# Properties of solution sets for Sobolev type fractional differential inclusions via resolvent family of operators

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**Abstract**

In this manuscript, by properties on some corresponding resolvent operators and techniques in multivalued analysis, we establish some results for solution sets of Sobolev type fractional differential inclusions in the Caputo and Riemann-Liouville fractional derivatives with order $1 < \alpha < 2$, respectively. We show that the solution sets are nonempty, compact, contractible and thus arcwise connected under some suitable conditions. We remark that our results are directly established through resolvent operators instead of subordination formulas usually applied, and the existence and compactness of $E^{-1}$ is not necessarily needed. Some applications are also given in the final.

**Keywords:** Fractional differential inclusions; Solution set; Resolvent operator.
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## 1 Introduction

Differential inclusions are usually applied to deal with differential equations with a discontinuous right-hand side or an inaccurately known right-hand side, which can be seen as a generalization of the notion of ordinary differential equations [18, 39]. On the other hand, differential inclusions are also closely related to control theory, for example, considering the following control problem

$$x' = f(x,u), u \in U,$$

where $u$ is known as a control parameter. It finds that the above control system and the following differential inclusion

$$x' \in f(x,U) = \bigcup_{u\in U} f(x,u)$$

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has the same trajectories. If the set of controls depends upon the state $x$, i.e. $U = U(x)$, then the following differential inclusion can be obtained

$$x' \in F(x, U(x)).$$

It is noted that the above mentioned equivalence between a control system and the corresponding differential inclusion plays a key role in establishing existence theorems in optimal control theory. Differential inclusion has found its wide applications to models arising in different disciplines, and thus it has been considerably investigated by lots of scholars in last decades, see for instance [1, 7, 9, 20, 21, 24, 25, 33, 39] and references therein. We especially address that in the monograph [20], properties of solution sets for various differential inclusions of integer order such as higher-order differential inclusions, neutral differential inclusions, hyperbolic differential inclusions and impulsive differential inclusions have been discussed in details.

Fractional calculus can be seen a generalization of the ordinary differentiation and integration to arbitrary non-integer order, which has been recognized as one of the most powerful tools to describe long-memory processes in the last decades. Many phenomena from physics, chemistry, mechanics, electricity et al can been modelled by ordinary and partial differential equations involving fractional derivatives, we refer to [2, 3, 4, 19, 29, 30, 38, 43, 44, 46, 48] and references therein for more details. We also notice that properties of solution sets for fractional differential inclusions have also been increasingly concerned recently, see for instance [11, 12, 45, 46] and references therein.

Motivated by the above mentioned work, the main purpose of this manuscript is to investigate properties of solution sets for Sobolev type semilinear fractional differential inclusions in Banach spaces. Concretely, let $A$ be a closed linear operator defined on a Banach space $(X, \|\cdot\|)$, $u_0, u_1 \in X$. Denote $\mathcal{P}(X) = \{Y \subseteq X : Y \neq \emptyset\}$. Now, we consider properties of solution sets for the following semilinear fractional differential inclusions of Sobolev type

$$\begin{cases} D_t^\alpha (Eu)(t) \in Au(t) + \mathcal{F}(t, u(t)), \\ \quad Eu(0) = u_0, \quad (Eu)'(0) = u_1 \end{cases} \tag{1.1}$$

and

$$\begin{cases} \quad D^\alpha (Eu)(t) \in Au(t) + \mathcal{F}(t, u(t)), \\ E(g_{2-\alpha} * u)(0) = u_0, (E(g_{2-\alpha} * u))'(0) = u_1, \end{cases} \tag{1.2}$$

where $t \in I := [0, b]$, the order $1 < \alpha < 2$, the notations $D_t^\alpha$ and $D^\alpha$ denote, respectively, the Caputo and Riemann-Liouville fractional derivatives, and the operator pair $(A, E)$ generates a resolvent family $\{S_{\alpha,\beta}^E(t)\}_{t \geq 0}$ (see definition below, Section 2.1) for suitable $\alpha, \beta > 0$, the multivalued term $\mathcal{F} : I \times X \to \mathcal{P}(X)$ and the function $g_\star(\cdot)$ is also specified by (2.1) in Section 2.

Sobolev type fractional differential inclusions are naturally applied to the control of dynamical system when the controlled system or the controller is described by a Sobolev type fractional differential equation. It is noted that there are already some interesting results on abstract fractional differential equations of Sobolev type with the order $0 < \alpha < 1$, see for example [8, 15, 16, 23, 26, 40, 41] and the references therein. The main techniques in these mentioned work are based upon the following assumptions: a). *the existence of*

$E^{-1}$ *as a bounded operator*, or b). $D(E) \subset D(A)$, *$E$ is bijective and $E^{-1} : X \to D(E)$ is a compact operator.* Under these circumstances, the change of variable $w(t) = Eu(t)$ or *subordination formulas* can be used to deal with solution representations and related problems. It should be pointed out that another method to deal with abstract Sobolev type fractional differential equation with the order $0 < \alpha < 1$ is developed in [26, 41], where solution representations are derived from *subordination formulas* of propagation family (see [27]) without the above assumptions a) and b).

To the best of our knowledge, properties of solution sets for general systems (1.1) and (1.2) in case $1 < \alpha < 2$ (and $E \neq \mathcal{I}$, identity operator) have not been addressed in the existing literature. In present paper, we shall deal with properties of solution sets for Eq. (1.1) and Eq. (1.2) respectively based upon properties on resolvent operator generated by the pair $(A, E)$ and techniques in multivalued analysis. We shall show that the solution sets are nonempty, compact, contractible and thus arcwise connected under some suitable conditions. We remark that our results are directly established through resolvent operators generated by the pair $(A, E)$ instead of subordination formulas usually applied, and thus previous assumptions a) or b) is not necessarily needed. Finally, some applications are also given to illustrate our main results.

The rest of this paper is organized as follows. Section 2 is involved in Preliminaries. Section 3 is devoted to investigate properties of solution sets for Eq. (1.1) and Eq. (1.2), respectively. Section 4 is involved in some applications, and Section 5 is Conclusions.

## 2 Preliminaries

In this section, we list some definitions, notations and recall some basic results which are used throughout this paper.

### 2.1 Basic results on fractional calculus and resolvent operator

In this subsection, we recall some basic results on fractional calculus and list some properties on fractional resolvent operators. Most of these results can be found in monographs [4, 19, 29, 48], papers [1, 2, 3, 5, 6, 11, 12, 14, 22, 28, 32, 33, 35, 36, 37, 38, 42, 43, 45] and references therein.

Let $(X, \|\cdot\|), Z$ be Banach spaces. We denote by $\mathcal{B}(X, Z)$ the space of all bounded linear operators from $X$ into $Z$, and denote by $\mathcal{B}(X)$ the space of all bounded linear operators from $X$ into itself. For a closed and linear operator $A : D(A) \subset X \to X$, where $D(A)$ is the domain of $A$, we denote by $\rho(A)$ its resolvent set and by $R(\lambda, A)$ its resolvent operator, that is, $R(\lambda, A) = (\lambda - A)^{-1}$ which is defined for all $\lambda \in \rho(A)$.

For $\mu > 0$, we define

$$g_\mu(t) = \begin{cases} \dfrac{t^{\mu-1}}{\Gamma(\mu)}, & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{2.1}$$

where $\Gamma(\cdot)$ is the Gamma function. We also define $g_0 \equiv \delta_0$, the Dirac delta. For $\mu > 0$,

$n = \lceil \mu \rceil$ denotes the smallest integer $n$ greater than or equal to $\mu$. The finite convolution of $f$ and $g$ is denoted by $(f * g)(t) = \displaystyle\int_0^t f(t-s)g(s)ds$.

**Definition 2.1** Let $\alpha > 0$. The $\alpha$-order Riemann-Liouville fractional integral of $u$ is defined by

$$J^\alpha u(t) := \int_0^t g_\alpha(t-s)u(s)ds, \quad t \geq 0.$$

Also, we define $J^0 u(t) = u(t)$. Because of the convolution properties, the integral operators $\{J^\alpha\}_{\alpha \geq 0}$ satisfy the following semigroup law: $J^\alpha J^\beta = J^{\alpha+\beta}$ for all $\alpha, \beta \geq 0$.

**Definition 2.2** Let $\alpha > 0$. The $\alpha$-order Caputo fractional derivative is defined

$$D_t^\alpha u(t) := \int_0^t g_{n-\alpha}(t-s)u^{(n)}(s)ds,$$

where $n = \lceil \alpha \rceil$.

**Definition 2.3** Let $\alpha > 0$. The $\alpha$-order Riemann-Liouville fractional derivative of $u$ is defined

$$D^\alpha u(t) := \frac{d^n}{dt^n} \int_0^t g_{n-\alpha}(t-s)u(s)ds,$$

where $n = \lceil \alpha \rceil$.

It is clear $D_t^m = D^m = \dfrac{d^m}{dt^m}$ if $\alpha = m \in \mathbb{N}$.

Let $\hat{f}$ (or $\mathcal{L}(f)$) denote the Laplace transform of $f$, we have the following facts for the fractional derivatives

$$\widehat{D^\alpha u}(\lambda) = \lambda^\alpha \hat{u}(\lambda) - \sum_{k=0}^{n-1} (g_{n-\alpha} * u)^{(k)}(0)\lambda^{n-1-k} \tag{2.2}$$

and

$$\widehat{D_t^\alpha u}(\lambda) = \lambda^\alpha \hat{u}(\lambda) - \sum_{k=0}^{n-1} u^{(k)}(0)\lambda^{\alpha-1-k}, \tag{2.3}$$

where $n = \lceil \alpha \rceil$ and $\lambda \in \mathbb{C}$. For $\alpha, \beta > 0$ and $z \in \mathbb{C}$, the generalized Mittag-Leffler function is defined by

$$e_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

and its Laplace transform $\mathcal{L}$ satisfies

$$\mathcal{L}(t^{\beta-1} e_{\alpha,\beta}(\rho t^\alpha))(\lambda) = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \rho}, \quad \rho \in \mathbb{C}, \mathrm{Re}\lambda > |\rho|^{1/\alpha}.$$

The *E-modified resolvent set of* $A$, $\rho_E(A)$, is defined by

$$\rho_E(A) := \{\lambda \in \mathbb{C} : (\lambda E - A) : D(A) \cap D(E) \to X$$
$$\text{is invertible and } (\lambda E - A)^{-1} \in \mathcal{B}(X, [D(A) \cap D(E)])\}.$$

The operator $(\lambda E - A)^{-1}$ is called the $E$-resolvent operator of $A$.

A strongly continuous family $\{T(t)\}_{t\geq 0} \subseteq \mathcal{B}(X)$ is said to be of type $(M, \omega)$ or *exponentially bounded* if there exist constants $M > 0$ and $\omega \in \mathbb{R}$, such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. Observe that, without loss of generality, we can assume $\omega > 0$ in the sequel.

**Definition 2.4** Let $A : D(A) \subseteq X \to X$, $E : D(E) \subseteq X \to X$ be closed linear operators defined on a Banach space $X$ satisfying $D(A) \cap D(E) \neq \{0\}$. Let $\alpha, \beta > 0$. We say that the pair $(A, E)$ is the generator of an $(\alpha, \beta)$-resolvent family, if there exist $\omega \geq 0$ and a strongly continuous function $S^E_{\alpha,\beta} : [0, \infty) \to \mathcal{B}(X)$ such that $S^E_{\alpha,\beta}(t)$ is exponentially bounded, $\{\lambda^\alpha : \text{Re}\lambda > \omega\} \subset \rho_E(A)$, and for all $x \in X$,

$$\lambda^{\alpha-\beta} (\lambda^\alpha E - A)^{-1} x = \int_0^\infty e^{-\lambda t} S^E_{\alpha,\beta}(t) x dt, \ \ \text{Re}\lambda > \omega.$$

In this case, $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ is called the $(\alpha, \beta)$-*resolvent family* generated by the pair $(A, E)$.

**Definition 2.5** The resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is to be *compact* if for every $t > 0$, the operator $S^E_{\alpha,\beta}(t)$ is a compact operator.

Next we give some results on the norm continuity and compactness of $S^E_{\alpha,\beta}(t)$ for given $\alpha, \beta > 0$. The proofs of these results can be conducted similarly to [36, Lemma 3.12, Theorem 3.14, Propositions 3.16-3.17, Proposition 7.1], we can also refer to [13] for details.

**Lemma 2.1** Let $\alpha > 0$ and $1 < \beta \leq 2$. Suppose that $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ is the $(\alpha, \beta)$-resolvent family of type $(M, \omega)$ generated by $(A, E)$. Then the function $t \mapsto S^E_{\alpha,\beta}(t)$ is continuous in $\mathcal{B}(X)$ for all $t > 0$.

**Lemma 2.2** Suppose that the pair $(A, E)$ generates an $(\alpha, \beta)$-resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ of type $(M, \omega)$. If $\gamma > 0$, then $(A, E)$ also generates an $(\alpha, \beta + \gamma)$-resolvent family of type $\left(\dfrac{M}{\omega^\gamma}, \omega\right)$.

**Lemma 2.3** Let $\alpha > 0$, $1 < \beta \leq 2$ and $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ be an $(\alpha, \beta)$-resolvent family of type $(M, \omega)$ generated by $(A, E)$. Then the following assertions are equivalent

i) $S^E_{\alpha,\beta}(t)$ is a compact operator for all $t > 0$.

ii) $(\mu E - A)^{-1}$ is a compact operator for all $\mu > \omega^{1/\alpha}$.

**Lemma 2.4** Let $1 < \alpha \leq 2$ and $\{S^E_{\alpha,\alpha}(t)\}_{t\geq 0}$ be an $(\alpha, \alpha)$-resolvent family of type $(M, \omega)$ generated by $(A, E)$. Then the following assertions are equivalent:

i) $S^E_{\alpha,\alpha}(t)$ is a compact operator for all $t > 0$.

ii) $(\mu E - A)^{-1}$ is a compact operator for all $\mu > \omega^{1/\alpha}$.

**Lemma 2.5** Let $1 < \alpha < 2$, and $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ be the $(\alpha,1)$-resolvent family of type $(M,\omega)$ generated by $(A,E)$. Suppose that $S^E_{\alpha,1}(t)$ is continuous in the uniform operator topology for all $t > 0$. Then the following assertions are equivalent

i) $S^E_{\alpha,1}(t)$ is a compact operator for all $t > 0$.

ii) $(\mu E - A)^{-1}$ is a compact operator for all $\mu > \omega^{1/\alpha}$.

**Lemma 2.6** Let $\frac{3}{2} < \alpha < 2$, and $\{S^E_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ be the $(\alpha,\alpha-1)$-resolvent family of type $(M,\omega)$ generated by $(A,E)$. Suppose that $S^E_{\alpha,\alpha-1}(t)$ is continuous in the uniform operator topology for all $t > 0$. Then the following assertions are equivalent

i) $S^E_{\alpha,\alpha-1}(t)$ is a compact operator for all $t > 0$.

ii) $(\mu E - A)^{-1}$ is a compact operator for all $\mu > \omega^{1/\alpha}$.

## 2.2 Basic results on multivalued analysis

In this subsection, we recall some basic definitions and lemmas on multivalued analysis. The following facts can be found in monographs [9, 10, 18, 20, 24, 25, 39], papers [5, 7, 21, 37, 42, 45] and references therein.

Let $(X, \|\cdot\|)$ be a Banach space. Denote $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{compact}\}$, and $\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{convex}\}$. The notation $L^1(I,X) = \{v : I \to X | v \text{ is Bochner integrable}\}$ on a compact interval $I$ of $\mathbb{R}$ with the norm $\|v\|_{L^1} = \int_I \|v(t)\|dt$.

A multivalued map $G : X \to \mathcal{P}(X)$ has convex (closed) values if $G(x)$ is convex (closed) for all $x \in X$. $G$ is bounded on bounded sets if $G(B) = \bigcup_{x\in B} G(x)$ is bounded in $X$ for all $B \in \mathcal{P}_b(X)$, i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$.

The multivalued map $G : X \to \mathcal{P}(X)$ is called upper semicontinuous (u.s.c.) on $X$ if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty, closed subset of $X$, and if for each open set $\mathbb{N}$ of $X$ containing $G(x_0)$, there exists an open neighborhood $\mathbb{N}_0$ of $x_0$ such that $G(\mathbb{N}_0) \subseteq \mathbb{N}$. $G$ is called lower semi-continuous (l.s.c.) if the set $\{x \in X : G(x) \bigcap \mathscr{A}\}$ is open for any open subset $\mathscr{A} \subseteq X$. Also, $G$ is said to be completely continuous if $G(B)$ is relatively compact for every $B \in \mathcal{P}_b(X)$. $G$ has a fixed point if there exists $x \in X$ such that $x \in G(x)$.

If the multivalued map $G$ is completely continuous with nonempty compact values, then $G$ is u.s.c. if and only if $G$ has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

**Definition 2.6** The multivalued map $G : I \times X \to \mathcal{P}(X)$ is said to be $L^1$-Carathéodory if
(i) $t \mapsto G(t,x)$ is measurable for each $x \in X$;
(ii) $u \mapsto G(t,x)$ is u.s.c on $X$ for almost all $t \in I$;
(iii) For each $r > 0$, there exists $\varphi_r \in L^1(I, \mathbb{R}_+)$ such that

$$\|G(t,x)\|_{\mathcal{P}(X)} := \sup\{\|v\| : v \in G(t,x)\} \leq \varphi_r(t),$$

for all $\|x\| \leq r$ and for a.e. $t \in I$.

**Definition 2.7** A space $X$ is said to be contractible if and only if there exists a point $x_0 \in X$ such that $\mathcal{I}_X \sim \psi$ (homotopically equivalent), where $\psi : X \to X$ is defined by $\psi(x) = x_0$ for each $x \in X$, and $\mathcal{I}$ denotes the identity operator.

We remark that a contractible space is arcwise connected.

**Lemma 2.7** Let $X$ be a Banach space. Let $G : I \times X \to \mathcal{P}_{cp,cv}(X)$ be an $L^1$-Carathéodory multivalued map with

$$S_{G,x} = \left\{ f \in L^1(I,X) : f(t) \in G(t, x(t)), \text{ for a.e. } t \in I \right\} \neq \emptyset,$$

and let $\Gamma$ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$, then the operator

$$\Gamma \circ S_G : C(I,X) \to \mathcal{P}_{cp,cv}(C(I,X)), \quad x \mapsto (\Gamma \circ S_G)(x) := \Gamma(S_{G,x})$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

**Lemma 2.8** Let $\{K_n\}_{n\in\mathbb{N}} \subset K \subset X$ be a subset of sequences where $K$ is compact in the separable Banach space $X$. Then

$$\overline{\text{conv}}\left(\limsup_{n\to\infty} K_n\right) = \bigcap_{n_0>0} \overline{\text{conv}}\left(\bigcup_{n\geq n_0} K_n\right).$$

**Lemma 2.9** Let $\Xi$ be a bounded and convex set in Banach space $\mathbb{X}$. $\Upsilon : \Xi \to \mathcal{P}(\Xi)$ is an u.s.c., condensing multivalued map. If for every $x \in \Xi$, $\Upsilon(x)$ is a closed and convex set in $\Xi$, then $\Upsilon$ has a fixed point in $\Xi$.

## 3 Properties of solution sets

In this section, we will prove our main results. We always assume that $X$ is a separable Banach space in the following.

### 3.1 The Caputo case–Eq. (1.1)

Let us list the following assumptions.

(A1) The pair $(A, E)$ generates the $(\alpha, 1)$-resolvent family $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ of type $(M, \omega)$, the operator $(\lambda^\alpha E - A)^{-1}$ is compact for all $\lambda^\alpha \in \rho_E(A)$ with $\lambda > \omega^{\frac{1}{\alpha}}$ and $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ is norm continuous for all $t > 0$.

(A2) $\mathcal{F} : I \times X \to \mathcal{P}_{cp,cv}(X)$ satisfies the following conditions:
(a) For a.e. $t \in I$, $\mathcal{F}(t, \cdot)$ is u.s.c, and for each $x \in X$, $\mathcal{F}(\cdot, x)$ is measurable. And for each $x \in C(I, X)$, $S_{\mathcal{F},x}$ is nonempty;
(b) There exists a function $\phi \in L^1(I, \mathbb{R}_+)$ such that

$$\|\mathcal{F}(t, x)\|_{\mathcal{P}} \leq \phi(t)\|x\|, \forall t \in I, x \in C(I, X).$$

**Definition 3.1** For each $u_0, u_1 \in X$, a function $u \in C(I, X)$ is said to be a mild solution to Eq. (1.1) if there exists $v \in L^1(I, X)$ such that $v(t) \in \mathcal{F}(t, u(t))$ a.e. on $I$ and $u$ verifies the following integral equation

$$u(t) = S^E_{\alpha,1}(t)u_0 + S^E_{\alpha,2}(t)u_1 + \int_0^t S^E_{\alpha,\alpha}(t-s)v(s)ds.$$

**Remark 3.1** (i) By the uniqueness of the Laplace transform, it is clear that the mild solution to Eq. (1.1) can expressed as

$$u(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)v(s)ds.$$

(ii) In view of Lemma 2.5, the condition (A1) implies $\{S^E_{\alpha,1}(t)\}$ is compact for all $t > 0$.

**Theorem 3.1** If assumptions (A1)-(A2) and the following relation

$$\frac{Me^{\omega b}}{\omega^{\alpha-1}}\|\phi\|_{L^1} < 1 \tag{3.1}$$

hold, then Eq. (1.1) admits at least one mild solution on $I$.

**Proof:** Consider the operator $N : C(I, X) \to \mathcal{P}(C(I, X))$ defined by

$$N(u) = \Big\{h \in C(I, X) : h(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)v(s)ds, t \in I\Big\},$$

where $v \in S_{\mathcal{F},u}$. Clearly, the fixed points of $N$ are mild solutions to Eq. (1.1). We shall show that $N$ satisfies all the hypothesis of Lemma 2.9. The proof will be given in several steps.
**Step 1.** There exists a positive number $r$ such that $N(B_r) \subseteq B_r$, where $B_r = \{u \in$

$C(I, X) : \|u\|_\infty \leq r\}$. If it is not true, then for each positive number $r$, there exists a function $u^r$ such that $h^r \in N(u^r)$ but $\|h^r(t)\| > r$ for some $t \in I$,

$$h^r(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)v^r(s)ds,$$

where $v^r \in S_{\mathcal{F},u^r}$. However, on the other hand, we have

$$\begin{aligned} r &< \left\| S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)v^r(s)ds \right\| \\ &\leq Me^{\omega t}\|u_0\| + \frac{M}{\omega}e^{\omega t}\|u_1\| + \frac{M}{\omega^{\alpha-1}}\int_0^t e^{\omega(t-s)}\phi(s)\|u\|ds \\ &\leq Me^{\omega b}\|u_0\| + \frac{M}{\omega}e^{\omega b}\|u_1\| + \frac{Mre^{\omega b}}{\omega^{\alpha-1}}\int_0^t e^{-\omega s}\phi(s)ds \\ &\leq Me^{\omega b}\|u_0\| + \frac{M}{\omega}e^{\omega b}\|u_1\| + \frac{Mre^{\omega b}}{\omega^{\alpha-1}}\|\phi\|_{L^1}. \end{aligned}$$

Dividing both sides by $r$ and and taking the lower limit as $r \to \infty$, we obtain

$$1 \leq \frac{Me^{\omega b}}{\omega^{\alpha-1}}\|\phi\|_{L^1},$$

which contradicts the relation (3.1).

**Step 2.** $N(u)$ is convex for each $u \in C(I, X)$.

Indeed, if $h_1, h_2 \in N(u)$, then there exist $v_1, v_2 \in S_{\mathcal{F},u}$ such that for each $t \in I$, we have

$$h_i(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)v_i(s)ds, i = 1, 2.$$

Let $\theta \in (0, 1)$. Then for each $t \in I$, we have

$$\begin{aligned} (\theta h_1 + (1-\theta)h_2)(t) &= S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 \\ &\quad + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)[\theta v_1(s) + (1-\theta)v_2(s)]ds. \end{aligned}$$

Since $\mathcal{F}$ has convex values and thus $S_{\mathcal{F},u}$ is convex, $\theta h_1 + (1-\theta)h_2 \in N(u)$.

**Step 3.** $N(u)$ is closed for each $u \in C(I, X)$.

Let$\{h_n\}_{n\geq 0} \in N(u)$ such that $h_n \to h$ in $C(I, X)$. Then $h \in C(I, X)$ and there exist $\{v_n\} \in S_{\mathcal{F},u}$ such that for each $t \in I$

$$h_n(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)v_n(s)ds.$$

Due to the fact that $\mathcal{F}$ has compact values, we may pass to a subsequence if necessary to get that $v_n$ converges to $v$ in $L^1(I, X)$ and hence $v \in S_{\mathcal{F},u}$. Then for each $t \in I$,

$$h_n(t) \to h(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)v(s)ds.$$

Thus, $h \in N(u)$.
**Step 4.** $N$ is u.s.c.
(i) $N(B_r)$ is obviously bounded.
(ii) $N(B_r)$ is equicontinuous.

Indeed, let $u \in B_r$, $h \in N(u)$ and take $t_1, t_2 \in I$ with $t_2 < t_1$. Then there exists a selection $v \in S_{\mathcal{F},u}$ such that

$$h(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)v(s)ds, t \in I.$$

Then

$$\begin{aligned} \|h(t_1) - h(t_2)\| &\le \| \left(S^E_{\alpha,1}(t_1) - S^E_{\alpha,1}(t_2)\right) u_0\| + \| \left((g_1 * S^E_{\alpha,1})(t_1) - (g_1 * S^E_{\alpha,1})(t_2)\right) u_1\| \\ &+ \int_{t_2}^{t_1} \|(g_{\alpha-1} * S^E_{\alpha,1})(t_1 - s)v(s)\|ds \\ &+ \int_0^{t_2} \| \left((g_{\alpha-1} * S^E_{\alpha,1})(t_1 - s) - (g_{\alpha-1} * S^E_{\alpha,1})(t_2 - s)\right) v(s)\|ds \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

For the term $I_1$, we have

$$I_1 \le \|(S^E_{\alpha,1}(t_1) - S^E_{\alpha,1}(t_2))\| \, \|u_0\|.$$

By the norm continuity of $S^E_{\alpha,1}(t)$ in assumption (A1), we get $\lim_{t_1 \to t_2} I_1 = 0$.
For the term $I_2$, we have $(g_1 * S^E_{\alpha,1})(t) = S^E_{\alpha,2}(t)$ for all $t \ge 0$ due to the uniqueness of the Laplace transform and Lemma 2.2. Meanwhile, the Lemma 2.1 implies that $(g_1 * S^E_{\alpha,1})(t)$ is continuous in $\mathcal{B}(X)$. Hence

$$I_2 \le \|(g_1 * S_{\alpha,1})(t_1) - (g_1 * S_{\alpha,1})(t_2)\| \, \|u_0\| \to 0, \text{ as } t_1 \to t_2.$$

For the term $I_3$, as $t_1 \to t_2$, we have

$$I_3 \le \frac{Me^{\omega b}}{\omega^{\alpha-1}} \int_{t_2}^{t_1} e^{-\omega s}\phi(s)\|u(s)\|ds \le \frac{Mre^{\omega b}}{\omega^{\alpha-1}} \int_{t_2}^{t_1} \phi(s)ds \to 0.$$

Finally for the term $I_4$, we have

$$\begin{aligned} I_4 &\le \int_0^{t_2} \left\| \left[(g_{\alpha-1} * S^E_{\alpha,1})(t_1 - s) - (g_{\alpha-1} * S^E_{\alpha,1})(t_2 - s)\right] \right\| \|v(s)\|ds \\ &\le \int_0^{t_2} \left\| \left[(g_{\alpha-1} * S^E_{\alpha,1})(t_1 - s) - (g_{\alpha-1} * S^E_{\alpha,1})(t_2 - s)\right] \right\| \phi(s)\|u(s)\|ds \\ &\le r\int_0^{t_2} \left\| \left[(g_{\alpha-1} * S^E_{\alpha,1})(t_1 - s) - (g_{\alpha-1} * S^E_{\alpha,1})(t_2 - s)\right] \right\| \phi(s)ds. \end{aligned}$$

Now taking into account that

$$\|(g_{\alpha-1} * S^E_{\alpha,1})(t_1 - \cdot) - (g_{\alpha-1} * S^E_{\alpha,1})(t_2 - \cdot)\|\phi(s) \le 2\frac{Me^{\omega b}}{\omega^{\alpha-1}}\phi(s) \in L^1(I, \mathbb{R}_+),$$

$(g_{\alpha-1} * S^E_{\alpha,1})(t) = S^E_{\alpha,\alpha}(t)$ for all $t \geq 0$ (see Lemma 2.2) and $S_{\alpha,\alpha}(t)$ is norm continuous (see Lemma 2.1), we have $(g_{\alpha-1} * S^E_{\alpha,1})(t_1 - s) - (g_{\alpha-1} * S^E_{\alpha,1})(t_2 - s) \to 0$ in $\mathcal{B}(X)$ as $t_1 \to t_2$. By the Lebesgue's dominated convergence theorem, we conclude $\lim_{t_1 \to t_2} I_4 = 0$.

(iii) $H(t) = \{h(t) : h(t) \in N(B_r)\}$ is relatively compact in $X$.

Clearly, $H(0)$ is relatively compact in $X$. For $u \in B_r$ and $v \in S_{\mathcal{F},u}$, we define

$$N_2(u) = \left\{ m \in C(I, X) : m(t) = \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)v(s)ds, t \in I \right\},$$

Now, let $0 < t \leq b$ and $\varepsilon$ be a real number satisfying $0 < \varepsilon < t$, we further introduce

$$m_\varepsilon(t) = \int_0^{t-\varepsilon} (g_{\alpha-1} * S^E_{\alpha,1})(t-s)v(s)ds, t \in I.$$

The assumption (A1) and Lemma 2.4 imply the compactness of $(g_{\alpha-1} * S^E_{\alpha,1})(t) = S^E_{\alpha,\alpha}(t)$ for all $t > 0$. Therefore the set $\mathcal{K}_\varepsilon := \{(g_{\alpha-1} * S_{\alpha,1})(t-s)v(s) : 0 \leq s \leq t-\varepsilon\}$ is compact for all $\varepsilon > 0$. Then $\overline{\mathrm{conv}(\mathcal{K}_\varepsilon)}$ is also a compact set by Mazur Theorem. In view of Mean-Value Theorem for the Bochner integrals, we have $m_\varepsilon(t) \in t\overline{\mathrm{conv}(\mathcal{K}_\varepsilon)}$ for all $t \in I$. Thus the set $M_\varepsilon(t) = \{m_\varepsilon(t) : m_\varepsilon(t) \in N_2(B_r)\}$ is relatively compact in $X$ for every $\varepsilon$, $0 < \varepsilon < t$. Moreover, for $m \in N(B_r)$,

$$\begin{aligned} \|m(t) - m_\varepsilon(t)\| &\leq \left\| \int_{t-\varepsilon}^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)v(s)ds \right\| \\ &\leq \frac{Mre^{\omega b}}{\omega^{\alpha-1}} \int_{t-\varepsilon}^t e^{-\omega s}\phi(s)ds. \end{aligned}$$

Since $s \mapsto e^{-\omega s}\phi(s)$ belong to $L^1([t-\varepsilon, t], \mathbb{R}_+)$, we conclude by the Lebesgue Dominated Convergence Theorem that $\lim_{\varepsilon \to 0} \|m(t) - m_\varepsilon(t)\| = 0$. Thus, let $\varepsilon \to 0$, we see that there are relatively compact sets arbitrarily $M_\varepsilon(t)$ close to the set $M(t) = \{m(t) : m(t) \in N_2(B_r)\}$. Hence, the set $M(t) = \{m(t) : m(t) \in N_2(B_r)\}$ is relatively compact in $X$. The compactness of $S^E_{\alpha,1}(t)$ and $(g_1 * S^E_{\alpha,1})(t) = S^E_{\alpha,2}(t)$ (see Lemma 2.5 and Lemma 2.3) imply that $H(t) = \{h(t) : h(t) \in N(B_r)\}$ is relatively compact in $X$. As a consequence of the above steps and the Arzela-Ascoli theorem, we can deduce that $N$ is completely continuous.

(iv) $N$ has a closed graph.

Let $u^n \to u^*, h^n \in N(u^n)$ and $h^n \to h^*$. We shall show that $h^* \in N(u^*)$. Now $h^n \in N(u^n)$ implies that there exists $v^n \in S_{\mathcal{F},u^n}$ such that for each $t \in I$

$$h^n(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)v^n(s)ds.$$

We need to prove that there exists $v^* \in S_{\mathcal{F},u^*}$ such that for each $t \in I$

$$h^*(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)v^*(s)ds.$$

Consider the linear continuous operator defined by

$$\begin{aligned} \Upsilon : L^1(I,X) &\to C(I,X), \\ v \mapsto (\Upsilon v)(t) &= \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)v(s)ds. \end{aligned}$$

From Lemma 2.7 it follows that $\Upsilon \circ S_{\mathcal{F}}$ is a closed graph operator. Moreover, we have

$$h^n(t) - S^E_{\alpha,1}(t)u_0 - (g_1 * S^E_{\alpha,1})(t)u_1 \in \Upsilon(S_{\mathcal{F},u^n}).$$

Since $u^n \to u^*$, it again follows from Lemma 2.7 that

$$h^*(t) - S^E_{\alpha,1}(t)u_0 - (g_1 * S^E_{\alpha,1})(t)u_1 \in \Upsilon(S_{\mathcal{F},u^*}).$$

Thus, there exists $v^* \in S_{\mathcal{F},u^*}$ such that

$$h^*(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)v^*(s)ds.$$

Therefore, $N$ is completely continuous and u.s.c. By the fixed point theorem Lemma 2.9, there exists a fixed point $u(\cdot)$ for $N$ on $B_r$ . Thus, Eq. (1.1) admits a mild solution.

For $u_0, u_1 \in X$, define the following set

$$\mathcal{S}(u_0,u_1) = \{u \in C(I,X) : u \text{ is a mild solution of Eq. (1.1)}\}.$$

**Theorem 3.2** Suppose $X$ is a reflexive Banach space. If assumptions (A1)-(A2) and the inequality (3.1) are satisfied, then the set $\mathcal{S}(u_0,u_1)$ is compact in $C(I,X)$.

**Proof:** In view of Theorem 3.1, the set $\mathcal{S}(u_0,u_1) \neq \emptyset$, and there exists $r > 0$ such that for each $u \in \mathcal{S}(u_0,u_1), \|u\|_\infty \leq r$. Owing to $N : C(I,X) \to \mathcal{P}(C(I,X))$ is completely continuous, the set $N(\mathcal{S}(u_0,u_1))$ is relatively compact. Considering the definition of $N$, we have $\mathcal{S}(u_0,u_1) \subset \overline{N(\mathcal{S}(u_0,u_1))}$. It remains to show that $\mathcal{S}(u_0,u_1)$ is closed.

Let $u^n \in \mathcal{S}(u_0,u_1)$ satisfy $\lim_{n\to\infty} u^n = u$. For each $n$, there exists $v^n \in S_{\mathcal{F},u^n}$ such that

$$u^n(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)v^n(s)ds.$$

Since $\mathcal{F}$ is $L^1$-Carathéodory with closed values, its selection set $S_{\mathcal{F},u}$ is closed and nonempty. Considering the space $X$ is reflexible, this selection set $S_{\mathcal{F},u}$ is weakly relatively compact due to [34, Theorem 6.4.6, Corollary 6.4.11], and hence sequentially weakly relatively compact by Eberlein's theorem (see [47]). Owing to (A2) and (3.1), $\|v^n(t)\| \leq \phi(t)r$. As a consequence, there exists a subsequence, still denoted by $\{v^n\}$, which converges weakly to some limit $v(\cdot) \in L^1$. According to Mazur Theorem, there exists a double sequence $\{c_{n,k}\}_{n,k\in\mathbb{N}}$ such that $\forall n \in \mathbb{N}$, $\exists k_0(n) \in \mathbb{N} : c_{n,k} = 0, \forall k \geq k_0(n)$, $\sum_{k=n}^{\infty} c_{n,k} = 1, \forall n \in \mathbb{N}$, and the sequence of convex combinations $\tilde{v}^n(\cdot) = \sum_{k=n}^{\infty} c_{n,k} v^k(\cdot)$ converges strongly to $v(\cdot) \in L^1$. By the facts that $\mathcal{F}$ takes convex values and Lemma 2.8, we get for a.e. $t \in I$

$$\begin{aligned} v(t) &\in \bigcap_{n\geq 1} \overline{\{\tilde{v}^k(t), k \geq n\}} \subset \bigcap_{n\geq 1} \overline{\text{conv}} \left\{ v^k(t), k \geq n \right\} \\ &\subset \bigcap_{n\geq 1} \overline{\text{conv}} \left\{ \bigcup_{k\geq n} \mathcal{F}\left(t, u^k(t)\right) \right\} = \overline{\text{conv}} \left( \limsup_{k\to\infty} \mathcal{F}\left(t, u^k(t)\right) \right). \end{aligned} \tag{3.2}$$

Note that $\mathcal{F}$ is u.s.c. with compact values, by [20, Lemma 6.48, Chapter 6], we obtain for a.e. $t \in I$

$$\limsup_{n\to\infty} \mathcal{F}\left(t, u^n(t)\right) = \mathcal{F}(t, u(t)).$$

This together with (3.2) implies that $v(t) \in \overline{\text{conv}} \mathcal{F}(t, u(t))$. Since $\mathcal{F}$ has closed convex values, we have $v(t) \in \mathcal{F}(t, u(t))$. Let

$$\tilde{u}(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)v(s)ds, \ t \in I.$$

From the properties of resolvent operators, we have

$$\begin{aligned} \|u^n(t) - \tilde{u}(t)\| &\leq \int_0^t \left\|(g_{\alpha-1} * S^E_{\alpha,1})(t-s)\right\| \|v^n(s) - v(s)\| ds \\ &\leq \frac{Me^{\omega b}}{\omega^{\alpha-1}} \int_0^t e^{-\omega s} \|v^n(s) - v(s)\| ds \\ &\leq \frac{Me^{\omega b}}{\omega^{\alpha-1}} \int_0^b \|v^n(s) - v(s)\| ds. \end{aligned}$$

Thus, by the Lebesgue dominated convergence theorem we obtain

$$\|u^n - \tilde{u}\|_\infty \leq \frac{Me^{\omega b}}{\omega^{\alpha-1}} \int_0^b \|v^n(s) - v(s)\| ds \to 0, \ \text{as } n \to \infty.$$

Therefore, $u(t) = \tilde{u}(t), t \in I$, which proves $\mathcal{S}(\cdot,\cdot) \in \mathcal{P}_{cp}(C(I, X))$.

**Theorem 3.3** Let $X$ be a reflexive Banach space. Suppose that conditions (A1)-(A2) and (3.1) are satisfied. Let $\mathcal{F} : I \times X \to \mathcal{P}_{cp,cv}(X)$ be an mLL-selectionable multivalued map. Then for each $u_0, u_1 \in X$, the solution set $\mathcal{S}(u_0, u_1)$ is contractible, and thus it is arcwise connected.

**Proof:** Let $f \subset \mathcal{F}$ be a measurable, locally Lipschitz selection and consider the following single valued equation

$$\begin{cases} D^\alpha_t (Eu)(t) = Au(t) + f(t, u(t)), \\ Eu(0) = u_0, \quad (Eu)'(0) = u_1. \end{cases} \tag{3.3}$$

Denote $\overline{u}(u_0, u_1)$ is the unique mild solution of Eq. (3.3). Taking into account conditions (A2) and (3.1), this solution exists on the whole interval $I$. Now we define the homotopy $h : \mathcal{S}(u_0, u_1) \times [0, 1] \to \mathcal{S}(u_0, u_1)$ as following

$$h(u, \tau)(t) = \begin{cases} u(t), & \text{for } 0 \leq t \leq \tau b, \\ \overline{u}(t), & \text{for } \tau b < t \leq b. \end{cases}$$

Particularly,

$$h(u, \tau) = \begin{cases} u, & \text{for } \tau = 1, \\ \overline{u}, & \text{for } \tau = 0. \end{cases}$$

Let $\{(u^n, \tau^n)\} \subset \mathcal{S}(u_0, u_1) \times [0, 1]$ be such that $(u^n, \tau^n) \to (u, \tau)$ as $n \to \infty$, we shall show $h$ is a continuous homotopy, i.e. $h(u^n, \tau^n) \to h(u, \tau)$ as $n \to \infty$ with

$$h(u^n, \tau^n)(t) = \begin{cases} u^n(t), & \text{for } 0 \leq t \leq \tau^n b, \\ \overline{u}(t), & \text{for } \tau^b < t \leq b. \end{cases}$$

Next we divide the proof into different cases:
**Case I.** If $\lim\limits_{n\to\infty} \tau^n = 0$, then from definition of $h$ we have

$$h(u, 0)(t) = \overline{u}(t), \ t \in I.$$

Thus,

$$\|h(u^n, \tau^n) - h(u, \tau)\| \leq \|u^n - \overline{u}\| = \sup_{t\in I}\{\|u^n(t) - \overline{u}(t)\|\},$$

which approaches 0 as $n \to \infty$. The case for $\lim_{n\to\infty} \tau^n = 1$ is treated similarly.
**Case II.** If $\lim\limits_{n\to\infty} \tau^n \neq 0$ and $\lim\limits_{n\to\infty} \tau^n = \tau < 1$, then the following appears:
(1) If $t \in [0, \tau b]$, then, from the fact $u^n \in \mathcal{S}(u_0, u_1)$ there exists $v^n \in S_{\mathcal{F},u^n}$ such that for $t \in [0, \tau b]$

$$u^n(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t - s)v^n(s)ds.$$

Arguing as in the proof of Theorem 3.2, we can obtain that there exists a subsequence, still denoted by $\{v^n\}$, which converges weakly to some limit $v(\cdot) \in L^1$, and $v(t) \in \overline{\text{conv}}\mathcal{F}(t, u(t))$. Since $\mathcal{F}$ has closed convex values, we have $v(t) \in \mathcal{F}(t, u(t))$. By the Lebesgue dominated convergence theorem, we obtain that for $t \in [0, \tau b]$

$$u(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t - s)v(s)ds.$$

(2) If $t \in (\tau b, b]$, then

$$h(u^n, \tau^n)(t) = h(u, \tau)(t) = \overline{u}(t).$$

Hence, $\|h(u^n, \tau^n)(t) - h(u, \tau)(t)\| \to 0$ as $n \to \infty$.

As a consequence of the above cases, we see that $h$ is continuous, and thus $\mathcal{S}(u_0, u_1)$ is contractible to the point $\overline{u}(u_0, u_1)$.

### 3.2 The Riemann-Liouville case–Eq. (1.2)

For Eq. (1.2), we need the following hypotheses.

(H1) Let $\frac{3}{2} < \alpha < 2$, and the pair $(A, E)$ generates the $(\alpha, \alpha-1)$-resolvent family $\{S^E_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ of type $(M, \omega)$, the operator $(\lambda^\alpha E - A)^{-1}$ is compact for all $\lambda^\alpha \in \rho_E(A)$ with $\lambda > \omega^{\frac{1}{\alpha}}$ and $\{S^E_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ is norm continuous for all $t > 0$.

**Definition 3.2** For each $u_0, u_1 \in X$, a function $u \in C(I, X)$ is said to be a mild solution to Eq. (1.2) if there exists $v \in L^1(I, X)$ such that $v(t) \in \mathcal{F}(t, u(t))$ a.e. on $I$ and $u$ verifies the following integral equation

$$u(t) = S^E_{\alpha,\alpha-1}(t)u_0 + S^E_{\alpha,\alpha}(t)u_1 + \int_0^t S^E_{\alpha,\alpha}(t-s)v(s)ds.$$

**Remark 3.2** (i) By the uniqueness of the Laplace transform, it is clear that the mild solution to Eq. (1.2) can expressed as

$$u(t) = S^E_{\alpha,\alpha-1}(t)u_0 + (g_1 * S^E_{\alpha,\alpha-1})(t)u_1 + \int_0^t (g_1 * S^E_{\alpha,\alpha-1})(t-s)v(s)ds.$$

(ii) In view of Lemma 2.6, the condition (H1) implies $\{S^E_{\alpha,\alpha-1}(t)\}$ is compact for all $t > 0$.

**Theorem 3.4** If assumptions (H1), (A2) and the following relation

$$\frac{Me^{\omega b}}{\omega}\|\phi\|_{L^1} < 1 \tag{3.4}$$

hold, then Eq. (1.2) admits at least one mild solution on $I$.

**Proof:** We define the operator $N : C(I, X) \to \mathcal{P}(C(I, X))$ as

$$N(u) = \Big\{h \in C(I, X) : h(t) = S^E_{\alpha,\alpha-1}(t)u_0 + (g_1 * S^E_{\alpha,\alpha-1})(t)u_1 + \int_0^t (g_1 * S^E_{\alpha,\alpha-1})(t-s)v(s)ds,\ v \in S_{\mathcal{F},u},\ t \in I\Big\}.$$

The remainder can be conducted similarly to Theorem 3.1.

We can conclude that there exists a positive number $r$ such that $N(B_r) \subseteq B_r$, and $N(u)$ is convex, closed for each $u \in C(I, X)$. Because $S^E_{\alpha,\alpha-1}(t)$ is norm continuous for all $t > 0$ (see (H1)) and $t \mapsto (g_1 * S^E_{\alpha,\alpha-1})(t)$ is also norm continuous by Lemma 2.1, we can similarly prove $N(B_r)$ is equicontinuous. The Lemma 2.3 implies the compactness of $(g_1 * S^E_{\alpha,\alpha-1})(t) = S^E_{\alpha,\alpha}(t)$ for all $t > 0$ and therefore the set $\{\int_0^t (g_1 * S^E_{\alpha,\alpha-1})(t-s)v(s)ds : v \in S_{\mathcal{F},u}, u \in B_r\}$ is relatively compact for all $t \in I$ (as in the proof of Theorem 3.1). On the other hand, from (H1) and Lemma 2.6, we get the compactness of $S^E_{\alpha,\alpha-1}(t)$ for all $t > 0$. Thus, we show the set $H(t) = \{h(t) : h(t) \in N(B_r)\}$ is relatively compact in $X$. By the Arzela-Ascoli theorem, we can deduce that $N$ is completely continuous. We can also show $N$ has a closed graph (see the proof of Theorem 3.1). In the final, we see $N$

is u.s.c. and satisfies Lemma 2.9, there exists a fixed point $u(\cdot)$ for $N$ on $B_r$ . Thus, Eq. (1.2) admits a mild solution.

For $u_0, u_1 \in X$, define the following set

$$\mathfrak{S}(u_0, u_1) = \{u \in C(I, X) : u \text{ is a mild solution of Eq. (1.2)}\}.$$

The following results involved in Eq. (1.2) can be proved the same as Theorems 3.2-3.3.

**Theorem 3.5** Suppose $X$ is a reflexive Banach space. If assumptions (H1), (A2) and the inequality (3.4) are satisfied, then the set $\mathfrak{S}(u_0, u_1)$ is compact in $C(I, X)$.

**Theorem 3.6** Let $X$ be a reflexive Banach space. Suppose that conditions (H1), (A2) and (3.4) are satisfied. Let $\mathcal{F} : I \times X \to \mathcal{P}_{cp,cv}(X)$ be an mLL-selectionable multivalued map. Then for each $u_0, u_1 \in X$, the solution set $\mathfrak{S}(u_0, u_1)$ is contractible, and thus it is arcwise connected.

## 4 Applications

As applications of the above results, we consider the following semilinear equation

$$\begin{cases} D_t^\alpha (Eu)(t) \in Au(t) + J^{2-\alpha}\mathcal{F}(t, u(t)), t \in I, \\ Eu(0) = u_0, (Eu)'(0) = u_1, \end{cases} \tag{4.1}$$

where $u_0, u_1 \in X$, $X$ is a separable Banach space, $1 < \alpha < 2$, $J^{2-\alpha}$ denotes the Riemann-Liouville fractional integral operator. Assume the pair $(A, E)$ generates the $(\alpha, 1)$-resolvent family $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$. The mild solution to Eq. (4.1) is given by

$$u(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_1 * S^E_{\alpha,1})(t - s)v(s)ds, v \in S_{\mathcal{F},u}, t \in I.$$

In view of Theorems 3.1-3.3, we have the following result for Eq. (4.1).

**Lemma 4.1** Let $X$ be a reflexive Banach space. Suppose that conditions (A1)-(A2) and relation and (3.4) hold. Let $\mathcal{F} : I \times X \to \mathcal{P}_{cp,cv}(X)$ be an mLL-selectionable multivalued map. Then for each $u_0, u_1 \in X$, the solution set $\mathcal{S}(u_0, u_1)$ of Eq. (4.1) is nonempty, compact, contractible, and thus arcwise connected.

On the other hand, for the semilinear equation in Riemann-Liouville fractional derivative

$$\begin{cases} D^\alpha (Eu)(t) \in Au(t) + J^{2-\alpha}\mathcal{F}(t, u(t)), t \in I, \\ (E(g_{2-\alpha} * u))(0) = u_0, (E(g_{2-\alpha} * u))'(0) = u_1, \end{cases} \tag{4.2}$$

where $u_0, u_1 \in X$, $X$ is a separable Banach space, $\frac{3}{2} < \alpha < 2$. Let the pair $(A, E)$ generate the $(\alpha, \alpha - 1)$-resolvent family $\{S^E_{\alpha,\alpha-1}(t)\}_{t\geq 0}$, then the mild solution to Eq. (4.2) can be written as

$$u(t) = S^E_{\alpha,\alpha-1}(t)u_0 + (g_1 * S^E_{\alpha,\alpha-1})(t)u_1 + \int_0^t (g_{3-\alpha} * S^E_{\alpha,\alpha-1})(t - s)v(s)ds, v \in S_{\mathcal{F},u}, t \in I.$$

Based upon Theorems 3.4-3.6, we can obtain the following result for Eq. (4.2).

**Lemma 4.2** Let $X$ be a reflexive Banach space. Suppose that conditions (H1), (A2) hold and $\frac{Me^{\omega b}}{\omega^{3-\alpha}}\|\phi\|_{L^1} < 1$. Let $\mathcal{F} : I \times X \to \mathcal{P}_{cp,cv}(X)$ be an mLL-selectionable multivalued map. Then for each $u_0, u_1 \in X$, the solution set $\mathfrak{S}(u_0, u_1)$ of Eq. (4.2) is nonempty, compact, contractible, and thus arcwise connected.

**Example 4.1** In the following, we end this paper with a simple example. Take $X = L^2[0,\pi], (t,x) \in [0,1] \times [0,\pi]$, consider the following problem

$$\begin{cases} D_t^\alpha [u(t,x) - u_{xx}(t,x)] = -u_{xx}(t,x) + f(t,u(t,x)), \\ u(t,0) = u(t,\pi) = 0, \quad t \in [0,1], \\ u(0,x) = u_0(x), \quad x \in [0,\pi], \\ u_t(0,x) = u_1(x), \quad x \in [0,\pi], \end{cases} \tag{4.3}$$

where $1 < \alpha < 2$, $f(t,u(t,x)) := \dfrac{e^{-t}u(t,x)}{(C_\alpha \pi^2 + t)(1 + u(t,x))}$, $C_\alpha > 0$ will be defined later.

Define the the operators $A : D(A) \subset X \to X$ and $E : D(E) \subset X \to X$ respectively by

$$\begin{cases} Au = -\dfrac{\partial^2 u}{\partial x^2} = -u_{xx}, \\ Eu = u - u_{xx}, \end{cases}$$

with the domain $D(E) = D(A) := \{u \in X : u \in H^2([0,\pi]), u(t,0) = u(t,\pi) = 0\}$. It is known that $A$ has discrete spectrum with eigenvalues of the form $n^2, n \in \mathbb{N}$, and the corresponding normalized eigenvectors are given by $u_n(s) := (\frac{2}{\pi})^{\frac{1}{2}} \sin(ns)$. Moreover, $\{u_n : n \in \mathbb{N}\}$ is an orthonormal basis for $X$, and thus $A$ and $E$ can be written as (see [31])

$$\begin{cases} Au = \displaystyle\sum_{n=1}^{\infty} n^2 \langle u, u_n \rangle u_n, u \in D(A), \\ Eu = \displaystyle\sum_{n=1}^{\infty} (1 + n^2) \langle u, u_n \rangle u_n, u \in D(E). \end{cases}$$

Thus, for any $u \in X$ and $\beta = 1$, we have

$$\begin{aligned} \lambda^{\alpha-1}(\lambda^\alpha E - A)^{-1}u &= \sum_{n=1}^{\infty} \frac{\lambda^{\alpha-1}}{\lambda^\alpha(1+n^2) + n^2} \langle u, u_n \rangle u_n \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2+1} \frac{\lambda^{\alpha-1}}{\lambda^\alpha + \frac{n^2}{n^2+1}} \langle u, u_n \rangle u_n \\ &= \int_0^\infty e^{-\lambda t} \sum_{n=1}^{\infty} \frac{1}{n^2+1} h_{\alpha,1}^n(t) dt \langle u, u_n \rangle u_n, \end{aligned} \tag{4.4}$$

where the function $h_{\alpha,1}^n(t) := e_{\alpha,1}\left(-\frac{n^2}{n^2+1} t^\alpha\right)$ satisfying $\hat{h}_{\alpha,1}^n(\lambda) = \dfrac{\lambda^{\alpha-1}}{\lambda^\alpha + \frac{n^2}{n^2+1}}$ for all $\lambda > 0$.

Therefore, the $(\alpha,1)$-resolvent family $\{S_{\alpha,1}^E(t)\}_{t \geq 0}$ generated by the pair $(A, E)$ can be given by

$$S^E_{\alpha,1}(t)u = \sum_{n=1}^{\infty} \frac{1}{n^2+1} h^n_{\alpha,1}(t)\langle u, u_n\rangle u_n, \text{ for all } u \in X.$$

From the continuity of $e_{\alpha,1}(\cdot)$ and the convergence of series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$, we can conclude that $S^E_{\alpha,1}(t)$ is norm continuous. From (4.4) and the fact $\lim_{n\to\infty} \frac{1}{n^2+1} \frac{\lambda^{\alpha-1}}{\lambda^\alpha + \frac{n^2}{n^2+1}} = 0$ for all $\lambda > 0$, we can also deduce that $(\lambda^\alpha E - A)^{-1}$ is a compact operator on the Hilbert space $X$. Furthermore, for each $u \in X$ we have (see[17])

$$\begin{aligned} \|S^E_{\alpha,1}(t)u\| &\leq \sum_{n=1}^{\infty} \frac{1}{n^2+1} |h^n_{\alpha,1}(t)| \|u\| \leq C_\alpha \sum_{n=1}^{\infty} \frac{1}{n^2+1} \frac{1}{1+\frac{n^2}{n^2+1}t^\alpha} \|u\| \\ &\leq C_\alpha \sum_{n=1}^{\infty} \frac{1}{n^2}\|u\| = C_\alpha \frac{\pi^2}{6}\|u\|, \end{aligned}$$

where $C_\alpha$ is a positive constant given in [17, Theorem 1]. Therefore, $S^E_{\alpha,1}(t)$ is of type $\left(C_\alpha \frac{\pi^2}{6}, 1\right)$, i.e. $M = C_\alpha \frac{\pi^2}{6}$ and $\omega = 1$.

Let $\mathcal{F} =: \{f\}$, $I := [0,1]$. We note that Eq. (4.3) can be rewritten in the abstract form (1.1). We also observe that in this case $\phi(t) := \frac{e^{-t}}{C_\alpha \pi^2 + t}$, $\|\phi\|_{L^1} \leq \frac{1}{C_\alpha \pi^2}$, $b = \omega = 1$ and thus $\frac{Me^{\omega b}}{\omega^{\alpha-1}} \|\phi\|_{L^1} := \frac{e}{6} < 1$. According to Theorems 3.1-3.3, the solution set of Eq. (4.3) is nonempty, compact, contractible and arcwise connected.

## 5 Conclusions.

In this paper, we treats properties of solution sets for Sobolev type fractional differential inclusions Eq. (1.1) and Eq. (1.2) with the order $1 < \alpha < 2$ respectively. We show that the solution sets are nonempty, compact, contractible and thus arcwise connected under some suitable conditions. Our main results are directly established through properties of resolvent operators generated by the operator pair $(A, E)$ instead of subordination formulas. In particular, the existence or compactness of an operator $E^{-1}$ is not necessarily needed here. Some applications are also presented to illustrate obtained results.
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