Weighted pseudo almost automorphic solutions to a semilinear fractional differential equation with Stepanov-like weighted pseudo almost automorphic nonlinear term

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Abstract

In this paper, we investigate some existence results of weighted pseudo almost automorphic solutions to a semilinear fractional differential equation in Banach spaces with Stepanov-like weighted pseudo almost automorphic nonlinear term. Our main results are based upon ergodicity and composition theorems of Stepanov-like weighted pseudo almost automorphic functions combined with fixed point techniques.

Keywords: Fractional differential equation, S^p-weighted pseudo almost automorphic function, Fixed point theorems.

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1 Introduction

The concept of almost automorphy was first introduced by Bochner in [5] in relation to some aspects of differential geometry, which can be seen as an important generalization of the classical almost periodicity in the sense of Bohr. After that, this concept has undergone several interesting, natural and powerful generalizations. The concept of pseudo almost automorphy was introduced by Liang et al. in [31] as a natural generalization of

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almost automorhy. Diagana [10] presented the concept of Stepanov-like pseudo almost automorphy which generalizes notations of pseduo almost automorphy and Stepanov-like almost automorphy by N'Guérékata and Pankov in [26]. Further, Blot et al. gave the concept of weighted pseudo almost automorphic functions with values in Banach spaces in [4]. Xia and Fan presented the notation of Stepanov-like (or S^p-) weighted pseudo almost automorphy in [30], which is more general than those of Stepanov-like pseudo almost automorphy [10] and weighted pseudo almost automorphy [26]. Chang, N'Guérékata et al. investigated some ergodic properties and composition theorems of Stepanov-like weighted pseudo almost automorphic functions in [32, 33]. Applications of above mentioned concepts to various differential and integro-differential equations have widely been investigated, see for instance [1, 2, 7, 11, 12, 13, 15, 17, 19, 23, 24, 25] and references therein.

In recently years, fractional equations have gained considerable importance due to their applications in various fields of the science, such as physics, mechanics, chemistry engineering, etc. Significant development has been made in ordinary and partial differential equations involving fractional derivatives, we refer to [16, 21, 22, 28, 29, 34, 35, 36] and references therein. Araya and Lizama in [3] investigated the existence and uniqueness of almost automorphic mild solutions to the semilinear equation

$$D_t^{\alpha} u(t) = Au(t) + f(t, u(t)),\ t \in \mathbb{R},\ 1 < \alpha < 2,$$

where A is a generator of an α-resolvent family and D_t is a Riemann Liouville fractional derivative. In [9], Cuevas and Lizama considered the following fractional differential equation:

$$D_t^{\alpha} u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t)),\ t \in \mathbb{R},\ 1 < \alpha < 2, \tag{1.1}$$

where A is a linear operator of sectorial negative type on a complex Banach space. Under suitable conditions on f, the authors proved the existence and uniqueness of an almost automorphic mild solution to Eq. (1.1). See also [6] for some existence results of weighted pseudo almost automorphic solutions for Eq. (1.1) with S^p-weighted pseudo almost automorphic coefficients. In a recent paper [27], Ponce studied the existence and uniqueness of bounded solutions for the following integro-differential fractional differential equation

$$D^{\alpha} u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t, u(t)), \quad t \in \mathbb{R} \tag{1.2}$$

where A is a closed linear operator defined on Banach space $\mathbb{X}$, $a \in L^1_{loc}(\mathbb{R}_+)$ is a scalar-valued kernel, $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ belongs to a closed subspace of the space of continuous and bounded functions satisfying some Lipschitz type conditions, and for $\alpha > 0$, the fractional derivative is understood in the sense of Weyl. Sufficient conditions are established for the existence and uniqueness of a continuous and bounded solution such as almost periodic (automorphic), pseudo-almost periodic (automorphic), asymptotically almost periodic (automorphic) and S-asymptotically ω-periodic solution. Thus, a natural question is: What is it about when the nonlinear term f is an Stepanov-like weighted pseudo almost automorphic function? The main purpose of this paper is to investigate some existence results of weighted pseudo almost automorphic solutions to Eq. (1.2) with Stepanov-like weighted pseudo almost automorphic nonlinear term f. Our main results are based upon ergodicity and composition theorems of Stepanov-like weighted pseudo almost automorphic functions in [32, 33] combined with fixed point techniques.

The paper is organized as follows. In Section 2, we recall some basic definitions, lemmas and preliminary results which will be used throughout this paper. In Section 3, we prove some existence results of weighted pseudo almost automorphic solutions to the problem (1.2) with Stepanov-like weighted pseudo almost automorphic nonlinear term f.

2 Preliminaries

In this section, we list some basic definitions, notations, lemmas and preliminary facts which are used in this paper. In the paper, we assume that $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ are two Banach spaces, let $BC(\mathbb{R}, \mathbb{X})$ (respectively $BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) stand for the class of all $\mathbb{X}$-valued bounded continuous functions from $\mathbb{R}$ into $\mathbb{X}$ (respectively, the class of all jointly bounded continuous functions from $\mathbb{R} \times \mathbb{Y}$ into $\mathbb{X}$). The space $BC(\mathbb{R}, \mathbb{X})$ equipped with the sup norm defined by $\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$ is a Banach space. The notation $\mathfrak{B}(\mathbb{X}, \mathbb{Y})$ stands for the space of bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ endowed uniform operator topology, and we abbreviate to $\mathfrak{B}(\mathbb{X})$, whenever $\mathbb{X} = \mathbb{Y}$.

Now we give some necessary definitions.

Definition 2.1 [27] Given a function $f : \mathbb{R} \to \mathbb{X}$, the Wely fractional integral of order $\alpha > 0$ is defined by

$$D^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} f(s) ds, \quad t \in \mathbb{R}$$

when this integral is convergent. The Wely fractional derivative $D^\alpha f$ of order $\alpha > 0$ is defined by

$$D^\alpha f(t) := \frac{d^n}{dt^n} D^{-(n-\alpha)} f(t), \quad t \in \mathbb{R}$$

where $n = [\alpha] + 1$.

It is known that $D^\alpha D^{-\alpha} f = f$ for any $\alpha > 0$, and $D^n = \frac{d^n}{dt^n}$ holds with $n \in \mathbb{N}$. See more details in [16] and [22].

Definition 2.2 [27] Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space $\mathbb{X}$, and $\alpha > 0$. Given $a \in L^1_{loc}(\mathbb{R}_+)$, the operator A is called the generator of an α-resolvent family, if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : [0,\infty) \to \mathfrak{B}(\mathbb{X})$ such that $\left\{ \frac{\lambda^\alpha}{1+\hat{a}(\lambda)} : Re\lambda > \omega \right\} \subset \bar{\rho}(A)$ and for all $x \in \mathbb{X}$,

$$(\lambda^\alpha - (1+\hat{a}(\lambda))A)^{-1}x = \frac{1}{1+\hat{a}(\lambda)}\left(\frac{\lambda^\alpha}{1+\hat{a}(\lambda)} - A\right)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x dt, \quad Re\lambda > 0,$$

where $\hat{a}$ denotes the Laplace transform of a, $\bar{\rho}(A)$ denotes the resolvent set of A. In this case, $S_\alpha(t)_{t\geq 0}$ is called the α-resolvent family generated by A.

Sufficient conditions for $\{S_\alpha(t)\}_{t\geq 0} \subset \mathfrak{B}(\mathbb{X})$ to be a resolvent family can be found from [8, 18, 20].

Definition 2.3 [25] A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost automorphic if every sequence of real numbers $\{s_n'\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$f(t) := \lim_{n\to\infty} g(t-s_n)$$

for each $t \in \mathbb{R}$. The collection of all such functions will be denoted by $AA(\mathbb{X})$.

Definition 2.4 [25] A continuous function $f : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$ is said to be almost automorphic if $f(t,x)$ is almost automorphic for each $t \in \mathbb{R}$ uniformly for all $x \in \mathbb{K}$, where $\mathbb{K}$ is any bounded subset of $\mathbb{Y}$. The collection of all such functions will be denoted by $AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$.

Now, let $\mathbb{U}$ denotes the set of all functions $\rho : \mathbb{R} \to (0,\infty)$ which are locally integrable over $\mathbb{R}$ such that $\rho > 0$ almost everywhere. For a given $r > 0$ and for each $\rho \in \mathbb{U}$, we set $m(r,\rho) := \int_{-r}^r \rho(t)dt$. Thus the space of weighted $\mathbb{U}_\infty$ is defined by

$$\mathbb{U}_\infty := \{\rho \in \mathbb{U} : \lim_{r\to\infty} m(r,\rho) = \infty\}.$$

Now for $\rho \in \mathbb{U}_\infty$ we define

$$PAA_0(\mathbb{X},\rho) := \{f \in BC(\mathbb{R},\mathbb{X}) : \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^r \|f(t)\|\rho(t)dt = 0\},$$

$$PAA_0(\mathbb{R}\times\mathbb{Y},\mathbb{X}) = \Big\{f \in C(\mathbb{R}\times\mathbb{Y},\mathbb{X}) : f(\cdot,y) \text{ is bounded for each } y \in \mathbb{Y}$$
$$\text{and } \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^r \|f(t,y)\|\,\rho(t)dt = 0 \text{ uniformly in } y \in \mathbb{Y}\Big\}.$$

Definition 2.5 [4] Let $\rho \in \mathbb{U}_\infty$. A function $f \in BC(\mathbb{R},\mathbb{X})$ (respectively, $f \in BC(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) is called weighted pseudo almost automorphic if it can be expressed as $f = g + \chi$, where $g \in AA(\mathbb{X})$ (respectively, $AA(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) and the function $\chi \in PAA_0(\mathbb{X},\rho)$ (respectively, $PAA_0(\mathbb{Y},\mathbb{X},\rho)$). We denote by $WPAA(\mathbb{R},\mathbb{X})$ (respectively, $WPAA(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) the set of all such functions.

Lemma 2.1 [23] Let $\rho \in \mathbb{U}_\infty$. Suppose that $PAA_0(\mathbb{X},\rho)$ is translation invariant, then the decomposition of weighted pseudo almost automorphic functions is unique.

Lemma 2.2 [23] Let $\rho \in \mathbb{U}_\infty$. If $PAA_0(\mathbb{X},\rho)$ is translation invariant, then $(WPAA(\mathbb{R},\mathbb{X}),\|\cdot\|_\infty)$ is Banach space.

Definition 2.6 [11] The Bochner transform $f^b(t,s), t\in\mathbb{R}, s\in[0,1]$, of a functions $f : \mathbb{R} \longrightarrow \mathbb{X}$ is defined by

$$f^b(t,s) := f(t+s).$$

Definition 2.7 [11] The Bochner transform $f^b(t,s,u), t\in\mathbb{R}, s\in[0,1], u\in\mathbb{X}$ of a functions $f : \mathbb{R}\times\mathbb{X} \longrightarrow \mathbb{X}$ is defined by

$$f^b(t,s,u) := f(t+s,u) \quad \text{for all} \quad u \in \mathbb{X}.$$

Definition 2.8 [11] Let $p \in [1,\infty)$, the space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \longrightarrow \mathbb{X}$ such that $f^b \in L^\infty(\mathbb{R}, L^p(0,1;\mathbb{X}))$. This is a Banach space with the norm

$$\|f\|_{S^p} = \left\|f^b\right\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p d\tau\right)^{\frac{1}{p}}.$$

Definition 2.9 [26] The space $AS^p(\mathbb{X})$ of Stepanov-like almost automorphic (or S^p-almost automorphic) functions consists of all $f \in BS^p(\mathbb{X})$ such that $f^b \in AA(L^p(0,1;\mathbb{X}))$. In other words, a function $f \in L^p_{loc}(\mathbb{R},\mathbb{X})$ is said to be S^p-almost automorphic if its Bochner transform $f^b : \mathbb{R} \longrightarrow L^p(0,1;\mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers $\{s_n^k\}_{n\in\mathbb{N}}$ there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $g \in L^p_{loc}(\mathbb{R},\mathbb{X})$ such that

$$\lim_{n\to\infty}\left(\int_t^{t+1} \|f(s+s_n)-g(s)\|^p ds\right)^{\frac{1}{p}} = 0$$

and

$$\lim_{n\to\infty}\left(\int_t^{t+1} \|g(s-s_n)-f(s)\|^p ds\right)^{\frac{1}{p}} = 0.$$

Definition 2.10 [26] A function $f : \mathbb{R} \times \mathbb{Y} \longrightarrow \mathbb{X}, (t,u) \longrightarrow f(t,u)$ with $f(\cdot,u) \in L^p_{loc}(\mathbb{R},\mathbb{X})$ for each $u \in \mathbb{Y}$, is said to be S^p-almost automorphic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{Y}$ if $t \longrightarrow f(t,u)$ is S^p-almost automorphic for each $u \in \mathbb{Y}$. That means, for every sequence of real numbers $\{s^k_n\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a functions $g(\cdot,u) \in L^p_{loc}(\mathbb{R},\mathbb{X})$ such that

$$\lim_{n\to\infty}\left(\int_t^{t+1}\|f(s+s_n,u)-g(s,u)\|^p ds\right)^{\frac{1}{p}} = 0$$

and

$$\lim_{n\to\infty}\left(\int_t^{t+1}\|g(s-s_n,u)-f(s,u)\|^p ds\right)^{\frac{1}{p}} = 0$$

pointwise on $\mathbb{R}$ and for each $u \in \mathbb{Y}$. We denote by $AS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ the set of all such functions.

Definition 2.11 [30, 32] Let $\rho \in \mathbb{U}_\infty$. A functions $BS^p(\mathbb{X})$ is said to be Stepanov-like weighted pseudo almost automorphic (or S^p-weighted pseudo almost automorphic) if it can be expressed as $f = g + \chi$, where $g \in AS^p(\mathbb{X})$ and $\chi^b \in PAA_0(L^p(0,1;\mathbb{X}))$. In other words, a function $f \in L^p_{loc}(\mathbb{R},\mathbb{X})$ is said to be Stepanov-like weighted pseudo almost automorphic relatively to the weighted $\rho \in \mathbb{U}_\infty$, if its Bochner transform $f^b : \mathbb{R} \longrightarrow L^p(0,1;\mathbb{X})$ is weighted pseudo almost automorphic in the sense that there exist two functions $g,\chi : \mathbb{R} \longrightarrow \mathbb{X}$ such that $f = g+\chi$, where $g \in AS^p(\mathbb{X})$ and $\chi^b \in PAA_0(L^p(0,1;\mathbb{X}),\rho)$. We denote by $WPAAS^p(\mathbb{R},\mathbb{X})$ the set of all such functions.

Definition 2.12 [30, 32] Let $\rho \in \mathbb{U}_\infty$. A functions $f : \mathbb{R}\times\mathbb{Y} \longrightarrow \mathbb{X}, (t,u) \longrightarrow f(t,u)$ with $f(\cdot,u) \in L^p_{loc}(\mathbb{R},\mathbb{X})$ for each $u \in \mathbb{Y}$ is said to be Stepanov-like weighted pseudo almost automorphic (or S^p-weighted pseudo almost automorphic) if it can be expressed as $f = g + \chi$, where $g \in AS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ and $\chi^b \in PAA_0(\mathbb{Y}, L^p(0,1;\mathbb{X}),\rho)$. We denote by $WPAAS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ the set of all such functions.

Remark 2.1 [30, 32] It is clear that if $1 \le p < q < \infty$ and $f \in L^q_{loc}(\mathbb{R},\mathbb{X})$ is S^q-almost automorphic, then f is S^p almost automorphic. Also if $f \in AA(\mathbb{X})$, then f is S^p-almost automorphic for any $1 \le p < \infty$.

Lemma 2.3 [30, 32] Let $\rho \in \mathbb{U}_\infty$ and assume that $PAA_0(L^p(0,1;\mathbb{X}),\rho)$ is translation invariant, then the decomposition of an S^p-weighted pseudo almost automorphic function is unique.

Lemma 2.4 [30, 33] Let $\rho \in \mathbb{U}_\infty$ be such that

$$\lim_{t\to\infty}\sup\frac{\rho(t+\iota)}{\rho(t)} < \infty \quad\text{and}\quad \lim_{r\to\infty}\sup\frac{m(r+\iota,\rho)}{m(r,\rho)} < \infty, \tag{2.1}$$

for every $\iota \in \mathbb{R}$, then spaces $WPAAS^p(\mathbb{R},\mathbb{X})$ and $PAA_0(L^p(0,1;\mathbb{X}),\rho)$ are translation invariant.

Lemma 2.5 [30, 32] If $f \in WPAA(\mathbb{R},\mathbb{X})$, then $f \in WPAAS^p(\mathbb{R},\mathbb{X})$ for each $1 \leq p < \infty$. In other words, $WPAA(\mathbb{R},\mathbb{X}) \subseteq WPAAS^p(\mathbb{R},\mathbb{X})$.

Lemma 2.6 [30, 32] Let $\rho \in \mathbb{U}_\infty$ satisfy the condition (2.1), then the space $WPAAS^p(\mathbb{R},\mathbb{X})$ equipped with the norm $\|\cdot\|_{S^p}$ is a Banach space.

Lemma 2.7 [32] Let $\rho \in \mathbb{U}_\infty$ and let $f = g + \chi \in WPAAS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$ with $g \in AS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$,$\chi^b \in PAA_0(\mathbb{X},L^p(0,1;\mathbb{X}),\rho)$. Assume that the following conditions are satisfied:

(i) $f(t,x)$ is Lipschitzian in $x \in \mathbb{X}$ uniformly in $t \in \mathbb{R}$; that is, there exist a constant $L > 0$ such that

$$\| f(t,x) - f(t,y) \| \leq L \| x - y \|$$

for all $x, y \in \mathbb{X}$ and $t \in \mathbb{R}$.

(ii) $g(t,x)$ is uniformly continuous in any bounded subset $K' \subseteq \mathbb{X}$ uniformly for $t \in \mathbb{R}$.

If $u = u_1 + u_2 \in WPAAS^p(\mathbb{R},\mathbb{X})$, with $u_1 \in AS^p(\mathbb{X})$, $u_2^b \in PAA_0(L^p(0,1;\mathbb{X}),\rho)$ and $K = \overline{\{u_1(t) : t \in \mathbb{R}\}}$ is compact, then $F : \mathbb{R} \longrightarrow \mathbb{X}$ defined by $F(\cdot) = f(\cdot,u(\cdot))$ belongs to $WPAAS^p(\mathbb{R},\mathbb{X})$.

Lemma 2.8 [32] Let $\rho \in \mathbb{U}_\infty$ and let $f = g + \chi \in WPAAS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$ with $g \in AS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $\chi^b \in PAA_0(\mathbb{X},L^p(0,1;\mathbb{X}),\rho)$. Assume that the following conditions are satisfied:

(i) there exists a nonnegative function $L(\cdot) \in BS^p(\mathbb{R})$ with $p > 1$ such that for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$

$$\left(\int_t^{t+1} \|f(s,u) - f(s,v)\|^p ds\right)^{\frac{1}{p}} \leq L(t)\,\|u - v\|\,;$$

(ii) $\rho \in L^q_{loc}(\mathbb{R})$ satisfies $\lim\limits_{T\to\infty} \sup \frac{T^{\frac{1}{p}} m_q(T,\rho)}{m(T,\rho)} < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $m_q(T,\rho) = \left(\int_{-T}^{T} \rho^q(t)dt\right)^{\frac{1}{q}}$;

(iii) $g(t,x)$ is uniformly continuous in any bounded subset $K' \subseteq \mathbb{X}$ uniformly for $t \in \mathbb{R}$.

If $u = u_1 + u_2 \in WPAAS^p(\mathbb{R},\mathbb{X})$, with $u_1 \in AS^p(\mathbb{X})$, $u_2^b \in PAA_0(L^p(0,1;\mathbb{X}),\rho)$ and $K = \overline{\{u_1(t) : t \in \mathbb{R}\}}$ is compact, then $F : \mathbb{R} \longrightarrow \mathbb{X}$ defined by $F(\cdot) = f(\cdot,u(\cdot))$ belongs to $WPAAS^p(\mathbb{R},\mathbb{X})$.

Lemma 2.9 [32] Let $\rho \in \mathbb{U}_\infty$ and let $f : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{X}$ be an S^p-weighted pseudo almost automorphic function. Supose that f satisfies the following conditions:

(i) $f(t,x)$ is uniformly continuous in any bounded subset $K^{'} \subseteq \mathbb{X}$ uniformly for $t \in \mathbb{R}$;

(ii) $g(t,x)$ is uniformly continuous in any bounded subset $K^{'} \subseteq \mathbb{X}$ uniformly for $t \in \mathbb{R}$;

(iii) For every bounded subset $K^{'} \subseteq \mathbb{X}$, $\{f(\cdot,x) : x \subseteq K^{'}\}$ is bounded in $WPAAS^p(\mathbb{R},\mathbb{X})$.

If $u = u_1 + u_2 \in WPAAS^p(\mathbb{R},\mathbb{X})$, with $u_1 \in AS^p(\mathbb{X})$, $u_2^b \in PAA_0(L^p(0,1;\mathbb{X}),\rho)$ and $K = \overline{\{u_1(t) : t \in \mathbb{R}\}}$ is compact, then $\digamma : \mathbb{R} \longrightarrow \mathbb{X}$ defined by $\digamma(\cdot) = f(\cdot, u(\cdot))$ belongs to $WPAAS^p(\mathbb{R},\mathbb{X})$.

Finally, we recall some properties on compactness criterion.

Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function such that $h(t) \geq 1$ for all $t \in \mathbb{R}$ and $h(t) \longrightarrow \infty$ as $\mid t \mid \longrightarrow \infty$. We consider the space:

$$C_h(\mathbb{X}) = \left\{ u \in C(\mathbb{R},\mathbb{X}) : \lim_{|t|\to\infty} \frac{u(t)}{h(t)} = 0 \right\}.$$

Endowed with the norm $\| u \|_h = \sup_{t\in\mathbb{R}} \frac{\|u(t)\|}{h(t)}$, it is a Banach space (see [15]).

Lemma 2.10 [15] A subset $\Xi \subseteq C_h(\mathbb{X})$ is a relatively compact set if it verifies the following conditions:

(c-1) The set $\Xi(t) = \{u(t) : u \in \Xi\}$ is relatively compact in $\mathbb{X}$ for each $t \in \mathbb{R}$.

(c-2) The Ξ is equicontinuous.

(c-3) For each $\epsilon > 0$ there exists $\mathbb{L} > 0$ such that $\|u(t)\| \leq \epsilon h(t)$ for all $u \in \Xi$ and all $|t| > \mathbb{L}$.

Lemma 2.11 [14] Let $\mathbb{D}$ be a closed convex subset of a Banach space $\mathbb{X}$ such that $0 \in \mathbb{D}$. Let $\Gamma : \mathbb{D} \to \mathbb{D}$ be a completely continuous map. Then the set $\{x \in \mathbb{D} : x = \lambda\Gamma(x), 0 < \lambda < 1\}$ is unbounded or the map Γ has a fixed point in $\mathbb{D}$.

3 Weighted pseudo almost automorphic solution

This section is mainly concerned with existence results of weighted pseudo almost automorphic mild solutions. We recall the definition of mild solutions to Eq.(1.2).

Definition 3.1 [27] Let $\alpha > 0$ and A be the generator of an α-resolvet family $\{S_\alpha(t)\}_{t\geq 0}$. A function $u \in C(\mathbb{R}, \mathbb{X})$ is called a mild solution to Eq. (1.2) if the function $s \mapsto S_\alpha(t-s)f(s, u(s))$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$ and

$$u(t) = \int_{-\infty}^{t} S_\alpha(t-s)f(s, u(s))ds.$$

In the sequel, we always assume that the weight ρ satisfies the condition (2.1). And now, we list the following basic assumptions.
(H1) Assume that A generates an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ such that $\|S_\alpha(t)\| \leq \varphi_\alpha(t)$, for all $t \geq 0$, where $\varphi_\alpha(\cdot) \in L^1(\mathbb{R}_+)$ is nonincreasing such that $\varphi_0 := \sum_{n=0}^{\infty} \varphi_\alpha(n) < \infty$.
(H2) The function $f \in WPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ $(p > 1)$, and there exist a constant $L_f > 0$, such that

$$\|f(t, x) - f(t, y)\| \leq L_f \|x - y\|$$

for all $t \in \mathbb{R}$ and for each $x, y \in \mathbb{X}$.
(H3) The function $f \in WPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ $(p > 1)$, and there exists a nonnegative function $L_f(\cdot) \in BS^p(\mathbb{R})$ with $p > 1$ such that

$$\|f(t, x) - f(t, y)\| \leq L_f(t) \|x - y\|$$

for all $t \in \mathbb{R}$ and each $x, y \in \mathbb{X}$.
(H4) Let $\rho \in L^q_{loc}(\mathbb{R})$ satisfy

$$\lim_{T\to\infty} \frac{T^{\frac{1}{p}} m_q(T, \rho)}{m(T, \rho)} < \infty$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $m_q(T, \rho) = (\int_{-T}^{T} \rho^q(t)dt)^{\frac{1}{q}}$.
(H5) The function $f = g + \psi \in WPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ $(p > 1)$, where $g \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ is uniformly continuous in any bounded subset $\mathbb{M} \subseteq \mathbb{X}$ uniformly in $t \in \mathbb{R}$ and $\psi^b \in PAA_0(L^p(0, 1; \mathbb{X}), \rho)$.

Lemma 3.1 Let $\rho \in \mathbb{U}_\infty$. Assume that A generates an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ satisfying the condition (H1). If $f \in WPAAS^p(\mathbb{R}, \mathbb{X})$ with $p > 1$, then

$$F(t) := \int_{-\infty}^{t} S_\alpha(t-s)f(s)ds \in WPAA(\mathbb{R}, \mathbb{X}),\ t \in \mathbb{R}.$$

Proof: Since $f \in WPAAS^p(\mathbb{R}, \mathbb{X})$, we have by Definition 2.11 that $f = g + \psi$, where $g \in AS^p(\mathbb{R}, \mathbb{X})$ and $\psi^b \in PAA_0(L^p(0, 1; \mathbb{X}), \rho)$. Consider for each $n = 1, 2, \cdots$, the integrals

$$F_n(t) = \int_{n-1}^{n} S_\alpha(r)f(t-r)dr$$

$$\begin{aligned}
&= \int_{n-1}^{n} S_\alpha(r)g(t-r)dr + \int_{n-1}^{n} S_\alpha(r)\psi(t-r)dr \\
&= X_n(t) + Y_n(t),
\end{aligned}$$

where $X_n(t) = \int_{n-1}^{n} S_\alpha(r)g(t-r)dr$, $Y_n(t) = \int_{n-1}^{n} S_\alpha(r)\psi(t-r)dr$. In order to prove that each F_n is a weighted pseudo almost automorphic function, we only need to verify $X_n \in AA(\mathbb{X})$ and $Y_n \in PAA_0(\mathbb{X},\rho)$ for each $n = 1, 2, \cdots$. Let us show that $X_n \in AA(\mathbb{X})$. We have

$$\begin{aligned}
\|X_n(t)\| &\leq \int_{t-n}^{t-n+1} \|S_\alpha(t-r)g(r)\|dr \\
&\leq \int_{t-n}^{t-n+1} \varphi_\alpha(t-r)\|g(r)\|dr \\
&\leq \varphi_\alpha(n-1)\int_{t-n}^{t-n+1} \|g(r)\|dr \\
&\leq \varphi_\alpha(n-1)\left(\int_{t-n}^{t-n+1} \|g(r)\|^p dr\right)^{\frac{1}{p}} \\
&\leq \varphi_\alpha(n-1)\|g\|_{S^p}.
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \varphi_\alpha(n-1) := \sum_{n=0}^{\infty} \varphi_\alpha(n) < \infty$, we deduce that from the well-known Weierstrass theorem that the series $\sum_{n=1}^{\infty} X_n(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$X(t) := \int_{-\infty}^{t} S_\alpha(t-r)g(r)dr = \sum_{n=1}^{\infty} X_n(t).$$

Clearly, $X(t) \in C(\mathbb{R},\mathbb{X})$ and $\|X(t)\| \leq \sum_{n=1}^{\infty} \|X_n(t)\| \leq \sum_{n=0}^{\infty} \varphi_\alpha(n)\|g\|_{S^p}$.

Since $g \in AS^p(\mathbb{R},\mathbb{X})$, then for every sequence of real numbers $\{s_n\}_{n\in\mathbb{N}}$, there exist a sequence $\{s_m\}_{m\in\mathbb{N}}$ and a function $\widetilde{g}(\cdot) \in L^p_{loc}(\mathbb{R},\mathbb{X})$ such that for each $t \in \mathbb{R}$

$$\lim_{m\to\infty}\left(\int_t^{t+1} \|g(s+s_m) - \widetilde{g}(s)\|^p ds\right)^{\frac{1}{p}} = 0 \text{ and } \lim_{m\to\infty}\left(\int_t^{t+1} \|\widetilde{g}(s-s_m) - g(s)\|^p ds\right)^{\frac{1}{p}} = 0.$$

Let $\widetilde{X}_n(t) = \int_{n-1}^{n} S_\alpha(r)\widetilde{g}(t-r)dr$. Then using the Hölder inequality, we have

$$\begin{aligned}
\|X_n(t+s_m) - \widetilde{X}_n(t)\| &= \left\|\int_{n-1}^{n} S_\alpha(r)[g(t+s_m-r) - \widetilde{g}(t-r)]dr\right\| \\
&\leq \int_{n-1}^{n} \varphi_\alpha(r)\|g(t+s_m-r) - \widetilde{g}(t-r)\|dr \\
&\leq \varphi_\alpha(n-1)\left(\int_{n-1}^{n} \|g(t+s_m-r) - \widetilde{g}(t-r)\|^p dr\right)^{\frac{1}{p}}.
\end{aligned}$$

Obviously, $\|X_n(t+s_m) - \widetilde{X}_n(t)\| \to 0$ as $m \to \infty$. Similarly, we can prove that

$$\lim_{m\to\infty} \|\widetilde{X}_n(t-s_m) - X_n(t)\| = 0.$$

Thus, we conclude that each $X_n \in AA(\mathbb{X})$ and consequently their uniform limit $X(t) \in AA(\mathbb{X})$.

Let us show that each $Y_n \in PAA_0(\mathbb{X}, \rho)$. For this, note that

$$\begin{aligned} \|Y_n(t)\| &\leq \int_{t-n}^{t-n+1} \|S_\alpha(t-r)\psi(r)\| dr \\ &\leq \int_{t-n}^{t-n+1} \varphi_\alpha(t-r)\|\psi(r)\| dr \\ &\leq \varphi_\alpha(n-1)\left(\int_{t-n}^{t-n+1} \|\psi(r)\|^p dr\right)^{\frac{1}{p}}. \end{aligned}$$

Then, for $T > 0$, we see that

$$\frac{1}{m(T,\rho)}\int_{-T}^{T} \|Y_n(t)\|\rho(t)dt \leq \varphi_\alpha(n-1)\frac{1}{m(T,\rho)}\int_{-T}^{T}\left(\int_{t-n}^{t-n+1} \|\psi(r)\|^p dr\right)^{\frac{1}{p}} \rho(t)dt.$$

Since $\psi^b \in PAA_0\left(L^p(0,1;\mathbb{X}), \rho\right)$, the above inequality leads to $Y_n \in PAA_0(\mathbb{X}, \rho)$ for each $n = 1, 2, \cdots$. Since $\| \psi \|_{S^p} \sum_{n=0}^{\infty} \varphi_\alpha(n) < \infty$, then we again deduce from Weierstrass test that the series $\sum_{n=1}^{\infty} Y_n(t)$ is uniformly convergent on $\mathbb{R}$. Hence

$$Y(t) := \int_{-\infty}^{t} S_\alpha(t-s)\psi(s)ds = \sum_{n=1}^{\infty} Y_n(t).$$

Applying $Y_n(t) \in PAA_0(\mathbb{X}, \rho)$ and the inequality

$$\begin{aligned} \frac{1}{m(T,\rho)}\int_{-T}^{T} \|Y(t)\|\,\rho(t)dt &\leq \frac{1}{m(T,\rho)}\int_{-T}^{T} \left\| Y(t) - \sum_{n=1}^{k} Y_n(t) \right\| \rho(t)dt \\ &+ \sum_{n=1}^{k} \frac{1}{m(T,\rho)}\int_{-T}^{T} \|Y_n(t)\|\,\rho(t)dt \\ &\longrightarrow 0, \end{aligned}$$

we obtain that the uniform limit $Y(\cdot) = \sum_{n=1}^{\infty} Y_n(t) \in PAA_0(\mathbb{X}, \rho)$. Therefore, $F(t) := X(t) + Y(t)$ is weighted pseudo almost automorphic in $t \in \mathbb{R}$. The proof is complete.

Theorem 3.1 Let $\rho \in \mathbb{U}_\infty$. Assume that (H1), (H2) and (H5) are satisfied. Then Eq. (1.2) has a unique mild solution in $WPAA(\mathbb{R}, \mathbb{X})$ provided

$$L_f \leq \|\varphi_\alpha\|_{L^1(\mathbb{R}_+)}^{-1} \tag{3.1}$$

Proof: Consider the nonlinear operator $\Gamma : WPAA(\mathbb{R},\mathbb{X}) \longrightarrow WPAA(\mathbb{R},\mathbb{X})$. Such that

$$(\Gamma x)(t) := \int_{-\infty}^{t} S_\alpha(t-s)f(s,u(s))ds.$$

First we need to prove Γ maps $WPAA(\mathbb{R},\mathbb{X})$ into itself. For every $x \in WPAA(\mathbb{R},\mathbb{X})$ has the form of $x = x_1 + x_2$ with $x_1 \in AA(\mathbb{X})$, the fact that the rang of almost automorphic functions is relatively compact, so the set $\{x_1(t) : t \in \mathbb{X}\}$ is relatively compact. By using Lemma 2.7, one can easily see that $f(\cdot, x(\cdot)) \in WPAAS^p(\mathbb{R},\mathbb{X})$. Therefore, from the proof of Lemma 3.1 we obtain that $(\Gamma x)(\cdot) \in WPAA(\mathbb{R},\mathbb{X})$. Thus, $\Gamma(WPAA(\mathbb{R},\mathbb{X})) \subseteq WPAA(\mathbb{R},\mathbb{X})$. The following is needed to prove Γ is a contraction, then for each $x, y \in WPAA(\mathbb{R},\mathbb{X})$, we have

$$\begin{aligned}
\|\Gamma x - \Gamma y\|_\infty &:= \sup_{t\in\mathbb{R}} \left\| \int_{-\infty}^{t} S_\alpha(t-s)[f(s,x(s)) - f(s,y(s))]ds \right\| \\
&\leq L_f \sup_{t\in\mathbb{R}} \int_0^\infty \|S_\alpha(s)\| \|x(t-s) - y(t-s)\| ds \\
&\leq L_f \|x-y\|_\infty \int_0^\infty \varphi_\alpha(s) ds \\
&\leq L_f \|\varphi_\alpha\|_{L^1(\mathbb{R}_+)} \|x-y\|_\infty.
\end{aligned}$$

From inequality (3.1), Γ is a contraction map. So by the Banach space fixed point theorem Γ has a unique fixed point $x(\cdot)$ in $WPAA(\mathbb{R},\mathbb{X})$. The proof is completed.

A different Lipschition condition is considered in the following result.

Theorem 3.2 Let $\rho \in \mathbb{U}_\infty$. Assume conditions (H1), (H3)–(H5) hold. Then Eq.(1.2) admits a unique mild solution in $WPAA(\mathbb{R},\mathbb{X})$ whenever

$$\|L_f\|_{S^p} \leq \varphi_0^{-1}. \tag{3.2}$$

Proof: Consider the nonlinear operator Γ given by

$$(\Gamma x)(t) := \int_{-\infty}^{t} S_\alpha(t-s)f(s,x(s))ds, \quad t \in \mathbb{R}.$$

First, let us prove that $\Gamma(WPAA(\mathbb{R},\mathbb{X})) \subseteq WPAA(\mathbb{R},\mathbb{X})$. For each $x \in WPAA(\mathbb{R},\mathbb{X})$ by the fact that the range of a almost automorphic function is relatively compact combined with Lemmas 2.5 and 2.8, we gain that the function $s \longmapsto f(s, x(s))$ is in $WPAAS^p(\mathbb{R},\mathbb{X})$. Furthermore, from Lemma 3.1 we know $(\Gamma x)(\cdot) \in WPAA(\mathbb{R},\mathbb{X})$, that is the operator Γ maps $WPAA(\mathbb{R},\mathbb{X})$ into $WPAA(\mathbb{R},\mathbb{X})$. Next, we prove that the operator Γ has a unique fixed point in $WPAA(\mathbb{R},\mathbb{X})$. Indeed, for each $t \in \mathbb{R}$, $x, y \in WPAA(\mathbb{R},\mathbb{X})$, we have

$$\|\Gamma x(t) - \Gamma y(t)\| = \left\| \int_{-\infty}^{t} S_\alpha(t-s)[f(s,x(s)) - f(s,y(s))]ds \right\|$$

$$
\begin{aligned}
&\leq \int_{-\infty}^{t} \|S_\alpha(t-s)\| \|f(s,x(s)) - f(s,y(s))\| ds \\
&\leq \int_{-\infty}^{t} \varphi_\alpha(t-s) \|f(s,x(s)) - f(s,y(s))\| ds \\
&\leq \sum_{n=1}^{\infty} \varphi_\alpha(n-1) \int_{t-n}^{t-n+1} L_f(s) ds \|x-y\|_\infty \\
&\leq \sum_{n=1}^{\infty} \varphi_\alpha(n-1) \left(\int_{t-n}^{t-n+1} \|L_f(s)\|^p ds \right)^{\frac{1}{p}} \|x-y\|_\infty \\
&\leq \phi_0 \|L_f\|_{S^p} \|x-y\|_\infty.
\end{aligned}
$$

Hence

$$
\|\Gamma x(t) - \Gamma y(t)\|_\infty \leq \phi_0 \|L_f\|_{S^p} \|x-y\|_\infty.
$$

Since $\phi_0 \|L_f\|_{S^p} < 1$ by the inequality (3.2) , Γ has a unique fixed point $x \in WPAA(\mathbb{R}, \mathbb{X})$. This finishes the proof.

We next investigate the existence of weighted pseudo almost automorphic mild solutions of Eq.(1.2) when the perturbation f is not necessarily Lipschitz continuous. For that, we require the following assumptions:

(H6) $f \in WPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ $(p > 1)$ and $f(t,x)$ is uniformly continuous in any bounded subset $\mathbb{M} \subseteq \mathbb{X}$ uniformly for $t \in \mathbb{R}$ and for every bounded subset $\mathbb{M} \subseteq \mathbb{X}$, $\{f(\cdot,x) : x \in \mathbb{M}\}$ is bounded in $WPAAS^p(\mathbb{X})$.

(H7) There exists a continuous nondecreasing function $W : [0,\infty) \to (0,\infty)$ such that

$$
\|f(t,x)\| \leq W(\|x\|) \text{ for all } t \in \mathbb{R} \text{ and } \ x \in \mathbb{X}.
$$

Theorem 3.3 Assume that A generates an α-resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ satisfying the condition (H1). Let $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ be a function that satisfies assumptions (H5)–(H7), and the following additional conditions:

(i) For each $c \geq 0$, the function $t \to \int_{-\infty}^{t} \varphi_\alpha(t-s) W(h(s))\, ds$ belongs to $BC(\mathbb{R})$. We set

$$
\beta() = \left\| \int_{-\infty}^{t} \varphi_\alpha(t-s) W(ch(s))\, ds \right\|_h
$$

(ii) For each $\varepsilon > 0$ there is $\delta > 0$ such that for every $u, v \in C_h(\mathbb{X})$, $\|u - v\|_h \leq \delta$ implies that

$$
\int_{-\infty}^{t} \varphi_\alpha(t-s) \|f(s,u(s)) - f(s,v(s))\| ds \leq \varepsilon
$$

for all $t \in \mathbb{R}$.

(iii) $\liminf_{\xi \to \infty} \frac{\xi}{\beta(\xi)} > 1$.

(iv) For all $c, d \in \mathbb{R}, c < d$, and $\Lambda > 0$, the set $\{f(s,x) : c \leq s \leq d, x \in C_h(\mathbb{X}), \|x\|_h \leq \Lambda\}$ is relatively compact in $\mathbb{X}$.
Then Eq.(1.2) has at least one weighted pseudo almost automorphic mild solution.

Proof: We define the nonlinear operator $\Gamma : C_h(\mathbb{X}) \to C_h(\mathbb{X})$ by

$$(\Gamma x)(t) := \int_{-\infty}^{t} S_\alpha(t-s)f(s,x(s))ds, \quad t \in \mathbb{R}.$$

We will show that Γ has a fixed point in $WPAA(\mathbb{R},\mathbb{X})$. For the sake of convenience, we divide the proof into several steps.

(I) For $x \in C_h(\mathbb{X})$, we have that

$$\|(\Gamma x)(t)\| \leq \int_{-\infty}^{t} \varphi_\alpha(t-s)W(\|x(s)\|)ds \leq \int_{-\infty}^{t} \varphi_\alpha(t-s)W\left(\|x\|_h h(s)\right)ds.$$

It follows from the condition (i) that Γ is well defined.

(II) The operator Γ is continuous. In fact, for any $\varepsilon > 0$, we take $\delta > 0$ involved in the condition (ii). If $x, y \in C_h(\mathbb{X})$ and $\|x - y\|_h \leq \delta$, then

$$\|(\Gamma x)(t) - (\Gamma y)(t)\| \leq \int_{-\infty}^{t} \varphi_\alpha(t-s)\|f(s,x(s)) - f(s,y(s))\|ds \leq \varepsilon,$$

which shows the assertion.

(III) We will show that Γ is completely continuous. We set $B_\Lambda(\mathbb{X})$ for the closed ball with center at 0 and radius Λ in the space $\mathbb{X}$. Let $V = \Gamma(B_\Lambda(C_h(\mathbb{X})))$ and $v = \Gamma(x)$ for $x \in B_\Lambda(C_h(\mathbb{X}))$. First, we will prove that $V(t)$ is a relatively compact subset of $\mathbb{X}$ for each $t \in \mathbb{R}$. It follows from the condition (i) that the function $s \to \varphi_\alpha(s)W(\Lambda h(t-s))$ is integrable on $[0,\infty)$. Hence, for $\varepsilon > 0$, we can choose $c \geq 0$ such that $\int_c^\infty \varphi_\alpha(s)W(\Lambda h(t-s))ds \leq \varepsilon$. Since

$$v(t) = \int_0^c S_\alpha(s)f(t-s,x(t-s))ds + \int_c^\infty S_\alpha(s)f(t-s,x(t-s))ds$$

and

$$\left\|\int_c^\infty S_\alpha(s)f(t-s,x(t-s))ds\right\| \leq \int_c^\infty \varphi_\alpha(s)W(\Lambda h(t-s))ds \leq \varepsilon,$$

we get $v(t) \in \overline{cc_0(N)} + B_\varepsilon(\mathbb{X})$, where $c_0(N)$ denotes the convex hull of N and $N = \{S_\alpha(s)f(\xi,x) : 0 \leq s \leq c, t-c \leq \xi \leq t, \|x\|_h \leq \Lambda\}$. Using the strong continuity of $S_\alpha(\cdot)$ and the property (iv) of f, we infer that N is a relatively compact set, and $V(t) \subseteq \overline{cc_0(N)} + B_\varepsilon(\mathbb{X})$, which establishes our assertion.

Second, we show that the set V is equicontinuous. In fact, we can decompose

$$v(t+s) - v(t) = \int_0^s S_\alpha(\sigma)f(t+s-\sigma, x(t+s-\sigma))d\sigma$$

$$+\int_0^c [S_\alpha(\sigma+s)-S_\alpha(\sigma)]f(t-\sigma,x(t-\sigma))d\sigma$$
$$+\int_c^\infty [S_\alpha(\sigma+s)-S_\alpha(\sigma)]f(t-\sigma,x(t-\sigma))d\sigma.$$

For each $\varepsilon>0$, we can choose $c>0$ and $\delta_1>0$ such that

$$\begin{aligned}&\left\|\int_0^s S_\alpha(\sigma)f(t+s-\sigma,x(t+s-\sigma))d\sigma+\int_c^\infty [S_\alpha(\sigma+s)-S_\alpha(\sigma)]f(t-\sigma,x(t-\sigma))d\sigma\right\|\\ \leq\;&\int_0^s \varphi_\alpha(\sigma)W(\Lambda h(t+s-\sigma))d\sigma+\int_c^\infty[\varphi_\alpha(\sigma+s)-\varphi_\alpha(\sigma)]W(\Lambda h(t-\sigma))d\sigma\\ \leq\;&\frac{\varepsilon}{2}\end{aligned}$$

for $s\leq\delta_1$. Moreover, since $\{f(t-\sigma,x(t-\sigma)):0\leq\sigma\leq c,x\in B_\Lambda(C_h(\mathbb{X}))\}$ is a relatively compact set and $S_\alpha(\cdot)$ is strongly continuous, we can choose $\delta_2>0$ such that $\|[S_\alpha(\sigma+s)-S_\alpha(\sigma)]f(t-\sigma,x(t-\sigma))\|\leq\frac{\varepsilon}{2c}$ for $s\leq\delta_2$. Combining these estimates, we get $\|v(t+s)-v(t)\|\leq\varepsilon$ for s small enough and independent of $x\in B_\Lambda(C_h(\mathbb{X}))$.

Finally, applying the condition (i), we can see that

$$\frac{\|v(t)\|}{h(t)}\leq\frac{1}{h(t)}\int_{-\infty}^t\varphi_\alpha(t-s)W(\Lambda h(s))ds\to 0,\quad |t|\to\infty,$$

and this convergence is independent of $x\in B_\Lambda(C_h(\mathbb{X}))$. Hence, by Lemma 2.10, V is a relatively compact set in $C_h(\mathbb{X})$.

(IV) Let us show assume that $x^\lambda(\cdot)$ is a solution of the equation $x^\lambda=\lambda\Gamma(x^\lambda)$ for some $0<\lambda<1$. We can estimate

$$\begin{aligned}\left\|x^\lambda(t)\right\|&=\lambda\left\|\int_{-\infty}^t S_\alpha(t-s)f(s,x^\lambda(s))ds\right\|\\&\leq\int_{-\infty}^t\varphi_\alpha(t-s)W(\|x^\lambda\|_h h(s))ds\\&\leq\beta(\|x^\lambda\|_h)h(t).\end{aligned}$$

Hence, we get

$$\frac{\|x^\lambda\|_h}{\beta(\|x^\lambda\|_h)}\leq 1$$

and combining with the condition (iii), we conclude that the set $\left\{x^\lambda:x^\lambda=\lambda\Gamma(x^\lambda),\lambda\in(0,1)\right\}$ is bounded.

(V) It follows from Lemma 2.5, (H5)–(H6) and Lemma 2.9 that the function $t\to f(t,x(t))$ belongs to $WPAAS^p(\mathbb{R},\mathbb{X})$, whenever $x\in WPAA(\mathbb{R},\mathbb{X})$. Moreover, from Lemma 3.1 we infer that $\Gamma(WPAA(\mathbb{R},\mathbb{X}))\subseteq WPAA(\mathbb{R},\mathbb{X})$ and noting that $WPAA(\mathbb{R},\mathbb{X})$

is a closed subspace of $C_h(\mathbb{X})$, consequently, we can consider $\Gamma : WPAA(\mathbb{R},\mathbb{X}) \to WPAA(\mathbb{R},\mathbb{X})$. Using properties (I)–(III), we deduce that this map is completely continuous. By Lemma 2.11, we infer that Γ has a fixed point $x \in WPAA(\mathbb{R},\mathbb{X})$. This completes the proof.

As a consequence of Theorem 3.3, we obtain the following corollary.

Corollary 3.1 Assume that A generates an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ satisfying condition (H1). Let $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ be a function that satisfies assumptions (H5)–(H6) and the Hölder type condition:

$$\|f(t,x) - f(t,y)\| \leq \gamma\|x-y\|^\tau, \quad 0<\tau<1,$$

for all $t \in \mathbb{R}$ and $x, y \in \mathbb{X}$, where $\tau > 0$ is a constant. Moreover, assume the following conditions:
(a) $f(t,0) = q$.
(b) $\sup_{t\in\mathbb{R}} \int_{-\infty}^t \varphi_\alpha(t-s)h(s)^\tau ds = \gamma_2 < \infty$.
(c) For all $c, d \in \mathbb{R}$, $c < d$, and $p > 0$, the set $\{f(s,x) : c \leq s \leq d, x \in C_h(\mathbb{X}), \|x\|_h \leq p\}$ is relatively compact in $\mathbb{X}$.
Then Eq.(1.2) has a weighted pseudo almost automorphic mild solution.

Proof: Let $\gamma_0 = \|q\|, \gamma_1 = \gamma$. We take $W(\xi) = \gamma_0 + \gamma_1\xi^\tau$. Then condition (H7) is satisfied. It follows from (b), we can see that function f satisfies (i) in Theorem 3.3. Note that for each $\varepsilon > 0$ there is $0 < \delta^\tau < \frac{\varepsilon}{\gamma_1\gamma_2}$ such that for every $x, y \in C_h(\mathbb{X})$, $\|x-y\|_h \leq \delta$ implies that $\int_{-\infty}^t \varphi_\alpha(t-s)\|f(s,x(s) - f(s,y(s))\|ds \leq \varepsilon$ for all $t \in \mathbb{R}$. The hypothesis (iii) in the statement of Theorem 3.3 can be easily verified using the definition of W. So by Theorem 3.3 we can prove Eq.(1.2) has a weighted pseudo almost automorphic mild solution.

Example 3.1 Let $A = -\varrho I$, $a(t) = \frac{\varrho}{4}\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $f(t,u) = u \sin \frac{1}{2+\cos t+\cos\sqrt{2}t} + e^{-|t|}\sin u$, where $0 < \alpha < 1$, $\varrho > 0$ and $f : \mathbb{X} \to \mathbb{X}$. From Eq. (1.1) we have

$$D^\alpha u(t) = -\varrho u(t) - \frac{\varrho^2}{4}\int_{-\infty}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds + f(t,u(t)), \quad t \in \mathbb{R}. \tag{3.3}$$

From [27, Example 4.17], it follows that A generates an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ such that

$$\hat{S}_\alpha(\lambda) = \frac{\lambda^\alpha}{(\lambda^\alpha + 2/\varrho)^2} = \frac{\lambda^{\alpha-\alpha/2}}{(\lambda^\alpha + 2/\varrho)}\cdot\frac{\lambda^{\alpha-\alpha/2}}{(\lambda^\alpha + 2/\varrho)}.$$

Thus, $S_\alpha(t) = (r*r)(t)$ with $r(t) = t^{\frac{\alpha}{2}-1}E_{\alpha,\frac{\alpha}{2}}(-\frac{\varrho}{2}t^\alpha)$, $E_{\alpha,\frac{\alpha}{2}}(\cdot)$ is the Mittag-Leffler function defined as in [21].

Note that the function $f \in WPAAS^p(\mathbb{R}, \mathbb{X})$ with weight $\rho(t) = |t|$ for $t \in \mathbb{R}$, and

$$||f(t,u) - f(t,v)|| \leq 2||u - v||.$$

Then we can conclude that there exists a unique mild solution $x(\cdot) \in WPAA(\mathbb{R}, \mathbb{X})$ of Eq.(3.3) by Theorem 3.1 provided $\|S_\alpha\| < \frac{1}{2}$. We remark that given $0 < \alpha < 1$, we can choose the number $\varrho > 0$ such that $\|S_\alpha\| < \frac{1}{2}$ as in the proof of [27, Lemma 3.9]

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