APPROXIMATE CONTROLLABILITY FOR FRACTIONAL DIFFERENTIAL EQUATIONS OF SOBOLEV TYPE VIA PROPERTIES ON RESOLVENT OPERATORS

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Abstract. This paper treats the approximate controllability of fractional differential systems of Sobolev type in Banach spaces. We first characterize properties on the norm continuity and compactness of some resolvent operators (also called solution operators). And then via the obtained properties on resolvent operators and fixed point technique, we give some approximate controllability results for Sobolev type fractional differential systems in the Caputo and Riemann-Liouville fractional derivatives with order $1 < \alpha < 2$, respectively. Particularly, the existence or compactness of an operator E^{-1} is not necessarily needed in our results.

1. Introduction

Let A and E be two closed linear operators defined on a Banach space X with domains $D(A)$ and $D(E)$, respectively. Consider the following fractional Sobolev (or also called *degenerate*) type system

$$D_t^\alpha (Eu)(t) = Au(t) + Bv(t) + f(t), \quad Eu(0) = u_0, \tag{1.1}$$

where $0 < \alpha < 1$, $t \in [0, b]$, $b > 0$, f is a continuous function, D_t^α denotes the Caputo fractional derivative, B is a bounded operator and the function v belongs to a Banach space of admissible control functions (see the definition in Section 4).

We observe that the change of variable $w(t) = Eu(t)$ reduces the system (1.1) to the fractional differential system

$$D_t^\alpha w(t) = Lw(t) + Bh(t) + g(t), \quad w(0) = w_0, \quad t \in [0, b], \tag{1.2}$$

where $h(t) = E^{-1}v(t)$, $g(t) = E^{-1}f(t)$, $L = AE^{-1}$ and $D(L) = E(D(A))$. Then, formally the approximate controllability of system (1.1) can be studied by using the system (1.2), which has been studied by several authors, see for instance [5, 14, 21, 30] and the references therein. However, we notice that this change of variable needs the existence of E^{-1} as a bounded operator, which in general is restrictive.

In case $0 < \alpha < 1$, the approximate controllability (and controllability) of fractional differential systems (for the Caputo fractional derivative) in the form of (1.1) has been studied by several authors by assuming that $D(E) \subset D(A)$, E is bijective and $E^{-1} : X \to D(E)$ is a compact operator. In this case, AE^{-1} is a bounded operator and generates a compact C_0-semigroup $T(t) = e^{AE^{-1}t}$, for $t \ge 0$, and there exist two characteristic solution operators $\mathcal{T}_E$ and $\mathcal{S}_E$ given by the following nice subordination formulas

$$\mathcal{T}_E(t) = \int_0^\infty E^{-1}\xi_\alpha(\theta)T(t^\alpha\theta)d\theta, \tag{1.3}$$

$$\mathcal{S}_E(t) = \alpha\int_0^\infty E^{-1}\theta\xi_\alpha(\theta)T(t^\alpha\theta)d\theta, \tag{1.4}$$

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where $t \geq 0$ and

$$\begin{aligned} \xi_\alpha(\theta) &= \frac{1}{\alpha}\theta^{-(1+\frac{1}{\alpha})}\varpi_\alpha(\theta^{-\frac{1}{\alpha}}), \\ \varpi_\alpha(\theta) &= \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha). \end{aligned}$$

By using the representations (1.3) and (1.4) several and interesting results on the approximate controllability of system (1.1) have been obtained for instance in [4, 9, 12, 14, 15, 20, 25, 27] and the references therein. Finally, we mention here that a different approach without the representations (1.3) and (1.4) was done in [8] by using the compactness of a resolvent family $S_\alpha(t)$ for $t > 0$.

On the other hand, in case of Riemann-Liouville fractional derivative with order $0 < \alpha \leq 1$, we refer to the work [19] where the authors study the approximate controllability of the system

$$D^\alpha u(t) = Au(t) + Bv(t) + f(t, u(t)),\ (g_{1-\alpha} * u)(0) = u_0,$$

where D^α denotes the Riemann-Liouville fractional derivative, $u_0 \in X$ and A is the generator of a norm continuous semigroup, by using again the representation (1.3)-(1.4). The case $1 < \alpha < 2$,

$$D^\alpha u(t) = Au(t) + Bv(t) + f(t, u(t)),\ (g_{2-\alpha} * u)(0) = 0,\ (g_{2-\alpha} * u)'(0) = u_1,$$

was considered in [21] assuming the existence of a norm continuous resolvent operator $S_{\alpha,\alpha}(t)$ satisfying $\hat{S}_{\alpha,\alpha}(\lambda) = (\lambda^\alpha - A)^{-1}$. We remark that in this case, the authors do not use a subordination formula as in case $0 < \alpha < 1$.

In this paper we study the approximate controllability of following Sobolev type fractional differential system

$$D_t^\alpha(Eu)(t) = Au(t) + Bv(t) + f(t, u(t)), \quad Eu(0) = u_0, \quad (Eu)'(0) = u_1, \tag{1.5}$$

and

$$D^\alpha(Eu)(t) = Au(t) + Bv(t) + f(t, u(t)),\ E(g_{2-\alpha} * u)(0) = u_0,\ (E(g_{2-\alpha} * u))'(0) = u_1, \tag{1.6}$$

where $t \in [0, b]$, the order $1 < \alpha < 2$, the notations D_t^α and D^α denote, respectively, the Caputo and Riemann-Liouville fractional derivatives, B is a bounded operator, v belongs to a Banach space of admissible control functions and the operators A and E generate a resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ for a suitable $\alpha, \beta > 0$. Here it is not necessarily assumed the existence of E^{-1}.

Through the Laplace transform, it is easy to see that the *mild* solution to the problems (1.5)-(1.6) can be respectively expressed by

$$u(t) = S^E_{\alpha,1}(t)u_0 + S^E_{\alpha,2}(t)u_1 + \int_0^t S^E_{\alpha,\alpha}(t-s)[Bv(s) + f(s, u(s))]ds, \tag{1.7}$$

and

$$u(t) = S^E_{\alpha,\alpha-1}(t)u_0 + S^E_{\alpha,\alpha}(t)u_1 + \int_0^t S^E_{\alpha,\alpha}(t-s)[Bv(s) + f(s, u(s))]ds, \tag{1.8}$$

where, for $\alpha, \beta > 0$, $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ is the *resolvent family* generated by (A, E) (see definition below, Section 2) which Laplace transform satisfies $\hat{S}^E_{\alpha,\beta}(\lambda) = \lambda^{\alpha-\beta}(\lambda^\alpha E - A)^{-1}$ for all $\lambda > 0$.

The approximate controllability of the system (1.5) in case where E is the the identity operator $E = \mathcal{I}$, was studied in [24] and [26] by assuming the compactness of $S^{\mathcal{I}}_{\alpha,1}(t), S^{\mathcal{I}}_{\alpha,2}(t)$ and $S^{\mathcal{I}}_{\alpha,\alpha}(t)$ for all $t > 0$. However, it is not completely clear in what conditions $S^{\mathcal{I}}_{\alpha,1}(t), S^{\mathcal{I}}_{\alpha,2}(t)$ and $S^{\mathcal{I}}_{\alpha,\alpha}(t)$ are compact operators. See also [5] for a finite dimensional approach.

To the best of our knowledge, the approximate controllability of general systems (1.5) and (1.6) in case $1 < \alpha < 2$ (and $E \neq \mathcal{I}$) have not been addressed in the existing literature. Motivated by the above mentioned works, in the present paper, we investigate the approximate controllability of Sobolev type fractional differential systems (1.5) and (1.6). Our approach relies on the norm continuity and compactness of resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ for suitable $\alpha, \beta > 0$, as well as some fixed point techniques. In particular, the existence or compactness of E^{-1} in common is not necessarily needed in our approximate controllability results.

The paper is organized as follows. The Section 2 gives the Preliminaries. In Section 3, we characterize the norm continuity and compactness of $S^E_{\alpha,\beta}(t)$ for $t > 0$ and suitable $\alpha, \beta > 0$. Section 4 treats the approximate controllability of system (1.5) in the Caputo fractional derivative. In Section 5 we consider the approximate controllability of system (1.6) in the Riemann-Liouville fractional derivative. Finally, the Section 6 is devoted to some applications.

2. Preliminaries

In this section, we list some definitions, notations and preliminary results which are used in this paper. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ Banach spaces. We denote by $\mathcal{B}(X,Y)$ the space of all bounded linear operators from X into Y, and denote by $\mathcal{B}(X)$ the space of all bounded linear operators from X into itself. For a closed and linear operator $T : D(T) \subset X \to X$, where $D(T)$ is the domain of T, we denote by $\rho(T)$ its resolvent set and by $R(\lambda, T)$ its resolvent operator, that is, $R(\lambda, T) = (\lambda - T)^{-1}$ which is defined for all $\lambda \in \rho(T)$.

Now, we review some definitions and results on fractional calculus. For $\mu > 0$, we define

$$g_\mu(t) = \begin{cases} \frac{t^{\mu-1}}{\Gamma(\mu)}, & t > 0 \\ 0, & t \leq 0, \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function. We also define $g_0 \equiv \delta_0$, the Dirac delta. For $\mu > 0$, $n = \lceil \mu \rceil$ denotes the smallest integer n greater than or equal to μ. The finite convolution of f and g is denoted by $(f * g)(t) = \int_0^t f(t-s)g(s)ds$.

Definition 2.1. [31] *Let $\alpha > 0$. The α-order Riemann-Liouville fractional integral of u is defined by*

$$J^\alpha u(t) := \int_0^t g_\alpha(t-s)u(s)ds, \quad t \geq 0.$$

Also, we define $J^0 u(t) = u(t)$. Because of the convolution properties, the integral operators $\{J^\alpha\}_{\alpha \geq 0}$ satisfy the following semigroup law: $J^\alpha J^\beta = J^{\alpha+\beta}$ for all $\alpha, \beta \geq 0$.

Definition 2.2. [31] *Let $\alpha > 0$. The α-order Caputo fractional derivative is defined*

$$D^\alpha_t u(t) := \int_0^t g_{n-\alpha}(t-s)u^{(n)}(s)ds,$$

where $n = \lceil \alpha \rceil$.

Definition 2.3. [31] *Let $\alpha > 0$. The α-order Riemann-Liouville fractional derivative of u is defined*

$$D^\alpha u(t) := \frac{d^n}{dt^n} \int_0^t g_{n-\alpha}(t-s)u(s)ds,$$

where $n = \lceil \alpha \rceil$.

We notice that if $\alpha = m \in \mathbb{N}$, then $D^m_t = D^m = \frac{d^m}{dt^m}$.

Throughout this paper we use the notation D^α_t and D^α to the α-fractional derivative of Caputo and Riemann-Liouville, respectively. The Riemann-Liouville derivative operator D^α satisfies

$$D^\alpha J^\alpha u(t) = u(t),$$

and

$$(J^\alpha D^\alpha)u(t) = u(t) - \sum_{k=0}^{n-1} (g_{n-\alpha} * u)^{(k)}(0) g_{\alpha+1+k-n}(t),$$

$n = \lceil \alpha \rceil$. On the other hand, the Caputo derivative operator D_t^α satisfies

$$D_t^\alpha J^\alpha u(t) = u(t),$$

and

$$(J^\alpha D_t^\alpha) u(t) = u(t) - \sum_{k=0}^{n-1} u^{(k)}(0) g_{k+1}(t).$$

For more detailed results on fractional calculus and fractional differential equations, we refer to [1, 2, 3, 13, 17, 22, 29, 31] and references therein.

If we denote by $\hat{f}$ (or $\mathcal{L}(f)$) to the Laplace transform of f, we have the following properties for the fractional derivatives

$$\widehat{D^\alpha u}(\lambda) = \lambda^\alpha \hat{u}(\lambda) - \sum_{k=0}^{n-1} (g_{n-\alpha} * u)^{(k)}(0) \lambda^{n-1-k} \tag{2.1}$$

and

$$\widehat{D_t^\alpha u}(\lambda) = \lambda^\alpha \hat{u}(\lambda) - \sum_{k=0}^{n-1} u^{(k)}(0) \lambda^{\alpha-1-k}, \tag{2.2}$$

where $n = \lceil \alpha \rceil$ and $\lambda \in \mathbb{C}$. For $\alpha, \beta > 0$ and $z \in \mathbb{C}$, the generalized Mittag-Leffler function is defined by

$$e_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

and its Laplace transform $\mathcal{L}$ satisfies

$$\mathcal{L}(t^{\beta-1} e_{\alpha,\beta}(\rho t^\alpha))(\lambda) = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \rho}, \quad \rho \in \mathbb{C}, \mathrm{Re}\lambda > |\rho|^{1/\alpha}.$$

The *E-modified resolvent set of* A, $\rho_E(A)$, is defined by

$$\rho_E(A) := \{\lambda \in \mathbb{C} : (\lambda E - A) : D(A) \cap D(E) \to X$$
$$\text{is invertible and } (\lambda E - A)^{-1} \in \mathcal{B}(X, [D(A) \cap D(E)])\}.$$

The operator $(\lambda E - A)^{-1}$ is called the E-resolvent operator of A.

A strongly continuous family $\{T(t)\}_{t \geq 0} \subseteq \mathcal{B}(X)$ is said to be of type (M, ω) or *exponentially bounded* if there exist constants $M > 0$ and $\omega \in \mathbb{R}$, such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. Observe that, without loss of generality, we can assume $\omega > 0$.

Definition 2.4. *Let $A : D(A) \subseteq X \to X$, $E : D(E) \subseteq X \to X$ be closed linear operators defined on a Banach space X satisfying $D(A) \cap D(E) \neq \{0\}$. Let $\alpha, \beta > 0$. We say that the pair (A, E) is the generator of an (α, β)-resolvent family, if there exist $\omega \geq 0$ and a strongly continuous function $S_{\alpha,\beta}^E : [0, \infty) \to \mathcal{B}(X)$ such that $S_{\alpha,\beta}^E(t)$ is exponentially bounded, $\{\lambda^\alpha : \mathrm{Re}\lambda > \omega\} \subset \rho_E(A)$, and for all $x \in X$,*

$$\lambda^{\alpha-\beta} E \left(\lambda^\alpha E - A\right)^{-1} x = \int_0^\infty e^{-\lambda t} S_{\alpha,\beta}^E(t) x dt, \;\; \mathrm{Re}\lambda > \omega.$$

In this case, $\{S_{\alpha,\beta}^E(t)\}_{t \geq 0}$ is called the (α, β)-resolvent family *generated by the pair (A, E).*

Finally, we recall the following results.

Theorem 2.5 (Mazur Theorem)**.** *If K is a compact subset of a Banach space X, then its convex closure $\overline{\mathrm{conv}(K)}$ is compact.*

Theorem 2.6 (Schauder's fixed point Theorem)**.** *Let C be a nonempty, closed, bounded and convex subset of a Banach space X. Suppose that $\Gamma : C \to C$ is a compact operator. Then Γ has at least a fixed point in C.*

Lemma 2.7. [28, Corollary 2.3] *Let (Ω,μ) be a measure space and $t\in\Omega\to T_t\in\mathcal{B}(X,Y)$ be a strongly integrable function, i.e.,*

$$Tx=\int_\Omega T_t x d\mu(t) \tag{2.3}$$

exists for all $x\in X$ as a Bochner integral and $\int_\Omega \|T_t\|d\mu(t)<\infty$. If μ-almost all T_t in (2.3) are compact, then T is compact.

Lemma 2.8. [10, Proposition 2.1] *Let X,Y be Banach spaces, let $\mathbb{S}:[0,\infty)\to\mathcal{B}(X,Y)$ be strongly continuous, and let $\mathrm{a}\in L^1_{loc}[0,\infty)$ be a scalar function, both a and $\mathbb{S}$ of finite exponential type. Then for every $\omega>\omega_0(\mathbb{S}),\omega_0(\mathrm{a})$ one has*

$$\lim_{N\to\infty}\frac{1}{2\pi i}\int_{\omega-iN}^{\omega+iN} e^{\lambda t}\widehat{(\mathrm{a}*\mathbb{S})}(\lambda)d\lambda=\mathrm{a}*\mathbb{S},$$

in $\mathcal{B}(X,Y)$, uniformly in t from compact subsets of $[0,\infty)$.

3. Norm continuity and compactness of $S^E_{\alpha,\beta}(t)$.

In this section we present some results on the norm continuity and compactness of $S^E_{\alpha,\beta}(t)$ for given $\alpha,\beta>0$.

Proposition 3.9. *Let $\alpha>0$ and $1<\beta\le 2$. Suppose that $\{S^E_{\alpha,\beta}(t)\}_{t\ge0}$ is the (α,β)-resolvent family of type (M,ω) generated by (A,E). Then the function $t\mapsto S^E_{\alpha,\beta}(t)$ is continuous in $\mathcal{B}(X)$ for all $t>0$.*

Proof. Firstly, let $1<\beta<2$. By the uniqueness of the Laplace transform, it is obvious to see that $S^E_{\alpha,\beta}(t)=(g_{\beta-1}*S^E_{\alpha,1})(t)$, for all $t>0$. Now, we take $0<t_0<t_1$. Then

$$\begin{aligned} S^E_{\alpha,\beta}(t_1)-S^E_{\alpha,\beta}(t_0) &= (g_{\beta-1}*S^E_{\alpha,1})(t_1)-(g_{\beta-1}*S^E_{\alpha,1})(t_0)\\ &= \int_{t_0}^{t_1} g_{\beta-1}(t_1-r)S^E_{\alpha,1}(r)dr+\int_0^{t_0}[g_{\beta-1}(t_1-r)-g_{\beta-1}(t_0-r)]S_{\alpha,1^E}(r)dr\\ &=: I_1+I_2. \end{aligned}$$

Since $\beta>1$, $g_\beta(0)=0$ and we obtain

$$\|I_1\| \le \int_{t_0}^{t_1} g_{\beta-1}(t_1-r)\|S^E_{\alpha,1}(r)\|dr\le Me^{\omega t_1}g_\beta(t_1-t_0)\to 0,\ \text{as } t_1\to t_0.$$

On the other hand,

$$\begin{aligned} \|I_2\| &\le \int_0^{t_0}|g_{\beta-1}(t_1-r)-g_{\beta-1}(t_0-r)|\|S^E_{\alpha,1}(r)\|dr\\ &\le Me^{\omega t_1}\int_0^{t_0}|g_{\beta-1}(t_1-r)-g_{\beta-1}(t_0-r)|dr\\ &= Me^{\omega t_1}\int_0^{t_0}|g_{\beta-1}(t_1-t_0+r)-g_{\beta-1}(r)|dr. \end{aligned}$$

Since $1<\beta<2$, the function $r\mapsto g_{\beta-1}(r)$ is decreasing in $[0,\infty)$ and therefore $g_{\beta-1}(r)-g_{\beta-1}(t_1-t_0+r)>0$, for all $r>0$, obtaining

$$\|I_2\|\le Me^{\omega t_1}[g_\beta(t_0)-g_\beta(t_1)+g_\beta(t_1-t_0)]\to 0, \text{as } t_1\to t_0.$$

Therefore $S^E_{\alpha,\beta}(t)$ is continuous for $1<\beta<2$.

Finally, if $\beta=2$, then the uniqueness of the Laplace transform implies

$$S^E_{\alpha,2}(t)x=(g_1*S^E_{\alpha,1})(t)x=\int_0^t S^E_{\alpha,1}(r)xdr,$$

for all $x \in X$. For $0 < t_0 < t_1$ we have

$$\|S^E_{\alpha,2}(t_1)x - S^E_{\alpha,2}(t_0)x\| \leq \int_{t_0}^{t_1} \|S^E_{\alpha,1}(r)x\|dr \leq Me^{\omega t_1}\|x\|(t_1 - t_0) \to \text{ as } t_1 \to t_0, \text{ for all } x \in X.$$

Therefore $\|S^E_{\alpha,2}(t_1) - S^E_{\alpha,2}(t_0)\| \to 0$ as $t_1 \to t_0$. □

Lemma 3.10. *Suppose that the pair (A, E) generates an (α, β)-resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ of type (M, ω). If $\gamma > 0$, then (A, E) also generates an $(\alpha, \beta + \gamma)$-resolvent family of type $\left(\frac{M}{\omega^\gamma}, \omega\right)$.*

Proof. Analogous to the proof of [23, Lemma 3.12]. □

Definition 3.11. *We say that the resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is* compact *if for every $t > 0$, the operator $S^E_{\alpha,\beta}(t)$ is a compact operator.*

The next result gives a compactness criteria of $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$. In what follows, we will assume that $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ is strongly continuous for all $\alpha, \beta > 0$.

Theorem 3.12. *Let $\alpha > 0$, $1 < \beta \leq 2$ and $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ be an (α, β)-resolvent family of type (M, ω) generated by (A, E). Then the following assertions are equivalent*

i) *$S^E_{\alpha,\beta}(t)$ is a compact operator for all $t > 0$.*
ii) *$(\mu E - A)^{-1}$ is a compact operator for all $\mu > \omega^{1/\alpha}$.*

Proof. Suppose that the resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t>0}$ is compact. Let $\lambda > \omega$ be fixed. Then we have

$$\lambda^{\alpha-\beta}(\lambda^\alpha E - A)^{-1} = \int_0^\infty e^{-\lambda t} S^E_{\alpha,\beta}(t)dt,$$

where the integral in the right-hand side exists in the Bochner sense, since $\{S^E_{\alpha,\beta}(t)\}_{t>0}$ is continuous in the uniform operator topology (by Proposition 3.9) we conclude that $(\lambda^\alpha E - A)^{-1}$ is a compact operator by by Lemma 2.7.

Conversely, let $t > 0$ be fixed. Assume that $1 < \beta < 2$. Owing to $\beta > 1$, it follows that $g_{\beta-1} \in L^1_{\text{loc}}[0, \infty)$ and therefore, by Lemma 2.8 we obtain

$$\lim_{N\to\infty} \frac{1}{2\pi i} \int_{\omega-iN}^{\omega+iN} e^{\lambda t}(\widehat{g_{\beta-1} * S^E_{\alpha,1}})(\lambda)d\lambda = (g_{\beta-1} * S^E_{\alpha,1})(t) = S^E_{\alpha,\beta}(t),$$

in $\mathcal{B}(X)$. Hence,

$$\frac{1}{2\pi i} \int_\Gamma e^{\lambda t}\lambda^{\alpha-\beta}(\lambda^\alpha E - A)^{-1}d\lambda = S^E_{\alpha,\beta}(t), \quad t > 0,$$

where Γ is the path consisting of the vertical line $\{\omega + is : s \in \mathbb{R}\}$. By hypothesis and Lemma 2.7, we conclude that $S^E_{\alpha,\beta}(t)$ is compact for all $\alpha > 0$ and $1 < \beta < 2$. Now, in case $\beta = 2$ we observe that in $\mathcal{B}(X)$ we have

$$\lim_{N\to\infty} \frac{1}{2\pi i} \int_{\omega-iN}^{\omega+iN} e^{\lambda t}(\widehat{g_1 * S^E_{\alpha,1}})(\lambda)d\lambda = (g_1 * S^E_{\alpha,1})(t) = S^E_{\alpha,2}(t),$$

by Lemma 2.8, and we conclude that $S^E_{\alpha,2}(t)$ is compact for all $t > 0$, analogously to case $1 < \beta < 2$. □

The Theorem 3.12 implies the following corollary.

Corollary 3.13. *Let $1 < \alpha \leq 2$ and $\{S^E_{\alpha,\alpha}(t)\}_{t\geq 0}$ be an (α, α)-resolvent family of type (M, ω) generated by (A, E). Then the following assertions are equivalent*

i) *$S^E_{\alpha,\alpha}(t)$ is a compact operator for all $t > 0$.*
ii) *$(\mu E - A)^{-1}$ is a compact operator for all $\mu > \omega^{1/\alpha}$.*

To obtain a compactness criteria of $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ and $\{S^E_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ (that is, in case $\beta = 1$ and $\beta = \alpha - 1$) we need an additional hypothesis: the norm continuity of $t \mapsto S^E_{\alpha,1}(t)$ and $t \mapsto S^E_{\alpha,\alpha-1}(t)$, respectively. Again, the proofs follow similarly to Theorem 3.12 and [23, Proposition 3.16 and 3.17].

Proposition 3.14. *Let $1 < \alpha < 2$, and $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ be the $(\alpha, 1)$-resolvent family of type (M, ω) generated by (A, E). Suppose that $S^E_{\alpha,1}(t)$ is continuous in the uniform operator topology for all $t > 0$. Then the following assertions are equivalent*

i) *$S^E_{\alpha,1}(t)$ is a compact operator for all $t > 0$.*
ii) *$(\mu E - A)^{-1}$ is a compact operator for all $\mu > \omega^{1/\alpha}$.*

Proposition 3.15. *Let $\frac{3}{2} < \alpha < 2$, and $\{S^E_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ be the $(\alpha, \alpha - 1)$-resolvent family of type (M, ω) generated by (A, E). Suppose that $S^E_{\alpha,\alpha-1}(t)$ is continuous in the uniform operator topology for all $t > 0$. Then the following assertions are equivalent*

i) *$S^E_{\alpha,\alpha-1}(t)$ is a compact operator for all $t > 0$.*
ii) *$(\mu E - A)^{-1}$ is a compact operator for all $\mu > \omega^{1/\alpha}$.*

4. Approximate Controllability. The Caputo case.

In this section we study the approximate controllability of the system, for the Caputo fractional derivative, given by

$$\left\{\begin{array}{lcl} D^\alpha_t(Eu)(t) & = & Au(t) + Bv(t) + f(t, u(t)), \quad t \in I := [0, b] \\ Eu(0) & = & u_0 \\ (Eu)'(0) & = & u_1, \end{array}\right. \tag{4.4}$$

where $u_0, u_1 \in X$, $1 < \alpha < 2$, $b > 0$ and A and E are closed linear operators defined on X which generates the $(\alpha, 1)$-resolvent family $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$, the state $u(\cdot)$ takes values in X and the control function $v(\cdot)$ is given in $\mathcal{V}$, the Banach space of admissible control functions, where

$$\mathcal{V} = L^2(I, V)$$

and V is a Banach space. The operator $B : V \to X$ is assumed to be a bounded linear operator. The function f will be specified later.

Definition 4.16. *For each $v \in \mathcal{V}$ and $u_0, u_1 \in X$, a function $u \in C(J, X)$ is said to be a mild solution to system (4.4) if*

$$u(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t - s)[Bv(s) + f(s, u(s))]ds,$$

for all $t \in [0, b]$.

Remark 4.17. *We notice that, from the uniqueness of the Laplace transform, it is easy to see that the mild solution to* (4.4) *can be written as*

$$u(t) \quad = \quad S^E_{\alpha,1}(t)u_0 + S^E_{\alpha,2}(t)u_1 + \int_0^t S^E_{\alpha,\alpha}(t - s)[Bv(s) + f(s, u(s))]ds, \quad t \in [0, b].$$

Definition 4.18. *Let u be a mild solution of the fractional system* (4.4) *corresponding to the control v. The system* (4.4) *is said to be* approximately controllable *on the interval I if for desired final state $x_0 \in X$ and $\varepsilon > 0$ there exists a control $v \in \mathcal{V}$ such that $\|u(b) - x_0\| < \varepsilon$. The set*

$$\mathcal{K}_b(f) := \{u(b) \in X : v \in \mathcal{V}, u \text{ is the mild solution of } (4.4) \text{ with control } v\}.$$

is called the reachable set of system (4.4).

We notice that the system (4.4) is approximately controllable on if and only if $\mathcal{K}_b(f)$ is dense in X.

Now, we introduce the following operators. The operator $\Gamma_0^b : X \to X$ is defined by

$$\Gamma_0^b = \int_0^b S_{\alpha,\alpha}^E(b-s)BB^*S_{\alpha,\alpha}^{E^*}(b-s)ds,$$

and $R(\nu,\Gamma_0^b)$ is defined from X into X by

$$R(\nu,\Gamma_0^b) = (\nu I + \Gamma_0^b)^{-1},$$

where B^* and $S_{\alpha,\alpha}^{E^*}(t)$ denote the adjoint operators of B and $S_{\alpha,\alpha}^E(t)$ respectively.

It will be shown that for every $\nu > 0$ and $x_0 \in X$ there exists a continuous function $u(\cdot) \in C(I,X)$ such that

$$\begin{aligned} u(t) &= S_{\alpha,1}^E(t)u_0 + (g_1 * S_{\alpha,1}^E)(t)u_1 + \int_0^t (g_{\alpha-1} * S_{\alpha,1}^E)(t-s)Bv(s)ds \\ &+ \int_0^t (g_{\alpha-1} * S_{\alpha,1}^E)(t-s)f(s,u(s))ds, \end{aligned} \tag{4.5}$$

where the function v is the control function defined by

$$v(t) = B^*S_{\alpha,\alpha}^{E^*}(b-t)R(\nu,\Gamma_0^b)p(u(\cdot)), \tag{4.6}$$

and

$$p(u(\cdot)) = x_0 - S_{\alpha,1}^E(b)u_0 - (g_1 * S_{\alpha,1}^E)(b)u_1 - \int_0^b (g_{\alpha-1} * S_{\alpha,1}^E)(b-s)f(s,u(s))ds.$$

Now, we assume the following hypotheses:

(H_1) The pair (A,E) generates the $(\alpha,1)$-resolvent family $\{S_{\alpha,1}^E(t)\}_{t\geq 0}$ of type (M,ω), the operator $(\lambda^\alpha E - A)^{-1}$ is compact for all $\lambda^\alpha \in \rho_E(A)$ and $\{S_{\alpha,1}^E(t)\}_{t\geq 0}$ is norm continuous for all $t > 0$.

(H_2) There exists a continuous function $\mu : I \to \mathbb{R}_+$ such that

$$\|f(t,u)\| \leq \mu(t)\|u\|, \quad \forall\, t \in I,\ u \in C(I,X).$$

Now, we shall prove that the operator $\mathcal{P} : C(I,X) \to C(I,X)$ defined by

$$\mathcal{P}u(t) := S_{\alpha,1}^E(t)u_0 + (g_1 * S_{\alpha,1}^E)(t)u_1 + \int_0^t (g_{\alpha-1} * S_{\alpha,1}^E)(t-s)[Bv(s) + f(s,u(s))]ds,$$

has a fixed point. For $R > 0$ we denote by B_R the set $B_R := \{u \in C(I,X) : \|u(t)\| \leq R, t \in I\}$.

Under the assumptions (H_1)-(H_2) we prove the following results.

Lemma 4.19. *Let $\rho > 0$ be given by*

$$\rho = \frac{Me^{\omega b}\|\mu\|_\infty b}{\omega^{\alpha-1}}\left[1 + \frac{Mb\|B\|e^{\omega b}}{\omega^{\alpha-1}}\frac{M\|B\|}{\nu\omega^{\alpha-1}}e^{\omega b}\right]. \tag{4.7}$$

If $\rho < 1$, then there exists a constant $R \geq \frac{C}{1-\rho}$, where

$$C = Me^{\omega b}\left[\|u_0\| + \frac{\|u_1\|}{\omega}\right] + \frac{Mb\|B\|e^{\omega b}}{\omega^{\alpha-1}}\frac{M\|B\|}{\nu\omega^{\alpha-1}}e^{\omega b}\left[\|x_0\| + Me^{\omega b}\|u_0\| + \frac{Me^{\omega b}}{\omega}\|u_1\|\right],$$

such that $\mathcal{P}B_R \subset B_R$.

Proof. Take $u \in B_R$. For all $t \in [0,b]$ we have

$$\begin{aligned}
\|\mathcal{P}u(t)\| &\leq \|S^E_{\alpha,1}(t)\|\,\|u_0\| + \|(g_1 * S^E_{\alpha,1})(t)\|\,\|u_1\| + \int_0^t \|(g_{\alpha-1} * S^E_{\alpha,1})(t-s)\|\,\|Bv(s) + f(s,u(s))\|ds \\
&\leq Me^{\omega t}\|u_0\| + \frac{Me^{\omega t}}{\omega}\|u_1\| + \frac{M}{\omega^{\alpha-1}}\int_0^t e^{\omega(t-s)}\|Bv(s)+f(s,u(s))\|ds \\
&\leq Me^{\omega b}\|u_0\| + \frac{Me^{\omega b}}{\omega}\|u_1\| + \frac{M\|B\|}{\omega^{\alpha-1}}\int_0^t e^{\omega(t-s)}\|v(s)\|ds + \frac{M\|u\|}{\omega^{\alpha-1}}\int_0^t e^{\omega(t-s)}\mu(s)ds \\
&\leq Me^{\omega b}\|u_0\| + \frac{Me^{\omega b}}{\omega}\|u_1\| + \frac{M\|B\|}{\omega^{\alpha-1}}\int_0^t e^{\omega(t-s)}\|v(s)\|ds + \frac{MRe^{\omega b}}{\omega^{\alpha-1}}\int_0^t e^{-\omega s}\mu(s)ds \\
&\leq Me^{\omega b}\|u_0\| + \frac{Me^{\omega b}}{\omega}\|u_1\| + \frac{M\|B\|}{\omega^{\alpha-1}}\int_0^t e^{\omega(t-s)}\|v(s)\|ds + \frac{MRe^{\omega b}}{\omega^{\alpha-1}}\|\mu\|_\infty b \\
&\leq Me^{\omega b}\|u_0\| + \frac{Me^{\omega b}}{\omega}\|u_1\| + \frac{M\|B\|e^{\omega b}}{\omega^{\alpha-1}}\int_0^b \|v(s)\|ds + \frac{MRe^{\omega b}}{\omega^{\alpha-1}}\|\mu\|_\infty b.
\end{aligned}$$

Next, we notice that the control v satisfies

$$\begin{aligned}
\|v(t)\| &\leq \frac{\|B\|M}{\nu\omega^{\alpha-1}}e^{\omega b}\|p(u(\cdot))\| \\
&\leq \frac{\|B\|M}{\nu\omega^{\alpha-1}}e^{\omega b}\left[\|x_0\| + Me^{\omega b}\|u_0\| + \frac{Me^{\omega b}}{\omega}\|u_1\| + \frac{MRe^{\omega b}}{\omega^{\alpha-1}}\|\mu\|_\infty b\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathcal{P}u(t)\| &\leq Me^{\omega b}\|u_0\| + \frac{Me^{\omega b}}{\omega}\|u_1\| + \frac{MRe^{\omega b}}{\omega^{\alpha-1}}\|\mu\|_\infty b \\
&+ \frac{Mb\|B\|e^{\omega b}}{\omega^{\alpha-1}}\frac{M\|B\|}{\nu\omega^{\alpha-1}}e^{\omega b}\left[\|x_0\| + Me^{\omega b}\|u_0\| + \frac{Me^{\omega b}}{\omega}\|u_1\| + \frac{MRe^{\omega b}}{\omega^{\alpha-1}}\|\mu\|_\infty b\right],
\end{aligned}$$

which implies $\mathcal{P}B_R \subset B_R$. □

Lemma 4.20. *For every $t \in I$, the set $H(t) := \{(\mathcal{P}u)(t) : u \in B_R\}$ is relatively compact.*

Proof. Obviously, $H(0)$ is relatively compact in X.

Now, we take $t > 0$. For $0 < \varepsilon < t$ we define on B_R the operator

$$(\mathcal{P}_2^\varepsilon u)(t) : = \int_0^{t-\varepsilon}(g_{\alpha-1} * S^E_{\alpha,1})(t-s)[Bv(s) + f(s,u(s))]ds.$$

The hypotheses (H_1) and Corollary 3.13 imply the compactness of $(g_{\alpha-1} * S^E_{\alpha,1})(t) = S^E_{\alpha,\alpha}(t)$ for all $t > 0$. Therefore the set $\mathcal{K}_\varepsilon := \{(g_{\alpha-1} * S_{\alpha,1})(t-s)[Bv(s) + f(s,u(s))] : u \in B_R, 0 \leq s \leq t-\varepsilon\}$ is compact for all $\varepsilon > 0$. Then $\overline{\mathrm{conv}(\mathcal{K}_\varepsilon)}$ is also a compact set by Theorem 2.5. The Mean-Value Theorem for the Bochner integrals (see [7, Corollary 8, p. 48]), implies

$$(\mathcal{P}_2^\varepsilon u)(t) \in t\overline{\mathrm{conv}(\mathcal{K}_\varepsilon)}, \text{ for all } t \in [0,b].$$

We conclude that the set $H_2^\varepsilon(t) := \{(\mathcal{P}_2^\varepsilon u)(t) : u \in B_R\}$ is relatively compact in X for all $\varepsilon > 0$. Next, we define the operator $\mathcal{P}_2$ on $C(I,X)$ by

$$(\mathcal{P}_2 u)(t) : = \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)[Bv(s) + f(s,u(s))]ds.$$

Now, observe that

$$\begin{aligned}\|(\mathcal{P}_2u)(t)-\mathcal{P}_2^{\varepsilon}u)(t)\| &\leq \int_{t-\varepsilon}^{t}\|(g_{\alpha-1}*S^E_{\alpha,1})(t-s)\|\,\|Bv(s)+f(s,u(s))\|ds\\ &\leq \frac{M\|B\|e^{\omega b}}{\omega^{\alpha-1}}\int_{t-\varepsilon}^{t}e^{-\omega s}\|v(s)\|ds+\frac{MRe^{\omega b}}{\omega^{\alpha-1}}\int_{t-\varepsilon}^{t}e^{-\omega s}\mu(s)ds\end{aligned}$$

Since $s\mapsto e^{-\omega s}\|v(s)\|$ and $s\mapsto e^{-\omega s}\mu(s)$ belong to $L^1([t-\varepsilon,t],\mathbb{R}_+)$ we conclude by the Lebesgue Dominated Convergence Theorem that

$$\lim_{\varepsilon\to 0}\|(\mathcal{P}_2u)(t)-(\mathcal{P}_2^{\varepsilon}u)(t)\|=0.$$

Therefore the set $\{\int_0^t(g_{\alpha-1}*S^E_{\alpha,1})(t-s)[Bv(s)+f(s,u(s))]ds : u\in B_R\}$ is relatively compact for all $t\in(0,b]$. The compactness of $S^E_{\alpha,1}(t)$ and $(g_1*S^E_{\alpha,1})(t)=S^E_{\alpha,2}(t)$ (see Proposition 3.14 and Theorem 3.12) imply that $\{(\mathcal{P}u)(t) : u\in B_R\}$ is relatively compact for each $t\in(0,b]$. □

Lemma 4.21. *The set $\{\mathcal{P}u : u\in B_R\}$ is equicontinuous.*

Proof. Take $u\in B_R$. For $0\leq t_2<t_1\leq b$ we have
$\|\mathcal{P}u(t_1)-\mathcal{P}u(t_2)\|\leq$

$$\begin{aligned}&\leq \|\left(S^E_{\alpha,1}(t_1)-S^E_{\alpha,1}(t_2)\right)u_0\|+\|\left((g_1*S^E_{\alpha,1})(t_1)-(g_1*S^E_{\alpha,1})(t_2)\right)u_1\|\\ &+\int_{t_2}^{t_1}\|(g_{\alpha-1}*S^E_{\alpha,1})(t_1-s)[Bv(s)+f(s,u(s))]\|ds\\ &+\int_0^{t_2}\|\left((g_{\alpha-1}*S^E_{\alpha,1})(t_1-s)-(g_1*S^E_{\alpha,1})(t_2-s)\right)[Bv(s)+f(s,u(s))]\|ds\\ &:= I_1+I_2+I_3+I_4.\end{aligned}$$

In I_1 we have

$$I_1\leq\|(S^E_{\alpha,1}(t_1)-S^E_{\alpha,1}(t_2))\|\,\|u_0\|.$$

By hypothesis, using the norm continuity of $S^E_{\alpha,1}(t)$, we obtain that $\lim_{t_1\to t_2}I_1=0$.

The uniqueness of the Laplace transform and Lemma 3.10 imply $(g_1*S^E_{\alpha,1})(t)=S^E_{\alpha,2}(t)$ for all $t\geq 0$ and the Proposition 3.9 implies that $(g_1*S^E_{\alpha,1})(t)$ is continuous in $\mathcal{B}(X)$. Therefore

$$I_2\leq\|(g_1*S^E_{\alpha,1})(t_1)-(g_1*S^E_{\alpha,1})(t_2)\|\,\|u_0\|\to 0,$$

as $t_1\to t_2$. On the other hand,

$$I_3\leq\frac{Me^{\omega b}}{\omega^{\alpha-1}}\int_{t_2}^{t_1}e^{-\omega s}[\|B\|\,\|v(s)\|+\mu(s)\|u(s)\|]ds\leq\frac{M\|B\|e^{\omega b}}{\omega^{\alpha-1}}\int_{t_2}^{t_1}e^{-\omega s}\|v(s)\|ds+\frac{MRe^{\omega b}}{\omega^{\alpha-1}}\|\mu\|_\infty(t_1-t_2),$$

and therefore (by (4.8)) we obtain $\lim_{t_1\to t_2}I_3=0$. Finally, to estimate I_4 we observe that

$$\begin{aligned}I_4 &\leq \int_0^{t_2}\|(g_{\alpha-1}*S^E_{\alpha,1})(t_1-s)-(g_{\alpha-1}*S^E_{\alpha,1})(t_2-s)\|\,\|Bv(s)+f(s,u(s))\|ds\\ &\leq \|B\|\int_0^{t_2}\|(g_{\alpha-1}*S^E_{\alpha,1})(t_1-s)-(g_{\alpha-1}*S^E_{\alpha,1})(t_2-s)\|\,\|v(s)\|ds\\ &\quad +R\int_0^{t_2}\mu(s)\|(g_{\alpha-1}*S^E_{\alpha,1})(t_1-s)-(g_{\alpha-1}*S^E_{\alpha,1})(t_2-s)\|\,\|u(s)\|ds.\end{aligned}$$

Since

$$\mu(\cdot)\|(g_{\alpha-1}*S^E_{\alpha,1})(t_1-\cdot)-(g_{\alpha-1}*S^E_{\alpha,1})(t_2-\cdot)\|\leq 2\frac{Me^{\omega b}}{\omega^{\alpha-1}}\mu(\cdot)\in L^1(I,\mathbb{R}),$$

$(g_{\alpha-1} * S^E_{\alpha,1})(t) = S^E_{\alpha,\alpha}(t)$ for all $t \geq 0$ (see Lemma 3.10) and $S_{\alpha,\alpha}(t)$ is norm continuous (see Proposition 3.9) we have that if $t_1 \to t_2$, then $(g_{\alpha-1} * S^E_{\alpha,1})(t_1 - s) - (g_{\alpha-1} * S^E_{\alpha,1})(t_2 - s) \to 0$ in $\mathcal{B}(X)$. By using (4.8) we obtain that the first integral in the above estimate goes to 0 as $t_1 \to t_2$ and the Lebesgue's dominated convergence theorem implies that the second integral goes also to 0 as $t_1 \to t_2$. We conclude that $\lim_{t_1\to t_2} I_4 = 0$ and therefore, $\{\mathcal{P}u : u \in B_R\}$ is an equicontinuous set. □

Theorem 4.22. *Assume that conditions* (H_1)-(H_2) *hold. If* ρ *given by* (4.7) *satisfies* $\rho < 1$, *then for every* $\nu > 0$ *and* $x_0 \in X$ *there exists at least one function* u *defined on* I *such that* u *satisfies* (4.5) *and* (4.6).

Proof. From Lemmata 4.19-4.21 and the Ascoli-Arzela theorem, it follows that the set $\{\mathcal{P}u : u \in B_R\}$ is relatively compact. We conclude that $\mathcal{P}$ is a compact operator on B_R. The Schauder fixed point theorem (Theorem 2.6) implies the existence of a fixed point to $\mathcal{P}$ on B_R. Now, we observe that any fixed point of $\mathcal{P}$ is a mild solution to problem (4.4) on I which satisfies $(\mathcal{P}u)(t) = u(t)$ for all $t \in [0, b]$. □

If we denote by u_ν the function given in Theorem 4.22, we will approximate any point $x_0 \in X$ by using the set $\{u_\nu : \nu > 0\}$. This will imply that the system (4.4) is approximately controllable. We need the following assumptions.

(H_3) $\nu R(\nu, \Gamma^b_0) \to 0$ as $\nu \to 0^+$ in the strong operator topology.

(H_4) The function $f : [0, b] \times X \to X$ is continuous and there exists a positive constant K such that

$$\|f(t, u)\| \leq K, \quad \forall\, t \in I,\ u \in C(I, X).$$

Theorem 4.23. *Assume that conditions* (H_1)-(H_3) *hold. If* ρ *given by* (4.7) *satisfies* $\rho < 1$, *then the system* (4.4) *is approximately controllable.*

Proof. The assumptions and Theorem 4.22 imply that for every $\nu > 0$ and $x_0 \in X$ there exists $u_\nu \in C(I, X)$ verifying (4.5) and (4.6). Moreover, an easy computation shows that

$$u_\nu(b) = x_0 - \nu R(\nu, \Gamma^b_0) p(u_\nu).$$

Now, the hypothesis (H_4) implies that

$$\int_0^b \|f(s, u_\nu(s))\|^2 ds \leq K^2 b$$

and therefore, the sequence $\{f(\cdot, u_\nu(\cdot)) : \nu > 0\}$ is bounded in $\mathcal{V}$, which implies there is a subsequence still denoted by $\{f(\cdot, u_\nu(\cdot)) : \nu > 0\}$ which converges weakly to some point $F(\cdot) \in \mathcal{V}$. Now, we write

$$w := x_0 - S^E_{\alpha,1}(b)u_0 - (g_1 * S^E_{\alpha,1})(b)u_1 - \int_0^b (g_{\alpha-1} * S^E_{\alpha,1})(b - s)F(s)ds.$$

Hence

$$\|p(x_\nu) - w\| \leq \left\| \int_0^b (g_{\alpha-1} * S^E_{\alpha,1})(b - s)[f(s, u_\nu(s)) - F(s)]ds \right\|.$$

Since $(g_{\alpha-1} * S^E_{\alpha,1})(t) = S^E_{\alpha,\alpha}(t)$ is a compact operator (see Corollary 3.13), the operator $g \to \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t - s)g(s)ds$ is compact, we obtain $\|p(u_\nu) - w\| \to 0$ as $\nu \to 0^+$. Moreover, the assumption (H_3) implies

$$\|u_\nu(b) - x_0\| \leq \|\nu R(\nu, \Gamma^b_0)w\| + \|\nu R(\nu, \Gamma^b_0)\|\|p(u_\nu) - w\| \to 0, \text{ as } \nu \to 0^+.$$

Therefore, the system (4.4) is approximately controllable. □

5. Approximate Controllability. The Riemann-Liouville case.

In this section we consider the system for the Riemann-Liouville fractional derivative

$$
\left\{
\begin{array}{rcll}
D^{\alpha}(Eu)(t) & = & Au(t)+Bv(t)+f(t,u(t)), & t\in[0,b]\\
(E(g_{2-\alpha}*u))(0) & = & u_0 & \\
(E(g_{2-\alpha}*u))'(0) & = & u_1, &
\end{array}
\right.
\tag{5.8}
$$

where $u_0,u_1\in X$, $3/2<\alpha<2$, A and E are closed linear operators defined on X, $B:V\to X$ is a bounded operator, and the control function v belongs to $\mathcal{V}$. Assume that (A,E) generates the $(\alpha,\alpha-1)$-resolvent family given by $\{S^E_{\alpha,\alpha-1}(t)\}_{t\geq 0}$.

Definition 5.24. *For each $v\in\mathcal{V}$ and $u_0,u_1\in X$, a function $u\in C(J,X)$ is said to be a mild solution to system (5.8) if*

$$u(t)=S^E_{\alpha,\alpha-1}(t)u_0+(g_1*S^E_{\alpha,\alpha-1})(t)u_1+\int_0^t(g_1*S^E_{\alpha,\alpha-1})(t-s)[Bv(s)+f(s,u(s))]ds,$$

for all $t\in[0,b]$.

From the uniqueness of the Laplace transform, it is easy to prove see that the mild solution to (5.8) can be written as

$$u(t)=S^E_{\alpha,\alpha-1}(t)u_0+S^E_{\alpha,\alpha}(t)u_1+\int_0^t S^E_{\alpha,\alpha}(t-s)Bv(s)ds+\int_0^t S^E_{\alpha,\alpha}(t-s)f(s,u(s))ds,$$

$t\in[0,b]$.

Next, we define the operator $\mathcal{Q}:C(I,X)\to C(I,X)$ by

$$\mathcal{Q}u(t):=S^E_{\alpha,\alpha-1}(t)u_0+(g_1*S^E_{\alpha,\alpha-1})(t)u_1+\int_0^t(g_1*S^E_{\alpha,\alpha-1})(t-s)[Bv(s)+f(s,u(s))]ds.$$

The definition of approximate controllability to system (5.8) is analogous to Definition 4.18 and we consider the same operators Γ_0^b and $R(\nu,\Gamma_0^b)$ in this case.

As in Section 4 we shall prove that for every $\nu>0$ and $x_0\in X$ there exists a continuous function $u(\cdot)\in C(I,X)$ such that

$$u(t) \;=\; S^E_{\alpha,\alpha-1}(t)u_0+(g_1*S^E_{\alpha,\alpha-1})(t)u_1+\int_0^t(g_1*S^E_{\alpha,\alpha-1})(t-s)[Bv(s)+f(s,u(s))]ds, \tag{5.9}$$

where the function v is the control function defined by

$$v(t) \;=\; B^*S^{E^*}_{\alpha,\alpha}(b-t)R(\nu,\Gamma_0^b)q(u(\cdot)), \tag{5.10}$$

and

$$q(u(\cdot))=x_0-S^E_{\alpha,\alpha-1}(b)u_0-(g_1*S^E_{\alpha,\alpha-1})(b)u_1-\int_0^b(g_1*S^E_{\alpha,\alpha-1})(b-s)f(s,u(s))ds.$$

Now, we assume the following hypothesis

- (H_5) The pair (A,E) generates the $(\alpha,\alpha-1)$-resolvent family $\{S^E_{\alpha,\alpha-1}(t)\}_{t\geq0}$ of type (M,ω), the operator $(\lambda^\alpha E-A)^{-1}$ is compact for all $\lambda^\alpha\in\rho_E(A)$ and $\{S^E_{\alpha,\alpha-1}(t)\}_{t\geq0}$ is norm continuous for all $t>0$.

The proof of the following Lemma follows similarly to Lemma 4.19.

Lemma 5.25. *Let $\tilde{\rho}>0$ given by*

$$\tilde{\rho}=\frac{Me^{\omega b}\|\mu\|_\infty b}{\omega}\left[1+\frac{Mb\|B\|e^{\omega b}}{\omega}\frac{M\|B\|}{\nu\omega}e^{\omega b}\right]. \tag{5.11}$$

If $\tilde{\rho} < 1$, then there exists a constant $\tilde{R} \geq \frac{\tilde{C}}{1-\tilde{\rho}}$, where

$$\tilde{C} = Me^{\omega b}\left[\|u_0\| + \frac{\|u_1\|}{\omega}\right] + \frac{Mb\|B\|e^{\omega b}}{\omega}\frac{M\|B\|}{\nu\omega}e^{\omega b}\left[\|x_0\| + Me^{\omega b}\|u_0\| + \frac{Me^{\omega b}}{\omega}\|u_1\|\right],$$

such that $\mathcal{Q}B_{\tilde{R}} \subset B_{\tilde{R}}$.

Lemma 5.26. *For every $t \in I$, the set $H(t) := \{(\mathcal{Q}u)(t) : u \in B_{\tilde{R}}\}$ is relatively compact.*

Proof. We give here only the details on the relatively compactness of $\{\mathcal{Q}u(t) : u \in B_{\tilde{R}}\}$ in X for each $t \in (0, b]$. The Theorem 3.12 implies the compactness of $(g_1 * S^E_{\alpha,\alpha-1})(t) = S^E_{\alpha,\alpha}(t)$ for all $t > 0$ and therefore the set $\{\int_0^t (g_1 * S^E_{\alpha,\alpha-1})(t-s)[Bv(s) + f(s, u(s))]ds : u \in B_{\tilde{R}}\}$ is relatively compact for all $t \in [0, b]$, see the proof of Lemma 4.20. On the other hand, from (H_5) and Proposition 3.15 we get the compactness of $S^E_{\alpha,\alpha-1}(t)$ for all $t > 0$. Therefore, $\{(\mathcal{Q}u)(t) : u \in B_{\tilde{R}}\}$ is relatively compact for each $t \in (0, b]$. □

Lemma 5.27. *The set $\{\mathcal{Q}u : u \in B_{\tilde{R}}\}$ is equicontinuous.*

Proof. Follows similarly to the proof of Lemma 4.21, because $S^E_{\alpha,\alpha-1}(t)$ is norm continuous for all $t > 0$ (see (H_5)) and $t \mapsto (g_1 * S^E_{\alpha,\alpha-1})(t)$ is also norm continuous by Proposition 3.9. □

From the hypotheses it is easy to show that $\mathcal{Q}$ is a compact operator on $B_{\tilde{R}}$. The proof of the following result is analogous to the proof of Theorem 4.22. We omit the details.

Theorem 5.28. *Assume that conditions (H_2), (H_5) hold. If $\tilde{\rho}$ given by (4.7) satisfies $\tilde{\rho} < 1$, then for every $\nu > 0$ and $x_0 \in X$ there exists at least one function u defined on I such that u satisfies (5.9) and (5.10).*

Finally, we present an approximate controllability result for the Riemann-Liouville case. The proof is similar to Theorem 4.22.

Theorem 5.29. *Assume that conditions (H_3)-(H_5) are satisfied. If $\tilde{\rho}$ defined by (5.11) satisfies $\tilde{\rho} < 1$, then the system (5.8) is approximately controllable on I.*

6. Applications

In this section, we give some applications of our results.

Example 6.30.

Consider the semilinear problem

$$\left\{\begin{array}{lcl} D^\alpha_t(Eu)(t) & = & Au(t) + J^{2-\alpha}[Bv(t) + f(t, u(t))], \quad t \in I := [0, b] \\ Eu(0) & = & u_0 \\ (Eu)'(0) & = & u_1, \end{array}\right. \tag{6.12}$$

where $u_0, u_1 \in X$, $J^{2-\alpha}$ denotes the Riemann-Liouville fractional integral operator, f is a suitable continuous function and B is a bounded operator. Assume the pair (A, E) generates the $(\alpha, 1)$-resolvent family $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$. The mild solution of (6.12) is given by

$$u(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_1 * S^E_{\alpha,1})(t-s)[Bv(s) + f(s, u(s))]ds, \quad t \in I.$$

The approximate controllability in case $0 < \alpha < 1$ and $E = \mathcal{I}$ (the identity operator) was recently studied in [8]. If $S^E_{\alpha,1}(t)$ is norm continuous, then $S^E_{\alpha,1}(t)$ is compact for all $t > 0$ (see Section 3). The proof of the next result follows similarly to Theorem 4.22. We omit the details.

Theorem 6.31. *Let $1<\alpha<2$. Let (A,E) be the generator of an $(\alpha,1)$-resolvent family $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ of type (M,ω) which is norm continuous. Suppose that $(\lambda^\alpha E-A)^{-1}$ is compact for all $\lambda>\omega^{1/\alpha}$. Let $\rho>0$ given by*

$$\rho=\begin{cases} \frac{Me^{\omega b}\|\mu\|_\infty b}{\omega}\left[1+\frac{M\|B\|e^{\omega b}b}{\omega}\right], & \text{if } \mathcal{V}=L^\infty(I,V),\\ \frac{Me^{\omega b}\|\mu\|_\infty b}{\omega}\left[1+\frac{M\|B\|e^{\omega b}b^{1/2}}{\omega}\right], & \text{if } \mathcal{V}=L^2(I,V).\end{cases}$$

Assume that the operator $W:\mathcal{V}\to D(E)$ defined by

$$Wv:=\int_0^b (g_1 * S^E_{\alpha,1})(b-s)Bv(s)ds$$

has a bounded right inverse operator $W^{-1}:D(E)\to\mathcal{V}$ and $\|W^{-1}\|\leq K$ for a suitable $K>0$.

If $\rho<1$, then under assumptions (H_3), the system (6.12) is approximately controllable in I.

Example 6.32.

Now, we consider the Riemann-Liouville fractional Cauchy system

$$\begin{cases} D^\alpha(Eu)(t) &= Au(t)+J^{2-\alpha}[Bv(t)+f(t,u(t))], \quad t\in I:=[0,b]\\ (E(g_{2-\alpha}*u))(0) &= u_0\\ (E(g_{2-\alpha}*u))'(0) &= u_1,\end{cases} \tag{6.13}$$

where $u_0,u_1\in X$, $3/2<\alpha<2$ and A,E are closed linear operators defined on X. We notice that the controllability of a system in the form of (6.13), in case $E=\mathcal{I}$ and $u_0=0$, was recently studied in [21] by assuming that A is the generator of an (α,α)-resolvent family.

The mild solution to problem (6.13) can be written as

$$u(t)=S^E_{\alpha,\alpha-1}(t)u_0+(g_1*S^E_{\alpha,\alpha-1})(t)u_1+\int_0^t (g_{3-\alpha}*S^E_{\alpha,\alpha-1})(t-s)[Bv(s)+f(s,u(s))]ds,$$

for all $t\in[0,b]$, where we assume that (A,E) generates the $(\alpha,\alpha-1)$-resolvent family $\{S^E_{\alpha,\alpha-1}(t)\}_{t\geq 0}$.

On the other hand, if $S^E_{\alpha,\alpha-1}(t)$ is norm continuous, then the Proposition 3.15 shows that $S^E_{\alpha,\alpha-1}(t)$ is compact for all $t>0$. Therefore, we have the following controllability result. We omit the proof.

Theorem 6.33. *Let $3/2<\alpha<2$. Let (A,E) be the generator of an $(\alpha,\alpha-1)$-resolvent family $\{S^E_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ of type (M,ω) which is norm continuous. Suppose that $(\lambda^\alpha E-A)^{-1}$ is compact for all $\lambda>\omega^{1/\alpha}$. Let $\rho>0$ given by*

$$\rho=\begin{cases} \frac{Me^{\omega b}\|\mu\|_\infty b}{\omega^{3-\alpha}}\left[1+\frac{M\|B\|e^{\omega b}b}{\omega^{3-\alpha}}\right], & \text{if } \mathcal{V}=L^\infty(I,V),\\ \frac{Me^{\omega b}\|\mu\|_\infty b}{\omega^{3-\alpha}}\left[1+\frac{M\|B\|e^{\omega b}b^{1/2}}{\omega^{3-\alpha}}\right], & \text{if } \mathcal{V}=L^2(I,V).\end{cases}$$

Assume that the operator $W:\mathcal{V}\to D(E)$ defined by

$$Wv:=\int_0^b (g_{3-\alpha} * S^E_{\alpha,\alpha-1})(b-s)Bv(s)ds$$

has a bounded right inverse operator $W^{-1}:D(E)\to\mathcal{V}$ and $\|W^{-1}\|\leq K$ for a suitable $K>0$.

If $\rho<1$, then under assumptions (H_3), the system (6.13) is controllable in I.

Example 6.34.

Take $X=V=L^2[0,\pi]$. Consider the following problem

$$\begin{cases} D^\alpha_t\left[u(t,x)-u_{xx}(t,x)\right]=-u_{xx}(t,x)+f(t,u(t,x))+Bv(t), \quad (t,x)\in[0,1]\times[0,\pi],\\ u(t,0)=u(t,\pi)=0, \quad t\in[0,1],\\ u(0,x)=u_0(x), \quad x\in[0,\pi],\\ u_t(0,x)=u_1(x), \quad x\in[0,\pi],\end{cases} \tag{6.14}$$

where $1 < \alpha < 2$. On X we consider the operators $A : D(A) \subset X \to X$ and $E : D(E) \subset X \to X$ given respectively by $Au := \frac{\partial^2 u}{\partial x^2} = u_{xx}$, and $Eu = u - u_{xx}$ with domain $D(E) = D(A) := \{u \in X : u \in H^2([0,\pi]), u(t,0) = u(t,\pi) = 0\}$. It is well known that A has discrete spectrum with eigenvalues of the form $-n^2, n \in \mathbb{N}$, and the corresponding normalized eigenfunctions are given by $u_n(s) := (\frac{2}{\pi})^{\frac{1}{2}} \sin(ns)$. In addition, $\{u_n : n \in \mathbb{N}\}$ is an orthonormal basis for X, and thus A and E can be written as (see [16]):

$$Au = \sum_{n=1}^{\infty} -n^2 \langle u, u_n\rangle u_n, u \in D(A),$$

$$Eu = \sum_{n=1}^{\infty} (1+n^2) \langle u, u_n\rangle u_n, u \in D(E).$$

Hence, for any $u \in X$ and $1 \le \beta \le 2$, we have

$$\begin{aligned}
\lambda^{\alpha-\beta}(\lambda^\alpha E - A)^{-1}u &= \sum_{n=1}^{\infty} \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha(1+n^2)+n^2} \langle u, u_n\rangle u_n \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2+1} \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha + \frac{n^2}{n^2+1}} \langle u, u_n\rangle u_n \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2+1} \int_0^\infty e^{-\lambda t} h^n_{\alpha,\beta}(t) dt \langle u, u_n\rangle u_n \\
&= \int_0^\infty e^{-\lambda t} \sum_{n=1}^{\infty} \frac{1}{n^2+1} h^n_{\alpha,\beta}(t) dt \langle u, u_n\rangle u_n,
\end{aligned}$$

where the function $h^n_{\alpha,\beta}(t) := t^{\beta-1} e_{\alpha,\beta}\left(-\frac{n^2}{n^2+1} t^\alpha\right)$ satisfies $\hat{h}^n_{\alpha,\beta}(\lambda) = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha + \frac{n^2}{n^2+1}}$ for all $\lambda > 0$. Therefore, the pair (A, E) generates the (α,β)-resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\ge 0}$ given by

$$S^E_{\alpha,\beta}(t)u = \sum_{n=1}^{\infty} \frac{1}{n^2+1} h^n_{\alpha,\beta}(t) \langle u, u_n\rangle u_n,$$

for all $u \in X$. In particular, if $\beta = 1$, then $S^E_{\alpha,1}(t) = \sum_{n=1}^\infty \frac{1}{n^2+1} h^n_{\alpha,1}(t) = \sum_{n=1}^\infty \frac{1}{n^2+1} e_{\alpha,1}\left(-\frac{n^2}{n^2+1} t^\alpha\right)$. We claim that $S^E_{\alpha,1}(t)$ is norm continuous. In fact, for each $t > s$ we have

$$\begin{aligned}
\|S^E_{\alpha,1}(t) - S^E_{\alpha,1}(s)\| &= \sup_{\|u\|\le 1} \|S^E_{\alpha,1}(t)u - S^E_{\alpha,1}(s)u\| \\
&\le \sup_{\|u\|\le 1} \sum_{n=1}^{\infty} \frac{1}{n^2+1} \left| e_{\alpha,1}\left(-\frac{n^2}{n^2+1} t^\alpha\right) - e_{\alpha,1}\left(-\frac{n^2}{n^2+1} s^2\right)\right| \|\langle u, u_n\rangle u\|,
\end{aligned}$$

which implies, because of the continuity of $e_{\alpha,1}(\cdot)$ and the convergence of $\sum_{n=1}^\infty \frac{1}{n^2+1}$ that $\|S^E_{\alpha,1}(t) - S^E_{\alpha,1}(s)\| \to 0$ as $t \to s$. Now, to see the compactness of $(\lambda^\alpha E - A)^{-1}$ we notice that in the representation

$$\lambda^{\alpha-1}(\lambda^\alpha E - A)^{-1}u = \sum_{n=1}^{\infty} \frac{1}{n^2+1} \frac{\lambda^{\alpha-1}}{\lambda^\alpha + \frac{n^2}{n^2+1}} \langle u, u_n\rangle u_n, \quad \text{for all } u \in X,$$

we have $\lim_{n\to\infty} \frac{1}{n^2+1} \frac{\lambda^{\alpha-1}}{\lambda^\alpha + \frac{n^2}{n^2+1}} = 0$ for all $\lambda > 0$, and therefore $\lambda^{\alpha-\beta}(\lambda^\alpha E - A)^{-1}$ is a compact operator on the Hilbert space X, which implies the compactness of $(\lambda^\alpha E - A)^{-1}$ for all $\lambda > 0$. Now, assume the condition (H_2). We consider here $B : V \to X$ defined by $B := k\mathcal{I}$ where $k > 0$ and $\mathcal{I}$ denotes the identity operator.

Next, we observe that for each $u \in X$ we have by [6]

$$\|S^E_{\alpha,1}(t)u\| \;\leq\; \sum_{n=1}^{\infty} \frac{1}{n^2+1}|h^n_{\alpha,1}(t)|\,\|u\| \leq C_\alpha \sum_{n=1}^{\infty} \frac{1}{n^2+1}\frac{1}{1+\frac{n^2}{n^2+1}t^\alpha}\,\|u\| \leq C_\alpha \sum_{n=1}^{\infty} \frac{1}{n^2}\|u\| = C_\alpha \frac{\pi^2}{6}\|u\|,$$

where C_α is a positive constant given in [6, Theorem 1]. Therefore, $S^E_{\alpha,1}(t)$ is of type $(C_\alpha \frac{\pi^2}{6}, 1)$, that is $M = C_\alpha \frac{\pi^2}{6}$ and $\omega = 1$. Finally, we define the function $f : [0,1] \times D(A) \to X$ by

$$f(t, u(t,x)) := \frac{e^{-t}u(t,x)}{(C_1+t)(1+u(t,x))},$$

where $C_1 > 0$ will be chosen later. We observe that in this case $\mu(t) = \frac{e^{-t}}{C_1+t}$, and $\|\mu\|_\infty = 1/C_1$ (see the notation in Section 4). Hence,

$$\rho = \frac{1}{C_1}\left(\frac{C_\alpha e\pi^2}{6} + \frac{C_\alpha k e^2\pi^4}{36}\right).$$

We take $C_1 > 0$ such that $\rho < 1$. We observe that, in particular, if $\alpha = 3/2$, then (by [11, Formula 7.7]) we have

$$|h^n_{\alpha,1}(t)| = e_{\alpha,1}\left(-\frac{n^2}{n^2+1}t^\alpha\right) = \frac{1}{\pi}\int_0^\infty \frac{s^{1/2}}{1+s^3}e^{-\left(\frac{n^2}{n^2+1}\right)^{\frac{2}{3}}ts}ds \leq \frac{1}{\pi}\int_0^\infty \frac{s^{1/2}}{1+s^3}ds = \frac{1}{3},$$

which means that $C_{3/2} = \frac{1}{3}$. If $k = 1$ we obtain

$$\rho = \frac{1}{C_1}\left(\frac{e\pi^2}{18} + \frac{e^2\pi^4}{108}\right) \doteq \frac{8.1549}{C_1}.$$

Thus, in this case, we can choose $C_1 = 9$. In this conditions, the system (6.14) can be written in the abstract form (4.4). Moreover, the assumptions (H_1)-(H_3) are valid, and therefore the system (6.14) is controllable in $[0,1]$ by Theorem 4.22.

6.1. **Conclusions.** In this paper, we obtain conditions implying the norm continuity and compactness of the family $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$. As consequence, we obtain results on the controllability of fractional systems of Sobolev type for the Caputo and Riemann-Liouville fractional derivatives. We remark that in our results, the existence or compactness of E^{-1} is not necessarily needed.

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