PROPERTIES OF VECTOR-VALUED τ-DISCRETE FRACTIONAL CALCULUS AND ITS CONNECTION WITH CAPUTO FRACTIONAL DERIVATIVES

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ABSTRACT. In this paper, for a given vector-valued sequence $(v^n)_{n\in\mathbb{N}_0}$, we study its discrete fractional derivative in the sense of Caputo for $0<\alpha<1$ and its connection with the Caputo fractional derivative. Moreover, we study the convergence of this Caputo fractional difference operator to the Caputo fractional derivative.

1. INTRODUCTION

In the last two decades, the theory and the applications of time-fractional differential equations have been a topic of great interest, see for instance, [2, 4, 12, 14, 15, 19, 20, 21, 23, 24, 25]. However, these continuous-time applications sometimes need to be studied, for practical purposes, as discrete problems.

The first investigations on difference of fractional order date back to Kuttner in 1957 (see [13]) and there are many different definitions of this concept. The study of existence, properties and applications of discrete fractional difference equations has attracted considerable attention of many researchers in the last years, see for instance [1, 3, 6, 7, 10, 18]. However, these articles focus mainly on scalar fractional difference equations. Very recently, C. Lizama in [16] introduced a new method to study on fractional difference equations in Banach spaces. See also [8, 9, 17] for related results.

For a given differentiable vector-valued function $u:\mathbb{R}_+\to X$, the Caputo fractional derivative of u of order α, with $0<\alpha<1$, is defined by $\partial_t^\alpha u(t):=(g_{1-\alpha}*u')(t)$, where for $\beta>0$, the function g_β is defined by $g_\beta(t):=\frac{t^{\beta-1}}{\Gamma(\beta)}$, and $*$ denotes the usual finite convolution: $(f*g)(t)=\int_0^t f(t-s)g(s)ds$. On the other hand, for $0<\alpha<1$ and a fixed time-size $\tau>0$, the Caputo fractional difference operator of a vector-valued sequence $(v^n)_{n\in\mathbb{N}_0}$ is defined by (see for instance [22])

$$({}_C\nabla^\alpha v)^n:=\nabla_\tau^{-(1-\alpha)}(\nabla_\tau^1 v)^n,\quad n\in\mathbb{N},$$

where $(\nabla_\tau^{-(1-\alpha)}v)^n:=\tau\sum_{j=0}^n k_\tau^{1-\alpha}(n-j)v^j$, $n\in\mathbb{N}_0$, and for $\beta>0$, $\nabla_\tau^1 v^n:=\frac{v^n-v^{n-1}}{\tau}$, and

$$k_\tau^\beta(n):=\frac{\tau^{\beta-1}\Gamma(\beta+n)}{\Gamma(\beta)\Gamma(n+1)},\quad n\in\mathbb{N}_0.$$

Intuitively, for a given $v:\mathbb{R}_+\to X$ and τ small enough, $({}_C\nabla^\alpha v)^n$ corresponds to an approximation of $\partial_t^\alpha v(t)$ at $t_n:=\tau n$, where the sequence $(v^n)_{n\in\mathbb{N}_0}$ is defined by $v^n=\int_0^\infty\rho_n^\tau(t)v(t)dt$ and $\rho_n^\tau(t):=e^{-\frac{t}{\tau}}\left(\frac{t}{\tau}\right)^n\frac{1}{\tau n!}$.

The properties of Caputo fractional derivatives and fractional differences are well-known, see for instance [3, 10, 12, 19] and the references therein. However, there are only some papers studying its connections. In this paper, we study the main properties of ${}_C\nabla^\alpha$ and its relations with the Caputo fractional derivative ∂_t^α for $0<\alpha<1$.

The paper is organized as follows. In Section 2 we give the preliminaries. In Section 3 we study the main properties of the discrete Caputo fractional derivative $({}_C\nabla^\alpha v)^n$ of a vector-valued sequence.

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Moreover, we study its connection with the Caputo fractional derivative ∂_t^α. In particular, we show that for a differentiable function $v:\mathbb{R}_+\to X$, it holds

$$\partial_t^\alpha v(t)=\lim_{\tau\to 0^+}\tau\sum_{n=0}^{\infty}\rho_n^\tau(t)\,{}_C\nabla_\tau^\alpha v^n,$$

for all $t\geq 0$, where $v^n=\int_0^\infty \rho_n^\tau(t)v(t)dt$. Finally, we study the convergence of $({}_C\nabla^\alpha v)^n$ to $\partial_t^\alpha v$ at $t_n=\tau n$ whenever $\tau\to 0^+$.

2. Preliminaries

In this section, we give some definitions which are used further in this paper. Let $\tau>0$ be fixed and $n\in\mathbb{N}_0$. The functions ρ_n^τ are defined by

$$\rho_n^\tau(t):=e^{-\frac{t}{\tau}}\left(\frac{t}{\tau}\right)^n\frac{1}{\tau n!},$$

for all $t\geq 0$, $n\in\mathbb{N}_0$. We notice that $\rho_n^\tau(t)\geq 0$ and the change of variables $s=t/\tau$ implies

$$\int_0^\infty \rho_n^\tau(t)dt=1,\qquad \text{for all}\qquad n\in\mathbb{N}_0.$$

For a given Banach space $X\equiv(X,\|\cdot\|)$, the space of all vector-valued sequences $v:\mathbb{N}_0\to X$ is denoted by $s(\mathbb{N}_0,X)$. The *backward Euler operator* $\nabla_\tau: s(\mathbb{N}_0,X)\to s(\mathbb{N}_0,X)$ is defined by

$$\nabla_\tau v^n:=\frac{v^n-v^{n-1}}{\tau},\quad n\in\mathbb{N}.$$

For $m\geq 2$, the *backward difference operator of order* m, $\nabla_\tau^m: s(\mathbb{N}_0,X)\to s(\mathbb{N}_0,X)$, is defined by

$$(\nabla_\tau^m v)^n:=\nabla_\tau^{m-1}(\nabla_\tau v)^n,\quad n\geq m,$$

where ∇_τ^1 is defined as $\nabla_\tau^1:=\nabla_\tau$, ∇_τ^0 as the identity operator, and for $n<m$, by $(\nabla_\tau^m v)^n:=0$. As in [10, Chapter 1, Section 1.5] we adopt the convention

$$\sum_{j=0}^{-k} v^j=0,\qquad \text{for all}\qquad k\in\mathbb{N}. \tag{2.1}$$

Moreover, by induction, we have that if $v\in s(\mathbb{N}_0,X)$, then

$$(\nabla_\tau^m v)^n=\frac{1}{\tau^m}\sum_{j=0}^{m}\binom{m}{j}(-1)^j v^{n-j},\quad n\in\mathbb{N}.$$

For a given $\alpha>0$, we define g_α as $g_\alpha(t):=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and the sequence $\{k_\tau^\alpha(n)\}_{n\in\mathbb{N}_0}$ by

$$k_\tau^\alpha(n):=\frac{\tau^{\alpha-1}\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)},\quad n\in\mathbb{N}_0,\ \alpha>0.$$

By [11, Formula 3.381-4, p. 346], we get

$$k_\tau^\alpha(n)=\int_0^\infty \rho_n^\tau(t)g_\alpha(t)dt,\quad n\in\mathbb{N}_0,\ \alpha>0. \tag{2.2}$$

In particular, we notice that $k_\tau^1(n)=1$ for all $n\in\mathbb{N}_0$.

Definition 2.1. [22] *Let $\alpha>0$. The α^{th}−fractional sum of $v\in\mathcal{F}(\mathbb{R};X)$ is defined by*

$$(\nabla_\tau^{-\alpha}v)^n:=\tau\sum_{j=0}^{n}k_\tau^\alpha(n-j)v^j,\quad n\in\mathbb{N}_0.$$

Definition 2.2. [22] *Let* $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}_0$. *The* Caputo fractional backward difference operator of order α, ${}_C\nabla^\alpha : \mathcal{F}(\mathbb{R}_+; X) \to \mathcal{F}(\mathbb{R}_+; X)$, *is defined by*

$$({}_C\nabla^\alpha v)^n := \nabla_\tau^{-(m-\alpha)}(\nabla_\tau^m v)^n, \quad n \in \mathbb{N},$$

where $m-1 < \alpha < m$.

Definition 2.3. *Let* $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}_0$. *The* Riemann-Liouville fractional backward difference operator of order α, ${}^R\nabla^\alpha : \mathcal{F}(\mathbb{R}_+; X) \to \mathcal{F}(\mathbb{R}_+; X)$, *is defined by*

$$({}^R\nabla^\alpha v)^n := \nabla_\tau^m (\nabla_\tau^{-(m-\alpha)} v)^n, \quad n \in \mathbb{N},$$

where $m-1 < \alpha < m$.

If $\alpha \in \mathbb{N}_0$, the operators ${}_C\nabla^\alpha$ and ${}^R\nabla^\alpha$ are defined as the backward difference operator ∇_τ^α.

For a given vector-valued sequence $\{v^n\}_{n\in\mathbb{N}_0}$ and a scalar sequence $c = (c^n)_{n\in\mathbb{N}_0}$, we define the discrete convolution $c \star v$ as

$$(c \star v)^n := \sum_{k=0}^{n} c^{n-k} v^k, \quad n \in \mathbb{N}_0.$$

Moreover, for scalar valued sequences $b = (b^n)_{n\in\mathbb{N}_0}$ and $c = (c^n)_{n\in\mathbb{N}_0}$, we define $(b \star c \star v)^n := (b \star (c \star v))^n$ for all $n \in \mathbb{N}_0$.

As in [22, Corollary 2.9] we can prove the following convolution property. If $\alpha, \beta > 0$, then

$$k_\tau^{\alpha+\beta}(n) = \tau \sum_{j=0}^{n} k_\tau^\alpha(n-j) k_\tau^\beta(j) = \tau(k_\tau^\alpha \star k_\tau^\beta)(n), \tag{2.3}$$

for all $n \in \mathbb{N}_0$. Given $s \in s(\mathbb{N}_0, X)$, its Z-transform, $\tilde{s}$, is defined by $\tilde{s}(z) := \sum_{j=0}^\infty z^{-j} s^j$, where $s^j := s(j)$ and $z \in \mathbb{C}$. We notice that the convergence of this series holds for $|z| > R$, where R is large enough. It is a well known fact that if $s_1, s_2 \in s(\mathbb{N}_0, X)$ and $\tilde{s_1}(z) = \tilde{s_2}(z)$ for all $|z| > R$ for some $R > 0$, then $s_1^j = s_2^j$ for all $j = 0, 1, ...$

3. Properties of discrete fractional derivative

In this section, we prove the main properties of the discrete fractional derivatives. The next proposition shows that $\nabla^{-\alpha}$ verifies a semigroup law.

Proposition 3.4. *If* $\alpha, \beta > 0$, *then* $\nabla_\tau^{-(\alpha+\beta)} v^n = \nabla_\tau^{-\alpha}(\nabla_\tau^{-\beta} v)^n$ *for all* $n \in \mathbb{N}_0$.

Proof. Let $n \in \mathbb{N}_0$. Then by (2.3) we get

$$\nabla_\tau^{-\alpha}(\nabla_\tau^{-\beta} v)^n = \nabla_\tau^{-\alpha}(\tau(k_\tau^\beta \star v)^n) = \tau^2(k_\tau^\alpha \star (k_\tau^\beta \star v))^n = \tau(k_\tau^{\alpha+\beta} \star v)^n = \nabla_\tau^{-(\alpha+\beta)} v^n.$$

□

Proposition 3.5. [22, Proposition 2.6] *If* $0 < \alpha < 1$ *and* $n \in \mathbb{N}_0$, *then*

(1) ${}_C\nabla^{\alpha+1} v^n = {}_C\nabla^\alpha(\nabla^1 v)^n$,
(2) ${}^R\nabla^{\alpha+1} v^n = \nabla^1({}^R\nabla^\alpha v)^n$, *and*
(3) ${}^R\nabla^\alpha(\nabla^1 v)^n = \nabla^1({}_C\nabla^\alpha v)^n$.

Moreover, ${}_C\nabla^{\alpha+1} v^n \neq {}_C\nabla^1({}_C\nabla^\alpha v)^n$, (see [22, Section 2]). The next result shows that ${}_C\nabla^\alpha$ is a left inverse of $\nabla^{-\alpha}$ but, in general, it is not a right inverse.

Proposition 3.6. *If* $0 < \alpha < 1$ *and* $n \in \mathbb{N}_0$, *then*

(1) ${}_C\nabla^\alpha(\nabla_\tau^{-\alpha} v)^n = v^n$.
(2) $\nabla_\tau^{-\alpha}({}_C\nabla^\alpha v)^n = v^n - v^0$.

Proof. Let $n \in \mathbb{N}$. Since $k_\tau^1(n) = 1$ for all $n \in \mathbb{N}_0$, by Proposition 3.4 we have

$$\begin{aligned} {}_C\nabla^\alpha(\nabla_\tau^{-\alpha} v)^n &= \nabla_\tau^{-(1-\alpha)}(\nabla_\tau^1(\nabla_\tau^{-\alpha} v)^n) \\ &= \frac{1}{\tau}\nabla_\tau^{-(1-\alpha)}(\nabla_\tau^{-\alpha} v^n - \nabla_\tau^{-\alpha} v^{n-1}) \\ &= \frac{1}{\tau}(\nabla_\tau^{-1} v^n - \nabla_\tau^{-1} v^{n-1}) \\ &= \frac{1}{\tau}\left(\tau\sum_{j=0}^{n} k_\tau^1(n-j)v^j - \tau\sum_{j=0}^{n-1} k_\tau^1(n-1-j)v^j\right) \\ &= v^n, \end{aligned}$$

for all $n \in \mathbb{N}$. By convention (2.1), the last equalities imply that ${}_C\nabla^\alpha(\nabla_\tau^{-\alpha} v)^0 = v^0$ and (1) holds for all $n \in \mathbb{N}_0$. To prove (2), as $k_\tau^1(n) = 1$ for all $n \in \mathbb{N}_0$, we have by Proposition 3.4 that

$$\nabla_\tau^{-\alpha}({}_C\nabla^\alpha v)^n = \nabla_\tau^{-\alpha}(\nabla_\tau^{-(1-\alpha)}\nabla_\tau^1 v^n) = \nabla_\tau^{-1}(\nabla_\tau^1 v)^n = \tau\sum_{j=0}^{n} k_\tau^1(n-j)\nabla_\tau^1 v^j = \sum_{j=1}^{n} v^j - v^{j-1} = v^n - v^0,$$

for all $n \in \mathbb{N}$. Now, if $n = 0$, then by definition $\nabla_\tau^{-\alpha}({}_C\nabla^\alpha v)^0 = 0$ and therefore, (2) holds for all $n \in \mathbb{N}_0$. □

Example 3.7. *If $0 < \alpha < 1$ and $\beta > 0$, then $\nabla_\tau^{-\alpha}(k_\tau^\beta)^n = k_\tau^{\alpha+\beta}(n)$. In fact, by* (2.3) *we have*

$$\nabla_\tau^{-\alpha}(k_\tau^\beta)^n = \tau\sum_{j=0}^{n} k_\tau^\alpha(n-j)k_\tau^\beta(j) = \tau(k_\tau^\alpha \star k_\tau^\beta)(n) = k_\tau^{\alpha+\beta}(n),$$

for all $n \in \mathbb{N}_0$.

Theorem 3.8. [22, Theorem 2.7] *Let $0 < \alpha < 1$. If $v : [0,\infty) \to X$ is differentiable and bounded, then for all $n \in \mathbb{N}$, we have*

$$\int_0^\infty \rho_n^\tau(t)\partial_t^\alpha v(t)dt = {}_C\nabla^\alpha v^n, \tag{3.4}$$

Example 3.9. *If $0 < \alpha < 1$ and $\beta > 1$, then $\nabla_\tau^\alpha(k_\tau^\beta)^n = k_\tau^{\beta-\alpha}(n)$. In fact, as $k_\tau^\beta(n) = \int_0^\infty \rho_n^\tau(t)g_\beta(t)dt$ (see* (2.2)*), by Theorem 3.8, we have*

$${}_C\nabla^\alpha(k_\tau^\beta)^n = \int_0^\infty \rho_n^\tau(t)\partial_t^\alpha g_\beta(t)dt,$$

for all $n \in \mathbb{N}_0$. Since $(g_\alpha * g_\beta)(t) = g_{\alpha+\beta}(t)$ for any $\alpha, \beta > 0$, we have $\partial_t^\alpha g_\beta(t) = (g_{1-\alpha} * g'_\beta)(t) = (g_{1-\alpha} * g_{\beta-1})(t) = g_{\beta-\alpha}(t)$, and therefore ${}_C\nabla^\alpha(k_\tau^\beta)^n = k_\tau^{\beta-\alpha}(n)$.

Proposition 3.10. *If $0 < \alpha < 1$ and $n \in \mathbb{N}$, then $\nabla_\tau^{-\alpha}(\nabla_\tau^1 v)^n = \nabla_\tau^1(\nabla_\tau^{-\alpha} v)^n - k_\tau^\alpha(n)v^0$.*

Proof. For all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\nabla_\tau^{-\alpha}(\nabla_\tau^1 v)^n &= \tau \sum_{j=0}^{n} k_\tau^\alpha(n-j)(\nabla_\tau^1 v^j) \\
&= \sum_{j=1}^{n} k_\tau^\alpha(n-j)(v^j - v^{j-1}) \\
&= \sum_{j=0}^{n} k_\tau^\alpha(n-j)v^j - \sum_{j=0}^{n-1} k_\tau^\alpha(n-1-j)v^j - k_\tau^\alpha(n)v^0 \\
&= (k_\tau^\alpha \star v)^n - (k_\tau^\alpha \star v)^{n-1} - k_\tau^\alpha(n)v^0 \\
&= \nabla_\tau^1(\tau(k_\tau^\alpha \star v)^n) - k_\tau^\alpha(n)v^0 \\
&= \nabla_\tau^1(\nabla_\tau^{-\alpha} v)^n - k_\tau^\alpha(n)v^0.
\end{aligned}$$

□

The next result relates the discrete Caputo and Riemann-Liouville fractional derivatives.

Proposition 3.11. *If $0 < \alpha < 1$ and $n \in \mathbb{N}$, then ${}_C\nabla_\tau^\alpha v^n = {}^R\nabla_\tau^\alpha v^n - k_\tau^{1-\alpha}(n)v^0$.*

Proof. Since $0 < 1-\alpha < 1$, by Proposition 3.10 we have

$${}_C\nabla_\tau^\alpha v^n = \nabla_\tau^{-(1-\alpha)}(\nabla_\tau^1 v)^n = \nabla_\tau^1(\nabla_\tau^{-(1-\alpha)} v)^n - k_\tau^{1-\alpha}(n)v^0 = {}^R\nabla_\tau^\alpha v^n - k_\tau^{1-\alpha}(n)v^0.$$

□

Similarly to [16, Theorem 3.1] we can prove the following assertion: If $(f^n)_{n\in\mathbb{N}_0}$ denotes the sequence defined by $f^n := \int_0^\infty \rho_n^\tau(t)f(t)dt$ for a given vector-valued function $f : \mathbb{R}_+ \to X$, then

$$\tilde{F}(z) = \frac{1}{\tau}\hat{f}\left(\frac{1}{\tau}\left(1 - \frac{1}{z}\right)\right), \quad |z| > 1,$$

where F denotes the sequence associated to $(f^n)_{n\in\mathbb{N}_0}$. As consequence, we have that for a given $\beta > 0$ the Z-transform of the sequence $(k_\tau^\beta(n))_{n\in\mathbb{N}_0}$ is given by

$$\widetilde{k_\tau^\beta}(z) = \tau^{\beta-1}\frac{z^\beta}{(z-1)^\beta} \tag{3.5}$$

for all $|z| > 1$.

We recall that for $0 < \alpha < 1$ and a differentiable function $f : \mathbb{R}_+ \to X$, the Laplace transform of the Caputo fractional derivative satisfies

$$\widehat{\partial_t^\alpha f}(\lambda) = \lambda^\alpha \hat{f}(\lambda) - \lambda^{\alpha-1}f(0). \tag{3.6}$$

The next theorem gives an analogous result for the Z-transform of the discrete Caputo fractional derivative of a sequence $(u^n)_{n\in\mathbb{N}_0}$.

Theorem 3.12. *Let $v^n := {}_C\nabla_\tau^\alpha u^n$. If $0 < \alpha < 1$, then the Z-transform of the sequence $(v^n)_{n\in\mathbb{N}_0}$ is given by*

$$\tilde{v}(z) = \frac{1}{\tau^\alpha}\left(\frac{z-1}{z}\right)^\alpha \tilde{u}(z) - \frac{1}{\tau^\alpha}\left(\frac{z-1}{z}\right)^{\alpha-1} u^0. \tag{3.7}$$

Proof. By definition and (3.5),

$$\begin{aligned}\tilde{v}(z) &= \sum_{n=0}^{\infty} v^n z^{-n} \\ &= \sum_{n=0}^{\infty} \nabla_\tau^{-(1-\alpha)} (\nabla_\tau^1 u)^n z^{-n} \\ &= \sum_{n=0}^{\infty} \left(\tau \sum_{j=0}^{n} k_\tau^{1-\alpha}(n-j) \nabla_\tau^1 u^j \right) z^{-n} \\ &= \left(\tau \sum_{n=0}^{\infty} k_\tau^{1-\alpha}(n) z^{-n} \right) \left(\sum_{n=0}^{\infty} (\nabla_\tau^1 u^n) z^{-n} \right) \\ &= \tau^{1-\alpha} \frac{z^{1-\alpha}}{(z-1)^{1-\alpha}} \left(\sum_{n=0}^{\infty} (\nabla_\tau^1 u^n) z^{-n} \right).\end{aligned}$$

Moreover, we have

$$\sum_{n=0}^{\infty} (\nabla_\tau^1 u^n) z^{-n} = \frac{1}{\tau} \sum_{n=1}^{\infty} (u^n - u^{n-1}) z^{-n} = \frac{1}{\tau} \left(\sum_{n=0}^{\infty} u^n z^{-n} - z^{-1} \sum_{n=0}^{\infty} u^n z^{-n} - u^0 \right) = \frac{1}{\tau} \left(\frac{z-1}{z} \tilde{u}(z) - u^0 \right).$$

Therefore,

$$\tilde{v}(z) = \tau^{1-\alpha} \frac{z^{1-\alpha}}{(z-1)^{1-\alpha}} \frac{1}{\tau} \left(\frac{z-1}{z} \tilde{u}(z) - u^0 \right) = \frac{1}{\tau^\alpha} \left(\frac{z-1}{z} \right)^\alpha \tilde{u}(z) - \frac{1}{\tau^\alpha} \left(\frac{z-1}{z} \right)^{\alpha-1} u^0.$$

□

Theorem 3.13. *Let $v : \mathbb{R}_+ \to X$ be a differentiable function and $0 < \alpha < 1$. For $v^n := \int_0^\infty \rho_n^\tau(r) v(r) dr$, we have*

$$\partial_t^\alpha v(t) = \lim_{\tau \to 0^+} \tau \sum_{n=0}^{\infty} \rho_n^\tau(t) \, {}_C\nabla_\tau^\alpha v^n,$$

for all $t \geq 0$.

Proof. Let $R_\tau^\alpha(t) := \tau \sum_{n=0}^{\infty} \rho_n^\tau(t) \, {}_C\nabla_\tau^\alpha v^n$. From [11, Formula 3.381-4] we have that the Laplace transform of ρ_n^τ verifies $\widehat{\rho}_n^\tau(\lambda) = \frac{1}{(1+\tau\lambda)^{n+1}}$. Then by Theorem 3.12 with $z = 1 + \tau\lambda$, we have

$$\widehat{R_\tau^\alpha}(\lambda) = \frac{\tau}{(1+\tau\lambda)} \sum_{n=0}^{\infty} {}_C\nabla_\tau^\alpha v^n (1+\tau\lambda)^{-n} = \frac{\tau}{(1+\tau\lambda)} \left[\frac{1}{\tau^\alpha} \left(\frac{\tau\lambda}{1+\tau\lambda} \right)^\alpha \tilde{v}(1+\tau\lambda) - \frac{1}{\tau^\alpha} \left(\frac{\tau\lambda}{1+\tau\lambda} \right)^{\alpha-1} v^0 \right].$$

Since $\tilde{v}(z) = \frac{1}{\tau} \hat{v} \left(\frac{1}{\tau} \left(\frac{z-1}{z} \right) \right)$ we have $\tilde{v}(1+\tau\lambda) = \frac{1}{\tau} \hat{v} \left(\frac{\lambda}{1+\tau\lambda} \right)$, and therefore

$$\widehat{R_\tau^\alpha}(\lambda) = \frac{\lambda^\alpha}{(1+\tau\lambda)^{\alpha+1}} \hat{v} \left(\frac{\lambda}{1+\tau\lambda} \right) - \lambda^{\alpha-1} \frac{1}{(1+\tau\lambda^\alpha)} v^0.$$

Moreover,

$$v^0 = \int_0^\infty \rho_0^\tau(r) v(r) dr = \frac{1}{\tau} \int_0^\infty e^{-\frac{r}{\tau}} v(r) dr = \frac{1}{\tau} \hat{v} \left(\frac{1}{\tau} \right),$$

for all $\tau > 0$. As $\lim_{\tau \to 0^+} \frac{1}{\tau} \hat{v} \left(\frac{1}{\tau} \right) = v(0)$, we conclude that

$$\lim_{\tau \to 0^+} \widehat{R_\tau^\alpha}(\lambda) = \lambda^\alpha \hat{v}(\lambda) - \lambda^{\alpha-1} v(0).$$

By the uniqueness of the Laplace transform and (3.6), we have $\lim_{\tau \to 0^+} R_\tau^\alpha(t) = \partial_t^\alpha v(t)$ for all $t \geq 0$. □

Now, for any fixed $t > 0$, we will consider the following path Γ_t : For $\frac{\pi}{2} < \theta < \pi$, we take ϕ such that $\frac{1}{2}\phi < \frac{\pi}{2}\alpha < \phi < \theta$. Next, we define Γ_t as the union $\Gamma_t^1 \cup \Gamma_t^2$, where

$$\Gamma_t^1 := \left\{ \frac{1}{t} e^{i\psi/\alpha} : -\phi < \psi < \phi \right\} \quad \text{and} \quad \Gamma_t^2 := \left\{ r e^{\pm i\phi/\alpha} : \frac{1}{t} \leq r \right\}.$$

1 From [22, Lemma 4.18] we have the following Lemma.

Lemma 3.14. *Let Γ_t be the complex path defined above. If $\mu \geq 0$, then there exist positive constants C_α, depending only on α, such that*

$$\int_{\Gamma_t} \left| \frac{e^{zt}}{z^\mu} \right| |dz| \leq C_\alpha t^{\mu-1}$$

for all $t > 0$, where

$$C_\alpha := \left(2\phi \int_{-\phi}^{\phi} e^{\cos(\psi/\alpha)} d\psi + \frac{2}{-\cos(\phi/\alpha)} \right).$$

Theorem 3.15. *Let $u : \mathbb{R}_+ \to X$ be bounded differentiable function, $0 < \alpha < 1$ and $\tau > 0$. Define the sequence $(v^n)_{n\in\mathbb{N}_0}$ as $v^n := \int_0^\infty \rho_n^\tau(r) v(r) dr$. Let $T > 0$ be fixed, $n \in \mathbb{N}$, $t_n = \tau n$ with $0 < t_n \leq T$. If v' is bounded and $v'' \in L^1(\mathbb{R}_+)$, then there exists a constant M_α, depending only on α, such that*

$$\|\partial_t^\alpha v(t_n) - {}_C\nabla_\tau^\alpha v^n\| \leq \tau^{1-\alpha} M_\alpha (\|v'(0)\| + \|v''\|_{L^1(\mathbb{R}_+)}).$$

Proof. Since $\int_0^\infty \rho_n^\tau(t) dt = 1$, by Theorem 3.8, we can write

$$\partial_t^\alpha v(t_n) - {}_C\nabla_\tau^\alpha v^n = \int_0^\infty \rho_n^\tau(t) [\partial_t^\alpha v(t_n) - \partial_t^\alpha v(t)] dt.$$

Let $\Gamma = \{\lambda \in \Gamma_t : \mathrm{Re}(\lambda) > 0\}$, where Γ_t is the path defined in Lemma 3.14. As the Laplace transform of $\partial_t^\alpha v(t)$ verifies $\widehat{\partial_t^\alpha v}(\lambda) = \lambda^\alpha \hat{v}(\lambda) - \lambda^{\alpha-1} v(0)$, we have by the inversion of the Laplace transform,

$$\partial_t^\alpha v(t_n) - \partial_t^\alpha v(t) = \frac{1}{2\pi i} \int_\Gamma (e^{\lambda t_n} - e^{\lambda t})(\lambda^\alpha \hat{v}(\lambda) - \lambda^{\alpha-1} v(0)) d\lambda.$$

Integrating by parts, we have

$$\partial_t^\alpha v(t_n) - \partial_t^\alpha v(t) = \frac{1}{2\pi i} \int_\Gamma \frac{(e^{\lambda t_n} - e^{\lambda t})}{\lambda} \frac{1}{\lambda^{1-\alpha}} v'(0) d\lambda + \frac{1}{2\pi i} \int_\Gamma \frac{(e^{\lambda t_n} - e^{\lambda t})}{\lambda} \frac{1}{\lambda^{1-\alpha}} \int_0^\infty e^{-\lambda s} v''(s) ds d\lambda.$$

By the mean value for complex valued functions, there exist t_0, t_1 with $0 < t_n < t_0 < t_1 < t$ such that

$$\frac{|e^{\lambda t} - e^{\lambda t_n}|}{|\lambda|} \leq (t - t_n) \left(|e^{t_0 \lambda}| + |e^{t_1 \lambda}| \right).$$

2 The hypotheses and Lemma 3.14 imply that

$$\begin{aligned}
\|\partial_t^\alpha v(t_n) - \partial_t^\alpha v(t)\| &\leq \frac{1}{2\pi} \int_\Gamma \frac{|e^{\lambda t_n} - e^{\lambda t}|}{|\lambda|} \frac{1}{|\lambda|^{1-\alpha}} \|v'(0)\| |d\lambda| \\
&\quad + \frac{1}{2\pi} \int_\Gamma \frac{|e^{\lambda t_n} - e^{\lambda t}|}{|\lambda|} \frac{1}{|\lambda|^{1-\alpha}} \int_0^\infty e^{-\mathrm{Re}(\lambda)s} \|v''(s)\| ds |d\lambda| \\
&\leq \frac{1}{2\pi} (t - t_n)(\|v'(0)\| + \|v''\|_{L^1(\mathbb{R}_+)}) \int_\Gamma \left[\frac{|e^{\lambda t_0}|}{|\lambda|^{1-\alpha}} + \frac{|e^{\lambda t_1}|}{|\lambda|^{1-\alpha}} \right] |d\lambda| \\
&\leq \frac{C_\alpha}{2\pi} (t - t_n)(\|v'(0)\| + \|v''\|_{L^1(\mathbb{R}_+)})(t_0^{-\alpha} + t_1^{-\alpha}) \\
&\leq \frac{C_\alpha}{\pi} (t - t_n)(\|v'(0)\| + \|v''\|_{L^1(\mathbb{R}_+)}) t_n^{-\alpha}.
\end{aligned}$$

Since $\int_0^\infty \rho_n^\tau(t)tdt = \tau(n+1)\int_0^\infty \rho_{n+1}^\tau(t)dt$, we obtain that $\int_0^\infty \rho_n^\tau(t)(t-t_n)dt = \tau$, and therefore,

$$\|\partial_t^\alpha v(t_n) - {}_C\nabla_\tau^\alpha v^n\| \le \int_0^\infty \rho_n^\tau(t)\|\partial_t^\alpha v(t_n) - \partial_t^\alpha v(t)\|dt \le \tau\frac{C_\alpha}{\pi}(\|v'(0)\| + \|v''\|_{L^1(\mathbb{R}_+)})t_n^{-\alpha}.$$

Thus,

$$\|\partial_t^\alpha v(t_n) - {}_C\nabla_\tau^\alpha v^n\| \le \tau^{1-\alpha}\frac{C_\alpha}{\pi}(\|v'(0)\| + \|v''\|_{L^1(\mathbb{R}_+)}) = \tau^{1-\alpha}M_\alpha(\|v'(0)\| + \|v''\|_{L^1(\mathbb{R}_+)}).$$

□

Corollary 3.16. *Under the assumptions of Theorem 3.15, we have*

$$\lim_{\tau\to 0^+}\|\partial_t^\alpha v(t_n) - {}_C\nabla_\tau^\alpha v^n\| = 0.$$

4. Examples

Given $\alpha > 0$ we have by [5, Appendix A] that if $u(t) = e^{\rho t}$, and $m = \lceil\alpha\rceil$, then

$$\partial_t^\alpha u(t) = \sum_{l=0}^\infty \frac{\rho^{l+m}t^{l+m-\alpha}}{\Gamma(l+1+m-\alpha)} = \rho^m t^{m-\alpha}E_{1,m-\alpha+1}(\rho t),$$

where for $p, q > 0$, $E_{p,q}$ is the Mittag-Leffler function defined by $E_{p,q}(z) = \sum_{j=0}^\infty \frac{z^j}{\Gamma(\alpha j+\beta)}$. Moreover, by [11, Formula 3.381-4] we have that

$$u^j = \int_0^\infty \rho_j^\tau(t)u(t)dt = \frac{1}{(1-\tau\rho)^{j+1}}$$

for all $j \ge 0$. On the other hand, for each $n \in \mathbb{N}$ and $\alpha = \frac{1}{2}$, by definition we have

$$({}_C\nabla^\alpha u)^n = \nabla_\tau^{-(1-\alpha)}(\nabla_\tau^1 u)^n = \tau\sum_{j=1}^n k_\tau^{1-\alpha}(n-j)\frac{u^j - u^{j-1}}{\tau} = \frac{\tau^{-\frac{1}{2}}}{\sqrt{\pi}}\sum_{j=1}^n \frac{\Gamma(\frac{1}{2}+n-j)}{\Gamma(n-j+1)}(u^j - u^{j-1}).$$

In Figure 1 we have $\partial_t^\alpha u(t)$ and its approximation $({}_C\nabla^\alpha u)^n$ on the interval $[0,1]$ for $1 \le n \le N$, $\alpha = \frac{1}{2}$ and $\rho = -\frac{1}{2}$. We consider $\tau = 1/N$ for $N = 30, N = 60$ and $N = 120$.

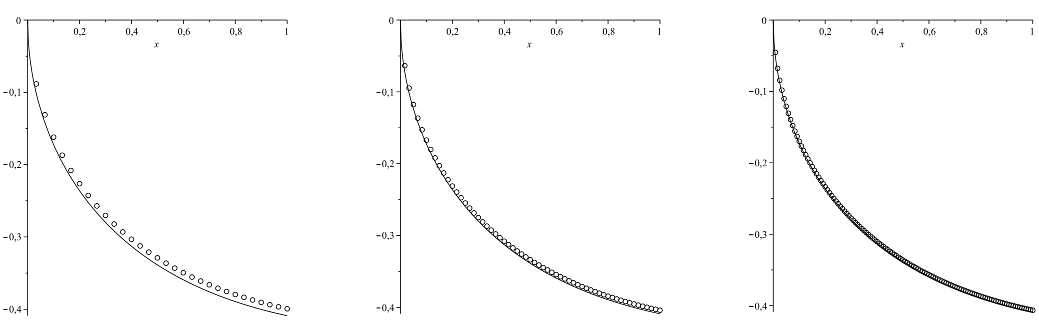

Figure 1. The fractional derivative $\partial_t^\alpha u(t)$ (line) and its approximation $({}_C\nabla^\alpha u)^n$ (circles) for $N = 30, N = 60$ and $N = 120$.

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