Sobolev type time fractional differential equations and optimal controls with the order in $(1, 2)$

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Abstract

This paper is mainly concerned with controlled time fractional differential equations of Sobolev type in Caputo and Riemann-Liouville fractional derivatives with the order in $(1, 2)$ respectively. By properties on some corresponding fractional resolvent operators family, we first establish sufficient conditions for the existence of mild solutions to these controlled time fractional differential equations of Sobolev type. And then we present the existence of optimal controls of systems governed by corresponding time fractional differential equations of Sobolev type via setting up approximating minimizing sequences of suitable functions twice.

Keywords: Fractional differential equations of Sobolve type; Resolvent operator; Feasible pairs, Optimal controls.

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1 Introduction

This paper mainly treats controlled time fractional differential equations of Sobolev type and optimal controls. Concretely, let $A : D(A) \subseteq X \to X$, $E : D(E) \subseteq X \to X$ be closed linear operators defined on a Banach space X with the norm $\|\cdot\|$, $x_0, x_1 \in X$. Now, we consider the following controlled fractional differential equations of Sobolev type

$$\begin{cases} D_t^\alpha (Ex)(t) = Ax(t) + f(t, x(t)) + \mathfrak{B}(t)u(t), \\ Ex(0) = Ex_0,\ (Ex)'(0) = Ex_1,\ u \in U_{ad}, \end{cases} \tag{1.1}$$

and

$$\begin{cases} D^\alpha (Ex)(t) = Ax(t) + f(t, x(t)) + \mathfrak{B}(t)u(t), \\ E(g_{2-\alpha} * x)(0) = Ex_0,\ (E(g_{2-\alpha} * x))'(0) = Ex_1,\ u \in U_{ad}, \end{cases} \tag{1.2}$$

where $t \in I := [0, b]$, the order $1 < \alpha < 2$, the notations D_t^α and D^α denote, respectively, the Caputo and Riemann-Liouville fractional derivatives, and the operator pair (A, E) generates a resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ (see definition below, Section 2) for suitable

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$\alpha, \beta > 0$, U_{ad} is a control set, nonlinear perturbation $f : I \times X \to X$ and the functions $\mathfrak{B}(\cdot)$, $g_\star(\cdot)$ will also specified in Section 2.

It is noted that there are already some fundamental results on abstract fractional differential equations of Sobolev type with the order $0 < \alpha < 1$, see for example [6, 9, 10, 14, 18, 31, 32] and the references therein. The main techniques in these mentioned work are based upon the following condition (C): *$D(E) \subset D(A)$, E is bijective and $E^{-1} : X \to D(E)$ is a compact operator.* Under this condition, the so-called *subordination formulas* can be applied to deal with solution representations and related problems. It should be mentioned that another method to deal with abstract time fractional differential equation of Sobolev type with the order $0 < \alpha < 1$ is developed in [18, 32], where solution representations are derived from *subordination formulas* of propagation family (see [20]) without the above condition (C).

Controlled fractional differential equations of Sobolev type are naturally applied to the control of dynamical system when the controlled system or the controller is described by a fractional differential equation of Sobolev type. We especially point out that in Refs [9, 10], some interesting results on optimal multi-controls and optimal multi-integral controls governed by fractional abstract evolution equations of Sobolev type have been established. The fractional derivative in Refs [9, 10] is understood in Caputo sense with the order $0 < \alpha < 1$, and solution operators are based upon subordination formulas under the condition (C). For more detailed results on optimal control theory and applications, we refer to [19, 37, 42, 43] and the references therein.

Notice the order $1 < \alpha < 2$, the above mentioned techniques are no longer directly applicable to (1.1) and (1.2). As far as we know, existence of solutions and optimal controls for controlled systems (1.1) and (1.2) in case $1 < \alpha < 2$ (and $E \neq \mathcal{I}$, identity operator) have not been addressed in the existing literature. In present paper, we shall first establish sufficient conditions for the existence of mild solutions to Eq. (1.1) and Eq. (1.2) respectively based upon properties on resolvent operator generated by the pair (A, E). And then we shall present the existence of optimal controls of systems governed by Eq. (1.1) or Eq. (1.2) via constructing approximating minimizing sequences of suitable functions twice. We remark that our results are directly established through resolvent operators generated by the pair (A, E), and thus previous condition (C) is not necessarily needed. Finally, some applications are also given to illustrate our main results.

The rest of this paper is organized as follows. Section 2 is involved in Preliminaries. Section 3 is devoted to investigate controlled time fractional differential equations of Sobolve type Eq. (1.1) and Eq. (1.2), respectively. Section 4 is involved in some applications.

2 Preliminaries

In this section, we list some definitions, notations and recall some basic results which are used throughout this paper. Most of these results can be found in monographs [3, 21, 41, 42], papers [1, 2, 4, 5, 8, 12, 15, 25, 26, 27, 28, 29, 30, 33, 34, 38, 39, 40] and references therein.

Let $(X, \|\cdot\|), Z$ be Banach spaces. We denote by $\mathcal{B}(X,Z)$ the space of all bounded linear operators from X into Z, and denote by $\mathcal{B}(X)$ the space of all bounded linear operators from X into itself. Let $C(I,X)$ be the Banach space of all continuous functions from I to X with the norm $\|x\|_\infty = \sup_{t\in I}\|u(t)\|$ and $L^p(I,X)(1 \leq p < +\infty)$ be the Banach space of all X-valued Bochner integrable functions defined on I with the norm $\|x\|_{L^p} = \left(\int_I \|x(t)\|^p dt\right)^{\frac{1}{p}}$.

For a closed and linear operator $A : D(A) \subset X \to X$, where $D(A)$ is the domain of A, we denote by $\rho(A)$ its resolvent set and by $R(\lambda, A)$ its resolvent operator, that is, $R(\lambda, A) = (\lambda - A)^{-1}$ which is defined for all $\lambda \in \rho(A)$. For $\mu > 0$, we define

$$g_\mu(t) = \begin{cases} \dfrac{t^{\mu-1}}{\Gamma(\mu)}, \quad t > 0, \\ 0, \quad t \leq 0, \end{cases} \tag{2.1}$$

where $\Gamma(\cdot)$ is the Gamma function. We also define $g_0 \equiv \delta_0$, the Dirac delta. For $\mu > 0$, $n = \lceil \mu \rceil$ denotes the smallest integer n greater than or equal to μ. The finite convolution of f and g is denoted by $(f * g)(t) = \int_0^t f(t-s)g(s)ds$.

Definition 2.1 Let $\alpha > 0$. The α-order Riemann-Liouville fractional integral of u is defined by

$$J^\alpha u(t) := \int_0^t g_\alpha(t-s)u(s)ds, \quad t \geq 0.$$

Also, we define $J^0 u(t) = u(t)$. Because of the convolution properties, the integral operators $\{J^\alpha\}_{\alpha\geq 0}$ satisfy the following semigroup law: $J^\alpha J^\beta = J^{\alpha+\beta}$ for all $\alpha, \beta \geq 0$.

Definition 2.2 Let $\alpha > 0$. The α-order Caputo fractional derivative is defined

$$D_t^\alpha u(t) := \int_0^t g_{n-\alpha}(t-s)u^{(n)}(s)ds,$$

where $n = \lceil \alpha \rceil$.

Definition 2.3 Let $\alpha > 0$. The α-order Riemann-Liouville fractional derivative of u is defined

$$D^\alpha u(t) := \frac{d^n}{dt^n}\int_0^t g_{n-\alpha}(t-s)u(s)ds,$$

where $n = \lceil \alpha \rceil$.

It is clear $D_t^m = D^m = \dfrac{d^m}{dt^m}$ if $\alpha = m \in \mathbb{N}$.

Let $\hat{f}$ (or $\mathcal{L}(f)$) denote the Laplace transform of f, we have the following facts for the fractional derivatives

$$\widehat{D^\alpha u}(\lambda) = \lambda^\alpha \hat{u}(\lambda) - \sum_{k=0}^{n-1}(g_{n-\alpha} * u)^{(k)}(0)\lambda^{n-1-k} \tag{2.2}$$

and

$$\widehat{D_t^\alpha u}(\lambda) = \lambda^\alpha \hat{u}(\lambda) - \sum_{k=0}^{n-1} u^{(k)}(0)\lambda^{\alpha-1-k}, \tag{2.3}$$

where $n = \lceil \alpha \rceil$ and $\lambda \in \mathbb{C}$. For $\alpha, \beta > 0$ and $z \in \mathbb{C}$, the generalized Mittag-Leffler function is defined by

$$e_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

and its Laplace transform $\mathcal{L}$ satisfies

$$\mathcal{L}(t^{\beta-1}e_{\alpha,\beta}(\rho t^\alpha))(\lambda) = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \rho}, \quad \rho \in \mathbb{C}, \mathrm{Re}\lambda > |\rho|^{1/\alpha}.$$

The *E-modified resolvent set of* A, $\rho_E(A)$, is defined by

$$\rho_E(A) := \{\lambda \in \mathbb{C} : (\lambda E - A) : D(A) \cap D(E) \to X \text{ is invertible and } (\lambda E - A)^{-1} \in \mathcal{B}(X, [D(A) \cap D(E)])\}.$$

The operator $(\lambda E - A)^{-1}$ is called the E-resolvent operator of A.

A strongly continuous family $\{T(t)\}_{t\geq 0} \subseteq \mathcal{B}(X)$ is said to be of type (M, ω) or *exponentially bounded* if there exist constants $M > 0$ and $\omega \in \mathbb{R}$, such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. Observe that, without loss of generality, we can assume $\omega > 0$ in the sequel.

Definition 2.4 Let $A : D(A) \subseteq X \to X$, $E : D(E) \subseteq X \to X$ be closed linear operators defined on a Banach space X satisfying $D(A) \cap D(E) \neq \{0\}$. Let $\alpha, \beta > 0$. We say that the pair (A, E) is the generator of an (α, β)-resolvent family, if there exist $\mu \geq 0$ and a strongly continuous function $S^E_{\alpha,\beta} : [0, \infty) \to \mathcal{B}(X)$ such that $S^E_{\alpha,\beta}(t)$ is exponentially bounded, $\{\lambda^\alpha : \mathrm{Re}\lambda > \mu\} \subset \rho_E(A)$, and for all $x \in X$,

$$\lambda^{\alpha-\beta} E\,(\lambda^\alpha E - A)^{-1} x = \int_0^\infty e^{-\lambda t} S^E_{\alpha,\beta}(t)x dt, \;\; \mathrm{Re}\lambda > \mu.$$

In this case, $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ is called the (α, β)-*resolvent family* generated by the pair (A, E).

It is easy to show (see [24, Proposition 3.1 and Lemma 2.2])) that if (A, E) generates an (α, β)-resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$, then it satisfies the following properties:

i) $S^E_{\alpha,\beta}(0)Ex = g_\beta(0)Ex$, for all $x \in D(E)$;

ii) $S^E_{\alpha,\beta}(t)x \in D(A) \cap D(E)$ and $S^E_{\alpha,\beta}(t)Ax = AS^E_{\alpha,\beta}(t)x$, $S^E_{\alpha,\beta}(t)Ex = ES^E_{\alpha,\beta}(t)x$, for all $x \in D(A) \cap D(E)$ and $t \geq 0$;

iii) $S^E_{\alpha,\beta}(t)Ex = g_\beta(t)Ex + \int_0^t g_\alpha(t-s)AS^E_{\alpha,\beta}(s)xds$, for all $x \in D(A) \cap D(E)$ and $t \geq 0$;

iv) $\int_0^t g_\alpha(t-s)S^E_{\alpha,\beta}(s)xds \in D(A)$ and $S^E_{\alpha,\beta}(t)Ex = g_\beta(t)Ex + A\int_0^t g_\alpha(t-s)S^E_{\alpha,\beta}(s)xds$, for all $x \in D(A)\cap D(E)$ and $t \geq 0$.

We notice that the Definition 2.4 corresponds to some well-known concepts in the literature. In fact, the case $S^E_{1,1}(t)$ corresponds to degenerate semigroups (see [13] and [16, Chapter 1, Section 1.5]), if $\alpha = 1, \beta = k+1$, then $S^E_{1,k+1}(t)$ is a degenerated k-integrated semigroup (see [16, Chapter 1, Section 1.5] and [7]), and $S^E_{2,1}(t)$ is a cosine degenerate family (see [16, Chapter 1, Section 1.7]). If $E = I$, then $S^I_{1,1}(t), S^I_{1,k+1}(t)$ and $S^I_{2,1}(t)$ correspond to a C_0-semigroup, a k-integrated semigroup and a cosine family, respectively. Finally, if $\beta = 1$, then $S^I_{\alpha,1}(t)$ is the α-resolvent family (also called the α-times resolvent family) for fractional differential equations.

Definition 2.5 The resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be *compact* if for every $t > 0$, the operator $S^E_{\alpha,\beta}(t)$ is a compact operator.

Next we give some results on the norm continuity and compactness of $S^E_{\alpha,\beta}(t)$ for given $\alpha, \beta > 0$. The proofs of these results can be conducted similarly to [28, Proposition 11, Lemma 12, Theorem 14, Corollary 15, Propositions 16-17], we can also refer to [8] for details.

Lemma 2.1 Let $\alpha > 0$ and $1 < \beta \leq 2$. Suppose that $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ is the (α,β)-resolvent family of type (M,ω) generated by (A,E). Then the function $t \mapsto S^E_{\alpha,\beta}(t)$ is continuous in $\mathcal{B}(X)$ for all $t > 0$.

Lemma 2.2 Suppose that the pair (A,E) generates an (α,β)-resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ of type (M,ω). If $\gamma > 0$, then (A,E) also generates an $(\alpha,\beta+\gamma)$-resolvent family of type $\left(\frac{M}{\omega^\gamma}, \omega\right)$.

Lemma 2.3 Let $\alpha > 0$, $1 < \beta \leq 2$ and $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ be an (α,β)-resolvent family of type (M,ω) generated by (A,E). Then the following assertions are equivalent

i) $S^E_{\alpha,\beta}(t)$ is a compact operator for all $t > 0$.

ii) $E(\mu E - A)^{-1}$ is a compact operator for all $\mu > \omega^{1/\alpha}$.

Lemma 2.4 Let $1 < \alpha \leq 2$ and $\{S^E_{\alpha,\alpha}(t)\}_{t\geq 0}$ be an (α,α)-resolvent family of type (M,ω) generated by (A,E). Then the following assertions are equivalent:

i) $S^E_{\alpha,\alpha}(t)$ is a compact operator for all $t > 0$.

ii) $E(\mu E - A)^{-1}$ is a compact operator for all $\mu > \omega^{1/\alpha}$.

Lemma 2.5 Let $1 < \alpha < 2$, and $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ be the $(\alpha,1)$-resolvent family of type (M,ω) generated by (A,E). Suppose that $S^E_{\alpha,1}(t)$ is continuous in the uniform operator topology for all $t > 0$. Then the following assertions are equivalent

i) $S^E_{\alpha,1}(t)$ is a compact operator for all $t > 0$.

ii) $E(\mu E - A)^{-1}$ is a compact operator for all $\mu > \omega^{1/\alpha}$.

Lemma 2.6 Let $\frac{3}{2} < \alpha < 2$, and $\{S^E_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ be the $(\alpha, \alpha-1)$-resolvent family of type (M, ω) generated by (A, E). Suppose that $S^E_{\alpha,\alpha-1}(t)$ is continuous in the uniform operator topology for all $t > 0$. Then the following assertions are equivalent

i) $S^E_{\alpha,\alpha-1}(t)$ is a compact operator for all $t > 0$.

ii) $E(\mu E - A)^{-1}$ is a compact operator for all $\mu > \omega^{1/\alpha}$.

Remark 2.1 If $E = \mathcal{I}$ (the identity operator), then the facts of above lemmas are reduced to the corresponding results in [28, Lemma 3.12, Theorem 3.14, Propositions 3.16-3.17, Proposition 7.1] with $S^{\mathcal{I}}_{\alpha,\beta}(t) = S_{\alpha,\beta}(t)$.

Finally, we recall the following results.

Lemma 2.7 If K is a compact subset of a Banach space X, then its convex closure $\overline{\text{conv}(K)}$ is compact.

Lemma 2.8 The closure and weak closure of a convex subset of a normed space are the same.

Lemma 2.9 Let $\mathbb{C}$ be a nonempty, closed, bounded and convex subset of a Banach space X. Suppose that $\Upsilon : \mathbb{C} \to \mathbb{C}$ is a compact operator. Then Υ has at least a fixed point in $\mathbb{C}$.

In what follows, we introduce the admissible control set as [9] and [42, pp.141]. Let Y be another separable reflexive Banach space from which the control u takes values. Let $1 < p < +\infty$ and $L^p(I, Y)$ denote the usual Banach space of all Y-valued Bochner integrable functions having p-th power summable norms. Denoted $\mathcal{P}(Y)$ by a class of nonempty closed and convex subsets of Y. We assume that the multivalued map $\mathcal{U} : I \to \mathcal{P}(Y)$ is graph measurable, $\mathcal{U}(\cdot) \subset \Xi$, where Ξ is a bounded set of Y. The admissible control set is defined as

$$U_{ad} = S^p_{\mathcal{U}} = \{u(\cdot) \in L^p(\Xi) | u(t) \in \mathcal{U}(t), \text{ a.e. } t \in I\}, 1 < p < +\infty.$$

Then $U_{ad} \neq \emptyset$, which can be found in [17].

3 Controlled fractional differential equations of Sobolev type

In this section, we will prove our main results.

3.1 The Caputo case–Eq. (1.1)

Let us list the following assumptions.

(A1) The pair (A, E) generates the $(\alpha, 1)$-resolvent family $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ of type (M, ω), the operator $E(\lambda^\alpha E - A)^{-1}$ is compact for all $\lambda^\alpha \in \rho_E(A)$ with $\lambda > \omega^{\frac{1}{\alpha}}$ and $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ is norm continuous for all $t > 0$.

(A2) $f : I \times X \to X$ satisfies the following conditions:
(a) For a.e. $t \in I$, $f(t, \cdot)$ is continous, and for each $x \in X$, $f(\cdot, x)$ is measurable;
(b) There exists a function $\phi \in L^1(I, \mathbb{R}_+)$ such that

$$\|f(t, x)\| \leq \phi(t)\|x\|, \forall t \in I, x \in X.$$

(A3) $\mathfrak{B} : I \to \mathcal{B}(Y, X)$ is essentially bounded, i.e. $\mathfrak{B} \in L^\infty(I, \mathcal{B}(Y, X))$.

(A4) There exists a constant $r > 0$ such that

$$Me^{\omega b}\left[\|x_0\| + \frac{1}{\omega}\|x_1\| + \frac{r}{\omega^{\alpha-1}}\|\phi\|_{L^1} + \frac{1}{\omega^{\alpha-1}}\|\mathfrak{B}u\|_{L^1}\right] \leq r.$$

According to the properties of the Laplace transform we can give the following definition of mild solution to problem (1.1).

Definition 3.1 For each $x_0, x_1 \in X$, a function $x \in C(I, X)$ is said to be a mild solution to Eq. (1.1) if it verifies the following integral equation

$$x(t) = S^E_{\alpha,1}(t)x_0 + S^E_{\alpha,2}(t)x_1 + \int_0^t S^E_{\alpha,\alpha}(t-s)[f(s, x(s)) + \mathfrak{B}(s)u(s)]ds.$$

Remark 3.1 (i) By the uniqueness of the Laplace transform, it is clear that the mild solution to Eq. (1.1) can expressed as

$$x(t) = S^E_{\alpha,1}(t)x_0 + (g_1 * S^E_{\alpha,1})(t)x_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)[f(s, x(s)) + \mathfrak{B}(s)u(s)]ds.$$

(ii) In view of Lemma 2.5, the condition (A1) implies $S^E_{\alpha,1}(t)$ is compact for all $t > 0$.
(iii) [42, pp.141] According to the assumption (A3) and the definition of the admissible set U_{ad}, it is concluded that $\mathfrak{B}u \in L^p(I, X)$ with $1 < p < \infty$ for all $u \in U_{ad}$. Thus, $\mathfrak{B}u \in L^1(I, X)$ and $\|\mathfrak{B}u\|_{L^1} < +\infty$.

For $r > 0$ we define $B_r := \{x \in C(I, X) : \|x(t)\| \leq r, t \in I\}$.

Theorem 3.1 If assumptions (A1)-(A4) hold, then Eq. (1.1) admits at least one mild solution on I.

Proof: Consider the operator $N : C(I,X) \to C(I,X)$ defined by

$$(Nx)(t) = S^E_{\alpha,1}(t)x_0 + (g_1 * S^E_{\alpha,1})(t)x_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)[f(s,x(s)) + \mathfrak{B}(s)u(s)]ds, t \in I.$$

Clearly, the fixed points of N are mild solutions to Eq. (1.1). We shall show that N admits a fixed point. The proof will be given in several steps.
Step I. N maps B_r into B_r.

$$\begin{aligned}
&\|(Nx)(t)\| \\
\leq\ & \left\| S^E_{\alpha,1}(t)x_0 + (g_1 * S^E_{\alpha,1})(t)x_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)[f(s,x(s)) + \mathfrak{B}(s)u(s)]ds \right\| \\
\leq\ & Me^{\omega t}\|x_0\| + \frac{M}{\omega}e^{\omega t}\|x_1\| + \frac{M}{\omega^{\alpha-1}}\int_0^t e^{\omega(t-s)}\left[\phi(s)\|x(s)\| + \|\mathfrak{B}(s)u(s)\|\right]ds \\
\leq\ & Me^{\omega b}\|x_0\| + \frac{M}{\omega}e^{\omega b}\|x_1\| + \frac{Mre^{\omega b}}{\omega^{\alpha-1}}\int_0^t e^{-\omega s}\phi(s)ds \\
& + \frac{Me^{\omega b}}{\omega^{\alpha-1}}\int_0^t e^{-\omega s}\|\mathfrak{B}(s)u(s)\|ds \\
\leq\ & Me^{\omega b}\|x_0\| + \frac{M}{\omega}e^{\omega b}\|x_1\| + \frac{Mre^{\omega b}}{\omega^{\alpha-1}}\|\phi\|_{L^1} + \frac{Me^{\omega b}}{\omega^{\alpha-1}}\|\mathfrak{B}u\|_{L^1} \\
\leq\ & Me^{\omega b}\left[\|x_0\| + \frac{1}{\omega}\|x_1\| + \frac{r}{\omega^{\alpha-1}}\|\phi\|_{L^1} + \frac{1}{\omega^{\alpha-1}}\|\mathfrak{B}u\|_{L^1}\right] \\
\leq\ & r.
\end{aligned}$$

By (A4), we conclude that $Nx \in B_r$.
Step II. N is continous in B_r.

Let $x_n, x \in B_r$ be such that $x_n \to x$ in B_r. In view of Lemma 2.2, we have

$$\begin{aligned}
\|(Nx_n)(t) - (Nx)(t)\| &\leq \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)[f(s,x_n(s)) - f(s,x(s))]ds \\
&\leq \frac{Me^{\omega b}}{\omega^{\alpha-1}}\int_0^t e^{-\omega s}\|f(s,x_n(s)) - f(s,x(s))\|ds \\
&\leq \frac{Me^{\omega b}}{\omega^{\alpha-1}}\int_0^t \phi(s)(\|x_n(s)\| + \|x(s)\|)ds \\
&\leq \frac{2rMe^{\omega b}}{\omega^{\alpha-1}}\int_0^t \phi(s)ds.
\end{aligned}$$

Note that the function $s \mapsto \phi(s)$ is integrable on I. By the Lebesgue Dominated Convergence Theorem $\int_0^t \|f(s,x_n(s)) - f(s,x(s))\|ds \to 0, n \to \infty$. Hence, N is continuous in B_r.
Step III. N is equicontinous.

Let $x \in B_r$, and take $0 \leq t_2 < t_1 \leq b$. Observe that

$$\|(Nx)(t_1) - (Nx)(t_2)\|$$

$$\begin{aligned}
&\le \; \| \left(S^E_{\alpha,1}(t_1) - S^E_{\alpha,1}(t_2)\right) x_0\| + \| \left((g_1 * S^E_{\alpha,1})(t_1) - (g_1 * S^E_{\alpha,1})(t_2)\right) x_1\| \\
&\quad + \int_{t_2}^{t_1} \|(g_{\alpha-1} * S^E_{\alpha,1})(t_1 - s)[f(s, x(s)) + \mathfrak{B}(s)u(s)]\| ds \\
&\quad + \int_0^{t_2} \| \left((g_{\alpha-1} * S^E_{\alpha,1})(t_1 - s) - (g_{\alpha-1} * S^E_{\alpha,1})(t_2 - s)\right) [f(s, x(s)) + \mathfrak{B}(s)u(s)]\| ds \\
&:= \; I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

For the term I_1, we have

$$I_1 \le \|(S^E_{\alpha,1}(t_1) - S^E_{\alpha,1}(t_2))\| \|x_0\|.$$

By the norm continuity of $S^E_{\alpha,1}(t)$ in assumption (A1), we get $\lim_{t_1 \to t_2} I_1 = 0$.
For the term I_2, we have $(g_1 * S^E_{\alpha,1})(t) = S^E_{\alpha,2}(t)$ for all $t \ge 0$ due to the uniqueness of the Laplace transform and Lemma 2.2. Meanwhile, the Lemma 2.1 implies that $(g_1 * S^E_{\alpha,1})(t)$ is continuous in $\mathcal{B}(X)$. Hence

$$I_2 \le \|(g_1 * S_{\alpha,1})(t_1) - (g_1 * S_{\alpha,1})(t_2)\| \, \|x_1\| \to 0, \text{ as } t_1 \to t_2.$$

For the term I_3, as $t_1 \to t_2$, we have

$$\begin{aligned}
I_3 \;&\le\; \frac{Me^{\omega b}}{\omega^{\alpha-1}} \int_{t_2}^{t_1} e^{-\omega s}[\phi(s)\|x(s)\| + \|\mathfrak{B}(s)u(s)\|] ds \\
&\le \frac{Mre^{\omega b}}{\omega^{\alpha-1}} \int_{t_2}^{t_1} \phi(s) ds + \frac{Me^{\omega b}}{\omega^{\alpha-1}} \int_{t_2}^{t_1} \|\mathfrak{B}(s)u(s)\| ds \to 0.
\end{aligned}$$

Finally for the term I_4, we have

$$\begin{aligned}
I_4 \;&\le\; \int_0^{t_2} \left\| \left[(g_{\alpha-1} * S^E_{\alpha,1})(t_1 - s) - (g_{\alpha-1} * S^E_{\alpha,1})(t_2 - s)\right] \right\| \|[f(s, x(s)) + \mathfrak{B}(s)u(s)]\| ds \\
&\le\; \int_0^{t_2} \left\| \left[(g_{\alpha-1} * S^E_{\alpha,1})(t_1 - s) - (g_{\alpha-1} * S^E_{\alpha,1})(t_2 - s)\right] \right\| \phi(s)\|x(s)\| ds \\
&\quad + \int_0^{t_2} \left\| \left[(g_{\alpha-1} * S^E_{\alpha,1})(t_1 - s) - (g_{\alpha-1} * S^E_{\alpha,1})(t_2 - s)\right] \right\| \|\mathfrak{B}(s)u(s)\| ds \\
&\le\; r \int_0^{t_2} \left\| \left[(g_{\alpha-1} * S^E_{\alpha,1})(t_1 - s) - (g_{\alpha-1} * S^E_{\alpha,1})(t_2 - s)\right] \right\| \phi(s) ds \\
&\quad + \int_0^{t_2} \left\| \left[(g_{\alpha-1} * S^E_{\alpha,1})(t_1 - s) - (g_{\alpha-1} * S^E_{\alpha,1})(t_2 - s)\right] \right\| \|\mathfrak{B}(s)u(s)\| ds.
\end{aligned}$$

Now taking into account that

$$\|(g_{\alpha-1} * S^E_{\alpha,1})(t_1 - \cdot) - (g_{\alpha-1} * S^E_{\alpha,1})(t_2 - \cdot)\|\phi(s) \le 2\frac{Me^{\omega b}}{\omega^{\alpha-1}}\phi(s) \in L^1(I, \mathbb{R}_+),$$

$$\left\| \left[(g_{\alpha-1} * S^E_{\alpha,1})(t_1 - s) - (g_{\alpha-1} * S^E_{\alpha,1})(t_2 - s)\right] \right\| \|\mathfrak{B}(s)u(s)\| \le 2\frac{Me^{\omega b}}{\omega^{\alpha-1}} \|\mathfrak{B}(s)u(s)\| \in L^1(I, \mathbb{R}_+),$$

$(g_{\alpha-1} * S^E_{\alpha,1})(t) = S^E_{\alpha,\alpha}(t)$ for all $t \geq 0$ (see Lemma 2.2) and $S_{\alpha,\alpha}(t)$ is norm continuous (see Lemma 2.1), we have $(g_{\alpha-1} * S^E_{\alpha,1})(t_1 - s) - (g_{\alpha-1} * S^E_{\alpha,1})(t_2 - s) \to 0$ in $\mathcal{B}(X)$ as $t_1 \to t_2$. By the Lebesgue's dominated convergence theorem, we conclude $\lim_{t_1 \to t_2} I_4 = 0$.

Step IV. The set $H(t) := \{(Nx)(t) : x \in B_r\}$ is relatively compact for every $t \in I$.

Clearly, $H(0)$ is relatively compact in X. For $x \in B_r$, we define the following operator

$$(N_2 x)(t) := \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)[f(s, x(s)) + \mathfrak{B}(s)u(s)]ds.$$

Now, let $0 < t \leq b$ and ε be a real number satisfying $0 < \varepsilon < t$, we further introduce

$$(N_2^\varepsilon x)(t) = \int_0^{t-\varepsilon} (g_{\alpha-1} * S^E_{\alpha,1})(t-s)[f(s, x(s)) + \mathfrak{B}(s)u(s)]ds.$$

The assumption (A1), Remark 3.1 (iii) and Lemma 2.4 imply the compactness of $(g_{\alpha-1} * S^E_{\alpha,1})(t) = S^E_{\alpha,\alpha}(t)$ for all $t > 0$. Therefore the set $\mathcal{K}_\varepsilon := \{(g_{\alpha-1} * S^E_{\alpha,1})(t-s)[f(s, x(s)) + \mathfrak{B}(s)u(s)] : x \in B_r, 0 \leq s \leq t - \varepsilon\}$ is compact for all $\varepsilon > 0$. Then $\overline{\mathrm{conv}(\mathcal{K}_\varepsilon)}$ is also a compact set by Lemma 2.7. In view of Mean-Value Theorem for the Bochner integrals, we have $(N_2^\varepsilon x)(t) \in t\overline{\mathrm{conv}(\mathcal{K}_\varepsilon)}$ for all $t \in I$. Thus the set $H_\varepsilon(t) = \{(N_2^\varepsilon x)(t) : x \in B_r\}$ is relatively compact in X for every ε, $0 < \varepsilon < t$. Moreover, for $x \in B_r$,

$$\begin{aligned} \|(N_2 x)(t) - (N_2^\varepsilon x)(t)\| &\leq \left\| \int_{t-\varepsilon}^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)[f(s, x(s)) + \mathfrak{B}(s)u(s)]ds \right\| \\ &\leq \frac{Me^{\omega b}}{\omega^{\alpha-1}} \int_{t-\varepsilon}^t e^{-\omega s}[r\phi(s) + \|\mathfrak{B}(s)u(s)\|]ds. \end{aligned}$$

Since $s \mapsto e^{-\omega s}[r\phi(s) + \|\mathfrak{B}(s)u(s)\|]$ belong to $L^1([t-\varepsilon, t], \mathbb{R}_+)$, we conclude by the Lebesgue Dominated Convergence Theorem that $\lim_{\varepsilon \to 0} \|(N_2 x)(t) - (N_2^\varepsilon x)(t)\| = 0$. Thus, the set $\left\{ \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)[f(s, x(s)) + \mathfrak{B}(s)u(s)]ds : x \in B_r \right\}$ is relatively compact for all $t \in (0, b]$. The compactness of $S^E_{\alpha,1}(t)$ and $(g_1 * S^E_{\alpha,1})(t) = S^E_{\alpha,2}(t)$ (see Lemma 2.5 and Lemma 2.3) imply that $H(t) := \{(Nx)(t) : x \in B_r\}$ is relatively compact in X.

As a consequence of the above steps and the Arzela-Ascoli theorem, we can deduce that N is a compact operator. By the fixed point theorem Lemma 2.9, there exists a fixed point $x(\cdot)$ for N on B_r. Thus, Eq. (1.1) admits a mild solution. This completes the proof.

Next, we consider the existence of optimal controls for Eq. (1.1). For any $u \in U_{ad}$, let $\mathcal{S}(u)$ denote all mild solutions to Eq. (1.1) in B_r.

Let $x^u \in B_r$ denote the mild solution to Eq. (1.1) corresponding to the control $u \in U_{ad}$, we consider the following limited Lagrange problem (LP):
Find $x^0 \in B_r \subseteq C(I, X)$ and $u^0 \in U_{ad}$ such that for all $u \in U_{ad}$, $J(x^0, u^0) \leq J(x^u, u)$, where

$$J(x^u, u) = \int_0^b \mathfrak{L}(t, x^u(t), u(t))dt,$$

and $x^0 \in B_r$ denotes the mild solution to Eq. (1.1) related to the control $u^0 \in U_{ad}$.

We need the following assumption.
(A5): The function $\mathfrak{L} : I \times X \times Y \to \mathbb{R} \bigcup \{\infty\}$ satisfies:

(1) The function $\mathfrak{L} : I \times X \times Y \to \mathbb{R}\bigcup\{\infty\}$ is Borel measurable;

(2) $\mathfrak{L}(t, \cdot, \cdot)$ is sequentially lower semicontinuous on $X \times Y$ for a.e. $t \in I$;

(3) $\mathfrak{L}(t, x, \cdot)$ is convex on Y for each $x \in X$ and a.e. $t \in I$;

(4) There exist constants $c \geq 0, d > 0$, ψ is nonnegative and $\psi \in L^1(I, \mathbb{R})$ such that

$$\mathfrak{L}(t, x, u) \geq \psi(t) + c\|x\| + d\|u\|_Y^p,$$

where $1 < p < \infty$. We remark that under the conditions of Theorem 3.1, a pair $(x(\cdot), u(\cdot))$ is feasible if it verifies Eq. (1.1) for $x(\cdot) \in B_r$, and if $(x^u(\cdot), u(\cdot))$ is feasible, then $x^u \in \mathcal{S}(u) \subset B_r$. We first list the following result.

Lemma 3.1 Assume that assumptions (A1) and (A3) hold and $1 < p < +\infty$. Define the operator Π by

$$(\Pi u)(\cdot) = \int_0^{\cdot} (g_{\alpha-1} * S_{\alpha,1}^E)(\cdot - s)\mathfrak{B}(s)u(s)ds,\ \forall u(\cdot) \in U_{ad} \subset L^p(I, Y).$$

Then, $\Pi : U_{ad} \subset L^p(I, Y) \to C(I, X)$ is compact. Moreover, if $u_n \in U_{ad}$ converges weakly to u as $n \to \infty$ in $L^p(I, Y)$, then $\Pi u_n \to \Pi u$ as $n \to \infty$.

Proof: The assumption (A1) and the Lemma 2.4 imply the compactness of $(g_{\alpha-1} * S_{\alpha,1}^E)(t) = S_{\alpha,\alpha}^E(t)$ for all $t > 0$. Conducted similarly as the proof of Theorem 3.1, we can show that Π is a compact operator. The convergence of $\Pi u_n \to \Pi u$ follows as in proof of Lemma 3.2 and Corollary 3.3 of Chapter 3 in [19].

Now, we can establish the existence of optimal controls for the problem (LP).

Theorem 3.2 Assume that conditions (A1)-(A5) hold. Then the problem (LP) admits at least one optimal feasible pair.

Proof: For any $u \in U_{ad}$, we define

$$J(u) = \inf_{x^u \in \mathcal{S}(u)} J(x^u, u).$$

If the set $\mathcal{S}(u)$ admits finite elements, there exists some $\tilde{x}^u \in \mathcal{S}(u)$ such that $J(\tilde{x}^u, u) = \inf_{x^u \in \mathcal{S}(u)} J(x^u, u) = J(u)$. If the set $\mathcal{S}(u)$ admits infinite elements and $\inf_{x^u \in \mathcal{S}(u)} J(x^u, u) = +\infty$, there is nothing to prove. We assume that $J(u) = \inf_{x^u \in \mathcal{S}(u)} J(x^u, u) < +\infty$. By (A5), we have $J(u) > -\infty$. We now divide the proof into the following steps.

Step 1. By the definition of the infimum, there exists a sequence $\{x_n^u\} \subseteq \mathcal{S}(u)$ satisfying $J(x_n^u, u) \to J(u)$ as $n \to \infty$. Considering $\{x_n^u, u\}$ is a sequence of feasible pairs, we have

$$\begin{aligned} x_n^u(t) &= S_{\alpha,1}^E(t)x_0 + (g_1 * S_{\alpha,1}^E)(t)x_1 \\ &\quad + \int_0^t (g_{\alpha-1} * S_{\alpha,1}^E)(t - s)[f(s, x_n^u(s)) + \mathfrak{B}(s)u(s)]ds, t \in I. \end{aligned} \tag{3.1}$$

Step 2. We show that there exists some $\tilde{x}^u \in \mathcal{S}(u)$ such that $J(\tilde{x}^u, u) = \inf_{x^u \in \mathcal{S}(u)} J(x^u, u) = J(u)$. To do this, we first prove that for each $u \in U_{ad}$, $\{x_n^u\}$ is relatively compact in $C(I, X)$. Note that

$$\begin{aligned} x_n^u(t) &= S_{\alpha,1}^E(t)x_0 + (g_1 * S_{\alpha,1}^E)(t)x_1 \\ &\quad + \int_0^t (g_{\alpha-1} * S_{\alpha,1}^E)(t-s)[f(s, x_n^u(s)) + \mathfrak{B}(s)u(s)]ds \\ &:= I_1 x_n^u + I_2 x_n^u + I_3 x_n^u. \end{aligned}$$

From (A1), Lemmas 2.3-2.5 and Steps III-IV in the proof of Theorem 3.1, we can conclude that $\{I_1 x_n^u\}$, $\{I_2 x_n^u\}$, $\{I_3 x_n^u\}$ are all precompact subsets of $C(I, X)$. In consequence, the set $\{x_n^u\}$ is precompact in $C(I, X)$ for $u \in U_{ad}$. Without loss of generality, we may assume that $x_n^u \to \tilde{x}^u$ in $C(I, X)$ for $u \in U_{ad}$ as $n \to \infty$. Let $n \to \infty$ in both sides of (3.1), by the Lebesgue Dominated Convergence Theorem, we obtain that

$$\begin{aligned} \tilde{x}^u(t) &= S_{\alpha,1}^E(t)x_0 + (g_1 * S_{\alpha,1}^E)(t)x_1 \\ &\quad + \int_0^t (g_{\alpha-1} * S_{\alpha,1}^E)(t-s)[f(s, \tilde{x}^u(s)) + \mathfrak{B}(s)u(s)]ds, t \in I, \end{aligned}$$

which implies that $\tilde{x}^u \in \mathcal{S}(u)$.

We claim that $J(\tilde{x}^u, u) = \inf_{x^u \in \mathcal{S}(u)} J(x^u, u) = J(u)$ for any $u \in U_{ad}$. In fact, owing to $C(I, X)$ is continuously embedded in $L^1(I, X)$, through the definition of a feasible pair, the assumption (A5) and Balder theorem, we have

$$J(u) = \lim_{n \to \infty} \int_0^b \mathfrak{L}(t, x_n^u(t), u(t))dt \geq \int_0^b \mathfrak{L}(t, \tilde{x}^u(t), u(t))dt = J(\tilde{x}^u, u) \geq J(u),$$

i.e. $J(\tilde{x}^u, u) = J(u)$. This shows that $J(u)$ admits its minimum at $\tilde{x}^u \in C(I, X)$ for each $u \in U_{ad}$.

Step 3. We show that there exists $u^0 \in U_{ad}$ such that $J(u^0) \leq J(u)$ for all $u \in U_{ad}$. If $\inf_{u \in U_{ad}} J(u) = +\infty$, there is nothing to prove. Assume that $\inf_{u \in U_{ad}} J(u) < +\infty$. Similarly to Step 1, we can prove that $\inf_{u \in U_{ad}} J(u) > -\infty$, and there exists a sequence $\{u_n\} \subseteq U_{ad}$ such that $J(u_n) \to \inf_{u \in U_{ad}} J(u)$ as $n \to \infty$. Since $\{u_n\} \subseteq U_{ad}$, $\{u_n\}$ is bounded in $L^p(I, Y)$ and $L^p(I, Y)$ is a reflexive Banach space for $1 < p < +\infty$, there exists a subsequence still denoted by $\{u_n\}$ weakly converges to some $u^0 \in L^p(I, Y)$ as $n \to \infty$. Note that U_{ad} is closed and convex, by Lemma 2.8 it follows that $u^0 \in U_{ad}$.

Suppose $\tilde{x}^{u_n}$ is the mild solution to Eq. (1.1) related to u_n, where $J(u_n)$ attains its minimum. Then $(\tilde{x}^{u_n}, u_n)$ is a feasible pair and verifies the following integral equation

$$\begin{aligned} \tilde{x}^{u_n}(t) &= S_{\alpha,1}^E(t)x_0 + (g_1 * S_{\alpha,1}^E)(t)x_1 \\ &\quad + \int_0^t (g_{\alpha-1} * S_{\alpha,1}^E)(t-s)[f(s, \tilde{x}^{u_n}(s)) + \mathfrak{B}(s)u_n(s)]ds, t \in I. \quad (3.2) \end{aligned}$$

Define

$$\begin{aligned}\Lambda_1\tilde{x}^{u_n}(t) &= S^E_{\alpha,1}(t)x_0, \quad \Lambda_2\tilde{x}^{u_n}(t) = (g_1 * S^E_{\alpha,1})(t)x_1,\\ \Lambda_3\tilde{x}^{u_n}(t) &= \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)f(s,\tilde{x}^{u_n}(s))ds,\\ \Lambda_4 u_n(t) &= \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)\mathfrak{B}(s)u_n(s)ds.\end{aligned}$$

Then

$$\tilde{x}^{u_n}(t) = \Lambda_1\tilde{x}^{u_n}(t) + \Lambda_2\tilde{x}^{u_n}(t) + \Lambda_3\tilde{x}^{u_n}(t) + \Lambda_4 u_n(t),\ t \in I.$$

From (A1), Lemmas 2.3-2.5 and similarly to Steps III-IV in the proof of Theorem 3.1, we can conclude that $\{\Lambda_1\tilde{x}^{u_n}\}$, $\{\Lambda_2\tilde{x}^{u_n}\}, \{\Lambda_3\tilde{x}^{u_n}\}$ are all relatively compact subsets of $C(I,X)$. Moreover, by Lemma 3.1, $\Lambda_4 u_n \to \Lambda_4 u^0$ in $C(I,X)$ as $n\to\infty$ and Λ_4 is compact. Thus, the set $\{\tilde{x}^{u_n}\} \subset C(I,X)$ is relatively compact, and there exists a subsequence still denoted by $\{\tilde{x}^{u_n}\}$, $\tilde{x}^{u^0} \in C(I,X)$ such that $\tilde{x}^{u_n} \to \tilde{x}^{u^0}$ in $C(I,X)$ as $n\to\infty$. Let $n\to\infty$ in both sides of (3.2), we have

$$\begin{aligned}\tilde{x}^{u^0}(t) &= S^E_{\alpha,1}(t)x_0 + (g_1 * S^E_{\alpha,1})(t)x_1\\ &\quad + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)[f(s,\tilde{x}^{u^0}(s)) + \mathfrak{B}(s)u^0(s)]ds, t \in I,\end{aligned}$$

which implies that $(\tilde{x}^{u^0}, u^0)$ is a feasible pair.

Since $C(I,X)$ is continuously embedded in $L^1(I,X)$, by the assumption (A5) and Balder theorem, we have

$$\begin{aligned}\inf_{u\in U_{ad}} J(u) &= \lim_{n\to\infty}\int_0^b \mathfrak{L}(t,\tilde{x}^{u_n}(t),u_n(t))dt \ge \int_0^b \mathfrak{L}(t,\tilde{x}^{u^0}(t),u^0(t))dt\\ &= J(\tilde{x}^{u^0},u^0) \ge \inf_{u\in U_{ad}} J(u).\end{aligned}$$

Therefore,

$$J(\tilde{x}^{u^0},u^0) = J(u^0) = \inf_{x^{u^0}\in\mathcal{S}(u^0)} J(x^{u^0},u^0).$$

Furthermore,

$$J(u^0) = \inf_{u\in U_{ad}} J(u),$$

i.e., J admits its minimum at $u^0 \in U_{ad}$. This finishes the proof.

3.2 The Riemann-Liouville case–Eq. (1.2)

For Eq. (1.2), we need the following hypotheses.

(H1) Let $\dfrac{3}{2} < \alpha < 2$, and the pair (A,E) generates the $(\alpha,\alpha-1)$-resolvent family $\{S^E_{\alpha,\alpha-1}(t)\}_{t\ge0}$ of type (M,ω), the operator $E(\lambda^\alpha E - A)^{-1}$ is compact for all $\lambda^\alpha \in \rho_E(A)$

with $\lambda > \omega^{\frac{1}{\alpha}}$ and $\{S^E_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ is norm continuous for all $t > 0$.
(H2) There exists a constant $r > 0$ such that

$$Me^{\omega b}\left[\|x_0\| + \frac{1}{\omega}\|x_1\| + \frac{r}{\omega}\|\phi\|_{L^1} + \frac{1}{\omega}\|\mathfrak{B}u\|_{L^1}\right] \leq r.$$

By using the properties of the Laplace transform we are able to give the following definition of mild solution to problem (1.2).

Definition 3.2 For each $x_0, x_1 \in X$, a function $x \in C(I,X)$ is said to be a mild solution to Eq. (1.2) if it verifies the following integral equation

$$x(t) = S^E_{\alpha,\alpha-1}(t)x_0 + S^E_{\alpha,\alpha}(t)x_1 + \int_0^t S^E_{\alpha,\alpha}(t-s)[f(s,x(s)) + \mathfrak{B}(s)u(s)]ds.$$

Remark 3.2 (i) By the uniqueness of the Laplace transform, it is clear that the mild solution to Eq. (1.2) can expressed as

$$x(t) = S^E_{\alpha,\alpha-1}(t)x_0 + (g_1 * S^E_{\alpha,\alpha-1})(t)x_1 + \int_0^t (g_1 * S^E_{\alpha,\alpha-1})(t-s)[f(s,x(s)) + \mathfrak{B}(s)u(s)]ds.$$

(ii) In view of Lemma 2.6, the condition (H1) implies $S^E_{\alpha,\alpha-1}(t)$ is compact for all $t > 0$.

Theorem 3.3 Let assumptions (H1)-(H2), (A2)-(A3) hold, then Eq. (1.2) has at least one mild solution on I.

Proof: We define the operator $N : C(I,X) \to C(I,X)$ as

$$\begin{aligned}(Nx)(t) &= S^E_{\alpha,\alpha-1}(t)x_0 + (g_1 * S^E_{\alpha,\alpha-1})(t)x_1 \\ &\quad + \int_0^t (g_1 * S^E_{\alpha,\alpha-1})(t-s)[f(s,x(s)) + \mathfrak{B}(s)u(s)]ds, t \in I.\end{aligned}$$

The remainder can be conducted similarly as the proof of Theorem 3.1. Since

$$\begin{aligned}&\|(Nx)(t)\| \\ \leq\ & \left\|S^E_{\alpha,\alpha-1}(t)x_0 + (g_1 * S^E_{\alpha,\alpha-1})(t)x_1 + \int_0^t (g_1 * S^E_{\alpha,\alpha-1})(t-s)[f(s,x(s)) + \mathfrak{B}(s)u(s)]ds\right\| \\ \leq\ & Me^{\omega t}\|x_0\| + \frac{M}{\omega}e^{\omega t}\|x_1\| + \frac{M}{\omega}\int_0^t e^{\omega(t-s)}\left[\phi(s)\|x(s)\| + \|\mathfrak{B}(s)u(s)\|\right]ds \\ \leq\ & Me^{\omega b}\|x_0\| + \frac{M}{\omega}e^{\omega b}\|x_1\| + \frac{Mre^{\omega b}}{\omega}\int_0^t e^{-\omega s}\phi(s)ds \\ & + \frac{Me^{\omega b}}{\omega}\int_0^t e^{-\omega s}\|\mathfrak{B}(s)u(s)\|ds \\ \leq\ & Me^{\omega b}\|x_0\| + \frac{M}{\omega}e^{\omega b}\|x_1\| + \frac{Mre^{\omega b}}{\omega}\|\phi\|_{L^1} + \frac{Me^{\omega b}}{\omega}\|\mathfrak{B}u\|_{L^1}\end{aligned}$$

$$
\begin{aligned}
&\leq \quad Me^{\omega b}\left[\|x_0\| + \frac{1}{\omega}\|x_1\| + \frac{r}{\omega}\|\phi\|_{L^1} + \frac{1}{\omega}\|\mathfrak{B}u\|_{L^1}\right] \\
&\leq \quad r.
\end{aligned}
$$

By (H2), we conclude that $Nx \in B_r$.

Because $S^E_{\alpha,\alpha-1}(t)$ is norm continuous for all $t > 0$ (see (H1)) and $t \mapsto (g_1 * S^E_{\alpha,\alpha-1})(t)$ is also norm continuous by Lemma 2.1, we can similarly prove $N(B_r)$ is equicontinuous. The Lemma 2.4 implies the compactness of $(g_1 * S^E_{\alpha,\alpha-1})(t) = S^E_{\alpha,\alpha}(t)$ for all $t > 0$ and therefore the set $\{\int_0^t (g_1 * S^E_{\alpha,\alpha-1})(t-s)[f(s,x(s)) + \mathfrak{B}(s)u(s)]ds : x \in B_r\}$ is relatively compact for all $t \in I$ (as in the proof of Theorem 3.1). On the other hand, from (H1) and Lemma 2.6, we get the compactness of $S^E_{\alpha,\alpha-1}(t)$ for all $t > 0$. Thus, we show the set $H(t) := \{(Nx)(t) : x \in B_r\}$ is relatively compact in X. By the Arzela-Ascoli theorem, we can deduce that N is a compact operator and by Lemma 2.9 there exists a fixed point $x(\cdot)$ for N on B_r. Thus, Eq. (1.2) admits a mild solution.

We also have the following result, which can be proved similarly to Lemma 3.1.

Lemma 3.2 Assume that assumptions (H1) and (A3) hold and $1 < p < +\infty$. Then the operator defined by

$$
(\Pi u)(\cdot) = \int_0^{\cdot} (g_1 * S^E_{\alpha,\alpha-1})(\cdot - s)\mathfrak{B}(s)u(s)ds,\ \forall u(\cdot) \in U_{ad} \subset L^p(I,Y)
$$

is compact. Moreover, if $u_n \in U_{ad}$ converges weakly to u as $n \to \infty$ in $L^p(I,Y)$, then $\Pi u_n \to \Pi u$ as $n \to \infty$.

Next, we consider the existence of optimal controls for Eq. (1.2). For any $u \in U_{ad}$, we still denote by $\mathcal{S}(u)$ all mild solutions to Eq. (1.2) in B_r. Let $x^u \in B_r$ denote the mild solution to Eq. (1.2) corresponding to the control $u \in U_{ad}$, we consider the following limited Lagrange problem (LP$'$):
Find $x^0 \in B_r \subseteq C(I,X)$ and $u^0 \in U_{ad}$ such that for all $u \in U_{ad}$, $J(x^0,u^0) \leq J(x^u,u)$, where

$$
J(x^u,u) = \int_0^T \mathfrak{L}(t,x^u(t),u(t))dt,
$$

and $x^0 \in B_r$ denotes the mild solution to Eq. (1.2) related to the control $u^0 \in U_{ad}$.

Theorem 3.4 Assume that conditions (H1)-(H2), (A2)-(A3) and (A5) hold. Then the problem (LP$'$) admits at least one optimal feasible pair.

Proof: From (H1), Lemmas 2.3-2.4, Lemma 2.6 and Lemma 3.2, we can complete the proof similarly to that of Theorem 3.2.

4 Some applications

In this section, we make some further discussions as applications. Let us consider special cases with $E = \mathcal{I}$, then Eq. (1.1) and Eq. (1.2) can be rewritten respectively in the following

$$
\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t,x(t)) + \mathfrak{B}(t)u(t), \\ \quad x(0) = x_0,\ x'(0) = x_1,\ u \in U_{ad}, \end{cases} \tag{4.1}
$$

and

$$\begin{cases} D^\alpha x(t) = Ax(t) + f(t, x(t)) + \mathfrak{B}(t)u(t), \\ (g_{2-\alpha} * x)(0) = x_0, \ (g_{2-\alpha} * x)'(0) = x_1, \ u \in U_{ad}. \end{cases} \tag{4.2}$$

Accordingly, we list the following assumptions.

(A1′) The operator A generates the $(\alpha, 1)$-resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$ of type (M, ω), the operator $(\lambda^\alpha - A)^{-1}$ is compact for all $\lambda^\alpha \in \rho(A)$ with $\lambda > \omega^{\frac{1}{\alpha}}$ and $\{S_{\alpha,1}(t)\}_{t\geq 0}$ is norm continuous for all $t > 0$.

(H1′) Let $\frac{3}{2} < \alpha < 2$, and the operator A generates the $(\alpha, \alpha - 1)$-resolvent family $\{S_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ of type (M, ω), the operator $(\lambda^\alpha - A)^{-1}$ is compact for all $\lambda^\alpha \in \rho(A)$ with $\lambda > \omega^{\frac{1}{\alpha}}$ and $\{S_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ is norm continuous for all $t > 0$.

For each $x_0, x_1 \in X$, the mild solution to Eq. (4.1) can be expressed as

$$x(t) = S_{\alpha,1}(t)x_0 + (g_1 * S_{\alpha,1})(t)x_1 + \int_0^t (g_{\alpha-1} * S_{\alpha,1})(t-s)[f(s, x(s)) + \mathfrak{B}(s)u(s)]ds,$$

and the mild solution to Eq. (4.2) can be expressed as

$$x(t) = S_{\alpha,\alpha-1}(t)x_0 + (g_1 * S_{\alpha,\alpha-1})(t)x_1 + \int_0^t (g_1 * S_{\alpha,\alpha-1})(t-s)[f(s, x(s)) + \mathfrak{B}(s)u(s)]ds.$$

From Remark 2.1 and [28, Proposition 11, Lemma 12, Theorem 14, Corollary 15, Propositions 16-17], we can obtain the following results.

Corollary 4.1 If assumptions (A1′), and (A2)-(A4) hold, then Eq. (4.1) admits at least one mild solution on I.

Corollary 4.2 Let assumptions (H1′)-(H2), and (A2)-(A3) hold, then Eq. (4.2) has at least one mild solution on I.

Corollary 4.3 Assume that conditions (A1′)-(A5) hold. Then the limited Lagrange problem related to Eq. (4.1) admits at least one optimal feasible pair.

Corollary 4.4 Let conditions (H1′)-(H2), (A2)-(A3) and (A5) hold. Then the limited Lagrange problem related to Eq. (4.2) admits at least one optimal feasible pair.

Now, we consider the following semilinear equation in the Caputo fractional derivatives

$$\begin{cases} D_t^\alpha (Ex)(t) = Ax(t) + J^{2-\alpha}[f(t, x(t)) + \mathfrak{B}(t)u(t)], t \in I, \\ Ex(0) = Ex_0, (Ex)'(0) = Ex_1, \end{cases} \tag{4.3}$$

where $x_0, x_1 \in X$, $1 < \alpha < 2$, $J^{2-\alpha}$ denotes the Riemann-Liouville fractional integral operator. Assume the pair (A, E) generates the $(\alpha, 1)$-resolvent family $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$. The mild solution to Eq. (4.3) is given by

$$x(t) = S^E_{\alpha,1}(t)x_0 + (g_1 * S^E_{\alpha,1})(t)x_1 + \int_0^t (g_1 * S^E_{\alpha,1})(t-s)[f(s, x(s)) + \mathfrak{B}(s)u(s)]ds, t \in I.$$

On the other hand, for the semilinear equation in Riemann-Liouville fractional derivative

$$\begin{cases} D^{\alpha}(Ex)(t) = Ax(t) + J^{2-\alpha}[f(t, x(t)) + \mathfrak{B}(t)u(t)], t \in I, \\ (E(g_{2-\alpha} * x))(0) = Ex_0, (E(g_{2-\alpha} * x))'(0) = Ex_1, \end{cases} \tag{4.4}$$

where $\frac{3}{2} < \alpha < 2$. Let the pair (A, E) generate the $(\alpha, \alpha-1)$-resolvent family $\{S^E_{\alpha,\alpha-1}(t)\}_{t\geq 0}$, then the mild solution to Eq. (4.4) can be written as

$$x(t) = S^E_{\alpha,\alpha-1}(t)x_0 + (g_1 * S^E_{\alpha,\alpha-1})(t)x_1 + \int_0^t (g_{3-\alpha} * S^E_{\alpha,\alpha-1})(t-s)[f(s, x(s)) + \mathfrak{B}(s)u(s)]ds, t \in I.$$

Note that

$$\left\|(g_1 * S^E_{\alpha,1})(t)\right\| \leq \frac{Me^{\omega t}}{\omega}, \text{ and } \left\|(g_{3-\alpha} * S^E_{\alpha,\alpha-1})(t)\right\| \leq \frac{Me^{\omega t}}{\omega^{3-\alpha}}.$$

According to proofs of Theorems 3.1-3.4, we can similarly obtain the following results.

Lemma 4.1 If assumptions (A1)-(A3) and (H2) hold, then Eq. (4.3) admits at least one mild solution on I.

Lemma 4.2 Assume that conditions (A1)-(A3), (H2) and (A5) hold. Then the limited Lagrange problem related to Eq. (4.3) admits at least one optimal feasible pair.

Lemma 4.3 Let assumptions (H1), (A2)-(A3) and the following condition
(H2′) There exists a constant $r > 0$ such that

$$Me^{\omega b}\left[\|x_0\| + \frac{1}{\omega}\|x_1\| + \frac{r}{\omega^{3-\alpha}}\|\phi\|_{L^1} + \frac{1}{\omega^{3-\alpha}}\|\mathfrak{B}u\|_{L^1}\right] \leq r.$$

hold. Then Eq. (4.4) has at least one mild solution on I.

Lemma 4.4 Assume that conditions (H1), (H2′), (A2)-(A3) and (A5) are satisfied. Then the limited Lagrange problem related to Eq. (4.4) admits at least one optimal feasible pair.

Example 4.1 In the following, we end this paper with a simple example. Take $X = L^2[0, \pi], (t, \xi) \in [0, 1] \times [0, \pi]$, consider the following problem

$$\begin{cases} D^{\alpha}_t \left[x(t, \xi) - \frac{\partial^2 x}{\partial \xi^2}(t, \xi)\right] = \frac{\partial^4 x}{\partial \xi^4}(t, \xi) + f(t, x(t, \xi)) + u(t, \xi), \\ x(t, 0) = x(t, \pi) = 0, \quad t \in [0, 1], \\ x(0, \xi) = x_0(\xi), \quad \xi \in [0, \pi], \\ x_t(0, \xi) = x_1(\xi), \quad \xi \in [0, \pi], \end{cases} \tag{4.5}$$

where $1 < \alpha < 2$, $f(t, x(t, \xi)) := \dfrac{e^{-t}x(t, \xi)}{(6 + t)(1 + |x(t, \xi)|)}$. Let $x(\cdot)(\xi) = x(\cdot, \xi)$, $\mathfrak{B}(\cdot)u(\cdot)(\xi) = u(\cdot, \xi)$, and

$$J(x, u) = \int_0^{\pi}\int_0^1 |x(t, \xi)|^2 dtd\xi + \int_0^{\pi}\int_0^1 |u(t, \xi)|^2 dtd\xi.$$

Define the the operators $A : D(A) \subset X \to X$ and $E : D(E) \subset X \to X$ respectively by

$$\begin{cases} Ax = -\dfrac{\partial^4 x}{\partial \xi^4}, \\ Ex = x - \frac{\partial^2 x}{\partial \xi^2}, \end{cases}$$

with the domain $D(E) = D(A) := \{x \in X : x \in H^4([0,\pi]), x(t,0) = x(t,\pi) = 0\}$. It is known that A has discrete spectrum with eigenvalues of the form $-n^4, n \in \mathbb{N}$, and the corresponding normalized eigenvectors are given by $x_n(s) := \left(\dfrac{2}{\pi}\right)^{\frac{1}{2}} \sin(ns)$. Moreover, $\{x_n : n \in \mathbb{N}\}$ is an orthonormal basis for X, and thus A and E can be written as (see [23])

$$\begin{cases} Ax = -\displaystyle\sum_{n=1}^{\infty} n^4 \langle x, x_n \rangle x_n, x \in D(A), \\ Ex = \displaystyle\sum_{n=1}^{\infty} (1+n^2) \langle x, x_n \rangle x_n, x \in D(E). \end{cases}$$

Thus, for any $x \in X$ and $\beta = 1$, we have

$$\begin{aligned} \lambda^{\alpha-1} E(\lambda^\alpha E - A)^{-1} x &= \sum_{n=1}^{\infty} \frac{\lambda^{\alpha-1}(1+n^2)}{\lambda^\alpha(1+n^2)+n^4} \langle x, x_n \rangle x_n \\ &= \sum_{n=1}^{\infty} \frac{\lambda^{\alpha-1}}{\lambda^\alpha + \frac{n^4}{n^2+1}} \langle x, x_n \rangle x_n \qquad (4.6) \\ &= \int_0^\infty e^{-\lambda t} \sum_{n=1}^{\infty} h^n_{\alpha,1}(t) dt \langle x, x_n \rangle x_n, \end{aligned}$$

where the function $h^n_{\alpha,1}(t) := e_{\alpha,1}\left(-\frac{n^4}{n^2+1} t^\alpha\right)$ satisfies $\widehat{h}^n_{\alpha,1}(\lambda) = \dfrac{\lambda^{\alpha-1}}{\lambda^\alpha + \frac{n^4}{n^2+1}}$ for all $\lambda > 0$. Therefore, the pair (A, E) generates the $(\alpha, 1)$-resolvent family $\{S^E_{\alpha,1}(t)\}_{t \geq 0}$ given by

$$S^E_{\alpha,1}(t)x = \sum_{n=1}^{\infty} h^n_{\alpha,1}(t) \langle x, x_n \rangle x_n, \text{ for all } x \in X.$$

From the continuity of $e_{\alpha,1}(\cdot)$, we can conclude that $S^E_{\alpha,1}(t)$ is norm continuous. From (4.6) and the fact $\lim_{n\to\infty} \dfrac{\lambda^{\alpha-1}}{\lambda^\alpha + \frac{n^4}{n^2+1}} = 0$ for all $\lambda > 0$, we can also deduce that $E(\lambda^\alpha E - A)^{-1}$ is a compact operator on the Hilbert space X. Furthermore, for each $x \in X$ we have (by [11])

$$\|S^E_{\alpha,1}(t)x\| \leq 2\|x\|.$$

Therefore, $S^E_{\alpha,1}(t)$ is of type $(2,1)$, i.e. $M = 2$ and $\omega = 1$.

Let $I := [0,1]$. We note that Eq. (4.5) can be rewritten in the abstract form (1.1). We also observe that in this case $\phi(t) := \dfrac{e^{-t}}{6+t}$, $\|\phi\|_{L^1} \leq \dfrac{1}{6}$, $b = \omega = 1$ and $\dfrac{Me^{\omega b}}{\omega^{\alpha-1}}\|\phi\|_{L^1} < \dfrac{e}{3} < 1$, thus we can choose a suitable constant r in (A4). According to Theorems 3.1-3.2, the Eq. (4.5) has a mild solution, and its corresponding limited Lagrange problem admits at least one optimal feasible pair.

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