Solvability of fractional differential inclusions with nonlocal initial conditions via resolvent family of operators

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Abstract

In this paper, we consider mild solutions to fractional differential inclusions with nonlocal initial conditions. The main results are proved under conditions that: (i) the multivalued term takes convex values with compactness of resolvent family of operators; (ii) the multivalued term takes nonconvex values with compactness of resolvent family of operators; and (iii) the multivalued term takes nonconvex values without compactness of resolvent family of operators, respectively.

Keywords: Fractional differential inclusions, Nonlocal initial condition, Fractional resolvent family, Mild solutions.

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1 Introduction

A differential inclusion is a generalization of the notion of an ordinary differential equation, which is often used to deal with differential equations with a discontinuous right-hand side or an inaccurately known right-hand side [14, 32]. Differential inclusions are also closely related to control theory, for instance, consider the control problem

$$x' = f(x, u), u \in U,$$

where u is known as a control parameter. It finds that the above control system and the following differential inclusion

$$x' \in f(x, U) = \bigcup_{u \in U} f(x, u)$$

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has the same trajectories. If the set of controls depends upon the state x, i.e. $U = U(x)$, then we can obtain the following differential inclusion

$$x' \in F(x, U(x)).$$

The above mentioned equivalence between a control system and the corresponding differential inclusion plays a key role in proving existence theorems in optimal control theory. Differential inclusion has found its wide applications to models arising in economics, sociology, and bio-ecology et al, and thus it has been considerably investigated by lots of scholars in last decades, see for instance [7, 8, 16, 18, 20, 22, 32] and references therein.

The concept of nonlocal initial condition has been introduced to extend the study of classical initial-valued problems. As indicated in [13], this notion can be more natural and more precise in describing nature phenomena than the classical notion since some additional information is taken into account. Nonlocal initial conditions for abstract differential inclusions, we can refer to [8, 18, 19, 25] and references therein.

The concept of fractional calculus appeared for the first time in a famous correspondence between G. A. de L'Hôspital and G.W. Leibniz, in 1695 (see Preface in [39]). After that, mangy mathematicians have devoted to further develop this theory. Fractional calculus can be seen a generalization of the ordinary differentiation and integration to arbitrary non-integer order, and has been recognized as one of the most powerful tools to describe long-memory processes in the last decades. Many phenomena from physics, chemistry, mechanics, electricity et al can been modelled by ordinary and partial differential equations involving fractional derivatives, we refer to [2, 3, 4, 15, 24, 30, 35, 38, 39] and references therein for more details. We also note that fractional differential inclusions have also been increasingly concerned, for instance [5, 6, 10, 12, 21, 23, 26, 29, 28, 31, 33, 36, 34, 37].

Very recently, some new properties on the compactness of resolvent family of operators related to fractional differential equations have been established in [27]. This new characterization of compactness of resolvent family of operators provides a new way to consider mild solutions of abstract fractional differential equations.

Let $(\mathbb{X}, \|\cdot\|)$ be a real Banach space and A be a closed and linear operator defined in Banach space $\mathbb{X}$. Let $\mathcal{P}(\mathbb{X}) = \{Y \subseteq \mathbb{X} : Y \neq \emptyset\}$. The notation $L^1(J, \mathbb{X}) = \{v : J \to \mathbb{X} | v \text{ is Bochner integrable}\}$ on a compact interval J of $\mathbb{R}$. In this paper, we consider the following abstract fractional differential inclusions with nonlocal initial conditions such as

$$\begin{aligned} D_t^\alpha x(t) &\in Ax(t) + J_t^{1-\alpha} F(t, x(t)), \ t \in J := [0, b] &(1.1)\\ x(0) &= x_0 + p(x), &(1.2)\end{aligned}$$

where $0 < \alpha < 1$, $J_t^\beta v(t) = \int_0^t \mathcal{G}_\beta(t-s)v(s)ds$ for $v \in L^1(J, \mathbb{X})$, $\mathcal{G}_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$ for $\beta > 0, t > 0$, and $\Gamma(\cdot)$ stands for the Gamma function, and

$$\begin{aligned} D_t^\alpha x(t) &\in Ax(t) + F(t, x(t)), \ t \in J &(1.3)\\ x(0) &= x_0 + p(x), x'(0) = x_1 + q(x) &(1.4)\end{aligned}$$

where $1 < \alpha < 2$, D_t^α is understood in Caputo sense, $x_0, x_1 \in \mathbb{X}$, $F : J \times \mathbb{X} \to \mathcal{P}(\mathbb{X})$, p, q are suitable continuous functions specified later.

We shall establish existence results of mild solutions to the above problems (1.1)-(1.2) and (1.3)-(1.4) under different cases: (i) the multivalued term F takes convex values with compactness of resolvent family of operators; (ii) the multivalued term F takes nonconvex values with compactness of resolvent family of operators; and (iii) the multivalued term F has nonconvex values without compactness of resolvent family of operators, respectively.

The rest of this paper is organized as follows. Section 2 is involved in Preliminaries. Section 3 is devoted to investigate mild solutions to the problems (1.1)-(1.2) and (1.3)-(1.4), respectively. And Section 4 is Conclusions.

2 Preliminaries

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space. Denote $\mathcal{P}_{cl}(\mathbb{X}) = \{Y \in \mathcal{P}(\mathbb{X}) : Y \text{closed}\}$, $\mathcal{P}_b(\mathbb{X}) = \{Y \in \mathcal{P}(\mathbb{X}) : Y \text{ bounded}\}$, $\mathcal{P}_{cp}(\mathbb{X}) = \{Y \in \mathcal{P}(\mathbb{X}) : Y \text{compact}\}$, and $\mathcal{P}_{cv}(\mathbb{X}) = \{Y \in \mathcal{P}(\mathbb{X}) : Y \text{convex}\}$. We also denote by $\mathcal{L}(\mathbb{X})$ the space of bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$.

A multivalued map $G : \mathbb{X} \to \mathcal{P}(\mathbb{X})$ has convex (closed) values if $G(x)$ is convex (closed) for all $x \in \mathbb{X}$. G is bounded on bounded sets if $G(B) = \bigcup_{x\in B} G(x)$ is bounded in $\mathbb{X}$ for all $B \in \mathcal{P}_b(\mathbb{X})$, i.e. $\sup_{x\in B} \{\sup \{\|y\| : y \in G(x)\}\} < \infty$.

The multivalued map $G : \mathbb{X} \to \mathcal{P}(\mathbb{X})$ is called upper semicontinuous (u.s.c.) on $\mathbb{X}$ if for each $x_0 \in \mathbb{X}$, the set $G(x_0)$ is a nonempty, closed subset of $\mathbb{X}$, and if for each open set $\mathbb{N}$ of $\mathbb{X}$ containing $G(x_0)$, there exists an open neighborhood $\mathbb{N}_0$ of x_0 such that $G(\mathbb{N}_0) \subseteq \mathbb{N}$. G is called lower semi-continuous (l.s.c.) if the set $\{x \in \mathbb{X} : G(x) \bigcap \mathscr{A}\}$ is open for any open subset $\mathscr{A} \subseteq \mathbb{X}$. Also, G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in \mathcal{P}_b(\mathbb{X})$. G has a fixed point if there exists $x \in \mathbb{X}$ such that $x \in G(x)$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e.,

$$x_n \to x_*, \; y_n \to y_*, \; y_n \in G(x_n) \text{ imply } y_* \in G(x_*).$$

The upper semicontinuous multivalued map G is said to be condensing if for any $B \in \mathcal{P}_b(\mathbb{X})$ with $\nu(B) \neq 0$, we have $\nu(G(B)) < \nu(B)$, where ν denotes the Kuratowski measure of noncompactness.

Definition 2.1 The multivalued map $G : J \times \mathbb{X} \to \mathcal{P}(\mathbb{X})$ is said to be L^1-Carathéodory if
(i) $t \mapsto G(t, x)$ is measurable for each $x \in \mathbb{X}$;
(ii) $u \mapsto G(t, x)$ is u.s.c on $\mathbb{X}$ for almost all $t \in J$;
(iii) For each $r > 0$, there exists $\varphi_r \in L^1(J, \mathbb{R}_+)$ such that

$$\|G(t, x)\|_{\mathcal{P}(\mathbb{X})} := \sup \{\|v\| : v \in G(t, x)\} \leq \varphi_r(t),$$

for all $\|x\| \leq r$ and for a.e. $t \in J$.

Lemma 2.1 Let $\mathbb{X}$ be a Banach space. Let $G : J \times \mathbb{X} \to \mathcal{P}_{cp,cv}(\mathbb{X})$ be an L^1-Carathéodory multivalued map with

$$S_{G,x} = \left\{f \in L^1(J, \mathbb{X}) : f(t) \in G(t, x(t)), \text{ for a.e. } t \in J\right\} \neq \emptyset,$$

and let Γ be a linear continuous mapping from $L^1(J,\mathbb{X})$ to $C(J,\mathbb{X})$, then the operator

$$\Gamma \circ S_G : C(J,\mathbb{X}) \to \mathcal{P}_{cp,cv}(C(J,\mathbb{X})), \quad x \mapsto (\Gamma \circ S_G)(x) := \Gamma(S_{G,x})$$

is a closed graph operator in $C(J,\mathbb{X}) \times C(J,\mathbb{X})$.

Let $\mathcal{A}$ be a subset of $J \times B$. $\mathcal{A}$ is $\mathcal{L} \times \mathcal{B}$ measurable if $\mathcal{A}$ belongs to the σ-algebra generated by all sets of the form $\mathcal{N} \times \mathcal{D}$, where $\mathcal{N}$ is Lebesgue measurable in J and $\mathcal{D}$ is Borel measurable in B. A subset $\mathcal{A}$ of $L^1(J,\mathbb{X})$ is decomposable if, for all $u, v \in \mathcal{A}$ and all measurable subsets $\mathcal{N}$ of J, the function $u\chi_{\mathcal{N}} + v\chi_{J-\mathcal{N}} \in \mathcal{A}$, where χ denotes the characteristic function.

Let $F : J \times \mathbb{X} \to \mathcal{P}_{cp}(\mathbb{X})$. Assign to F the multivalued operator

$$\mathcal{F} : C(J,\mathbb{X}) \to \mathcal{P}(L^1(J,\mathbb{X}))$$

by letting

$$\mathcal{F}(x) = S_{F,x} = \{v \in L^1(J,\mathbb{X}) : v(t) \in F(t,x(t)) \text{ for a.e. } t \in J\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated to F.

Definition 2.2 [8] Let $\mathbb{Y}$ be a separable metric space and let $N : \mathbb{Y} \to \mathcal{P}(L^1(J,\mathbb{X}))$ be a multivalued operator. We say that N has property (BC) if
(1) N is lower semicontinuous (l.s.c.);
(2) N has nonempty closed and decomposable values.

Definition 2.3 [8] Let $F : J \times \mathbb{X} \to \mathcal{P}_{cp}(\mathbb{X})$. F is called to be of lower semicontinuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is l.s.c. and has nonempty closed and decomposable values.

Lemma 2.2 [11] Let $\mathbb{Y}$ be a separable metric space and let $N : \mathbb{Y} \to \mathcal{P}(L^1(J,\mathbb{X}))$ be a multivalued operator with property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $f : \mathbb{Y} \to L^1(J,\mathbb{X})$ such that $f(x) \in N(x)$ for every $x \in \mathbb{Y}$.

Let $(\mathbb{X}, d)$ be a metric space induced by the normed space $(\mathbb{X}, \|\cdot\|)$. Let $H_d : \mathcal{P}(\mathbb{X}) \times \mathcal{P}(\mathbb{X}) \to \mathbb{R}_+ \bigcup \{\infty\}$ be defined as

$$H_d(C,D) = \max\left\{\sup_{c\in C} d(c,D), \sup_{d\in D} d(C,d)\right\},$$

where $d(c,D) = \inf_{d\in D} d(c,d)$, $d(C,d) = \inf_{c\in C} d(c,d)$. Then $(\mathcal{P}_{b,cl}(\mathbb{X}), H_d)$ is a metric space and $(\mathcal{P}_{cl}(\mathbb{X}), H_d)$ is a generalized (complete) metric space.

Definition 2.4 [8] A multivalued operator $G : \mathbb{X} \to \mathcal{P}_{cl}(\mathbb{X})$ is called
(i) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H_d(G(x), G(y)) \le \gamma d(x,y), \text{ for each } x, y \in \mathbb{X};$$

(ii) a contraction if it is γ-Lipschitz with $\gamma < 1$.

For more detailed results on multivalued maps and differential inclusions, we refer to [8, 14, 20, 22, 32]. We now give some important properties of resolvent family of operators.

Definition 2.5 [39] Let $\alpha > 0$. The α-order Caputo fractional derivative of v is defined as

$$D_t^{\alpha} v(t) := \int_0^t \mathcal{G}_{m-\alpha}(t-s) v^{(m)}(s) ds,$$

where $m = \lceil \alpha \rceil$.

Definition 2.6 [27] Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space $\mathbb{X}$ and $\alpha > 0$. We call A the generator of an $(\alpha, 1)$-resolvent family if there exists $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to \mathcal{L}(\mathbb{X})$ such that $\{\lambda^\alpha : \mathrm{Re}\lambda > \omega\} \subseteq \rho(A)$ and

$$\lambda^{\alpha-1} \left(\lambda^\alpha - A\right)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \ \mathrm{Re}\lambda > \omega, x \in \mathbb{X}.$$

In this case, the family $\{S_\alpha(t)\}_{t\geq 0}$ is called an $(\alpha, 1)$-resolvent family generated by A.

Definition 2.7 [27] Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space $\mathbb{X}$ and $1 \leq \alpha \leq 2$. We say that A is the generator of an (α, α)-resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $R_\alpha : \mathbb{R}_+ \to \mathcal{L}(\mathbb{X})$ such that $\{\lambda^\alpha : \mathrm{Re}\lambda > \omega\} \subseteq \rho(A)$ and

$$\left(\lambda^\alpha - A\right)^{-1} x = \int_0^\infty e^{-\lambda t} R_\alpha(t) x dt, \ \mathrm{Re}\lambda > \omega, x \in \mathbb{X}.$$

In this case, the family $\{R_\alpha(t)\}_{t\geq 0}$ is called an (α, α)-resolvent family generated by A.

Recall that a strongly continuous family $\{T(t)\}_{t\geq 0} \subseteq \mathcal{L}(\mathbb{X})$ is said to be of type (M, ω) if there exist constants $M > 0$ and $\omega \in \mathbb{R}$, such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.

Lemma 2.3 [27, Theorem 3.1] Let $0 < \alpha \leq 1$ and $\{S_\alpha(t)\}_{t\geq 0}$ be an $(\alpha, 1)$-resolvent family of type (M, ω) generated by A. Suppose that $S_\alpha(t)$ is continuous in the uniform operator topology for all $t > 0$. Then the following assertions are equivalent
(i) $S_\alpha(t)$ is a compact operator for all $t > 0$.
(ii) $(\mu - A)^{-1}$ is a compact operator for all $\mu > \omega^{\frac{1}{\alpha}}$.

Lemma 2.4 [27, Theorem 3.5] Let $1 < \alpha \leq 2$ and A be the generator of an $(\alpha, 1)$-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ of type (M, ω). Then A generates an (α, α)-resolvent family $\{R_\alpha(t)\}_{t\geq 0}$ of type $\left(\frac{M}{\alpha-1}, \omega\right)$ and the following assertions are equivalent
(i) $R_\alpha(t)$ is a compact operator for all $t > 0$.
(ii) $(\mu - A)^{-1}$ is a compact operator for all $\mu > \omega^{\frac{1}{\alpha}}$.

Next, we list some well-known fixed point theorems.

Lemma 2.5 [14] Let Ξ be a bounded, convex and closed subsets of a Banach space $\mathbb{X}$ and let $\Upsilon : \Xi \to \Xi$ be a condensing map. Then, Υ has a fixed point in Ξ.

Lemma 2.6 [14] Let Ξ be a bounded and convex set in Banach space $\mathbb{X}$. $\Upsilon : \Xi \to \mathcal{P}(\Xi)$ is an u.s.c., condensing multivalued map. If for every $x \in \Xi$, $\Upsilon(x)$ is a closed and convex set in Ξ, then Υ has a fixed point in Ξ.

Lemma 2.7 (see [8, Theorem 1.11]) Let $(\mathbb{X}, d)$ be a metric space. If $G : \mathbb{X} \to \mathcal{P}_{cl}(\mathbb{X})$ is a contraction, then $Fix(G) \neq \emptyset$, where $Fix(G)$ denotes the fixed point set of G.

3 Mild solutions to fractional differential inclusions

In this section, we shall investigate some existence results for mild solutions to the equation (1.1)-(1.2) and the equation (1.3)-(1.4). We shall prove our main results under conditions that: (i) the multivalued term takes convex values with compactness of resolvent family of operators; (ii) the multivalued term takes nonconvex values with compactness of resolvent family of operators; and (iii) the multivalued term takes nonconvex values without compactness of resolvent family of operators, respectively.

For the problem (1.1)-(1.2), according to [27], we have the following definition.

Definition 3.1 Let A be the generator of an $(\alpha, 1)$-resolvent family $S_\alpha(t)$, the mild solutions of the problem (1.1)-(1.2) is defined as following

$$x(t) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(t-s)v(s)ds, v \in S_{F,x}, t \in J.$$

We list the following assumptions:

(A1) A generates an $(\alpha, 1)$-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ of type (M, ω). $(\lambda^\alpha - A)^{-1}$ is compact for all $\lambda > \omega$, and $S_\alpha(t)$ is continuous in the uniform operator topology for all $t > 0$.

(A2) $F : J \times \mathbb{X} \to \mathcal{P}_{cp,cv}(\mathbb{X})$ satisfies the following conditions:

(a) For a.e. $t \in J$, $F(t, \cdot)$ is u.s.c, and for each $x \in \mathbb{X}$, $F(\cdot, x)$ is measurable. And for each $x \in C(J, \mathbb{X})$, $S_{F,x}$ is nonempty;

(b) There exists a function $\phi \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t,x)\|_{\mathcal{P}} \leq \phi(t)\|x\|, \forall t \in J, x \in \mathbb{X}.$$

(A3) $p : C(J, \mathbb{X}) \to C(J, \mathbb{X})$ is continuous and there exists $L_p > 0$ such that

$$\|p(x) - p(y)\| < L_p\|x - y\|, \ \forall x, y \in C(J, \mathbb{X}).$$

Remark 3.1 (i) Of concern useful criteria for the continuity of $S_\alpha(t)$ in the uniform operator topology, one can refer to the work [17]. For instance, this property holds true for the class of analytic resolvent.

(ii) According to Lemma 2.3, the condition (A1) implies $S_\alpha(t)$ is compact for all $t > 0$.

Theorem 3.1 If conditions (A1)-(A3) hold, then the problem (1.1)-(1.2) admits at least one mild solution on J provided that

$$\tilde{M}(L_p + \|\phi\|_{L^1}) < 1, \tag{3.1}$$

where $\tilde{M} = \max\{M, Me^{\omega b}\}$.

Proof: Consider the operator $N : C(J, \mathbb{X}) \to \mathcal{P}(C(J, \mathbb{X}))$ defined by

$$N(x) = \left\{ h \in C(J, \mathbb{X}) : h(t) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(t-s)v(s)ds, t \in J \right\},$$

where $v \in S_{F,x}$. Clearly, the fixed points of N are mild solutions to (1.1)-(1.2). We shall show that N satisfies all the hypothesis of Lemma 2.6. The proof will be given in several steps.

Step 1. There exists a positive number r such that $N(B_r) \subseteq B_r$, where $B_r = \{x \in C(J, \mathbb{X}) : \|x\|_\infty \leq r\}$. If it is not true, then for each positive number r, there exists a function x^r such that $h^r \in N(x^r)$ but $\|h^r(t)\| > r$ for some $t \in J$,

$$h^r(t) = S_\alpha(t)[x_0 + p(x^r)] + \int_0^t S_\alpha(t-s)v^r(s)ds,$$

where $v^r \in S_{F,x^r}$. However, on the other hand, we have

$$\begin{aligned} r &< \left\| S_\alpha(t)[x_0 + p(x^r)] + \int_0^t S_\alpha(t-s)v^r(s)ds \right\| \\ &\leq \tilde{M}(\|x_0\| + \|p(x^r)\|) + \tilde{M} \int_0^t |\phi(s)| \|x\| ds \\ &\leq \tilde{M}\|x_0\| + \tilde{M}(L_p\|x^r\| + \|p(0)\|) + \tilde{M}\|\phi\|_{L^1} r \\ &\leq \tilde{M}(x_0 + \|p(0)\|) + \tilde{M}(L_p + \|\phi\|_{L^1})r. \end{aligned}$$

Dividing both sides by r and and taking the lower limit as $r \to \infty$, we obtain

$$1 \leq \tilde{M}(L_p + \|\phi\|_{L^1}),$$

which contradicts the relation (3.1).

Step 2. $N(x)$ is convex for each $x \in C(J, \mathbb{X})$.

Indeed, if $h_1, h_2 \in N(x)$, then there exist $v_1, v_2 \in S_{F,x}$ such that for each $t \in J$, we have

$$h_i(t) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(t-s)v_i(s)ds, i = 1, 2.$$

Let $\theta \in (0, 1)$. Then for each $t \in J$, we have

$$(\theta h_1 + (1-\theta)h_2)(t) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(t-s)[\theta v_1(s) + (1-\theta)v_2(s)]ds.$$

Because $S_{F,x}$ is convex (since F has convex values), $\theta h_1 + (1-\theta)h_2 \in N(x)$.

Step 3. $N(x)$ is closed for each $x \in C(J, \mathbb{X})$.

Let$\{h_n\}_{n\geq 0} \in N(x)$ such that $h_n \to h$ in $C(J, \mathbb{X})$. Then $h \in C(J, \mathbb{X})$ and there exist $\{v_n\} \in S_{F,x}$ such that for each $t \in J$

$$h_n(t) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(t-s)v_n(s)ds.$$

Due to the fact that F has compact values, we may pass to a subsequence if necessary to get that v_n converges to v in $L^1(J, X)$ and hence $v \in S_{F,x}$. Then for each $t \in J$,

$$h_n(t) \to h(t) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(t-s)v(s)ds.$$

Thus, $h \in N(x)$.

Step 4. N is u.s.c. and condensing.

Now, we decompose N as N_1+N_2 as

$$\begin{aligned} (N_1x)(t) &= S_\alpha(t)[x_0 + p(x)] \\ N_2(x) &= \left\{ m \in C(J, \mathbb{X}) : m(t) = \int_0^t S_\alpha(t-s)v(s)ds, t \in J \right\}. \end{aligned}$$

We only need to prove that N_1 is a contraction and N_2 is completely continuous.

To show that N_1 is a contraction, for arbitrary $x_1, x_2 \in B_r$ and each $t \in J$, we have from (A3)

$$\|N_1(x_1)(t) - N_1(x_2)(t)\| = \|S_\alpha(t)[p(x_1) - p(x_2)]\| \leq \tilde{M}L_p\|x_1 - x_2\|_\infty,$$

Thus

$$\|N_1(x_1) - N_1(x_2)\|_\infty \leq \tilde{M}L_p\|x_1 - x_2\|_\infty.$$

From the relation (3.1), we conclude that N_1 is a contraction.

Next, we show that N_2 is u.s.c. and condensing.

(i) $N_2(B_r)$ is obviously bounded.

(ii) $N_2(B_r)$ is equicontinuous.

Indeed, Let $x \in B_r$, $m \in N_2(x)$ and take $t_1, t_2 \in J$ with $t_2 < t_1$. Then there exists a selection $v \in S_{F,x}$ such that

$$m(t) = \int_0^t S_\alpha(t-s)v(s)ds, t \in J.$$

Then

$$\begin{aligned} \|m(t_1) - m(t_2)\| &\leq \int_{t_2}^{t_1} \|S_\alpha(t_1 - s)v(s)\|ds \\ &\quad + \int_0^{t_2} \|[S_\alpha(t_1 - s) - S_\alpha(t_2 - s)]v(s)\|ds \\ &= I_1 + I_2. \end{aligned}$$

For the term I_1, as $t_1 \to t_2$, we have

$$I_1 \leq \int_{t_2}^{t_1} \tilde{M}\phi(s)\|x(s)\|ds \leq \tilde{M}r \int_{t_2}^{t_1} \phi(s)ds \to 0.$$

Next for the term I_2, we have

$$\begin{aligned} I_2 &\leq \int_0^{t_2} \|[S_\alpha(t_1-s) - S_\alpha(t_2-s)]\|\|v(s)\|ds \\ &\leq \int_0^{t_2} \|[S_\alpha(t_1-s) - S_\alpha(t_2-s)]\|\phi(s)\|x(s)\|ds \\ &\leq r\int_0^{t_2} \|[S_\alpha(t_1-s) - S_\alpha(t_2-s)]\|\phi(s)ds. \end{aligned}$$

Now take into account that

$$\|[S_\alpha(t_1-\cdot) - S_\alpha(t_2-\cdot)]\|\phi(s) \leq 2\tilde{M}\phi(s) \in L^1(J, \mathbb{R}_+),$$

and $S_\alpha(t_1-s) - S_\alpha(t_2-s) \to 0$ in $\mathcal{L}(\mathbb{X})$, as $t_1 \to t_2$ (see (A1)). By the Lebesgue's dominated convergence theorem, $I_2 \to 0$ as $t_1 \to t_2$.

(iii) $V(t) = \{m(t) : m(t) \in N_2(B_r)\}$ is relatively compact in $\mathbb{X}$.

For $t = 0$, the conclusion obviously holds. Let $0 < t \leq b$ and ε be a real number satisfying $0 < \varepsilon < t$. For $x \in B_r$ and $v \in S_{F,x}$ such that

$$m(t) = \int_0^t S_\alpha(t-s)v(s)ds, t \in J.$$

Define

$$m_\varepsilon(t) = \int_0^{t-\varepsilon} S_\alpha(t-s)v(s)ds, t \in J.$$

In view of (A1) and Lemma 2.3, we have $S_\alpha(t)$ is compact for $t > 0$. Therefore, the set $\mathcal{K} = \{S_\alpha(t-s)v(s), 0 \leq s \leq t-\varepsilon\}$ is relatively compact. Then $\overline{conv\mathcal{K}}$ is compact. Considering $m_\varepsilon(t) \in t\overline{conv\mathcal{K}}$ for all $t \in J$, the set $V_\varepsilon(t) = \{m_\varepsilon(t) : m_\varepsilon(t) \in N_2(B_r)\}$ is relatively compact in $\mathbb{X}$ for every ε, $0 < \varepsilon < t$. Moreover, for $m \in N(B_r)$,

$$\begin{aligned} \|m(t) - m_\varepsilon(t)\| &\leq \left\| \int_{t-\varepsilon}^t S_\alpha(t-s)v(s)ds \right\| \\ &\leq \tilde{M}r \int_{t-\varepsilon}^t \phi(s)ds. \end{aligned}$$

Therefore, let $\varepsilon \to 0$, we see that there are relatively compact sets arbitrarily close to the set $V(t) = \{m(t) : m(t) \in N_2(B_r)\}$. Hence, the set $V(t) = \{m(t) : m(t) \in N_2(B_r)\}$ is relatively compact in $\mathbb{X}$.

As a consequence of the above steps and the Arzela-Ascoli theorem, we can deduce that N_2 is completely continuous.

(iv) N_2 has a closed graph.

Let $x_n \to x_*, m_n \in N_2(x_n)$ and $m_n \to m_*$. We shall show that $m_* \in N_2(x_*)$. Now $m_n \in N_2(x_n)$ implies that there exists $v_n \in S_{F,x_n}$ such that

$$m_n(t) = \int_0^t S_\alpha(t-s)v_n(s)ds, t \in J.$$

We must prove that there exists $v_* \in S_{F,x_*}$ such that

$$m_*(t) = \int_0^t S_\alpha(t-s)v_*(s)ds, t \in J.$$

Consider the linear continuous operator defined by

$$\Gamma : L^1(J,\mathbb{X}) \to C(J,\mathbb{X})), \quad v \mapsto (\Gamma v)(t) = \int_0^t S_\alpha(t-s)v(s)ds.$$

From Lemma 2.1 it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have $m_n(t) \in \Gamma(S_{F,x_n})$.

Since $x_n \to x_*$ and $m_n \to m_*$, it follows again from Lemma 2.1 that $m_*(t) \in \Gamma(S_{F,x_*})$. That is, there must exists $v_* \in S_{F,x_*}$ such that

$$m_*(t) = \int_0^t S_\alpha(t-s)v_*(s)ds, t \in J.$$

Therefore, N_2 is u.s.c. On the other hand, N_1 is a contraction, hence $N = N_1 + N_2$ is u.s.c. and condensing. By the fixed point theorem Lemma 2.6, there exists a fixed point $x(\cdot)$ for N on B_r . Thus, the problem (1.1)-(1.2) admits a mild solution. ■

Replace the condition (A2)(b) by

(b′) There exist a constant $\tau \in (0,1)$ and a function $\phi \in L^1(J,\mathbb{R}_+)$ such that

$$\|F(t,x)\|_{\mathcal{P}} \le \phi(t)(\|x\|)^\tau, \forall t \in J, x \in C(J,\mathbb{X}).$$

From the above proof of Theorem 3.1, we can obtain the following result.

Corollary 3.1 If conditions (A1)-(A2)(a) and (A2)(b′)-(A3) hold, then the problem (1.1)-(1.2) admits at least one mild solution on J provided that

$$\tilde{M}L_p < 1. \tag{3.2}$$

For the problem (1.3)-(1.4). We first consider the following equation

$$\begin{aligned} D_t^\alpha x(t) &= Ax(t) + v(t), \ t \in J, \\ x(0) &= x_0, x'(0) = x_1, \end{aligned}$$

where $1 < \alpha < 2, v \in L^1(J,\mathbb{X})$. By Laplace transform, we have

$$\lambda^\alpha \widehat{x}(\lambda) - \lambda^{\alpha-1}x(0) - \lambda^{\alpha-2}x'(0) = A\widehat{x}(\lambda) + \widehat{v}(\lambda).$$

That is

$$\begin{aligned}\widehat{x}(\lambda) &= \lambda^{\alpha-1}(\lambda^{\alpha}-A)^{-1}x_0+\lambda^{\alpha-2}(\lambda^{\alpha}-A)^{-1}x_1+(\lambda^{\alpha}-A)^{-1}\widehat{v}(\lambda)\\ &= \widehat{S_{\alpha}}(\lambda)x_0+(\widehat{\mathcal{G}_1 * S_{\alpha}})(\lambda)x_1+(\widehat{R_{\alpha}*v})(\lambda).\end{aligned}$$

Thus, we have

$$x(t)=S_{\alpha}(t)x_0+\int_0^t S_{\alpha}(\theta)x_1 d\theta+\int_0^t R_{\alpha}(t-s)v(s)ds.$$

Now, we can give the following definition.

Definition 3.2 Let $1<\alpha<2$ and A be the generator of an $(\alpha,1)$-resolvent family $\{S_{\alpha}(t)\}_{t\geq 0}$ of type (M,ω). Then A generates an (α,α)-resolvent family $\{R_{\alpha}(t)\}_{t\geq 0}$ of type $\left(\frac{M}{\alpha-1},\omega\right)$ and the mild solution of the problem (1.3)-(1.4) can be given as

$$x(t)=S_{\alpha}(t)[x_0+p(x)]+\int_0^t S_{\alpha}(\theta)[x_1+q(x)]d\theta+\int_0^t R_{\alpha}(t-s)v(s)ds, v\in S_{F,x}, t\in J.$$

Let us list the following basic assumptions:
(A4) Let $1<\alpha<2$ and A generates an $(\alpha,1)$-resolvent family $\{S_{\alpha}(t)\}_{t\geq 0}$ of type (M,ω). $(\lambda^{\alpha}-A)^{-1}$ is compact for all $\lambda>\omega$.
(A5) $q: C(J,\mathbb{X})\to C(J,\mathbb{X})$ is continuous and there exists $L_q>0$ such that

$$\|q(x)-q(y)\|<L_q\|x-y\|,\ \forall x,y\in C(J,\mathbb{X}).$$

Remark 3.2 If (A4) holds, according to Lemma 2.4, A generates an (α,α)-resolvent family $\{R_{\alpha}(t)\}_{t\geq 0}$ of type $\left(\frac{M}{\alpha-1},\omega\right)$ and $R_{\alpha}(t)$ is a compact operator for all $t>0$. And from the proof of [27, Theorem 3.5], $R_{\alpha}(t)$ is continuous in the uniform operator topology for all $t>0$.

Theorem 3.2 If conditions (A2)-(A5) hold, then the problem (1.3)-(1.4) admits at least one mild solution on J provided that

$$\tilde{M}(L_p+bL_q+\|\phi\|_{L^1})<1, \tag{3.3}$$

where $\tilde{M}=\max\left\{\frac{M}{\alpha-1},\frac{M}{\alpha-1}e^{\omega b}\right\}$.

Proof: Consider the operator $N: C(J,\mathbb{X})\to\mathcal{P}(C(J,\mathbb{X}))$ defined by

$$\begin{aligned}N(x) &= \left\{h\in C(J,\mathbb{X}): h(t)=S_{\alpha}(t)[x_0+p(x)]+\int_0^t S_{\alpha}(\theta)[x_1+q(x)]d\theta\right.\\ &\quad \left.+\int_0^t R_{\alpha}(t-s)v(s)ds, t\in J\right\},\end{aligned}$$

where $v \in S_{F,x}$. Clearly, the fixed points of N are mild solutions to (1.1)-(1.2). We shall show that N satisfies all the hypothesis of Lemma 2.6. The proof will be given in several steps.

Step 1. There exists a positive number r such that $N(B_r) \subseteq B_r$, where $B_r = \{x \in C(J,\mathbb{X}) : \|x\|_\infty \le r\}$. If it is not true, then for each positive number r, there exists a function x^r such that $h^r \in N(x^r)$ but $\|h^r(t)\| > r$ for some $t \in J$,

$$h^r(t) = S_\alpha(t)[x_0 + p(x^r)] + \int_0^t S_\alpha(\theta)[x_1 + q(x^r)]d\theta + \int_0^t R_\alpha(t-s)v^r(s)ds,$$

where $v^r \in S_{F,x^r}$. However, on the other hand, we have

$$\begin{aligned} r &< \left\| S_\alpha(t)[x_0 + p(x^r)] + \int_0^t S_\alpha(\theta)[x_1 + q(x^r)]d\theta \right\| + \left\| \int_0^t R_\alpha(t-s)v^r(s)ds \right\| \\ &\le \tilde{M}(\|x_0\| + \|p(x^r)\|) + b\tilde{M}(\|x_1\| + \|q(x^r)\|) + \tilde{M}\int_0^t |\phi(s)|\|x\|ds \\ &\le \tilde{M}\|x_0\| + \tilde{M}(L_p\|x^r\| + \|p(0)\|) + b\tilde{M}\|x_1\| + b\tilde{M}(L_q\|x^r\| + \|q(0)\|) + \tilde{M}\|\phi\|_{L^1}r \\ &\le \tilde{M}(x_0 + \|p(0)\| + b\|x_1\| + b\|q(0)\|) + \tilde{M}(L_p + bL_q + \|\phi\|_{L^1})r. \end{aligned}$$

Dividing both sides by r and and taking the lower limit as $r \to \infty$, we obtain

$$1 \le \tilde{M}(L_p + bL_q + \|\phi\|_{L^1}),$$

which contradicts the relation (3.3).

Step 2. $N(x)$ is convex for each $x \in C(J,\mathbb{X})$.

Indeed, if $h_1, h_2 \in N(x)$, then there exist $v_1, v_2 \in S_{F,x}$ such that for each $t \in J$, we have

$$h_i(t) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(\theta)[x_1 + q(x)]d\theta + \int_0^t R_\alpha(t-s)v_i(s)ds, i = 1, 2.$$

Let $\delta \in (0,1)$. Then for each $t \in J$, we have

$$\begin{aligned} (\delta h_1 + (1-\delta)h_2)(t) &= S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(\theta)[x_1 + q(x)]d\theta \\ &\quad + \int_0^t R_\alpha(t-s)[\delta v_1(s) + (1-\delta)v_2(s)]ds. \end{aligned}$$

Because $S_{F,x}$ is convex (since F has convex values), $\delta h_1 + (1-\delta)h_2 \in N(x)$.

Step 3. $N(x)$ is closed for each $x \in C(J,\mathbb{X})$.

Let$\{h_n\}_{n\ge 0} \in N(x)$ such that $h_n \to h$ in $C(J,\mathbb{X})$. Then $h \in C(J,\mathbb{X})$ and there exist $\{v_n\} \in S_{F,x}$ such that for each $t \in J$

$$h_n(t) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(\theta)[x_1 + q(x)]d\theta + \int_0^t R_\alpha(t-s)v_n(s)ds.$$

Due to the fact that F has compact values, we may pass to a subsequence if necessary to get that v_n converges to v in $L^1(J,X)$ and hence $v \in S_{F,x}$. Then for each $t \in J$,

$$h_n(t) \to h(t) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(\theta)[x_1 + q(x)]d\theta + \int_0^t R_\alpha(t-s)v(s)ds.$$

Thus, $h \in N(x)$.

Step 4. N is u.s.c. and condensing.

Now, we decompose N as N_1+N_2 as

$$\begin{aligned}(N_1x)(t) &= S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(\theta)[x_1 + q(x)]d\theta \\ N_2(x) &= \left\{ m \in C(J,\mathbb{X}) : m(t) = \int_0^t R_\alpha(t-s)v(s)ds, t \in J \right\}.\end{aligned}$$

We only need to prove that N_1 is a contraction and N_2 is completely continuous.

To show that N_1 is a contraction, for arbitrary $x_1, x_2 \in B_r$ and each $t \in J$, we have from (A3) and (A5)

$$\begin{aligned} & \|N_1(x_1)(t) - N_1(x_2)(t)\| \\ \le\ & \|S_\alpha(t)[p(x_1) - p(x_2)]\| + \left\| \int_0^t S_\alpha(\theta)[q(x_1) - q(x_2)]d\theta \right\| \\ \le\ & \tilde{M} L_p \|x_1 - x_2\|_\infty + b\tilde{M} L_q \|x_1 - x_2\|_\infty, \end{aligned}$$

Thus

$$\|N_1(x_1) - N_1(x_2)\|_\infty \le \tilde{M}(L_p + bL_q)\|x_1 - x_2\|_\infty.$$

From the relation (3.3), we conclude that N_1 is a contraction.

Next, we show that N_2 is u.s.c. and condensing.

(i) $N_2(B_r)$ is obviously bounded.

(ii) $N_2(B_r)$ is equicontinuous.

Indeed, Let $x \in B_r$, $m \in N_2(x)$ and take $t_1, t_2 \in J$ with $t_2 < t_1$. Then there exists a selection $v \in S_{F,x}$ such that

$$m(t) = \int_0^t R_\alpha(t-s)v(s)ds, t \in J.$$

Then

$$\begin{aligned} \|m(t_1) - m(t_2)\| &\le \int_{t_2}^{t_1} \|R_\alpha(t_1 - s)v(s)\|ds \\ &\quad + \int_0^{t_2} \|[R_\alpha(t_1 - s) - R_\alpha(t_2 - s)]v(s)\|ds \\ &= I_1 + I_2. \end{aligned}$$

For the term I_1, as $t_1 \to t_2$, we have

$$I_1 \le \int_{t_2}^{t_1} \tilde{M}\phi(s)\|x(s)\|ds \le \tilde{M}r \int_{t_2}^{t_1} \phi(s)ds \to 0.$$

Next for the term I_2, we have

$$\begin{aligned} I_2 &\leq \int_0^{t_2} \|[R_\alpha(t_1-s)-R_\alpha(t_2-s)]\|\|v(s)\|ds \\ &\leq \int_0^{t_2} \|[R_\alpha(t_1-s)-R_\alpha(t_2-s)]\|\phi(s)\|x(s)\|ds \\ &\leq r\int_0^{t_2} \|[R_\alpha(t_1-s)-R_\alpha(t_2-s)]\|\phi(s)ds. \end{aligned}$$

Now take into account that

$$\|[R_\alpha(t_1-\cdot)-R_\alpha(t_2-\cdot)]\|\phi(s) \leq 2\tilde{M}\phi(s) \in L^1(J,\mathbb{R}_+),$$

and $R_\alpha(t_1-s)-R_\alpha(t_2-s) \to 0$ in $\mathcal{L}(\mathbb{X})$, as $t_1 \to t_2$ (see (A4)). By the Lebesgue's dominated convergence theorem, $I_2 \to 0$ as $t_1 \to t_2$.

(iii) $V(t) = \{m(t) : m(t) \in N_2(B_r)\}$ is relatively compact in $\mathbb{X}$.

For $t = 0$, the conclusion obviously holds. Let $0 < t \leq b$ and ε be a real number satisfying $0 < \varepsilon < t$. For $x \in B_r$ and $v \in S_{F,x}$ such that

$$m(t) = \int_0^t R_\alpha(t-s)v(s)ds, t \in J.$$

Define

$$m_\varepsilon(t) = \int_0^{t-\varepsilon} R_\alpha(t-s)v(s)ds, t \in J.$$

In view of (A4) and Lemma 2.4, we have $R_\alpha(t)$ is compact for $t > 0$. Therefore, the set $\mathcal{K} = \{R_\alpha(t-s)v(s), 0 \leq s \leq t-\varepsilon\}$ is relatively compact. Then $\overline{conv\mathcal{K}}$ is compact. Considering $m_\varepsilon(t) \in t\overline{conv\mathcal{K}}$ for all $t \in J$, the set $V_\varepsilon(t) = \{m_\varepsilon(t) : m_\varepsilon(t) \in N_2(B_r)\}$ is relatively compact in $\mathbb{X}$ for every ε, $0 < \varepsilon < t$. Moreover, for $m \in N(B_r)$,

$$\begin{aligned} \|m(t)-m_\varepsilon(t)\| &\leq \left\|\int_{t-\varepsilon}^t R_\alpha(t-s)v(s)ds\right\| \\ &\leq \tilde{M}r\int_{t-\varepsilon}^t \phi(s)ds. \end{aligned}$$

Therefore, let $\varepsilon \to 0$, we see that there are relatively compact sets arbitrarily close to the set $V(t) = \{m(t) : m(t) \in N_2(B_r)\}$. Hence, the set $V(t) = \{m(t) : m(t) \in N_2(B_r)\}$ is relatively compact in $\mathbb{X}$.

As a consequence of the above steps and the Arzela-Ascoli theorem, we can deduce that N_2 is completely continuous.

(iv) N_2 has a closed graph.

Let $x_n \to x_*, m_n \in N_2(x_n)$ and $m_n \to m_*$. We shall show that $m_* \in N_2(x_*)$. Now $m_n \in N_2(x_n)$ implies that there exists $v_n \in S_{F,x_n}$ such that

$$m_n(t) = \int_0^t R_\alpha(t-s)v_n(s)ds, t \in J.$$

We must prove that there exists $v_* \in S_{F,x_*}$ such that

$$m_*(t) = \int_0^t R_\alpha(t-s)v_*(s)ds, t \in J.$$

Consider the linear continuous operator defined by

$$\Gamma : L^1(J,\mathbb{X}) \to C(J,\mathbb{X})), \quad v \mapsto (\Gamma v)(t) = \int_0^t R_\alpha(t-s)v(s)ds.$$

From Lemma 2.1 it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have $m_n(t) \in \Gamma(S_{F,x_n})$.

Since $x_n \to x_*$ and $m_n \to m_*$, it follows again from Lemma 2.1 that $m_*(t) \in \Gamma(S_{F,x_*})$. That is, there must exist $v_* \in S_{F,x_*}$ such that

$$m_*(t) = \int_0^t R_\alpha(t-s)v_*(s)ds, t \in J.$$

Therefore, N_2 is u.s.c. On the other hand, N_1 is a contraction, hence $N = N_1 + N_2$ is u.s.c. and condensing. By the fixed point theorem Lemma 2.6, there exists a fixed point $x(\cdot)$ for N on B_r . Thus, the problem (1.1)-(1.2) admits a mild solution. ■

According to the above proof of Theorem 3.2, we can also have the following result.

Corollary 3.2 If conditions (A2)(a), (A2)(b′) and (A3)-(A5) hold, then the problem (1.1)-(1.2) admits at least one mild solution on J provided that

$$\tilde{M}(L_p + bL_q) < 1. \tag{3.4}$$

Next we consider the problems (1.1)-(1.2) and (1.3)-(1.4) when the multivalued map F takes nonconvex values with compactness of resolvent family of operators. Let $\mathbb{X}$ be a separable Banach space $\mathbb{X}$. We list the following condition:
(C1) $F : J \times \mathbb{X} \to \mathcal{P}_{cp}(\mathbb{X})$ satisfies
(I) $(t,x) \mapsto F(t,x)$ is $\mathcal{L} \times \mathcal{B}$ measurable;
(II) $x \mapsto F(t,x)$ is l.s.c for a.e. $t \in J$.

Theorem 3.3 Suppose hypotheses (A1), (A2)(b), (C1) and (A3) are satisfied. Then the problem (1.1)-(1.2) admits at least one mild solution on J if the condition (3.1) holds.

Proof: Hypotheses (A2)(b) and (C1) imply that F is of l.s.c. type. In view of Lemma 2.2, there exists a continuous function $f : C(J,\mathbb{X}) \to L^1(J,\mathbb{X})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C(J,\mathbb{X})$. Now consider the following equation

$$\begin{aligned} D_t^\alpha x(t) &= Ax(t) + J_t^{1-\alpha} f(x)(t),\ t \in J && (3.5)\\ x(0) &= x_0 + p(x), && (3.6) \end{aligned}$$

Notice that if $x \in C(J, \mathbb{X})$ is a solution of the problem (3.5)-(3.6), then x is also a solution of the problem (1.1)-(1.2). Next, we transform the problem (3.5)-(3.6) into a fixed point problem by defining $N : C(J, \mathbb{X}) \to C(J, \mathbb{X})$ as

$$N(x) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(t-s)f(x)(s)ds, t \in J.$$

We shall show that N satisfies all the hypothesis of Lemma 2.5. The proof will be given in several steps.

Step 1. There exists a positive number r such that $N(B_r) \subseteq B_r$, where $B_r = \{x \in C(J, \mathbb{X}) : \|x\|_\infty \leq r\}$.

This can be conducted similarly as **Step 1.** in the proof of Theorem 3.1.

We decompose N as N_1+N_2 as

$$\begin{aligned} N_1(x)(t) &= S_\alpha(t)[x_0 + p(x)] \\ N_2(x)(t) &= \int_0^t S_\alpha(t-s)f(x)(s)ds. \end{aligned}$$

Step 2. N_2 is continuous on B_r.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ in B_r. Then

$$\begin{aligned} & \|N_2(x_n)(t) - N_2(x)(t)\| \\ \leq & \int_0^t \|S_\alpha(t-s)\|\|f(x_n)(s) - f(x)(s)\|ds \\ \leq & \tilde{M} \int_0^t \phi(s)[\|x_n(s)\| + \|x(s)\|]ds \\ \leq & 2r\tilde{M} \int_0^t \phi(s)ds. \end{aligned}$$

Note that $\phi \in L^1(J, \mathbb{R}_+)$, $\int_0^t \|f(x_n)(s) - f(x)(s)\|ds \to 0, n \to \infty$ by the Lebesgue's Dominated Convergence Theorem. Hence, N_2 is continuous.

Step 3. N is condensing.

Similarly conducted as the proof of Theorem 3.1, we can prove that N_1 is a contraction and N_2 is completely continuous.

From the above three steps, we can complete the proof via Lemma 2.5.

Theorem 3.4 Suppose hypotheses (C1), (A2)(b) and (A3)-(A5) are satisfied. Then the problem (1.3)-(1.4) admits at least one mild solution on J if the condition (3.3) holds.

Proof: Deduced as the proof of Theorem 3.3, we can transform the problem (1.3)-(1.4) into a single-valued problem. We define $N = N_1 + N_2 : C(J, \mathbb{X}) \to C(J, \mathbb{X})$ as

$$\begin{aligned} N_1(x)(t) &= S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(\theta)[x_1 + q(x)]d\theta \\ N_2(x)(t) &= \int_0^t R_\alpha(t-s)f(x)(s)ds. \end{aligned}$$

Similarly conducted as the proof of Theorems 3.2 and 3.3, we can prove that N_1 is a contraction and N_2 is completely continuous. Thus, Lemma 2.5 can be applied to complete the proof. ■

Similarly, from proofs of Theorems 3.3 and 3.4, we have the following results.

Corollary 3.3 Suppose hypotheses (A1), (A2)(b′), (C1) and (A3) are satisfied. Then the problem (1.1)-(1.2) admits at least one mild solution on J if the condition (3.2) holds.

Corollary 3.4 Suppose hypotheses (C1), (A2)(b′) and (A3)-(A5) are satisfied. Then the problem (1.3)-(1.4) admits at least one mild solution on J if the condition (3.4) holds.

In the following, we give some results when the multivalued map F has nonconvex values without compactness of resolvent family of operators. Let us list the following assumptions:
(A6) $F : J \times \mathbb{X} \to \mathcal{P}_{cp}(\mathbb{X})$ satisfies the following conditions:
(1) $F : J \times \mathbb{X} \to \mathcal{P}_{cp}(\mathbb{X}) : (\cdot, x) \mapsto F(\cdot, x)$ is measurable for each $x \in \mathbb{X}$;
(2) There exists a function $l \in L^1(J, \mathbb{R}+)$ such that

$$\begin{aligned} H_d(F(t,x_1), F(t,x_2)) &\leq l(t)\|x_1 - x_2\|, \text{ for a.e. } t \in J, \forall x_1, x_2 \in \mathbb{X}, \\ d(0, F(t,0)) &\leq l(t), \text{ for a.e. } t \in J. \end{aligned}$$

Remark 3.3 [8] Owing to (A6)(1), for each $x \in C(J, \mathbb{X})$, F has a measurable selection, thus $S_{F,x} \neq \emptyset$.

Theorem 3.5 Let A be the generator of an $(\alpha, 1)$-resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ of type (M, ω). Assume that conditions (A3) and (A6) are satisfied, then the problem (1.1)-(1.2) admits at least one mild solution on J provided that

$$\tilde{M}\left(L_p + \|l\|_{L^1}\right) < 1, \tag{3.7}$$

where $\tilde{M} = \max\left\{M, Me^{\omega b}\right\}$.

Proof: Transform the problem (1.1)-(1.2) into a fixed point problem. Let the multivalued operator $N : C(J, \mathbb{X}) \to \mathcal{P}(C(J, \mathbb{X}))$ be defined as in Theorem 3.1. We shall prove that N admits at leas one fixed point. We divide the proof into two steps.

Step 1. For each $x \in C(J, \mathbb{X})$, $N(x) \in \mathcal{P}_{cl}(C(J, \mathbb{X}))$.

This can be proved just as Step 3 in the proof of Theorem 3.1.

Step 2. For each $x, \tilde{x} \in C(J, \mathbb{X})$, there exists a constant $0 < \gamma < 1$ such that $H_d(N(x), N(\tilde{x})) \leq \gamma\|x - \tilde{x}\|_\infty$.

Let $x, \tilde{x} \in C(J, \mathbb{X})$ and $h \in N(x)$. Then there exists $v \in S_{F,x}$ such that for each $t \in J$

$$h(t) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(t-s)v(s)ds.$$

From (A6)(2), we have

$$H_d(F(t,x(t)),F(t,\tilde{x}(t))) \le l(t)\|x(t)-\tilde{x}(t)\|.$$

Thus there exists $w \in S_{F,\tilde{x}}$ such that

$$\|v(t)-w(t)\| \le l(t)\|x(t)-\tilde{x}(t)\|,\ t\in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{X})$ defined as

$$W(t) = \{w \in \mathbb{X} : \|v(t)-w(t)\| \le l(t)\|x(t)-\tilde{x}(t)\|\}.$$

Because $U(t) = W(t)\bigcap F(t,\tilde{x})$ is measurable (see [9, Proposition III. 4]), there exists a function $\tilde{v}(t)$, which is a measurable selection for U. Hence, $\tilde{v}(t) \in F(t,\tilde{x}(t))$ and

$$\|v(t)-\tilde{v}(t)\| \le l(t)\|x(t)-\tilde{x}(t)\|,\ t\in J.$$

For each $t \in J$, we now define

$$\tilde{h}(t) = S_\alpha(t)[x_0+p(\tilde{x})] + \int_0^t S_\alpha(t-s)\tilde{v}(s)ds.$$

Then for each $t\in J$, we have

$$\begin{aligned}\|h(t)-\tilde{h}(t)\| &\le \|S_\alpha(t)[p(x(t))-p(\tilde{x})(t)]\| \\ &\quad + \left\|\int_0^t S_\alpha(t-s)[v(s)-\tilde{v}(s)]ds\right\| \\ &\le \tilde{M}L_p\|x-\tilde{x}\|_\infty + \tilde{M}\int_0^t l(s)ds\|x-\tilde{x}\|_\infty \\ &\le \tilde{M}\left[L_p+\|l\|_{L^1}\right]\|x-\tilde{x}\|_\infty.\end{aligned}$$

Thus,

$$\|h-\tilde{h}\|_\infty \le \tilde{M}\left[L_p+\|l\|_{L^1}\right]\|x-\tilde{x}\|_\infty.$$

By an analogous relation, obtained by interchanging the roles of $\tilde{x}$ and x, we can obtain

$$H_d(N(x),N(\tilde{x})) \le \tilde{M}\left[L_p+\|l\|_{L^1}\right]\|x-\tilde{x}\|_\infty.$$

Owing to relation (3.7), we conclude that N is a contraction. Thus, by Lemma 2.7, N admits a fixed point, which just is one mild solution to the problem (1.1)-(1.2). ■

Theorem 3.6 Let $1<\alpha<2$ and A generates an $(\alpha,1)$-resolvent family $\{S_\alpha(t)\}_{t\ge 0}$ of type (M,ω). Suppose that conditions (A3), (A5) and (A6) are satisfied, then the problem (1.3)-(1.4) has at least one mild solution on J provided that

$$\tilde{M}(L_p+bL_q+\|l\|_{L^1}) < 1, \tag{3.8}$$

where $\tilde{M} = \max\left\{\frac{M}{\alpha-1}, \frac{M}{\alpha-1}e^{\omega b}\right\}$.

Proof: Transform the problem (1.3)-(1.4) into a fixed point problem. Let the multivalued operator $N : C(J, \mathbb{X}) \to \mathcal{P}(C(J, \mathbb{X}))$ be defined as in Theorem 3.2. We shall prove that N admits at leas one fixed point. We divide the proof into two steps.

Step 1. For each $x \in C(J, \mathbb{X})$, $N(x) \in \mathcal{P}_{cl}(C(J, \mathbb{X}))$.

This can be proved just as Step 3 in the proof of Theorem 3.2.

Step 2. N is a contraction.

Let $x, \tilde{x} \in C(J, \mathbb{X})$ and $h \in N(x)$. Then there exists $v \in S_{F,x}$ such that for each $t \in J$

$$h(t) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(\theta)[x_1 + q(x)]d\theta + \int_0^t R_\alpha(t-s)v(s)ds.$$

From (A6)(2), we have

$$H_d(F(t, x(t)), F(t, \tilde{x}(t))) \leq l(t)\|x(t) - \tilde{x}(t)\|.$$

Thus there exists $w \in S_{F,\tilde{x}}$ such that

$$\|v(t) - w(t)\| \leq l(t)\|x(t) - \tilde{x}(t)\|, \ t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{X})$ defined as

$$W(t) = \{w \in \mathbb{X} : \|v(t) - w(t)\| \leq l(t)\|x(t) - \tilde{x}(t)\|\}.$$

Because $U(t) = W(t) \bigcap F(t, \tilde{x})$ is measurable (see [9, Proposition III. 4]), there exists a function $\tilde{v}(t)$, which is a measurable selection for U. Hence, $\tilde{v}(t) \in F(t, \tilde{x}(t))$ and

$$\|v(t) - \tilde{v}(t)\| \leq l(t)\|x(t) - \tilde{x}(t)\|, \ t \in J.$$

For each $t \in J$, we now define

$$\tilde{h}(t) = S_\alpha(t)[x_0 + p(\tilde{x})] + \int_0^t S_\alpha(\theta)[x_1 + q(\tilde{x}))]d\theta + \int_0^t R_\alpha(t-s)\tilde{v}(s)ds.$$

Then for each $t \in J$, we have

$$\begin{aligned}
\|h(t) - \tilde{h}(t)\| \quad &\leq \quad \|S_\alpha(t)[p(x(t)) - p(\tilde{x})(t)]\| \\
&\quad + \int_0^t S_\alpha(\theta)[q(x(t))) - q(\tilde{x}(t)))]d\theta \\
&\quad + \left\| \int_0^t R_\alpha(t-s)[v(s) - \tilde{v}(s)]ds \right\| \\
&\leq \quad \tilde{M}L_p\|x - \tilde{x}\|_\infty + b\tilde{M}L_q\|x - \tilde{x}\|_\infty \\
&\quad + \tilde{M}\int_0^t l(s)ds\|x - \tilde{x}\|_\infty \\
&\leq \quad \tilde{M}\left[L_p + bL_q + \|l\|_{L^1}\right]\|x - \tilde{x}\|_\infty.
\end{aligned}$$

Thus,

$$\|h - \tilde{h}\|_\infty \leq \tilde{M}\left[L_p + bL_q + \|l\|_{L^1}\right]\|x - \tilde{x}\|_\infty.$$

By an analogous relation, obtained by interchanging the roles of $\tilde{x}$ and x, we can obtain

$$H_d(N(x), N(\tilde{x})) \leq \tilde{M}\left[L_p + bL_q + \|l\|_{L^1}\right]\|x - \tilde{x}\|_\infty.$$

Owing to relation (3.8), we conclude that N is a contraction. Thus, by Lemma 2.7, N admits a fixed point, which just is one mild solution to the problem (1.3)-(1.4). ∎

Example 3.1 As a simple application, we consider the following equations

$$D_t^\alpha u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + g(t, u(t,x)),\ (t,x) \in [0,1] \times [0,\pi], \tag{3.9}$$

$$u(t,0) = u(t,\pi) = 0, u_x'(t,0) = u_x'(t,\pi) = 0,\ t \in [0,1], \tag{3.10}$$

$$u(0,x) = \textstyle\sum_{k=1}^n a_k u(t,x) + u_0(x),\ u'(0,x) = \sum_{k=1}^n b_k u'(t,x) + u_1(x), \tag{3.11}$$

where $1 < \alpha < 2$, $a_k, b_k \in \mathbb{R}, n \in \mathbb{N}$. Let $\mathbb{X} = L^2([0,\pi])$ and consider the operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ defined by $D(A) := \{u \in \mathbb{X} : u \in H^2([0,\pi]), u(0) = u(\pi)\}$ and for $u \in D(A)$, $Au := \dfrac{\partial^2 u}{\partial x^2}$. Define the functions $g : [0,1] \times D(A) \to \mathbb{X}$, and $p, q : D(A) \to \mathbb{X}$ by

$$g(t, u(t,x)) := \frac{e^{-t}u(t,x)}{(6+t)(1+u(t,x))},\ p(u)(x) := \sum_{k=1}^n a_k u(t,x),\ q(u)(x) := \sum_{k=1}^n b_k u'(t,x).$$

It is well-known that A generates a compact and analytic (and hence norm continuous for all $t > 0$) C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\mathbb{X}$ such that $\|T(t)\| \leq 1$. Now, we can extract an (α,α)-resolvent family $\{R_\alpha(t)\}_{t\geq 0}$ of type $(1,1)$ (see [1]). Meanwhile, the compactness of $T(t)$ implies that $(\lambda^\alpha - A)^{-1}$ is compact.

Let $F =: \{g\}$, $J := [0,1]$. We note that the above problem (3.9)-(3.11) can be rewritten in the abstract form (1.3)-(1.4). Furthermore, we assume that $\sum_{k=1}^n |a_k| \leq \dfrac{1}{6}$, $\sum_{k=1}^n |b_k| \leq \dfrac{1}{6}$. We also observe that in this case

$$\phi(t) := \frac{e^{-t}}{6+t},\ b = \tilde{M} = 1,\ \text{and}\ L_p = \frac{1}{6},\ L_q = \frac{1}{6},\ \|\phi\|_{L^1} \leq \frac{1}{6}.$$

According to Theorem 3.2, the problem (3.9)-(3.11) has at least one mild solution on J.

4 Conclusions

In this paper, we establish some sufficient conditions to guarantee the existence of mild solutions to abstract fractional differential inclusions with nonlocal initial conditions under conditions that: (i) the multivalued term takes convex values with compactness of resolvent family of operators; (ii) the multivalued term takes nonconvex values with compactness of resolvent family of operators; and (iii) the multivalued term takes nonconvex values without compactness of resolvent family of operators, respectively. The main results are based upon theories of resolvent family of operators, multivalued analysis and fixed point approach.

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