\| \le N_{1,\theta}\,e^{-\nu_{1,\theta}n}\,\|x\|\ .$$

Proof of Part II. We study the asymptotic behavior of $\Phi(\theta,\,\cdot\,)x$ when $x\in X_{2,\mathcal{F}}(\theta)$.

Let $n \in \mathbb{N}$, $k \in \mathbb{N}^*$ and $x \in X_{2,\mathcal{F}}(\theta) \setminus \{0\}$ and consider the sequence

$$f : \mathbb{N} \to X \quad , \quad f(m) = \delta_{n+k}(m) \frac{\Phi(\theta, m)x}{\|\Phi(\theta, n+k)x\|} \ . \tag{5.9}$$

We have that $f \in \mathcal{E}(X)$ with $\|f\|_{\mathcal{E}(X)} = \beta_{\mathcal{E}}(0)$. On the one hand,

(5.10)

$$\begin{aligned} -\sum_{j=m+1}^{\infty} \delta_{n+k}(j) \frac{\Phi(\theta, m)x}{\|\Phi(\theta, n+k)x\|} &= -\sum_{j=0}^{\infty} \delta_{n+k}(j) \frac{\Phi(\theta, m)x}{\|\Phi(\theta, n+k)x\|} + \sum_{j=0}^{m} \delta_{n+k}(j) \frac{\Phi(\sigma(\theta, j)m-j)\Phi(\theta, j)x}{\|T(n+k)x\|} \\ &= \Phi(\theta, m) \left(\frac{-x}{\|\Phi(\theta, n+k)x\|} \right) + \sum_{j=0}^{m} \Phi(\sigma(\theta, j)m-j) f(k) = p_f(m; y, \theta) \ , \end{aligned}$$

for all $m \in \mathbb{N}$ (where $y = \frac{-x}{\|\Phi(\theta,n+k)x\|} \in X_{2,\mathcal{F}}(\theta)$), while on the other hand

$$-\sum_{j=m+1}^{\infty} \delta_{n+k}(j) \frac{\Phi(\theta, m)x}{\|\Phi(\theta, n+k)x\|} = \begin{cases} -\frac{\Phi(\theta,m)x}{\|T(n+k)x\|} & , \ m < n+k \\ 0 & , \ m \geq n+k \end{cases} \ , \tag{5.11}$$

which implies that $p_f(\,\cdot\,; y, \theta) \in \mathcal{F}(X)$. Then, $\|p_f(m; y, \theta)\| \leq \frac{K(\theta)}{\beta_{\mathcal{F}}(0)} \|f\|_{\mathcal{E}(X)}$ and therefore,

$$\|\Phi(\theta, m)x\| \leq K(\theta) \frac{\beta_{\mathcal{E}}(0)}{\beta_{\mathcal{F}}(0)} \|\Phi(\theta, n+k)x\| \ , \text{ for all } 0 \leq m < n+k \ . \tag{5.12}$$

Taking now $m = n = 0$ we can write down

$$\|\Phi(\theta, k)x\| \geq \frac{\beta_{\mathcal{F}}(0)}{K(\theta)\beta_{\mathcal{E}}(0)} \|x\| \ , \text{ for all } k \in \mathbb{N}^* \text{ and } x \in X_{2,\mathcal{F}}(\theta) \ .$$

For $t \geq 0$, take $k = [t] + 1$ to evaluate: $Me^{\omega} \|\Phi(\theta, t)x\| \geq \|\Phi(\theta, k)x\| \geq \frac{\beta_{\mathcal{F}}(0)}{K(\theta)\beta_{\mathcal{E}}(0)} \|x\|$. Denoting $C_{2,\theta} := \beta_{\mathcal{F}}(0)(K(\theta)\beta_{\mathcal{E}}(0)Me^{\omega})^{-1}$ we have

$$\|\Phi(\theta, t)x\| \geq C_{2,\theta} \|x\| \ , \text{ for all } t \geq 0 \text{ and } x \in X_{2,\mathcal{F}}(\theta) \ . \tag{5.13}$$

For $n \in \mathbb{N}$, $j > n$, take $k = j - n$ and $m = n$ in (5.12) to obtain $K(\theta) \frac{\beta_{\mathcal{E}}(0)}{\beta_{\mathcal{F}}(0)} \|\Phi(\theta, j)x\| \geq \|\Phi(\theta, n)x\|$. Taking $C'_{2,\theta} := 1 + K(\theta)\beta_{\mathcal{E}}(0)(\beta_{\mathcal{F}}(0))^{-1}$ we have that

$$C'_{2,\theta} \|\Phi(\theta, j)x\| \geq \|\Phi(\theta, n)x\| \ , \text{ for all } j \geq n \text{ and } x \in X_{2,\mathcal{F}}(\theta) \ . \tag{5.14}$$

Since n, k and x were arbitrarily taken (see the begining of Part II), we have that

(5.15)

$$\begin{aligned} \frac{\|\Phi(\theta, n)x\|}{\|\Phi(\theta(n+k)x\|} \sum_{j=n}^{n+k-1} \delta_j(m) &\leq C'_{2,\theta} \sum_{j=n}^{n+k-1} \frac{\|\Phi(\theta, j)x\|}{\|\Phi(\theta, n+k)x\|} \delta_j(m) = C'_{2,\theta} \sum_{j=n}^{n+k-1} \|p_f(j; y, \theta)\| \delta_j(m) \\ &\leq C'_{2,\theta} \|p_f(m; y, \theta)\| \ , \text{ for all } m \in \mathbb{N}, \end{aligned}$$

and therefore $\frac{\|\Phi(\theta,n)x\|}{\|\Phi(\theta,n+k)x\|} \chi_{\{n,n+1,\ldots,n+k-1\}} \leq C'_{2,\theta} \|p_f(\,\cdot\,; y, \theta)\|$. It follows that

$$\| \frac{\|\Phi(\theta, n)x\|}{\|\Phi(\theta, n+k)x\|} R^n \chi_{\{0,1,\ldots,k-1\}} \|_{\mathcal{F}} \leq C'_{2,\theta} \| \, \|p_f(\,\cdot\,; y, \theta)\| \, \|_{\mathcal{F}} \ ,$$

or equivalently,

$$\frac{\|\Phi(\theta, n)x\|}{\|\Phi(\theta, n+k)x\|} \beta_{\mathcal{F}}(k-1) \leq C'_{2,\theta} \|p_f(\,\cdot\,; y, \theta)\|_{\mathcal{F}(X)} \ . \tag{5.16}$$

Using the fact that $\|p_f(\,\cdot\,;y,\theta)\|_{\mathcal{F}(X)} \le K(\theta)\|f\|_{\mathcal{E}(X)}$, we obtain that

$$\|\Phi(\theta,n+k)x\| \ge \frac{\beta_{\mathcal{F}}(k-1)}{C'_{2,\theta}K(\theta)\beta_{\mathcal{E}}(0)}\|\Phi(\theta,n)x\| \ , \text{ for all } n\in\mathbb{N},\ k\in\mathbb{N}^* \text{ and } x\in X_{2,\mathcal{F}}(\theta) \ . \tag{5.17}$$

If $\ell_0^\infty \not\subset \mathcal{F}$, then β_F is not bounded and therefore there exists $n_0\in\mathbb{N}^*$ such that

$$\eta := \frac{\beta_{\mathcal{F}}(n_0-1)}{C'_{2,\theta}K(\theta)\beta_{\mathcal{E}}(0)} > 1 \quad \text{and} \quad \|\Phi(\theta,n+n_0)x\| \ge \eta\|\Phi(\theta,n)x\| \ , \text{ for all } n\in\mathbb{N},\ x\in X_{2,\mathcal{F}}(\theta) \ .$$

If $\ell_0^\infty \subset \mathcal{F}$, then $E\neq\ell^1$ and therefore there exists $h\in\mathcal{E}\setminus\ell^1$. Consider γ as in (5.5), and for $n,k\in\mathbb{N}$, $x\in X_{2,\mathcal{F}}(\theta)\setminus\{0\}$ we define

$$g:\mathbb{N}\to X \quad , \quad g(m)=\chi_{\{n,n+1,\dots,n+k\}}(m)|R^nh(m)|\frac{\Phi(\theta,m)x}{\|\Phi(\theta,n+k)x\|} \tag{5.18}$$

Since $\|g(m)\| \le C'_{2,\theta}|(R^nh)(m)|$ for each $m\in\mathbb{N}$, we have that $g\in\mathcal{E}(X)$ with $\|g\|_{\mathcal{E}(X)} \le C'_{2,\theta}\|R^nh\|_{\mathcal{E}} = C'_{2,\theta}\|h\|_{\mathcal{E}}$. On the one hand,

$$\begin{aligned}
&-\sum_{j=m+1}^{\infty}\chi_{\{n,\dots,n+k\}}(j)|R^nh(j)|\frac{\Phi(\theta,m)x}{\|\Phi(\theta,n+k)x\|}\\
&\qquad = -\sum_{j=0}^{\infty}\chi_{\{n,\dots,n+k\}}(j)|R^nh(j)|\frac{\Phi(\theta,m)x}{\|\Phi(\theta,n+k)x\|}\\
&\qquad\quad +\sum_{j=0}^{m}\Phi(\sigma(\theta,j),m-j)\chi_{\{n,\dots,n+k\}}(j)|R^nh(j)|\frac{\Phi(\theta,j)x}{\|\Phi(\theta,n+k)x\|}\\
&\qquad = \Phi(\theta,m)\left(\gamma(k)\frac{-x}{\|\Phi(\theta,n+k)\|x\|}\right)+\sum_{j=0}^{m}\Phi(\sigma(\theta,j),m-j)g(j)\\
&\qquad = p_g(m;z,\theta) \quad ,
\end{aligned}\tag{5.19}$$

for all $m\in\mathbb{N}$ (where $z:=\gamma(k)\frac{-x}{\|\Phi(\theta,n+k)\|x\|}\in X_{2,\mathcal{F}}$), while on the other hand

$$-\sum_{j=m+1}^{\infty}\chi_{\{n,\dots,n+k\}}(j)|R^nh(j)|\frac{\Phi(\theta,m)x}{\|\Phi(\theta,n+k)x\|} = \begin{cases} -\gamma(k)\frac{\Phi(\theta,m)x}{\|\Phi(\theta,n+k)x\|} & ,\ m<n\\ -(\gamma(k)-\gamma(m-n))\frac{\Phi(\theta,m)x}{\|\Phi(\theta,n+k)x\|} & ,\ n\le m<n+k\\ 0 & ,\ m\ge n+k\end{cases} \ . \tag{5.20}$$

From (5.19) and (5.20), it follows that $p_g(\,\cdot\,;z,\theta)$ has finite support, thus it belongs to $\mathcal{F}(X)$. Therefore, $\|p_g(n;z,\theta)\| \le \frac{K(\theta)}{\beta_{\mathcal{F}}(0)}\|g\|_{\mathcal{E}(X)}$ and using $\|g\|_{\mathcal{E}(X)} \le C'_{2,\theta}\|h\|_{\mathcal{E}}$ we obtain

$$(\gamma(k)-|h(0)|)\frac{\|\Phi(\theta,n)x\|}{\|\Phi(\theta,n+k)x\|} \le \frac{K(\theta)C'_{2,\theta}}{\beta_{\mathcal{F}}(0)}\|h\|_{\mathcal{E}} \ .$$

Thus,

$$\|\Phi(\theta,n+k)x\| \ge \frac{\beta_{\mathcal{F}}(0)}{K(\theta)C'_{2,\theta}\|h\|_{\mathcal{E}}}(\gamma(k)-|h(0)|)\,\|\Phi(\theta,n)x\| \ , \text{ for all } n,k\in\mathbb{N}\,,\ x\in X_{2,\mathcal{F}}(\theta) \ . \tag{5.21}$$

and there exists $n_0\in\mathbb{N}^*$ such that

$$\eta := \frac{\beta_{\mathcal{F}}(0)(\gamma(n_0)-|h(0)|)}{K(\theta)C'_{2,\theta}\|h\|_{\mathcal{E}}} > 1 \quad \text{and} \quad \|\Phi(\theta,n+n_0)x\| \ge \eta\|\Phi(\theta,n)x\| \text{for all } n\in\mathbb{N}\,,\ x\in X_{2,\mathcal{F}}(\theta) \ .$$

In both cases, we obtained the existence of some $n_0 \in \mathbb{N}^*$ and a constant $\eta > 1$ satisfying the condition (ii) from Lemma 4.27, while the first condition is assured by (5.13). In consequence, there exist $N'_{2,\theta}$, $\nu_{2,\theta} > 0$ such that

$$\|\Phi(\theta, n)x\| \geq N'_{2,\theta}\, e^{\nu_{2,\theta}\, n}\, \|x\| \ , \text{ for all } n \in \mathbb{N} \text{ and } x \in X_{2,\mathcal{F}}(\theta) \ .$$

To obtain the above property in the continuous case, simply take $N_{2,\theta} = (M\, e^{\omega})^{-1}\, N'_{2,\theta}$. For $t \geq 0$, put $n = [t] + 1$ and note that $Me^{\omega}\|\Phi(\theta, t)x\| \geq \|\Phi(\theta, n)x\|$ to obtain

$$\|\Phi(\theta, t)x\| \geq N_{2,\theta}\, e^{\nu_{2,\theta}\, t}\, \|x\| \ , \text{ for all } t \geq 0 \text{ and } x \in X_{2,\mathcal{F}}(\theta) \ .$$

If we take $N := \max\{\, N_{1,\theta} \, , \, 1/N_{2,\theta} \,\} > 0$ and $\nu := \min\{\, \nu_{1,\theta} \, , \, \nu_{2,\theta} \,\} > 0$, the conclusion follows immediately.

Corollary 5.34. *Let $\mathcal{E}$, $\mathcal{F}$ be two sequence Schäffer spaces such that $\ell^1 \neq \mathcal{E}$ or $\ell^\infty_0 \not\subset \mathcal{F}$, Φ an exponentially bounded cocycle over the semiflow σ and $\theta \in \Theta$. If the pair $(\mathcal{E}, \mathcal{F})$ is θ-admissible to Φ, then $X_{1,\mathcal{F}}(\theta) = X_1(\theta)$.*

Proof. It follows from Theorem 5.33 and Remark 4.24. □

Definition 5.35. *Let Φ be a cocycle over a semiflow σ. We say that the pair $(\mathcal{E}, \mathcal{F})$ is*

(i) uniformly admissible to Φ *if it is θ-admissible to Φ for all $\theta \in \Theta$ and* $K := \sup_{\theta\in\Theta} K(\theta) < \infty$*;*

(ii) uniformly admissible to Φ at the point $\theta_0 \in \Theta$ *if it is θ-admissible to Φ for all $\theta \in \mathcal{O}_\sigma(\theta_0)$ and $K_{\theta_0} := \sup_{\theta\in\mathcal{O}_\sigma(\theta_0)} K(\theta) < \infty$.*

Remark 5.36. *If the pair $(\mathcal{E}, \mathcal{F})$ (with the same property as in the Theorem 5.33 above) is uniformly admissible (respectively, uniformly admissible at the point $\theta_0 \in \Theta$) to the exponentially bounded cocycle Φ over a semiflow σ, then Φ has a no past exponential dichotomy on Θ (respectively, a no past exponential dichotomy at the point θ_0). The argument for this statement relies on the proof of Theorem 5.33, since all intermediate constants that occur (namely, $C_{1,\theta}$, $C'_{1,\theta}$, $N'_{1,\theta}$, $C_{2,\theta}$, $C'_{2,\theta}$, $N'_{2,\theta}$) may be evaluated to upper/lower values independent of $K(\theta)$ on Θ (respectively, on $\mathcal{O}_\sigma(\theta_0)$). With this strategy in mind, note that all the obtained constants, namely $N_{1,\theta}$, $N_{2,\theta}$, $\nu_{1,\theta}$, $\nu_{2,\theta}$) may be replaced by constants like N_1, N_2, ν_1, ν_2 which are independent of $\theta \in \Theta$ (respectively, of $\theta \in \mathcal{O}_\sigma(\theta_0)$).*

The next two results follows immediately from the proof of Theorem 5.33 and Remark 5.36.

Corollary 5.37. *Let $\mathcal{E}$, $\mathcal{F}$ be two sequence Schäffer spaces such that $\ell^1 \neq \mathcal{E}$ or $\ell^\infty_0 \not\subset \mathcal{F}$, Φ an exponentially bounded cocycle over the semiflow σ and $\theta_0 \in \Theta$. If the pair $(\mathcal{E}, \mathcal{F})$ is uniformly admissible to Φ at the point θ_0, then Φ has a no past exponential dichotomy at the point θ_0.*

Corollary 5.38. *Let $\mathcal{E}$, $\mathcal{F}$ be two sequence Schäffer spaces such that $\ell^1 \neq \mathcal{E}$ or $\ell^\infty_0 \not\subset \mathcal{F}$, Φ an exponentially bounded cocycle over the semiflow σ. If the pair $(\mathcal{E}, \mathcal{F})$ is uniformly admissible to Φ, then Φ has a no past exponential dichotomy on Θ.*

To show that condition "$E \neq \ell^1$ or $\ell^\infty_0 \not\subset \mathcal{F}$" in the statement of Theorem 5.33 is essential we give the following trivial example.

Example 5.39. *Let $X = \mathbb{R}$ and consider a semiflow σ on $\Theta = \mathbb{R}_+$ (see Example 3.13). Define $\Phi(\theta, t) = I_{\mathbb{R}}$ for any $(\theta, t) \in \Theta \times \mathbb{R}_+$. Clearly, Φ is a cocycle over σ independent of $\theta \in \Theta$, therefore fix $\theta \in \Theta$. If $f \in \ell^1$, then there exists $x = -\sum_{k=0}^{\infty} f(k) \in \mathbb{R}$ (the series being absolutely convergent) such that $p_f(\,\cdot\,; x, \theta) \in \ell^\infty_0(X)$, for all $\theta \in \Theta$. Therefore the pair (ℓ^1, ℓ^∞_0) is θ-admissible to Φ, but one can easily check that no matter how we choose the constants $N(\theta)$, $\nu(\theta) > 0$ the conclusion of Theorem 5.33 does not hold.*

Remark 5.40. *In [5] it was introduced the order relation between pairs of Banach spaces: (B_1, D_1) is said to be stronger than (B, D) (or the later pair is weaker than the former) if $B \hookrightarrow B_1$ and $D_1 \hookrightarrow D$. In the case of sequence Schäffer spaces the two inclusions with continuous injection are equivalent with the corresponding algebraic inclusions (see Proposition 2.9), so the pair (ℓ^1, ℓ_0^∞) is stronger than $(\mathcal{E}, \mathcal{F})$ if $\mathcal{E} \subset \ell^1$ and $\ell_0^\infty \subset \mathcal{F}$ or, equivalently (considering Proposition 2.8), $\mathcal{E} = \ell^1$ and $\ell_0^\infty \subset \mathcal{F}$. Therefore, the hypothesis "$E \neq \ell^1$ or $\ell_0^\infty \not\subset \mathcal{F}$" can be replaced by "the pair (ℓ^1, ℓ_0^∞) is not stronger than the pair $(\mathcal{E}, \mathcal{F})$".*

Dropping off the restriction over the sequence Schäffer spaces $\mathcal{E}$, $\mathcal{F}$, the proof of the Theorem 5.33 still provides useful information. Of course, for Corollary 5.41 one can formulate analogous results as those of Corollary 5.37 and Corollary 5.38, but we will omit them.

Corollary 5.41. *Let $\mathcal{E}, \mathcal{F}$ be two sequence Schäffer spaces, Φ an exponentially bounded cocycle over the semiflow σ and $\theta \in \Theta$. If the pair $(\mathcal{E}, \mathcal{F})$ is θ-admissible to Φ, then there exist $C_{1,\theta}, C_{2,\theta} > 0$ such that*

$$\|\Phi(\theta, t)x\| \leq C_{1,\theta}\|x\| \ , \ \text{for all } t \geq 0 \text{ and } x \in X_{1,\mathcal{F}}(\theta) \ ,$$
$$\|\Phi(\theta, t)x\| \geq C_{2,\theta}\|x\| \ , \ \text{for all } t \geq 0 \text{ and } x \in X_{2,\mathcal{F}}(\theta) \ .$$

Proof. Note that in the proof of Theorem 5.33, we do not use the hypothesis "$E \neq \ell^1$ or $\ell_0^\infty \not\subset \mathcal{F}$" to prove the relations (5.2) and (5.13). □

Corollary 5.42. *Let $p, q \in [1, \infty]$ such that $(p, q) \neq (1, \infty)$, Φ an exponentially bounded cocycle over the semiflow σ and $\theta \in \Theta$. If the pair (ℓ^p, ℓ^q) is θ-admissible to Φ, then there exist two constants $N = N(\theta)$, $\nu = \nu(\theta) > 0$ such that*

$$\|\Phi(\theta, t)x\| \leq N e^{-\nu t}\|x\| \ , \ \text{for all } t \in \mathbb{R}_+ \text{ and } x \in X_{1,\mathcal{F}}(\theta)$$
$$\|\Phi(\theta, t)x\| \geq \frac{1}{N} e^{\nu t}\|x\| \ , \ \text{for all } t \in \mathbb{R}_+ \text{ and } x \in X_{2,\mathcal{F}}(\theta) \ .$$

Proof. We have the equivalence (due to Proposition 2.10 and the fact that $\beta_{\ell^q}(n) = (n+1)^{\frac{1}{q}}$ for each $n \in \mathbb{N}$): $\ell_0^\infty \subset \ell^q$ if and only if $q = \infty$. Thus, given $(p, q) \neq (1, \infty)$ the hypothesis of Theorem 5.33 is satisfied and the conclusion is immediate. □

5.2. **Characterizations of exponential dichotomies.** With Theorem 5.43 and Theorem 5.44 we prove that if we impose the "$\Phi(\theta, t)$-invariance" (in the sense of Remark 4.25) of $X_{2,\mathcal{F}}(\cdot)$, we can deduce the invertibility of $\Phi(\theta, t)_| : X_2(\theta) \to X_2(\sigma(\theta, t))$.

Theorem 5.43. *Let $\mathcal{E}$, $\mathcal{F}$ be two sequence Schäffer spaces such that $\ell^1 \neq \mathcal{E}$ or $\ell_0^\infty \not\subset \mathcal{F}$, Φ an exponentially bounded cocycle over the semiflow σ and $\theta_0 \in \Theta$. If*

(i) *$Q_{\mathcal{F}}(\sigma(\theta, t))\Phi(\theta, t)Q_{\mathcal{F}}(\theta) = \Phi(\theta, t)Q_{\mathcal{F}}(\theta)$, for all $(\theta, t) \in \mathcal{O}_\sigma(\theta_0) \times \mathbb{R}_+$ and*
(ii) *$(\mathcal{E}, \mathcal{F})$ is uniformly admissible to Φ at the point θ_0,*

then $\Phi(\theta, t)_| : X_2(\theta) \to X_2(\sigma(\theta, t))$ is invertible, for all $(\theta, t) \in \mathcal{O}_\sigma(\theta_0) \times \mathbb{R}_+$. Therefore, Φ has an exponential dichotomy at the point θ_0.

Proof. The condition (i) assures us that the operator $\Phi(\theta, t)_| : X_2(\theta) \to X_2(\sigma(\theta, t))$ is corectly codefined for each $t \geq 0$ and from Corrolary 5.37 we have that Φ has a no past exponential dichotomy in θ_0.

Let $(\theta, t) \in \mathcal{O}_\sigma(\theta_0) \times \mathbb{R}_+$, $n_0 = [t] + 1$, $y \in X_2(\sigma(\theta, t)$ and consider the sequence

$$f : \mathbb{N} \to X \quad , \quad f(n) = -\delta_{n_0}(n)\Phi(\sigma(\theta, t), n_0 - t)y \ , \tag{5.22}$$

which is in $\mathcal{E}(X)$ with $\|f\|_{\mathcal{E}(X)} = \beta_{\mathcal{E}}(0)\|\Phi(\sigma(\theta,t),n_0-t)y\|$. Then, according to Proposition 5.30, there exists a unique $x \in X_2(\theta)$ such that $p_f(\,\cdot\,;x,\theta) \in \mathcal{F}(X)$. Since,

$$\begin{aligned} p_f(n;x,\theta) &= \Phi(\theta,n)x - \sum_{k=0}^{n} \delta_{n_0}(k)\Phi(\sigma(\theta,k),n-k)f(k) \\ &= \Phi(\theta,n)x - \Phi(\sigma(\theta,n_0),n-n_0)\Phi(\sigma(\theta,t),n_0-t)y \\ &= \Phi(\theta,n)x - \Phi(\sigma(\theta,t),n-t)y \\ &= \Phi(\sigma(\theta,t),n-t)(\Phi(\theta,t)x-y) \quad , \end{aligned} \tag{5.23}$$

for all $n \in \mathbb{N}$, $n \geq n_0$. We have that $\Phi(\theta,t)x\,,\, y \in X_2(\sigma(\theta,t))$ and

$$\|p_f(n;x,\theta)\| \geq \frac{1}{N}e^{\nu(n-t_0)}\|\Phi(\theta,t)x-y\|\ ,\ \text{for all } n \geq n_0$$

(where $N, \nu > 0$ are given by Theorem 5.33).

If we assume that $\Phi(\theta,t)x - y \neq 0$, then $\lim_{n\to\infty}\|p_f(n;x,\theta)\| = \infty$ contradicting $p_f(\,\cdot\,;x,\theta) \in \ell^\infty(X)$. It follows that $\Phi(\theta,t)x = y$. We proved that $\Phi(\theta,t)_|$ is onto and since the one-to-one property was already proved in the Proposition 5.29, we have that it is invertible.

Since only the invertibility of operators $\Phi(\theta,t)$ restricts the no past exponential dichotomy to be an exponential dichotomy, we completed the proof. □

Theorem 5.44. *Let $\mathcal{E}$, $\mathcal{F}$ be two sequence Schäffer spaces such that $\ell^1 \neq \mathcal{E}$ or $\ell_0^\infty \not\subset \mathcal{F}$, Φ an exponentially bounded cocycle over the semiflow σ. If*

(i) *$Q_{\mathcal{F}}(\sigma(\theta,t))\Phi(\theta,t)Q_{\mathcal{F}}(\theta) = \Phi(\theta,t)Q_{\mathcal{F}}(\theta)$, for all $(\theta,t) \in \Theta\times\mathbb{R}_+$ and*

(ii) *$(\mathcal{E},\mathcal{F})$ is uniformly admissible to Φ,*

then $\Phi(\theta,t)_| : X_2(\theta) \to X_2(\sigma(\theta,t))$ is invertible, for all $(\theta,t) \in \Theta\times\mathbb{R}_+$. Therefore, Φ has an exponential dichotomy on Θ.

Proof. It is similar with the proof of Theorem 5.43. □

In what follows we try to answer concerns regarding the converse of what was obtained with Theorem 5.43, respectively Theorem 5.44. It will be clear (see Example 5.53 below) that for a cocycle Φ that has exponential dichotomy, not every pair $(\mathcal{E},\mathcal{F})$ of sequence Schäffer spaces is admissible to Φ.

Lemma 5.45. *If the exponentially bounded cocycle Φ (over a semiflow σ) has a no past exponential dichotomy, then $\sup_{\theta\in\Theta}\|P(\theta)\| < \infty$ (where $\{P(\theta)\}_{\theta\in\Theta}$ is provided by the Definition 4.18).*

Proof. We take $x_1 \in Im\,P(\theta)$ and $x_2 \in Ker\,P(\theta)$ with $||x_1|| = ||x_2|| = 1$. Recall that the angular distance between $Im\,P(\theta)$ and $Ker\,P(\theta)$ is defined by

$$\gamma[ImP(\theta),KerP(\theta)] = \inf\{\|\frac{x}{||x||} - \frac{y}{||y||}\| \,:\, x \in ImP(\theta), y \in KerP(\theta), x,y \neq 0\}.$$

But

$$||x_1 - x_2|| \geq \frac{1}{Me^{\omega t}}||\Phi(\theta,t)x_2 - \Phi(\theta,t)x_1|| \geq \frac{1}{Me^{\omega t}}\Big(Ne^{\nu t} - \frac{1}{N}e^{-\nu t}\Big).$$

Choose $t_0 > 0$ such that $Ne^{\nu t_0} - \frac{1}{N}e^{-\nu t_0} = \psi_0 > 0$. Then

$$||x_1 - x_2|| \geq \psi = \frac{\psi_0}{Me^{\omega t_0}},$$

and thus $\gamma[Im\,P(\theta), Ker\,P(\theta)] \geq \psi$, for all $\theta \in \Theta$. Taking into account that

$$\frac{1}{||P(\theta||} \leq \gamma[Im\,P(\theta), Ker\,P(\theta)] \leq \frac{2}{||P(\theta)||}$$

(see [18, (11.D), p. 8]) it follows that $\sup\limits_{\theta\in\Theta}||P(\theta)|| < \infty$.

□

Lemma 5.46. *If the exponentially bounded cocycle Φ (over a semiflow σ) has a no past exponential dichotomy at the point $\theta_0 \in \Theta$, then $\sup_{t\geq 0}\|P_{\theta_0}(t)\| < \infty$ (where $\{P_{\theta_0}(t)\}_{t\geq 0}$ is provided by the Definition 4.21).*

Proof. It is similar with the proof of Lemma 5.45. □

Remark 5.47. *If $\mathcal{E}$, $\mathcal{F}$ are sequence Schäffer spaces such that $L\mathcal{E} \subset \mathcal{F}$, then $\mathcal{E} \subset \mathcal{F}$ and $\|Lf\|_{\mathcal{F}} \leq \|f\|_{\mathcal{F}}$ for all $f \in \mathcal{E}$.*

Proof. Let $f \in \mathcal{E}$. Then, $Rf \in \mathcal{E}$ and therefore $f = L(Rf) \in \mathcal{F}$ and $Lf \in \mathcal{F}$. Since, $|RLf(m)| \leq |f(m)|$ for each $m \in \mathbb{N}$, we deduce that $\|Lf\|_{\mathcal{F}} = \|RLf\|_{\mathcal{F}} \leq \|f\|_{\mathcal{F}}$. □

Theorem 5.48. *Let $\mathcal{E}$, $\mathcal{F}$ be two sequence Schäffer spaces such that $L(\mathcal{E}\cup\mathcal{F}) \subset \mathcal{F}$. If the cocycle Φ (over the semiflow σ) has an exponential dichotomy at the point $\theta_0 \in \Theta$, then the pair $(\mathcal{E},\mathcal{F})$ is uniformly admissible to Φ at the point θ_0.*

Proof. Let $\{P_{\theta_0}(t)\}_{t\geq 0}$ be the family of projectors provided by the Definition 4.20. We have that there exist $N, \nu > 0$ such that

$$\|\Phi(\sigma(\theta_0,t_0),t)P_{\theta_0}(t_0)x\| \leq Ne^{-\nu t}\|P_{\theta_0}(t_0)x\| \quad \text{and}$$

$$\|\Phi(\sigma(\theta_0,t_0),t)_{|}^{-1}Q_{\theta_0}(t_0+t)x\| \leq Ne^{-\nu t}\|Q_{\theta_0}(t_0+t)x\| \ ,$$

for all $t_0, t \in \mathbb{R}_+$ and $x \in X$ (where $Q_{\theta_0}(t) = I - P_{\theta_0}(t)$). From Lemma 5.46, we have that $C_P := \sup_{t\geq 0}\|P_{\theta_0}(t)\| < \infty$ and $C_Q := \sup_{t\geq 0}\|Q_{\theta_0}(t)\| < \infty$.

We fix arbitrarily $t_0 \in \mathbb{R}_+$ in order to prove the θ-admissibility of the pair $(\mathcal{E},\mathcal{F})$ to Φ, where $\theta = \sigma(\theta_0,t_0)$. For the simplicity of notation, put $P(t) = P_{\theta_0}(t_0+t)$ and $Q(t) = Q_{\theta_0}(t_0+t)$ for all $t \geq 0$. Let $f \in \mathcal{E}(X)$ and consider the sequence $\varphi : \mathbb{N} \to X$ by

$$\varphi(n) = \sum_{k=0}^{n}\Phi(\sigma(\theta,k),n-k)P(k)f(k) \ - \sum_{k=n+1}^{\infty}\Phi(\sigma(\theta,n),k-n)_{|}^{-1}Q(k)f(k) \ . \tag{5.24}$$

which is corectly defined, since $f \in \ell^{\infty}(X)$ and

$$\begin{aligned}\sum_{k=n+1}^{\infty}\|\Phi(\sigma(\theta,n),k-n)_{|}^{-1}Q(k)f(k)\| &\leq \frac{1}{N}\sum_{k=n+1}^{\infty}e^{-\nu(k-n)}\|Q(k)\|\,\|f(k)\| \\ &\leq \sup_{k\in\mathbb{N}}\|Q(k)\|\,\|f\|_{\infty}\,\frac{e^{-\nu}}{N(1-e^{-\nu})} \leq C_Q\,\frac{e^{-\nu}}{N(1-e^{-\nu})}\,\|f\|_{\infty} \ ,\end{aligned}$$

for all $n \in \mathbb{N}$. Also,

$$\begin{aligned}\|\varphi(n)\| &\leq N\sup_{t\geq 0}\|P(t)\|\sum_{k=0}^{n}e^{-\nu(n-k)}\|f(k)\| \ + \ N^{-1}\sup_{t\geq 0}\|Q(t)\|\sum_{k=n+1}^{\infty}e^{-\nu(k-n)}\|f(k)\| \\ &\leq C\left(\sum_{k=0}^{n}e^{-\nu k}\|R^k f\|(n) + \sum_{k=1}^{\infty}e^{-\nu k}\|L^k f\|(n)\right) \ ,\end{aligned} \tag{5.25}$$

for all $n \in \mathbb{N}$ (where $C := NC_P + N^{-1}C_Q$). ¿From the hypothesis and Remark 5.47 we have that $f, R^k f, L^k f \in \mathcal{F}(X)$ with $\|L^k f\|_{\mathcal{F}(X)} \le \|f\|_{\mathcal{F}(X)}$ for all $k \in \mathbb{N}$. Therefore,

$$\begin{aligned}\sum_{k=1}^{\infty} \| e^{-\nu k}\|L^k f\| \|_{\mathcal{F}} \le \sum_{k=1}^{\infty} e^{-\nu k}\|f\|_{\mathcal{F}(X)} = \frac{e^{-\nu}}{1-e^{-\nu}}\|f\|_{\mathcal{F}(X)} \ ,\\ \sum_{k=0}^{\infty} \| e^{-\nu k}\|R^k f\| \|_{\mathcal{F}} = \sum_{k=0}^{\infty} e^{-\nu k}\|f\|_{\mathcal{F}(X)} = \frac{1}{1-e^{-\nu}}\|f\|_{\mathcal{F}(X)}\end{aligned} \tag{5.26}$$

and thus $g := \sum_{k=0}^{n} e^{-\nu k}\|R^k f\| + \sum_{k=1}^{\infty} e^{-\nu k}\|L^k f\|$ exists as an element in $\mathcal{F}$.

¿From (5.25) we have that $\|\varphi(n)\| \le Cg(n)$ for all $n \in \mathbb{N}$ and therefore, $\varphi \in \mathcal{F}(X)$. Note that for $k \ge n$, since $\Phi(\theta,k) = \Phi(\sigma(\theta,n),k-n)\Phi(\theta,n)$ we have that $\Phi(\sigma(\theta,n),k-n)_{|}^{-1} = \Phi(\theta,n)\Phi(\theta,k)_{|}^{-1}$. Then,

$$\begin{aligned}\varphi(n) &= \sum_{k=0}^{n} \Phi(\sigma(\theta,k),n-k)P(k)f(k) \ - \ \sum_{k=0}^{\infty} \Phi(\theta,n)\Phi(\theta,k)_{|}^{-1}Q(k)f(k) \ + \\ &\qquad + \sum_{k=0}^{n} \Phi(\sigma(\theta,k),n-k)\Phi(\theta,k)\Phi(\theta,k)_{|}^{-1}Q(k)f(k)\\ &= \Phi(\theta,n)\left(-\sum_{k=0}^{\infty}\Phi(\theta,k)^{-1}Q(k)f(k)\right) + \sum_{k=0}^{n}\Phi(\sigma(\theta,k),n-k)f(k)\\ &= p_f(n;x,\theta) \ , \ \text{ for all } n \in \mathbb{N} \ ,\end{aligned} \tag{5.27}$$

where $x = -\sum_{k=0}^{\infty}\Phi(\theta,k)^{-1}Q(k)f(k) \in X_2(\theta)$ (the series being absolutely convergent). It follows that there exists $x \in X_2(\theta)$ such that $p_f(\,\cdot\,;x,\theta) = \varphi \in \mathcal{F}(X)$.

We proved that the pair $(\mathcal{E},\mathcal{F})$ is θ-admissible to Φ, for any $\theta \in \mathcal{O}_\sigma(\theta_0)$. Note that

$$\|\varphi(n)\| \le NC_P\frac{1}{1-e^{-\nu}}\|f\|_\infty + N^{-1}C_Q\frac{e^{-\nu}}{1-e^{-\nu}}\|f\|_\infty \le \frac{C}{\beta_{\mathcal{E}}(0)(1-e^{-\nu})}\|f\|_{E(X)} \ ,$$

which implies that the pair $(\mathcal{E},\mathcal{F})$ is uniformly admissible to Φ at the point θ_0.

□

Remark 5.49. *Let us examine more closely the condition $L(\mathcal{E}\cup\mathcal{F}) \subset \mathcal{F}$ and the proof of the above theorem. The condition is in fact equivalent with $L\mathcal{F} \subset \mathcal{F}$ and $\mathcal{E} \subset \mathcal{F}$ (see Remark 5.47). If we keep $\mathcal{E} \subset \mathcal{F}$ and prove (in some other setting) that $\varphi \in \mathcal{E}$ the argument presented above still works. Such a new setting is given by $L\mathcal{E} \subset \mathcal{E}$ (the space $\mathcal{E}$ is invariant under the left shift), since the series defining the element g will be absolutely convergent in $\mathcal{E}$.*

In the following result we put all the pieces together to provide a necessary and sufficient condition for pointwise exponential dichotomies of an exponentially bounded cocycle Φ over a semiflow σ.

Corollary 5.50. *Let Φ be an exponentially bounded cocycle over the semiflow σ, $\theta_0 \in \Theta$ and $\mathcal{E}$, $\mathcal{F}$ be two sequence Schäffer spaces such that*

(*i*) $\Phi(\sigma(\theta_0,t_0),t)P_{\theta_0}(t_0) = P_{\theta_0}(t_0+t)\Phi(\sigma(\theta_0,t_0),t)$, *for all* $t_0, t \in \mathbb{R}_+$;
(*ii*) $\mathcal{E} \subset \mathcal{F}$;
(*iii*) $L\mathcal{E} \subset \mathcal{E}$ *or* $L\mathcal{F} \subset \mathcal{F}$;
(*iv*) $\ell^1 \ne \mathcal{E}$ *or* $\ell_0^\infty \not\subset \mathcal{F}$.

Then, Φ has an exponential dichotomy at the point θ_0 if and only if the pair $(\mathcal{E},\mathcal{F})$ is uniformly admissible to Φ at the point θ_0.

Proof. The *necessity* follows from Theorem 5.48 and Remark 5.49, while the *sufficiency* follows from Theorem 5.43. □

Theorem 5.51. *Let $\mathcal{E}$, $\mathcal{F}$ be two sequence Schäffer spaces such that $L(\mathcal{E}\cup\mathcal{F})\subset\mathcal{F}$. If the cocycle Φ (over the semiflow σ) has an exponential dichotomy on Θ, then the pair $(\mathcal{E},\mathcal{F})$ is uniformly admissible to Φ.*

Proof. It results in a similar way as the proof of Theorem 5.48.

Indeed, let $\{P(\theta)\}_{\theta\in\Theta}$ be the family of projectors provided by the Definition 4.17. We have that there exist $N,\nu>0$ such that

$$\|\Phi(\theta,t)P(\theta)x\|\leq Ne^{-\nu t}\|P(\theta)x\| \quad \text{and} \quad \|\Phi(\theta,t)_{|}^{-1}Q(\sigma(\theta,t))x\|\leq Ne^{-\nu t}\|Q(\sigma(\theta,t))x\| \ ,$$

for all $(\theta,t)\in\Theta\times\mathbb{R}_+$ and $x\in X$ (where $Q(\theta)=I-P(\theta)$). ¿From Lemma 5.45, we have that $C_P:=\sup_{\theta\in\Theta}\|P(\theta)\|<\infty$ and $C_Q:=\sup_{\theta\in\Theta}\|Q(\theta)\|<\infty$.

Now fix arbitrarily $\theta\in\Theta$ and for $f\in\mathcal{E}(X)$, consider the sequence $\varphi:\mathbb{N}\to X$ by

$$\varphi(n)=\sum_{k=0}^{n}\Phi(\sigma(\theta,k),n-k)P(\sigma(\theta,k))f(k) \ - \sum_{k=n+1}^{\infty}\Phi(\sigma(\theta,n),k-n)_{|}^{-1}Q(\sigma(\theta,k))f(k) \ .$$

The rest of the proof follows using similar arguments to those employed in the above mentioned proof.

□

Corollary 5.52. *Let Φ be an exponentially bounded cocycle over the semiflow σ and $\mathcal{E}$, $\mathcal{F}$ be two sequence Schäffer spaces such that*

(*i*) *$\Phi(\theta,t)P(\theta)=P(\sigma(\theta,t))\Phi(\theta,t)$, for all $(\theta,t)\in\Theta\times\mathbb{R}_+$;*
(*ii*) *$\mathcal{E}\subset\mathcal{F}$;*
(*iii*) *$L\mathcal{E}\subset\mathcal{E}$ or $L\mathcal{F}\subset\mathcal{F}$;*
(*iv*) *$\ell^1\neq\mathcal{E}$ or $\ell^\infty_0\not\subset\mathcal{F}$.*

Then, Φ has an exponential dichotomy on Θ if and only if the pair $(\mathcal{E},\mathcal{F})$ is uniformly admissible to Φ.

Proof. The *necessity* follows from Theorem 5.51 and Remark 5.49, while Theorem 5.44 proves the *sufficiency.* □

The following example will convince us that the condition $\mathcal{E}\subset\mathcal{F}$ in Theorem 5.48, Theorem 5.51 cannot be dropped.

Example 5.53. *Let Θ be a metric space and σ a semiflow over Θ and take the state space as $(\mathbb{R},|\cdot|)$. Consider the cocycle Φ over the semiflow σ by $\Phi(\theta,t):\mathbb{R}\to\mathbb{R}$, $\Phi(\theta,t)x=e^{-t}x$ for all $x\in\mathbb{R}$. Since the cocycle is independent of $\theta\in\Theta$, the following computations will be carried out for an arbitrarily fixed $\theta_0\in\Theta$.*

We have that $\|\Phi(\theta_0,t)x\|=e^{-t}|x|$ for all $t\geq 0$ and $x\in\mathbb{R}$. Therefore, Φ is exponentially stable at the point θ_0 (thus it has an exponential dichotomy at every point θ_0).

Consider the sequence $f:\mathbb{N}\to\mathbb{R}$, $f(n)=\frac{1}{n+1}$ with the property: $f\in\ell^2\setminus\ell^1$. For any $x\in\mathbb{R}$ we have that

$$p_f(n;x,\theta_0)=\Phi(\theta_0,n)x+\sum_{k=0}^{n}\Phi(\sigma(\theta_0,k),n-k)f(k)=e^{-n}x+\sum_{k=0}^{n}e^{-(n-k)}\frac{1}{k+1} \ ,$$

which implies $|p_f(n;x,\theta_0)|\geq\frac{1}{n+1}-e^{-n}|x|$, for all $n\in\mathbb{N}$. Then, $\sum_{n=0}^{\infty}|p_f(n;x,\theta_0)|\geq\sum_{n=0}^{\infty}\frac{1}{n+1}-\frac{|x|}{1-e^{-1}}$. We have that, $p_f(\,\cdot\,;x,\theta_0)\notin\ell^1$ for any $x\in X$ and $\theta_0\in\Theta$. Thus, the pair (ℓ^2,ℓ^1) is not θ_0-admissible to Φ.

Remark 5.54. *It is worth to note that so far, it has been extensively analyzed the asymptotic behavior of exponentially bounded, strongly continuous cocycles over flows. The main results in this direction are focused on the characterization of exponential dichotomy of an exponentially bounded, strongly continuous cocycles over a flow in terms of SackerSell spectral properties* [30] *or the hyperbolicity of the associated evolution semigroups and their generators* [11]. *In particular, a characterization of exponential dichotomy for cocycles over flows was given in* [30] *assuming the dimension of the unstable manifold to be finite. Meanwhile, in* [11] *a characterization is given through the hyperbolicity of the associated evolution semigroup and its generator. Another characterization in* [3] *uses a discrete cocycle over a discretized flow. In this paper we made an attempt to characterize the exponential dichotomy in a more general setting and we did consider an exponentially bounded cocycle over a semiflow, i.e. there is only a semiflow on the base space. This setting is particularly appropriate in the infinite-dimensional case since in this case the dynamical systems restricted to invariant manifolds are only semiflows in general.*

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