FRACTIONAL DIFFERENTIAL EQUATIONS OF SOBOLEV TYPE WITH SECTORIAL OPERATORS

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Abstract. This paper treats the asymptotic behavior of resolvent operators of Sobolev type and its applications to the existence and uniqueness of mild solutions to fractional functional evolution equations of Sobolev type in Banach spaces. We first study the asymptotic decay of some resolvent operators (also called solution operators) and next, by using fixed point results, we obtain the existence and uniqueness of solutions to a class of Sobolev type fractional differential equation. We notice that, the existence or compactness of an operator E^{-1} is not necessarily needed in our results.

1. Introduction

In this paper we study the existence of bounded mild solutions to the semilinear fractional differential equation of Sobolev type in the form

$$\partial_t^\alpha (Eu)(t) = Au(t) + \partial_t^{\alpha-\beta}(Ef)(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where A and E are closed linear operators defined on a Banach space X, $\alpha, \beta > 0$, the function f belongs to a closed subspace of the space of continuous and bounded functions, and ∂_t^α denotes the Weyl fractional derivative.

Fractional differential equations describe several physical and biological processes. Some examples include studies in electrochemistry, electromagnetism, viscoelasticity, heredity of materials, rheology, among other. See, for instance [1, 22, 25, 32] for further details.

The existence (and uniqueness) of mild solutions to fractional differential equations of Sobolev type (also called degenerate) has been studied in the last years by several authors. See for instance [14, 15, 20, 23, 27, 28, 37] and the references therein. Sobolev type differential equations describes several partial differential equations arising in physics and applied sciences. For example, if $A = \Delta$ is the Laplacian and $E = m$ is the multiplication operator by a function $m(x)$, then model in the form of (1.1) describes the infiltration of water in unsaturated porous media. See for instance [8, 19, 33] for further details.

The equation (1.1) has been considered in some cases. For instance, if $1 < \alpha < 2, \beta = 1$, A is a sectorial operator and $E = I$ (the identity operator on X), then we get the equation

$$\partial_t^\alpha u(t) = Au(t) + \partial_t^{\alpha-1} f(t), \quad t \in \mathbb{R}, \tag{1.2}$$

which has been widely studied in the last years, see for instance [2, 9, 10, 11, 26, 36, 41, 42] and references therein.

More explicitly, if A is ω-sectorial operators (see the definition below) defined on a Banach space X, $1 < \alpha < 2$, $\beta = 1$, $\omega < 0$ and $E = I$, then A generates a *resolvent family* $\{R_\alpha(t)\}_{t\geq 0}$ which decays in norm as $\frac{1}{1+|\omega|t^\alpha}$ (see [12]) and the solution to (1.2) is given in terms of this resolvent family by

$$u(t) = \int_{-\infty}^t R_\alpha(t-s)f(s)ds, \quad t \in \mathbb{R},$$

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where the Laplace transform of $R_\alpha(t)$ verifies $\hat{R}_\alpha(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}$.

We observe that the equation (1.2) can be viewed as the limiting initial value problem

$$\left\{\begin{array}{rcl} v'(t) &=& \displaystyle\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Av(s) + f(t), \quad t \geq 0 \\ v(0) &=& v_0, \quad v_0 \in X, \end{array}\right. \tag{1.3}$$

in the sense that the solution to (1.3) goes to the solution of (1.2) as $t \to \infty$, because the mild solution of (1.3) is given by

$$v(t) = R_\alpha(t)v_0 + \int_0^t R_\alpha(t-s)f(s)ds, \quad t \geq 0.$$

On the other hand, if $1 < \alpha < 2, \beta = 1$ and $E \neq I$, by using the change of variable $w(t) = Eu(t)$ we get from equation (1.1) the fractional differential system

$$\partial_t^\alpha w(t) = Lw(t) + \partial_t^{\alpha-1} g(t), \quad t \in \mathbb{R}, \tag{1.4}$$

where $L = AE^{-1}$ with domain $D(L) = E(D(A))$ and $g(t) = Ef(t)$. Then, formally the equations (1.2) and (1.4) are the same. However, in this change of variable we need the existence of E^{-1} as a bounded operator, which in general could be restrictive. In some previous works, to establish the existence of mild solutions to Sobolev type differential equations some assumptions on operators A and E are considered:

i) $D(A) \subseteq D(E)$ and A admits a continuous inverse operator A^{-1} [16, 17],
ii) $D(A) \subseteq D(E)$ and E has the bounded inverse [19],
iii) $D(E) \subseteq D(A)$ and E has the compact inverse [5, 6].

In this paper, we introduce a Sobolev type resolvent family (also called characteristic solution operators) $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ which allows to write the solution to equation (1.1) as

$$u(t) = \int_{-\infty}^t S^E_{\alpha,\beta}(t-s)f(s)ds, \quad t \in \mathbb{R}.$$

We give conditions on operators A, E and on the parameters α, β implying the existence of $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$. Moreover, we prove that if $1 < \alpha < 2$ and $\beta \geq 1$ are such that $\alpha - \beta + 1 > 0$ and A is an ω-sectorial operator with respect to E (see the definition in Section 2) then, the norm of $S^E_{\alpha,\beta}(t)$ behaves as $\frac{t^{\beta-1}}{1+|\omega|t^\alpha}$. With this results, we study the existence and uniqueness of almost periodic, almost automorphic (and others) mild solutions to (1.1). We notice that in this paper it is not assumed the existence or compactness of the inverse E^{-1} as well as any assumption on the relation between $D(A)$ and $D(E)$.

The paper is organized as follows. The Section 2 gives some preliminaries and we study the existence and the asymptotic behavior of $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$. In Section 3, we study the existence and uniqueness of mild solutions to the semilinear equation

$$\partial_t^\alpha (Eu)(t) = Au(t) + \partial_t^{\alpha-\beta}(Ef)(t, u(t)), \quad t \in \mathbb{R},$$

where the pair (A, E) generates the (α, β)-resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$. Finally, the Section 4 gives some applications.

2. Asymptotic behavior of Sobolev type resolvents.

Let $(X, \|\cdot\|)$ be a Banach space. We denote by $\mathcal{B}(X)$ the space of all bounded and linear operators from X into X. If A is a closed linear operator on X we denote by $\rho(A)$ the resolvent set of A and $R(\lambda, A) = (\lambda - A)^{-1}$ the resolvent operator of A defined for all $\lambda \in \rho(A)$ and $[D(A)]$ denotes the domain of A equipped with the graph norm.

For $1 \leq p < \infty$, $L^p(\mathbb{R}_+, X)$ denotes the space of all Bochner measurable functions $g : \mathbb{R}_+ \to X$ such that

$$\|g\|_p := \left(\int_0^\infty \|g(t)\|^p dt\right)^{1/p} < \infty.$$

We recall that a strongly continuous family $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be exponentially bounded if there exist two constants $M > 0$ and $w \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{wt}$, for all $t > 0$.

A closed operator A, defined on a Banach space $(X, \|\cdot\|)$, is said to be *ω-sectorial with respect to E of angle ϕ*, if there exist $\phi \in [0, \pi/2)$ and $\omega \in \mathbb{R}$ such that its *E-resolvent operator* $(\lambda E - A)^{-1}$ exists in the sector

$$\omega + \Sigma_\phi := \left\{\omega + \lambda : \lambda \in \mathbb{C}, |\arg(\lambda)| < \frac{\pi}{2} + \phi\right\} \setminus \{\omega\}$$

and

$$\|(\lambda E - A)^{-1}E\| \leq \frac{K}{|\lambda - \omega|}, \quad \lambda \in \omega + \Sigma_\phi.$$

A class of such operators are the operators A which are 0-sectorial with respect to E, see [39, Chapter 3]. See moreover [18, 38].

Definition 2.1. *Let A, E be closed and linear operators with domain $D(A) \cap D(E) \neq \{0\}$ defined on a Banach space X, and $\alpha, \beta > 0$. We say that the pair (A, E) is the generator of an (α, β)-resolvent family, if there exist $\tilde{\omega} \geq 0$ and a strongly continuous function $S^E_{\alpha,\beta} : [0, \infty) \to \mathcal{B}([D(E)], X)$ such that $\{\lambda^\alpha : \text{Re}\lambda > \tilde{\omega}\} \subset \rho_E(A)$ and for all $x \in D(E)$,*

$$\lambda^{\alpha-\beta}(\lambda^\alpha E - A)^{-1}Ex = \int_0^\infty e^{-\lambda t} S^E_{\alpha,\beta}(t)x dt, \ \ \text{Re}\lambda > \tilde{\omega},$$

where $\rho_E(A) := \{\mu \in \mathbb{C} : (\mu E - A)^{-1}\text{is invertible and } (\mu E - A)^{-1}\text{is bounded}\}$. In this case, $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ is called the (α, β)-resolvent family *generated by (A, E).*

We define for all $t \geq 0$ the function $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$. It is easy to show (see [30, Proposition 3.1 and Lemma 2.2])) that if (A, E) generates an (α, β)-resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$, then it satisfies the following properties:

i) $S^E_{\alpha,\beta}(0)E = g_\alpha(0)E$;

ii) $(g_\alpha * S^E_{\alpha,\beta})(t)x \in D(A) \cap D(E)$ and $ES^E_{\alpha,\beta}(t)x = g_\beta(t)Ex + A\int_0^t g_\alpha(t-s)S^E_{\alpha,\beta}(s)xds$, for all $x \in D(E)$ and $t \geq 0$.

The next generation result, analogous to the Hille-Yosida Theorem for C_0-semigroups, can be obtained similarly to [30, Theorem 3.4]. See also [3] and [13].

Theorem 2.2. *Let A be a closed linear operator defined in a Banach space X. Then the following assertions are equivalent.*

(1) *The pair (A, E) generates an (α, β)-resolvent family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ satisfying $\|S_{\alpha,\beta}(t)\| \leq Me^{\mu t}$ for all $t \geq 0$ and for some constants $M > 0$ and $\mu \in \mathbb{R}$.*

(2) *There exist constants $\mu \in \mathbb{R}$ and $M > 0$ such that $\lambda^\alpha \in \rho_E(A)$ for all λ with $\lambda > \mu$ and $H(\lambda) := \lambda^{\alpha-\beta}(\lambda^\alpha E - A)^{-1}E$ satisfies the estimates*

$$\|H^{(n)}(\lambda)\| \leq \frac{Mn!}{(\lambda - \mu)^{n+1}},$$

for all $\lambda > \mu$ and $n \in \mathbb{N}_0$.

The next result gives conditions on operators A and E in order to generate an (α, β)-resolvent family.

Theorem 2.3. *Let $1 < \alpha < 2$ and $\beta \geq 1$ such that $\alpha - \beta + 1 > 0$. Assume that A is an ω-sectorial operator with respect to E of angle $0 \leq \phi < (\alpha - 1)\frac{\pi}{2}$, where $\omega < 0$. Then the pair (A, E) generates the (α, β)-resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$.*

Proof. For $\lambda = re^{i\theta}$ with $|\theta| < \pi/2$ and $r > 0$, we define $g(\lambda) = \lambda^\alpha$. We observe that

$$\arg(g(re^{i\theta})) = \operatorname{Im}\log(g(re^{i\theta})) = \operatorname{Im}\int_0^\theta \frac{d}{dt}\log(g(re^{it}))dt = \operatorname{Im}\int_0^\theta \frac{g'(re^{it})ire^{it}}{g(re^{it})}dt$$

with

$$\frac{\lambda g'(\lambda)}{g(\lambda)} = \alpha.$$

Therefore

$$|\arg(g(re^{i\theta}))| \le \alpha|\theta| < (\alpha-1)\frac{\pi}{2} + \frac{\pi}{2}.$$

We conclude that $\lambda^\alpha \in \Sigma_{(\alpha-1)\frac{\pi}{2}}$ for all $\mathrm{Re}\lambda > 0$. From the above, we have that $H(\lambda) = \lambda^{\alpha-\beta}(\lambda^\alpha E - A)^{-1}E$ is well defined and satisfies

$$\|\lambda H(\lambda)\| \le \frac{K|\lambda|^{\alpha-\beta+1}}{|\lambda^\alpha - \omega|} \le M_1 \quad \text{for all} \quad \mathrm{Re}(\lambda) > 0,$$

where M_1 is a positive constant. On the other hand,

$$\begin{aligned}\|\lambda^2 H'(\lambda)\| &\le |\alpha-\beta|\,\|\lambda H(\lambda)\| + \alpha\|\lambda H(\lambda)\|\,\|\lambda^\alpha(\lambda^\alpha E - A)^{-1}E\| \\ &\le |\alpha-\beta|\,\|\lambda H(\lambda)\| + \alpha\|\lambda H(\lambda)\|\frac{|\lambda^\alpha|}{|\lambda^\alpha-\omega|} \\ &\le M_2,\end{aligned}$$

for all $\mathrm{Re}\lambda > 0$ and a constant $M_2 > 0$. By using [40, Proposition 0.1] and Theorem 2.2 we obtain that (A, E) generates a resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\ge0}$. □

Remark 2.4. *If $0 < \alpha < 1$, then the method of proof given in Theorem 2.3 does not allow to prove that the pair (A, E) generates an (α, β)-resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\ge0}$.*

The next result gives an asymptotic behavior of the resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\ge0}$ and is one of the main Theorem in this paper.

Theorem 2.5. *Let $1 < \alpha < 2$ and $\beta \ge 1$ such that $\alpha - \beta + 1 > 0$. Let A and E closed linear operators on X, $D(A) \cap D(E) \ne \{0\}$. Suppose A is an ω-sectorial operator with respect to E of angle $0 \le \phi < (\alpha-1)\frac{\pi}{2}$, where $\omega < 0$. Then, there exists a constant $M > 0$ depending only on α and β such that the resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\ge0}$ generated by (A, E) satisfies*

$$\|S^E_{\alpha,\beta}(t)\| \le \frac{Mt^{\beta-1}}{1+|\omega|t^\alpha}, \tag{2.5}$$

for all $t > 0$.

Proof. We exploit some ideas of [24]. Since A is ω-sectorial with respect to E of angle $0 \le \phi < (\alpha-1)\frac{\pi}{2}$, we have by Theorem 2.3 that the pair (A, E) generates an (α, β)-resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\ge0}$. As in the proof of Theorem 2.3 we have that $\lambda^\alpha \in \rho_E(A)$ for all $\lambda^\alpha \in \Sigma_\phi$. Moreover, the Laplace transform of $S^E_{\alpha,\beta}(t)$ satisfies $\widehat{S^E_{\alpha,\beta}}(\lambda) = \lambda^{\alpha-\beta}(\lambda^\alpha E - A)^{-1}E$ for all $\lambda^\alpha \in \rho_E(A)$. The inversion theorem of the Laplace transform implies

$$S^E_{\alpha,\beta}(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{\alpha-\beta}(\lambda^\alpha E - A)^{-1}E d\lambda, \tag{2.6}$$

where γ is a suitable positively oriented path. Now, we define γ as the path whose support Γ is given by

$$\Gamma := \{\lambda : \lambda \in \mathbb{C}, \lambda^\alpha \text{ belongs to the boundary of } B_{\frac{1}{t^\alpha}}, t > 0\},$$

where $B_{\frac{1}{t^\alpha}}$ is given by

$$B_{\frac{1}{t^\alpha}} := \left\{\frac{1}{t^\alpha} + \Sigma_\theta\right\} \cup \{\omega + \Sigma_\phi\},$$

and $\phi < \theta < \frac{\pi}{2}$. Note that, with this path γ the function $S^E_{\alpha,\beta}(t)$ given in (2.6), is well-defined.

Since A is ω-sectorial with respect to E of angle $0 \le \phi < (\alpha-1)\frac{\pi}{2}$, it follows that

$$\|(\lambda E - A)^{-1}E\| \le \frac{K}{|\lambda - \omega|},$$

for all $\lambda \in \mathbb{C}$ with $\lambda \in \omega + \Sigma_\phi, \lambda \ne \omega$.

Now, we split γ into two paths, γ_1, γ_2, whose supports Γ_1 and Γ_2 are given by

$$\Gamma_1 := \Gamma \cap \overline{\left\{\frac{1}{t^\alpha} + \Sigma_\theta\right\}} \quad \text{and} \quad \Gamma_2 = \Gamma \cap \overline{\{\omega + \Sigma_\phi\}},$$

Therefore, $\Gamma = \Gamma_1 \cup \Gamma_2$ and $S^E_{\alpha,\beta}(t) = I_1(t) + I_2(t)$, where

$$I_j(t) := \frac{1}{2\pi i}\int_{\gamma_j} e^{\lambda t}\lambda^{\alpha-\beta}(\lambda^\alpha E - A)^{-1}E d\lambda, \quad j = 1, 2.$$

First, we estimate $\|I_1(t)\|$. We define $\lambda_{\min}$ as the complex $\lambda \in \mathbb{C}$ such that $\mathrm{Im}(\lambda) > 0$, and $|\lambda^\alpha_{\min} - \omega| = \mathrm{dist}(L, \omega)$, where L in the line passing by $(\frac{1}{t^\alpha}, 0)$ and the intersection of Γ_1 and Γ_2. For $\lambda \in \Gamma_1$ and $\omega < 0$ we have that

$$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right) = \frac{|\lambda^\alpha_{\min} - \omega|}{|\omega| + \frac{1}{t^\alpha}} \le \frac{|\lambda^\alpha - \omega|}{|\omega| + \frac{1}{t^\alpha}}.$$

Therefore, if $\lambda \in \Gamma_1$ then

$$\frac{1}{|\lambda^\alpha - \omega|} \le \frac{t^\alpha}{\cos(\theta)(1 + |\omega| t^\alpha)}.$$

Hence,

$$\begin{aligned}
\|I_1(t)\| &\le \frac{K}{2\pi}\frac{t^\alpha}{\cos(\theta)(1 + t^\alpha|\omega|)}\int_{\gamma_1} |e^{\lambda t}||\lambda|^{\alpha-\beta}|d\lambda| \\
&\le \frac{K}{\pi}\frac{t^\alpha}{\cos(\theta)(1 + t^\alpha|\omega|)}\int_0^\infty e^{-t\sin(\theta)s}s^{(\alpha-\beta)}ds \\
&= \frac{K}{\pi}\frac{t^\alpha}{\cos(\theta)(1 + t^\alpha|\omega|)}\frac{\Gamma(\alpha-\beta+1)}{\sin(\theta)^{\alpha-\beta+1}t^{\alpha-\beta+1}} \\
&\le \frac{M_1 t^{\beta-1}}{1 + t^\alpha|\omega|}.
\end{aligned}$$

Next, we estimate $\|I_2(t)\|$. Let z_t be the intersection point between the boundary of $\frac{1}{t^\alpha} + \Sigma_\theta$ and $\omega + \Sigma_\phi$. We notice that for all $\lambda \in \Gamma_2$, we have (by using the law of sines)

$$|z_t - \omega| = \frac{|\omega| + \frac{1}{t^\alpha}}{\sin(\theta - \phi)}\cos(\theta), \quad t > 0.$$

Hence, if $\lambda \in \Gamma_2$, then

$$\frac{1}{|\lambda^\alpha - \omega|} \le \frac{\sin(\theta-\phi)}{\cos(\theta)\cos(\phi)}\frac{t^\alpha}{(1 + |\omega| t^\alpha)}.$$

Therefore, there exists a constant $C > 0$ (depending only on θ and ϕ) such that

$$\begin{aligned} \|I_2(t)\| &\leq \frac{KC}{2\pi}\frac{t^\alpha}{(1+t^\alpha|\omega|)}\int_{\gamma_2}|e^{\lambda t}||\lambda|^{\alpha-\beta}|d\lambda| \\ &\leq \frac{KC}{\pi}\frac{t^\alpha}{(1+t^\alpha|\omega|)}\int_0^\infty e^{-t\sin(\phi)s}s^{(\alpha-\beta)}ds \\ &= \frac{K}{\pi}\frac{t^\alpha}{(1+t^\alpha|\omega|)}\frac{\Gamma(\alpha-\beta+1)}{\sin(\phi)^{\alpha-\beta+1}t^{\alpha-\beta+1}} \\ &\leq \frac{M_2t^{\beta-1}}{1+t^\alpha|\omega|}. \end{aligned}$$

Therefore, there exists a constant M depending only on α and β such that

$$\|S^E_{\alpha,\beta}(t)\| \leq \|I_1(t)\| + \|I_2(t)\| \leq \frac{Mt^{\beta-1}}{1+|\omega|t^\alpha}, \quad t > 0.$$

□

3. Bounded mild solutions on the real line

In this section, we study the existence of mild solutions to a Sobolev type fractional differential equations defined on the real line. We recall that for a given function $g : \mathbb{R} \to X$, the *Weyl fractional integral* of order $\alpha > 0$ is defined by

$$\partial_t^{-\alpha}g(t) := \frac{1}{\Gamma(\alpha)}\int_{-\infty}^t (t-s)^{\alpha-1}g(s)ds, \quad t \in \mathbb{R},$$

when this integral is convergent. The *Weyl fractional derivatives* $\partial^\alpha g$ of order $\alpha > 0$ is defined by

$$\partial_t^\alpha g(t) := \frac{d^n}{dt^n}\partial_t^{-(n-\alpha)}g(t), \quad t \in \mathbb{R},$$

where $n = [\alpha]+1$, and $[\alpha]$ denotes the integer part of α. It is known that $\partial_t^\alpha\partial_t^{-\alpha}g = g$ for any $\alpha > 0$, and $\partial_t^n = \frac{d^n}{dt^n}$ holds with $n \in \mathbb{N}$. See [34] for further details.

Now, we recall some definitions of some subspaces of continuous functions. The Banach space of all bounded and continuous functions is defined by $BC(X) := \{f : \mathbb{R} \to X : \|f\|_\infty := \sup_{t\in\mathbb{R}} \|f(t)\| < \infty\}$. On the other hand, $P_T(X) := \{f \in BC(X) : f(t+T) = f(t), \text{ for all } t \in \mathbb{R}\}$ defines the space of all vector-valued $T-$periodic functions.

By $AP(X)$ we denote the space of all almost periodic functions, which consists of all $f \in BC(X)$ such that for every $\varepsilon > 0$ there exists $l > 0$ such that for every subinterval of $\mathbb{R}$ of length l contains at least one point τ such that $\|f(t+\tau) - f(t)\|_\infty \leq \varepsilon$. A function $f \in BC(X)$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$f(t) = \lim_{n\to\infty} g(t-s_n), \quad \text{for each } t \in \mathbb{R}.$$

This Banach space is denoted by $AA(X)$.

Next, we consider the set $C_0(X) := \{f \in BC(X) : \lim_{|t|\to\infty} ||f(t)|| = 0\}$, and define the space of asymptotically periodic functions as $AP_T(X) := P_T(X)\oplus C_0(X)$. Analogously, the space of asymptotically almost periodic functions is defined by,

$$AAP(X) := AP(X) \oplus C_0(X),$$

and the space of asymptotically almost automorphic functions,

$$AAA(X) := AA(X) \oplus C_0(X).$$

For more details on this function spaces, we refer to reader to [31, 35].

Throughout, we will use the notation $\mathcal{N}(X)$ to denote any of the above spaces. Finally, for a Banach space Y, we define the set $\mathcal{N}(\mathbb{R}\times X;Y)$ which consists of all functions $f:\mathbb{R}\times X\to Y$ such that $f(\cdot,x)\in\mathcal{N}(Y)$ uniformly for each $x\in B$, where B is any bounded subset of X.

Now, we consider the following Sobolev type linear fractional differential equation

$$\partial_t^{\alpha}(Eu)(t)=Au(t)+\partial_t^{\alpha-\beta}(Ef)(t),\quad t\in\mathbb{R}. \tag{3.7}$$

This equation, in case $\beta=1$ and $E=I$ (the identity operator) has been widely studied in the last years, see for instance [2, 9, 10, 11, 42] and references therein.

Assume that (A,E) is the generator of an (α,β)-resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ which is *uniformly integrable,* which means that

$$\int_0^{\infty}\|S^E_{\alpha,\beta}(t)\|dt<\infty.$$

Given $f\in\mathcal{N}(X)$, let $\Phi(t)$ be the function defined by

$$\Phi(t):=\int_{-\infty}^{t}S^E_{\alpha,\beta}(t-s)f(s)ds,\quad t\in\mathbb{R}. \tag{3.8}$$

If $f(t)\in D(E)$ for all $t\in\mathbb{R}$, then $\Phi(t)\in D(E)$ for all $t\in\mathbb{R}$ (see [4, Proposition 1.1.7]). Since $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ is uniformly integrable, we get $\Phi\in\mathcal{N}([D(E)])$ by [31, Theorem 3.3]. Take $n=[\alpha]+1$ and assume the existence of $\partial_t^{\alpha}(E\Phi)$. From the Fubini's theorem we get

$$\begin{aligned}
\partial_t^{\alpha}(E\Phi)(t) &= \frac{d^n}{dt^n}\int_{-\infty}^{t}g_{n-\alpha}(t-s)E\Phi(s)ds\\
&= \frac{d^n}{dt^n}\int_{-\infty}^{t}g_{n-\alpha}(t-s)\int_{-\infty}^{s}ES^E_{\alpha,\beta}(s-r)f(r)drds\\
&= \frac{d^n}{dt^n}\int_{-\infty}^{t}g_{n-\alpha}(t-s)\int_{-\infty}^{s}\left[g_\beta(s-r)Ef(r)+A(g_\alpha*S^E_{\alpha,\beta})(s-r)f(r)\right]drds\\
&= \frac{d^n}{dt^n}\int_{-\infty}^{t}g_{n-\alpha}(t-s)\partial^{-\beta}Ef(s)ds+\\
&\qquad\frac{d^n}{dt^n}\int_{-\infty}^{t}g_{n-\alpha}(t-s)\int_{-\infty}^{s}A\int_0^{s-r}g_\alpha(s-r-v)S^E_{\alpha,\beta}(v)f(r)dvdrds\\
&= \partial_t^{\alpha-\beta}(Ef)(t)+\frac{d^n}{dt^n}\int_{-\infty}^{t}g_{n-\alpha}(t-s)\int_{-\infty}^{s}A\int_r^{s}g_\alpha(s-w)S^E_{\alpha,\beta}(w-r)f(r)dwdrds\\
&= \partial_t^{\alpha-\beta}(Ef)(t)+\frac{d^n}{dt^n}\int_{-\infty}^{t}g_{n-\alpha}(t-s)\int_{-\infty}^{s}A\int_{-\infty}^{w}g_\alpha(s-w)S^E_{\alpha,\beta}(w-r)f(r)drdwds\\
&= \partial_t^{\alpha-\beta}(Ef)(t)+\frac{d^n}{dt^n}\int_{-\infty}^{t}g_{n-\alpha}(t-s)\int_{-\infty}^{s}g_\alpha(s-w)A\Phi(w)dwds\\
&= \partial_t^{\alpha-\beta}(Ef)(t)+A\Phi(t),
\end{aligned}$$

for all $t\in\mathbb{R}$, which means that Φ is a (strong) solution to equation (3.7).

In general, we only have $\Phi(t)\in X$ or that $\partial_t^{\alpha}(E\Phi)$ does not exists. We introduce the following definition of solution.

Definition 3.6. *Let $\alpha,\beta>0$. Assume that (A,E) generates an (α,β)-resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$. A continuous function $u\in C(\mathbb{R},X)$ is called a* mild solution *to equation (3.7) if the function $s\mapsto$*

$S^E_{\alpha,\beta}(t-s)f(s)$ is integrable on $(-\infty,t)$ for each $t\in\mathbb{R}$ and

$$u(t)=\int_{-\infty}^t S^E_{\alpha,\beta}(t-s)f(s)ds,\quad t\in\mathbb{R}.$$

Lemma 3.7. *If $1\le\beta<\alpha<2$ and (A,E) is an ω-sectorial operator of angle $0\le\phi<(\alpha-1)\frac{\pi}{2}$, where $\omega<0$, then $\{S^E_{\alpha,\beta}(t)\}_{t\ge0}$ is uniformly integrable.*

Proof. The condition $1\le\beta<\alpha<2$ implies $\alpha-\beta+1>0$ and by using the estimate (2.5) we have

$$\int_0^\infty \|S^E_{\alpha,\beta}(t)\|dt\le M\int_0^\infty\frac{t^{\beta-1}}{1+|\omega|t^\alpha}dt=M\frac{\left(\frac{1}{|\omega|}\right)^{\beta-1/\alpha}}{\alpha|\omega|^{1/\alpha}}\int_0^\infty\frac{u^{(\beta-\alpha)/\alpha}}{1+u}du=\frac{M}{\alpha|\omega|^{\beta/\alpha}}\mathbf{B}\left(\frac{\beta}{\alpha},1-\frac{\beta}{\alpha}\right),$$

where $\mathbf{B}(\cdot,\cdot)$ denotes the Beta function. Since that $\frac{\beta}{\alpha}>0$ and $1-\frac{\beta}{\alpha}>0$, then we have the claim. $\square$

The next result gives conditions on operators A and E to ensure the existence of bounded mild solutions to the linear equation (3.7).

Theorem 3.8. *Let $1\le\beta<\alpha<2$ and assume that (A,E) is an ω-sectorial operator of angle $0\le\phi<(\alpha-1)\frac{\pi}{2}$, where $\omega<0$. If $f\in\mathcal{N}([D(E)])$, then the equation (3.7) has a unique mild solution $u\in\mathcal{N}([D(E)])$.*

Proof. By Theorem 2.3, the pair (A,E) generates an (α,β)-resolvent $\{S^E_{\alpha,\beta}(t)\}_{t\ge0}$, which is uniformly integrable by Lemma 3.7. Next, if $f\in\mathcal{N}([D(E)])$, then u given by $u(t):=\int_{-\infty}^t S^E_{\alpha,\beta}(t-s)f(s)ds$ is well defined and by [31, Theorem 3.3], the function u belongs to $\mathcal{N}([D(E)])$, and therefore u defines a mild solution of (3.7). The uniqueness it is easy to prove. $\square$

Now, we consider the semilinear differential equation

$$\partial_t^\alpha(Eu)(t)=Au(t)+\partial_t^{\alpha-\beta}Ef(t,u(t)),\quad t\in\mathbb{R},\tag{3.9}$$

where $\alpha,\beta>0$, (A,E) is the generator of an (α,β)-resolvent family. We define the concept of mild solution to equation (3.9) as follows.

Definition 3.9. *Let $\alpha,\beta>0$. Assume that (A,E) generates an (α,β)-resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\ge0}$. A function $u\in C(\mathbb{R},X)$ is called a* mild solution *to equation (3.9) if the function $s\mapsto S^E_{\alpha,\beta}(t-s)f(s,u(s))$ is integrable on $(-\infty,t)$ for each $t\in\mathbb{R}$ and*

$$u(t)=\int_{-\infty}^t S^E_{\alpha,\beta}(t-s)f(s,u(s))ds,\quad t\in\mathbb{R}.$$

The following is the main result in this section.

Theorem 3.10. *Let $1\le\beta<\alpha<2$ and assume that (A,E) is an ω-sectorial operator of angle $0\le\phi<(\alpha-1)\frac{\pi}{2}$, where $\omega<0$. If $f\in\mathcal{N}(\mathbb{R}\times X,[D(E)])$ satisfies*

$$||f(t,x)-f(t,y)||\le L||x-y||,\ \text{for all } t\in\mathbb{R},\ \text{and } x,y\in X,\tag{3.10}$$

where $L<\frac{\alpha}{M}|\omega|^{\beta/\alpha}\mathbf{B}\left(\frac{\beta}{\alpha},1-\frac{\beta}{\alpha}\right)^{-1}$, and M is the constant given in Theorem 2.5, and $\mathbf{B}(\cdot,\cdot)$ denotes the Beta function, then the equation (3.9) has a unique mild solution $u\in\mathcal{N}([D(E)])$.

Proof. Define the operator $F:\mathcal{N}([D(E)])\to\mathcal{N}([D(E)])$ by

$$(F\Phi)(t):=\int_{-\infty}^t S^E_{\alpha,\beta}(t-s)f(s,\Phi(s))\,ds,\quad t\in\mathbb{R}.\tag{3.11}$$

By the composition Theorem [31, Theorem 4.1], the function $s \mapsto f(s, \Phi(s))$ belongs to $\mathcal{N}([D(E)])$ and by Lemma 3.7 and [31, Theorems 3.3], $F\Phi \in \mathcal{N}([D(E)])$, which means that F is well defined. For $\Phi_1, \Phi_2 \in \mathcal{N}([D(E)])$ and $t \in \mathbb{R}$, we have:

$$\begin{aligned}
||(F\Phi_1)(t) - (F\Phi_2)(t)|| &\leq \int_{-\infty}^{t} ||S^E_{\alpha,\beta}(t-s)[f(s,\Phi_1(s)) - f(s,\Phi_2(s))]||ds \\
&\leq \int_{-\infty}^{t} L\|S^E_{\alpha,\beta}(t-s)\| \cdot \|\Phi_1(s) - \Phi_2(s)\|ds \\
&\leq L\|\Phi_1 - \Phi_2\|_\infty \int_0^\infty \|S^E_{\alpha,\beta}(r)\|dr \\
&\leq \frac{LM}{\alpha}|\omega|^{-\beta/\alpha}\mathbf{B}\left(\frac{\beta}{\alpha}, 1 - \frac{\beta}{\alpha}\right)\|\Phi_1 - \Phi_2\|_\infty.
\end{aligned}$$

This proves that F is a contraction, and thus by the Banach fixed point theorem there exists a unique $u \in \mathcal{N}([D(E)])$ such that $Fu = u$. $\square$

Theorem 3.11. *Let $1 \leq \beta < \alpha < 2$ and assume that (A, E) is an ω-sectorial operator of angle $0 \leq \phi < (\alpha - 1)\frac{\pi}{2}$, where $\omega < 0$. If $f \in \mathcal{N}(\mathbb{R} \times X, [D(E)])$ satisfies*

$$||f(t,x) - f(t,y)|| \leq L(t)\|x - y\|, \text{ for all } t \in \mathbb{R}, \text{ and } x, y \in X, \tag{3.12}$$

where $L \in L^1(\mathbb{R})$. Then the equation (3.9) has a unique mild solution $u \in \mathcal{N}([D(E)])$.

Proof. Notice that if $t \geq 1$, then $\|S^E_{\alpha,\beta}(t)\| \leq \frac{M}{|\omega|}\frac{1}{t^{\alpha-\beta+1}} \leq \frac{M}{|\omega|}$, and if $0 \leq t \leq 1$, then $\|S^E_{\alpha,\beta}(t)\| \leq \frac{M}{1+|\omega|t^\alpha} \leq M$. Therefore, $\|S^E_{\alpha,\beta}(t)\| \leq N$, where $N = \max\{M, \frac{M}{|\omega|}\}$.

Define the operator F as in (3.11). For $\Phi_1, \Phi_2 \in \mathcal{N}([D(E)])$ and $t \in \mathbb{R}$ we have:

$$\begin{aligned}
\|(F\Phi_1)(t) - (F\Phi_2)(t)\| &\leq \int_{-\infty}^{t} \|S^E_{\alpha,\beta}(t-s)[f(s,\Phi_1(s)) - f(s,\Phi_2(s))]\|ds \\
&\leq N\|\Phi_1 - \Phi_2\|_\infty \int_0^\infty L(t-\tau)d\tau \\
&= N\|\Phi_1 - \Phi_2\|_\infty \int_{-\infty}^{t} L(s)ds.
\end{aligned}$$

In general we get

$$\begin{aligned}
\|(F^n\Phi_1)(t) - (F^n\Phi_2)(t)\| &\leq \|\Phi_1 - \Phi_2\|_\infty \frac{N^n}{(n-1)!}\left(\int_{-\infty}^{t} L(s)\left(\int_{-\infty}^{s} L(\tau)d\tau\right)^{n-1} ds\right) \\
&\leq \|\Phi_1 - \Phi_2\|_\infty \frac{N^n}{n!}\left(\int_{-\infty}^{t} L(s)ds\right)^n \\
&\leq \|\Phi_1 - \Phi_2\|_\infty \frac{(\|L\|_1 N)^n}{n!}.
\end{aligned}$$

Hence, since $\frac{(\|L\|_1 N)^n}{n!} < 1$ for n sufficiently large, by the contraction principle F has a unique fixed point $u \in \mathcal{N}([D(E)])$. $\square$

4. An Example

We consider the following problem

$$\partial_t^\alpha(m(x)u) = \Delta u + \partial_t^{\alpha-\beta}(m(x)f(t,x)), \text{ in } \mathbb{R} \times \Omega \tag{4.13}$$

$$u = 0, \text{ in } \mathbb{R} \times \partial\Omega, \tag{4.14}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $m(x) \geq 0$ is a given measurable bounded function on Ω and f is a function on $\mathbb{R}\times\Omega$. We notice that if $m(x) = 0$ over a non empty subset of Ω, then the inverse of the multiplication operator E defined by $Eu(t,x) := m(x)u(t,x)$ is unbounded.

We consider this problem in the space $X = H^{-1}(\Omega)$. By [7, p.38] there exists a constant $C > 0$ such that

$$\|(Ez-\Delta)^{-1}E\| \leq \frac{C}{1+|z|},$$

for all $\mathrm{Re}(z) \geq -c(1+|\mathrm{Im}(z)|)$, where c is a positive constant. Take $\omega = -c < 0$. Observe that $|z-\omega| \leq |z| + c \leq K(|z|+1)$ for a positive constant $K \in \mathbb{N}$. Hence, if $\mathrm{Re}(z) \geq -c(1+|\mathrm{Im}(z)|)$, then

$$\|(Ez-\Delta)^{-1}E\| \leq \frac{C}{1+|z|} \leq \frac{KC}{|z-\omega|}.$$

Observe that $\mathrm{Re}(z) \geq -c(1+|\mathrm{Im}(z)|)$ represents the right hand side sector of the complex plane bounded by $\gamma_1(t) = -c - te^{i\arctan\left(\frac{1}{c}\right)}$ and $\gamma_2(t) = -c - te^{-i\arctan\left(\frac{1}{c}\right)}$, for all $t \geq 0$. Therefore, the operator $A := \Delta$ is a $(-c)$-sectorial operator with respect to E of angle $\phi = \arctan\left(\frac{1}{c}\right)$.

If $\frac{2}{\pi}\left(\arctan\left(\frac{1}{c}\right) + \frac{\pi}{2}\right) < \alpha < 2$ and $\beta \geq 1$ such that $1 \leq \beta < \alpha < 2$, then by Theorem 2.3 the pair (A,E) generates an (α,β)-resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ which satisfies (by Theorem 2.5 and Lemma 3.7) that the function $t \mapsto S^E_{\alpha,\beta}(t)$ belongs to $L^1(\mathbb{R}_+, X)$.

Therefore, if $f \in \mathcal{N}(\mathbb{R}\times X, X)$ satisfies the condition (3.12), then the problem (4.13)-(4.14) has a unique mild solution $u \in \mathcal{N}(X)$.

Now, if $f(t,v)(s) = \gamma b(t)\sin(v(s))$ for all $v \in X$ and $s,t \in \mathbb{R}$ with $b \in AA(\mathbb{R})$ and $\gamma \in \mathbb{R}$. We notice that $t \mapsto f(t,v)$ is almost automorphic in t for each $v \in X$. Moreover, for $u, v \in X$ there exists a constant $D := D(\Omega)$ (by the Poincaré's inequality) such that

$$\|f(t,u) - f(t,v)\|^2_X \leq D\|f(t,u)-f(t,v)\|^2_{L^2(\Omega)} \leq D\gamma^2\|b\|^2_\infty\|u-v\|^2_X.$$

If $L := \sqrt{D}\gamma\|b\|_\infty$, then we can choose $\gamma \in \mathbb{R}$ such that

$$L < \frac{\alpha}{KC}|c|^{\beta/\alpha}\mathbf{B}\left(\frac{\beta}{\alpha}, 1-\frac{\beta}{\alpha}\right)^{-1},$$

and then the problem (4.13)-(4.14) has a unique mild solution $u \in AA(X)$.

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