WELL–POSEDNESS, REGULARITY, AND ASYMPTOTIC BEHAVIOR OF THE CONTINUOUS AND DISCRETE SOLUTIONS OF LINEAR FRACTIONAL INTEGRO–DIFFERENTIAL EQUATIONS WITH ORDER VARYING IN TIME

EDUARDO CUESTA AND RODRIGO PONCE

Abstract. The well–posedness of abstract time evolution fractional integro–differential equations of variable order $u(t) = u_0 + \partial^{-\alpha(t)} Au(t) + f(t)$, as well as the asymptotic behavior as $t \to +\infty$, and the regularity of its solutions are studied. Moreover, the asymptotic behavior of the discrete solution provided by a numerical method based on convolution quadratures, inherited from the behavior of the continuous solution, is also presented. Here A plays the role of a linear operator of sectorial type.

Several definitions proposed in the literature for the fractional integral of variable order are discussed, and the differences between the solutions provided for each of them are numerically illustrated. In particular, the definition we have finally chosen for this work is the one based on the Laplace transform, and the reasons for this choice are also discussed.

1. Introduction

In the last decades fractional calculus has become a very active field of research in the framework of evolution phenomena due mainly to the fact that many of these phenomena, classically described by means of evolution equations involving integer order derivatives and/or integrals, have turned out to be better suited if non integer integrals/derivatives are introduced (see [17, 22, 25, 27, 35, 48] and references therein). The prototype linear fractional equation with constant order for evolution phenomena can be written in integral form as

$$u(x,t) = u_0(x) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (Au)(x,s)\,\mathrm{d}s + f(x,t), \qquad t > 0, \quad x \in \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^m$ is the spatial domain, A is a closed linear operator acting on $(Au)(x,t)$ in a convenient functional set, and the integral term stands for the fractional integral in the sense of Riemann–Liouville with order of integration (or viscosity parameter) $\alpha \in \mathbb{R}$, $1 < \alpha < 2$.

More recently fractional models involving non constant order of integration (viscosity function, $\alpha = \alpha(t)$ or $\alpha = \alpha(x,t)$) have received special attention as a natural extension of those with constant order [1, 3, 10, 11, 24, 26, 31, 38, 45, 46, 47, 49].

2010 *Mathematics Subject Classification.* 45A05, 45E10, 45N05, 65R20, 65J08, 65J10.

Key words and phrases. Fractional integrals; Banach spaces; variable order; convolution quadratures.

The first author has been supported by Spanish Ministerio de Economía y Competitividad under the Research Grant MTM2014-54710-P.

The second author has been supported by Fondecyt Grant 11130619.

The main reason is that practical applications modeling has demonstrated that the freedom to choose a variable order of integration/derivation instead of the constant one allows us a finer tuning in mathematical modeling. In other words, with variable order of integration more accurate modeling can be achieved. This fact applies over a large variety of applications: control theory [18, 23, 36], mechanical engineering [12], image processing [14], remote sensing [50], physical applications [5, 6], and some others [4, 40, 43, 44]. But also theoretical properties of such kind of equations attracted the interest of researchers [2, 27], in this way it is interesting to show how these properties differ from the ones satisfied by the fractional equations with constant order.

One can accept that for linear equations of type (1), i.e. for A standing for a constant, some properties, as for instance the existence and uniqueness of solution, can be easily proven. However, these properties are in general not straightforwardly extended to abstract formulations of these equations as the one considered in the present work. Nonetheless there exists a extensive literature related to non local equations in time, in particular related to Volterra equations and the well–posedness, but to our knowledge, there are not general results establishing appropriate conditions on the viscosity function $\alpha(t)$ to ensure the well–posedness of that equations within a framework of general operators setting. That is why we devote this work to the study of some relevant properties of fractional equations of variable order.

To be more precise, the first contribution of this work is to establish conditions, as weak as possible, for the viscosity functions $\alpha(t)$ in order to guarantee the well–posedness of linear abstract evolution equations of fractional type with order varying in time in a very general functional setting as it is the framework of complex Banach spaces. Once these conditions are stated we show the regularity exhibited by the solution as well as the asymptotic behavior as the time goes to infinity. These results are accompanied by the study of the asymptotic behavior of the numerical solution provided by backward Euler based convolution quadrature method, which is inherited from the asymptotic behavior of the continuous solution, and it is accompanied as well by several numerical experiments illustrating the theoretical results. Notice that the viscosity functions $\alpha(t)$ can be assumed to be depending on spatial variables, however in the present framework this dependence is meaningless.

This paper is organized as follows. Section 2 is devoted to present a preliminar discussion on some of definitions existing in the literature for the fractional integrals with order varying in time, and to motivate our choice from all those mentioned. In Section 3 we establish the conditions under which we can ensure the well–posedness of initial value problems of fractional type with order varying in time, and we prove the well–posedness under such assumptions. In Sections 4 and 5 we show the regularity of the analytic solution at time level $t = 0$, and the asymptotic behavior as time goes to infinity respectively. In Section 6 we set a time discretization based on the backward Euler convolution quadrature and we show how the asymptotic behavior as the number of steps goes to infinity is inherited from the asymptotic behavior of the analytic solution. Finally, in Section 7 we illustrate numerically how the results of the fractional integration significantly depends on the definition we choose, and moreover we illustrate the behavior of the solutions of initial boundary value problems of fractional type with order varying in time depending on the choice of $\alpha(t)$.

2. Background on the fractional integrals with order varying in time

The study of fractional integration with order varying in time $\alpha(t) > 0$, $t \geq 0$, can be addressed in different manners depending on the definition one adopts.

Assume that the order of integration/derivation ranges between 1 and 2, i.e. $1 < \alpha < 2$. A natural generalization of the Riemann–Liouville definition

$$\partial^{-\alpha} g(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\,\mathrm{d}s, \qquad t > 0, \tag{2}$$

seems to be the one directly obtained by replacing in (2) the constant order α by a function $\alpha(t)$. In fact, for $\alpha : (0, +\infty) \to (1, 2)$, this definition reads

$$\partial^{-\alpha(t)} g(t) := \int_0^t \frac{(t-s)^{\alpha(t-s)-1}}{\Gamma(\alpha(t-s))} g(s)\,\mathrm{d}s, \qquad t > 0. \tag{3}$$

Definition (3) stands for a convolution integral $(k * g)(t)$, where the convolution kernel is given by

$$k(t) = \frac{t^{\alpha(t)-1}}{\Gamma(\alpha(t))}, \qquad t > 0. \tag{4}$$

However, in classical operational calculus, the numerical solution of equations involving convolution terms, e.g. the equation (3), requires the evaluation of the Laplace transform K of the convolution kernel k, namely $K = \mathcal{L}k$, instead of k itself. In that case, the analysis of (3) in terms of the Laplace transform seems to be, in general, difficult if not unaffordable, since an explicit expression of the Laplace transform of k, is in general not explicitly reachable.

Other definitions can be found in the literature, for example [23, 46]

$$\partial^{-\alpha(t)} g(t) := \frac{1}{\Gamma(\alpha(t))} \int_0^t (t-s)^{\alpha(s)-1} g(s)\,\mathrm{d}s, \qquad t > 0, \tag{5}$$

or, with a more general formulation [37]

$$\partial^{-\alpha(\cdot,\cdot)} g(t) := \int_0^t \frac{(t-s)^{\alpha(t,s)-1}}{\Gamma(\alpha(t,s))} g(s)\,\mathrm{d}s, \qquad t > 0. \tag{6}$$

Other definitions can be proposed from the definitions above.

Notice that the analysis of fractional integrations of variable order according definitions (5) and (6) neither looks like a straightforward matter since these equations have not convolution structure, excepting might be (6) with a convenient choice of $\alpha(\cdot,\cdot)$.

The definition of fractional integral with order varying in time we will adopt in the present work was basically proposed in [49], and the main feature is that it is given in terms of the Laplace transform of the convolution kernel itself. In fact, let $\tilde{\alpha}(z)$ be the Laplace transform of $\alpha(t)$, then the fractional integral of order $\alpha(t)$ is defined as the convolution integral

$$\partial^{-\alpha(t)} g(t) := \int_0^t k(t-s) g(s)\,\mathrm{d}s, \qquad t > 0, \tag{7}$$

where

$$k(t) := (\mathcal{L}^{-1}K)(t), \quad t > 0, \quad \text{with} \quad K(z) := \frac{1}{z^{z\tilde{\alpha}(z)}}, \quad z \in \mathcal{D}(K) \subset \mathbb{C}. \tag{8}$$

We will discuss below the domain of definition for K, $\mathcal{D}(K)$, or, equivalently, the domain of definition of the function $\tilde{\alpha}(z)$.

Note that the definition of fractional integral of variable order according (7)-(8) has been already used in practical instances, in particular in models related to image processing [14] where only K (the Laplace transform of k) was necessary, i.e. k was not explicitly required at all.

Note also that definition (7)-(8) is consistent with (2) in the sense that, if the viscosity function $\alpha(t)$ is constant, i.e. $\alpha(t)$ =const., then the convolution kernel turns out to be the one in (2)

$$k(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \qquad t > 0.$$

Certainly the main advantage of definition (7)-(8) versus definitions of type (3), (5), or (6) (the last one depending on the choice of α), comes out when one addresses the solvability of an abstract integral equation of type

$$u(t) = u_0 + \int_0^t k(t-s)(Au)(s)\,\mathrm{d}s, \qquad t > 0, \tag{9}$$

where A stands for a linear operator in an abstract functional setting X, $A : D(A) \subset X \to X$, e.g. A standing for the Laplacian operator Δ in $\mathbb{R}^n$.

In the following sections the solvability of fractional integral equations (9) of variable order in time in the sense of to the definition (7)-(8) is addressed, as well as some properties the solution holds.

3. Well–posedness of the fractional initial value problem

Let X be a complex Banach space, and let $\alpha(t)$ be a function $\alpha : (0, +\infty) \to (1, 2)$. Consider the abstract integral equation of fractional type of variable order

$$u(t) = u_0 + \partial^{-\alpha(t)}(Au)(t) = u_0 + \int_0^t k(t-s)(Au)(s)\,\mathrm{d}s, \qquad t > 0, \tag{10}$$

where $A : D(A) \subset X \to X$ is a linear and closed operator of sectorial type in X, $u_0 \in X$ is the initial data, and k is a convolution kernel defined as in (7)–(8), for a given complex function $\tilde{\alpha}(z)$ defined in certain domain $D \subset \mathbb{C}$ to be discussed below.

Recall that a linear and closed operator A is of *sectorial type*, or *θ–sectorial* [21, 39] in X, if there exist $0 < \theta < \pi/2$, $\omega \in \mathbb{R}$, and $M > 0$ such that

$$\|(zI - A)^{-1}\|_{X\to X} \le \frac{M}{|z-\omega|}, \qquad \text{for} \qquad z \notin \omega_+ + S_\theta, \tag{11}$$

where I stands for the identity operator in X, $\omega_+ = \max\{\omega, 0\}$, and

$$\omega_+ + S_\theta := \{\omega_+ + \xi : \xi \in \mathbb{C}, |\arg(-\xi)| < \theta\}, \tag{12}$$

Henceforth, for the simplicity of notation, and if not confusing, for $B \in \mathcal{L}(X)$, we will write $\|B\|$ instead of $\|B\|_{X\to X}$.

For the simplicity of the notation as well, hereafter we will denote

$$g(z) := z\tilde{\alpha}(z), \quad g_R(z) := \operatorname{Re} g(z), \quad g_I(z) := \operatorname{Im} g(z), \quad \text{and} \quad h(z) = z^{g(z)},$$

for $z \in D \subset \mathbb{C}$. Therefore the Laplace transform of k can be written as

$$K(z) = \frac{1}{h(z)}.$$

Assume that there exist constants $D_\alpha, m_1, m_2, \varepsilon, R > 0$, $0 < \varepsilon^* < 1$, and $0 < \theta < \pi/2$, satisfying the following assumptions:

(H1) The function $\alpha(t)$ admits Laplace transform $\tilde{\alpha}(z)$ in the complex domain $\mathrm{Re}\, z \geq D_\alpha$.

(H2) The real part of $g(z)$, $g_R(z)$, is bounded by

$$1 < m_1 \leq g_R(z) \leq m_2 < 2, \quad \text{and} \quad \frac{m_2\pi}{2} < \varepsilon^*(\pi - \theta).$$

(it is expected that $\varepsilon^* \approx 1$)

(H3) The imaginary part of $g(z)$, $g_I(z)$, is bounded, and there holds, for $\mathrm{Im} z \leq 0$,

$$|\log|z|\, g_I(z)| < (1 - \varepsilon^*)(\pi - \theta).$$

(A1) The operator A is θ–sectorial for some $\omega \in \mathbb{R}$, $\omega < D_\alpha$, with θ satisfying

$$0 < \theta < \pi - \frac{m_2\pi}{2} - \max_{\rho \geq R} \frac{\log(\rho)}{\rho^\varepsilon},$$

where R is assumed to be large enough.

Notice that, the closer m_2 is to 2, the more restricted is θ (closer to $\pi/2$ is). On the contrary, the closer is θ to $\pi/2$, the closer are m_1 and m_2 to 1, as expected in view of behavior in the case $\alpha(t) =$const.

Examples of functions $\alpha(t)$ satisfying (H1)–(H3) are,

$$\frac{c \sin t}{2} + \frac{3}{2}, \quad \frac{c \cos t}{2} + \frac{3}{2}, \quad c\, \mathrm{e}^{-t} + 1, \ldots, \quad \text{with} \quad 0 < c < 1,$$

and/or e.g. piece–wise constant functions conveniently defined.

Moreover, examples of θ–sectorial operators are the Laplacian operator Δ in $\mathbb{R}^n$, fractional powers of the Laplacian Δ^β ($\beta > 0$), or in finite dimensional operators such as matrices $\Delta_h \in \mathcal{M}_{M\times M}(\mathbb{R})$, in particular the ones coming out from most classical discretizations of Δ.

Prior to state the existence and uniqueness result, we look for a convenient representation of the time evolution operator associated to the solution of (10). In fact, since by Assumption (H1) the function $\alpha(t)$ admits Laplace transform, we can write (10) in the frequency domain as

$$U(z) = \frac{u_0}{z} + K(z)AU(z), \qquad z \notin \omega_+ + S_\theta,$$

where $U(z)$ stands for the Laplace transform of the solution $u(t)$ of (10), and K is the Laplace transform of k. Therefore, according the notation stated above, $U(z)$ reads in terms of the resolvent of A as

$$U(z) = \frac{1}{z}(I - K(z)A)^{-1}u_0 = \frac{h(z)}{z}\,(h(z)I - A)^{-1}u_0. \tag{13}$$

The well–posedness of (10) requires the existence of a bounded evolution operator $E(t)$ such that the generalized solution may be written as

$$u(t) = E(t)u_0, \qquad t > 0. \tag{14}$$

Let us show that $E(t)$ exists and, by virtue of the inversion formula of the Laplace transform, and Assumptions (H1)–(H3) and (A1), the evolution operator admits the expression

$$E(t) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{tz} \frac{h(z)}{z}\,(h(z)I - A)^{-1}\, \mathrm{d}z, \qquad t > 0, \tag{15}$$

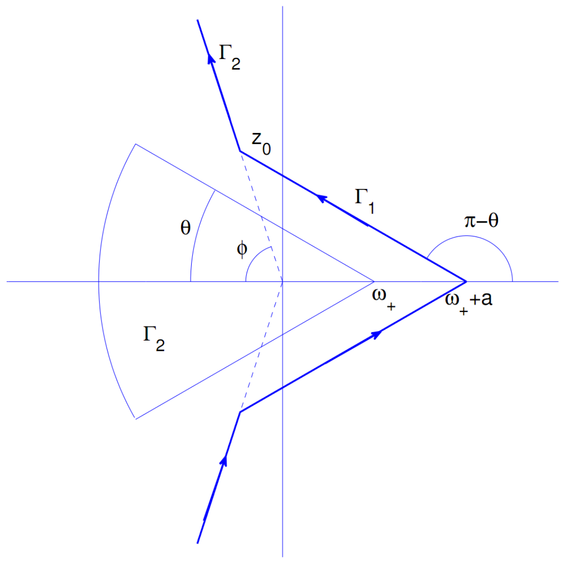

FIGURE 1. Complex paths Γ_1 and Γ_2

where Γ is a convenient complex path connecting $-\infty$i and $+\infty$i with increasing imaginary part (i.e. positively oriented), and surrounding the sector $\omega_+ + S_\theta$.

Therefore, the key point is to find one of such complex paths providing a convergent integral in (15). In this regard define the complex paths (see Fig. 1)

$$\begin{aligned} \Gamma_1 \quad &: \quad \gamma_1(\rho) := \omega_+ + a + \rho \mathrm{e}^{\pm \mathrm{i}(\pi-\theta)}, \qquad 0 \le \rho \le \rho_0, \\ \Gamma_2 \quad &: \quad \gamma_2(\rho) := \rho \mathrm{e}^{\pm \mathrm{i}\varphi}, \qquad \rho \ge \rho_1, \end{aligned}$$

where

$$\rho_0 = \frac{\omega_+ + a}{\cos(\theta)\left(1 - \frac{\tan(\theta)}{\tan(\pi-\varphi)}\right)}, \qquad \rho_1 = |z_0| = \frac{\omega_+ + a}{\cos(\pi-\varphi)\left(\frac{\tan(\pi-\varphi)}{\tan(\theta)} - 1\right)}, \tag{16}$$

z_0 and $\bar{z}_0$ are the intersection points of Γ_1 and Γ_2, i.e $z_0 = \omega_+ + a + \rho_0 \mathrm{e}^{\mathrm{i}(\pi-\theta)} = \rho_1 \mathrm{e}^{\mathrm{i}\varphi}$, a is positive constant (we will see below the role a plays), and $\pm$ means that we are considering both parts of the paths, the one located in the upper half–complex plane joint with the symmetric one. The angle φ satisfies

$$m_2 \frac{\pi}{2} < \varphi \le \varepsilon^*(\pi - \theta) < \pi - \theta,$$

and without loss of generality we can set $\varphi = \varepsilon^*(\pi-\theta)$. Notice that φ exists thanks to Assumption (A1).

Now define the continuous complex path positively oriented Γ consisting of the union of Γ_1^{1/m_2}, and Γ_2^{1/m_2}, where Γ_j^{1/m_2} is defined as $(\gamma_j(\rho))^{1/m_2}$, for $j = 1, 2$, and show that the integral (15) along Γ is convergent. This will prove that the evolution operator $E(t)$ is well–defined.

Consider first Γ_1^{1/m_2}. The following discussion will focus only on the part of Γ_1^{1/m_2} lying in the upper half–complex plane $\operatorname{Im} z \ge 0$, and the same applies for Γ_2^{1/m_2}. In the lower half–complex plane $\operatorname{Im} z \le 0$ the proof easily follows.

We have to prove that if $z \in \Gamma_1^{1/m_2}$, then $h(z) \notin \omega_+ + S_\theta$. To this end, we set $z \in \Gamma_1^{1/m_2}$, and denote,

$$\xi = |\xi| \mathrm{e}^{\mathrm{i}\eta} = \omega_+ + a + \rho \mathrm{e}^{\mathrm{i}(\pi-\theta)}, \quad \text{for certain} \quad \rho > 0, \text{ and } 0 \leq \eta \leq \varphi,$$

such that $z = \xi^{1/m_2}$. Then, there is $\tilde{\eta}$ such that

$$h(z) = |h(z)| \mathrm{e}^{\mathrm{i}\tilde{\eta}},$$

and there is also $\tau \geq 0$ such that

$$\tilde{z} = |\tilde{z}| \mathrm{e}^{\mathrm{i}\tilde{\eta}} = \omega_+ + \tau \mathrm{e}^{\mathrm{i}(\pi-\theta)}.$$

belongs to the boundary of $\omega_+ + S_\theta$, and has the same argument as $h(z)$. It is straightforward to prove that

$$|\tilde{z}| = \frac{\omega_+ \sin(\pi-\theta)}{\sin(\pi-\theta-\tilde{\eta})},$$

$$|h(z)| = \left(\frac{(\omega_+ + a)\sin(\pi-\theta)}{\sin(\pi-\theta-\eta)} \right)^{g_R(z)/m_2} \mathrm{e}^{-\eta g_I(z)/m_2},$$

and

$$\tilde{\eta} = \frac{1}{m_2} \left(g_I(z) \log \left(\frac{(\omega_+ + a)\sin(\pi-\theta)}{\sin(\pi-\theta-\eta)} \right) + \eta g_R(z) \right).$$

Hence, the proof reduces to show that

$$|h(z)| > |\tilde{z}|, \quad z \in \Gamma_1^{1/m_2}. \tag{17}$$

Here we discuss several cases. First, if $\omega_+ = 0$, then inequality (17) obviously holds. On the other hand, if $\omega_+ > 0$ and $g_I = 0$, then $\tilde{\eta} = \eta g_R(z)/m_2$ and therefore inequality (17) reduces to

$$\frac{\omega_+ \sin(\pi-\theta)}{\sin\left(\pi-\theta-\eta\frac{g_R(z)}{m_2}\right)} < \left(\frac{(\omega_+ + a)\sin(\pi-\theta)}{\sin(\pi-\theta-\eta)} \right)^{g_R(z)/m_2}.$$

In that case, $g_R(z) = m_1 = m_2$ leads to $\alpha(t)$ constant, and the last inequality straightforwardly holds. However, without additional assumptions, it is easy to find a naive function $g_R(z)$ (non constant, e.g. with $\eta = 0$) for which this equality does not satisfy. Therefore, in the general case (i.e. with $g_I(z)$ not necessarily 0) with $\omega_+ > 0$ additional assumptions are required, e.g. assumptions of type $g_I(z)/m_2$ being small enough joint with $\omega_+ < 1$. Since additional assumptions for $\omega_+ > 0$ look like very unrealistic, now and hereafter we will assume that $\omega_+ = 0$, i.e. $\omega \leq 0$.

Consider now Γ_2^{1/m_2}, and $z \in \Gamma_2^{1/m_2}$. Therefore $z = (\rho \mathrm{e}^{\mathrm{i}\varphi})^{1/m_2}$, for certain $\rho \geq \rho_1$. In that case we have to prove that $\mathrm{Arg}(h(z)) < \pi - \theta$, but this is straightforward in view of inequality

$$|\mathrm{Arg}(h(z))| \leq |\log|z|\, g_I(z)| + \frac{\varphi}{m_2} g_R(z),$$

and Assumption (H3). In fact

$$|\log|z|\, g_I(z)| + \frac{\varphi}{m_2} g_R(z) < (1-\varepsilon^*)(\pi-\theta) + \varepsilon^*(\pi-\theta) = \pi - \theta,$$

and therefore $h(z) \notin S_\theta$.

On the other hand, by virtue of (H2),

$$\operatorname{Arg} z = \frac{\varphi}{m_2} = \frac{\varepsilon^*(\pi-\theta)}{m_2} > \frac{\pi}{2},$$

and therefore Γ_2^{1/m_2} falls inside the half–complex plane $\operatorname{Re} z < 0$.

In conclusion, we are now in a position to prove the following theorem where, under Assumptions (H1)–(H3) and (A1), we prove that the integral (15) is convergent, and equivalently the Problem (10) is well–posed.

Theorem 1. *Let $\alpha(t)$ be a function belonging to $L^1(0,+\infty)$ satisfying (H1)–(H3). Assume also that the operator A satisfies (A1). Then the problem (10) is well–posed.*

Proof. The proof reduces to state that the integral in (15) is convergent for which we consider the positively oriented complex path Γ consisting of the union of Γ_1^{1/m_2} and Γ_2^{1/m_2} defined above. Therefore we can write

$$E(t) = \sum_{j=1}^{2} I_j, \quad \text{where} \quad I_j = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_j^{1/m_2}} \mathrm{e}^{tz} \frac{h(z)}{z} \left(h(z)I - A\right)^{-1} \mathrm{d}z, \quad j = 1, 2.$$

Now we prove that I_1 and I_2 are bounded. First of all, for $0 < t \le T$, and regarding the properties of Γ_1^{1/m_2} stated above, we have

$$\begin{aligned}
\|I_1\| &= \left\| \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1^{1/m_2}} \mathrm{e}^{tz} \frac{h(z)}{z} (h(z)I - A)^{-1} \, \mathrm{d}z \right\| \\
&\le \frac{1}{2\pi} \int_{\Gamma_1^{1/m_2}} \left|\mathrm{e}^{tz}\right| \left|\frac{h(z)}{z}\right| \|(h(z)I - A)^{-1}\| \, |\,\mathrm{d}z| \\
&\le \frac{M}{2\pi} \int_{\Gamma_1^{1/m_2}} \left|\frac{\mathrm{e}^{tz}}{z}\right| |\,\mathrm{d}z| \\
&\le \frac{M \mathrm{e}^{aT/m_2}}{2\pi} \int_{\Gamma_1^{1/m_2}} \frac{1}{|z|} \, |\,\mathrm{d}z| \\
&\le \frac{M \mathrm{e}^{aT/m_2} \rho_0}{\pi a \sin(\varphi) m_2}. \qquad (18)
\end{aligned}$$

On the other hand, for $0 < t \le T$, and regarding the properties of Γ_2^{1/m_2} stated above, we obtain

$$\begin{aligned}
\|I_2\| &= \left\| \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_2^{1/m_2}} \mathrm{e}^{tz} z^{g(z)-1} (z^{g(z)} I - A)^{-1} \, \mathrm{d}z \right\| \\
&\le \frac{M}{2\pi} \int_{\Gamma_2^{1/m_2}} \left|\frac{\mathrm{e}^{tz}}{z}\right| |\,\mathrm{d}z| \\
&\le \frac{M}{2\pi} \int_{\Gamma_2^{1/m_2}} \frac{\mathrm{e}^{t|z|\cos(\varphi/m_2)}}{|z|} \, |\,\mathrm{d}z|. \qquad (19)
\end{aligned}$$

Therefore, since $\cos(\varphi/m_2) < 0$ by hypothesis (H2), the integral (19) is convergent, and leads to the boundness of I_2, and consequently joint with (18) the boundness of $E(t)$, for $0 < t \le T$.

In conclusion, the generalized solution of (10) exists even if u_0 merely belongs to X, i.e. even if u_0 is not in $D(A)$, and moreover admits an unique representation in terms of the evolution operator (15). Therefore, the problem (10) is well–posed. □

Remark 2. *If $u_0 \in D(A)$, then since A commutes with the associated resolvent, proceeding as in Theorem 1, it can be proved that,*

$$\begin{aligned}\|Au(t)\| &= \left\|\frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{zt}\frac{h(z)}{z}(h(z)I-A)^{-1}Au_0\,\mathrm{d}z\right\| \\ &\le \frac{1}{2\pi}\left\|\int_\Gamma \mathrm{e}^{zt}\frac{h(z)}{z}(h(z)I-A)^{-1}\,\mathrm{d}z\right\|\,\|Au_0\| \\ &\le C\|Au_0\|.\end{aligned}$$

Therefore $u(t) \in D(A)$, for $t > 0$, and (14) is a genuine solution of (10).

Remark 3. *We have shown that the evolution operator $E(t)$, $t > 0$, associated to (10) admits an integral representation (15) defined along a suitable complex path. However, such a complex path, in particular the one we set Γ, cannot be strictly placed in the half–complex plane $\mathrm{Re}\, z < 0$ since the integrand does not admit holomorphic extension to the real line $\mathrm{Re}\, z \le 0$ ($\mathrm{Im}\, z = 0$). Therefore, no exponential decay will be obtained as one could expect having in mind the theory of classical C_0-semigroups.*

4. Continuous solution: Regularity at $t = 0$

The regularity at $t = 0^+$ has been already studied for $\alpha(t) =$const. in [13]. In this Section we extend the study to the case of α depending on time, i.e. $\alpha(t)$ instead of α.

Let δ be a positive non–zero constant, and define the function set

$$\mathcal{F}_\delta := \left\{ f : (0,T] \to X, \text{ meassurable} : \sup_{0<t\le T} \|t^\delta f(t)\| < +\infty \right\}, \tag{20}$$

equipped with the norm

$$|||f|||_\delta := \sup_{0<t\le T} \|t^\delta f(t)\|, \qquad f \in \mathcal{F}_\delta. \tag{21}$$

Therefore we have the next result.

Theorem 4. *Let $u(t)$ be the solution of (10) under Assumptions (H1)–(H3), and (A1). If $u_0 \in D(A)$, then the derivative of $u(t)$, $u'(t)$, is bounded for $0 < t \le T$, and $u'(t) = O(t^{m_1-1})$ as $t \to 0^+$.*

Proof. We first notice that from the resolvent identity (13) it follows that the operator A conmutes with the evolution operator, that is, $E(t)A\xi = AE(t)\xi$, for all $\xi \in D(A)$.

Let Γ be again the complex path defined in Section 2. So, given $\xi \in D(A)$ we can write

$$\begin{aligned} u(t) &= E(t)\xi = \frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{zt}\frac{h(z)}{z}(h(z)I-A)^{-1}\xi\,\mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{zt}\left(\frac{\xi}{z}+\frac{1}{z}(h(z)I-A)^{-1}A\xi\right)\mathrm{d}z \\ &= \xi + \frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{zt}\frac{1}{z}(h(z)I-A)^{-1}A\xi\,\mathrm{d}z,\end{aligned}$$

for $t > 0$. Note that last equality has been reached by making use of the following fact,

$$\frac{h(z)}{z}(h(z)I - A)^{-1}\xi = \left(\frac{h(z)}{z} + A - A\right)(h(z)I - A)^{-1}\xi = \xi + \frac{1}{z}\,(h(z) - A)^{-1}A\xi.$$

Therefore, if we denote the derivative of $E(t)$ respect to t as $E'(t)$, then we have

$$u'(t) = E'(t)\xi = \frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{zt}(h(z) - A)^{-1}A\xi\,\mathrm{d}z.$$

For the convenience of the proof we consider here in the definition of Γ the parameter a as a function depending on time, in fact a takes the value $1/t^{m_2}$.

So we have,

$$u'(t) = \sum_{j=1}^{2} I_j, \qquad \text{where} \qquad I_j = \frac{1}{2\pi \mathrm{i}}\int_{\Gamma_j^{1/m_2}} \mathrm{e}^{zt}(h(z)I - A)^{-1}A\xi\,\mathrm{d}z,$$

and where each individual integral will be studied separately. Note that, according the definition of Γ, we have that $h(\Gamma) \subset \mathbb{C} \setminus \{\omega_+ + S_\theta\}$, and therefore, for $j = 1, 2$, the choice of Γ allows us to apply the sectorial property (18)–(19) of A.

We consider two separated cases, with the same notation as in Theorem 1: $\omega = 0$, and $\omega < 0$. Now and hereafter in the proof we assume that $C > 0$ is a generic positive constants independent of t.

We first set $\omega = 0$. Note that, by assumption (H3) there exist $c_m, C_M > 0$ such that

$$c_m \le \mathrm{e}^{-\mathrm{Arg}(z)g_I(z)} \le C_M. \tag{22}$$

If $z \in \Gamma_1^{1/m_2}$, then $z = (1/t^{m_2} + \rho\mathrm{e}^{\mathrm{i}(\pi-\theta)})^{1/m_2}$, $0 \le \rho \le \rho_0$. In that case, and having in mind that $t \to 0^+$, we have

$$|\mathrm{e}^{zt}| \le \mathrm{e}^{(1/t^{m_2})^{1/m_2}t} = \mathrm{e}, \tag{23}$$

and there exists $C > 0$ such that

$$\begin{aligned} |h(z)| &= |z|^{g_R(z)}\mathrm{e}^{-\mathrm{Arg}(z)g_I(z)} = \left(\frac{1}{t^{m_2}} + \rho\mathrm{e}^{\mathrm{i}(\pi-\theta)}\right)^{g_R(z)/m_2}\mathrm{e}^{-\mathrm{Arg}(z)g_I(z)} \\ &\ge \left(\frac{1}{t^{m_2}\sin(\theta)}\right)^{m_1/m_2} c_m \ge \frac{c_m C}{t^{m_1}}. \end{aligned} \tag{24}$$

On the other hand,

$$\mathrm{long}(\Gamma_1^{1/m_2}) \le C\,\left(\frac{1/t^{m_2}}{\cos(\theta)\left(1 - \frac{\tan(\theta)}{\tan(\pi-\varphi)}\right)}\right)^{1/m_2}. \tag{25}$$

Therefore, regarding (23)–(25) we have

$$\begin{aligned}\|I_1\| &\leq \frac{M}{2\pi}\int_{\Gamma_1^{1/m_2}}\left|\frac{\mathrm{e}^{zt}}{h(z)}\right| |\,\mathrm{d}z| \\ &\leq \frac{M\mathrm{e}t^{m_1}}{2\pi c_m \sin(\theta)^{m_1/m_2}}\int_{\Gamma_1^{1/m_2}} |\,\mathrm{d}z| \\ &= \frac{M\mathrm{e}t^{m_1}}{2\pi c_m \sin(\theta)^{m_1/m_2}}\mathrm{long}(\Gamma_1^{1/m_2}) \\ &\leq \frac{M\mathrm{e}Ct^{m_1-1}}{\pi c_m \sin(\theta)^{m_1/m_2}}.\end{aligned}$$

If $z \in \Gamma_2^{1/m_2}$, then $z = (\rho \mathrm{e}^{\mathrm{i}\varphi})^{1/m_2}$, $\rho \geq \rho_1$, where ρ_1 was computed in (16). In that case, there exists $C > 0$ such that

$$\begin{aligned}|h(z)| &= |z|^{g_R(z)}\mathrm{e}^{-\mathrm{Arg}(z)g_I(z)} \geq c_m \rho_1^{g_R(z)/m_2} \\ &= \frac{c_m(1/t)^{g_R(z)}}{\left(\cos(\pi-\varphi)\left(\frac{\tan(\pi-\varphi)}{\tan(\theta)}-1\right)\right)^{g_R(z)/m_2}} \geq \frac{C}{t^{m_1}}.\end{aligned} \tag{26}$$

On the other hand,

$$|\,\mathrm{d}z| = \frac{\rho^{1/m_2-1}}{m_2}\,\mathrm{d}\rho. \tag{27}$$

Therefore, by (26), and (27) we have

$$\begin{aligned}\|I_2\| &\leq \frac{M}{2\pi}\int_{\Gamma_2^{1/m_2}}\left|\frac{\mathrm{e}^{zt}}{h(z)}\right| |\,\mathrm{d}z| \\ &\leq \frac{CM}{2\pi}\int_{\rho_1}^{+\infty} \mathrm{e}^{\rho^{1/m_2}\cos(\varphi/m_2)}\frac{t^{m_1}\rho^{1/m_2-1}}{m_2}\,\mathrm{d}\rho \qquad (\nu = \rho^{1/m_2}) \\ &\leq \frac{CMt^{m_1}}{2\pi}\int_{\rho_1^{1/m_2}}^{+\infty}\mathrm{e}^{\nu\cos(\varphi/m_2)}\,\mathrm{d}\nu \\ &\leq \frac{CMt^{m_1}}{2\pi}\frac{\mathrm{e}^{\cos(\varphi/m_2)/t}}{|\cos(\varphi/m_2)|} \\ &\leq Ct^{m_1-1}.\end{aligned}$$

The statement for $\omega = 0$ is now straightforward.

If $\omega < 0$, then we have

$$\|u'(t)\| \leq \frac{M}{2\pi}\int_\Gamma \left|\frac{\mathrm{e}^{zt}}{h(z)-\omega}\right| |\,\mathrm{d}z| \|A\xi\|,$$

In this case, the bound

$$\begin{aligned}|h(z)-\omega| \geq |h(z)| - |\omega| &= \left|\frac{1}{t^{m_2}} + \rho \mathrm{e}^{\mathrm{i}(\pi-\theta)}\right|^{g_R(z)/m_2} \mathrm{e}^{-\mathrm{Arg}(z)g_I(z)} - |\omega| \\ &\geq \frac{c_m\sin(\theta)^{m_1/m_2}}{t^{m_1}} - |\omega|,\end{aligned}$$

and the same ideas as for $\omega = 0$, lead to the bound

$$\|u'(t)\| \leq \frac{Ct^{m_1-1}}{c_m\sin(\theta)^{m_1/m_2} - t^{m_1}|\omega|},$$

and the proof of the theorem follows. □

As occurs in classical abstract parabolic ordinary differential equations, no regularity in time is expected for no regular initial data u_0. In fact, for u_0 merely belonging to X, we have next the Corollaries.

Corollary 5. *Let $u(t)$ be the solution of (10) satisfying (H1)–(H3), and (A1). For any u_0 belonging to X, u' belongs to $\mathcal{F}_1$.*

The proof of Corollary 5 follows the ideas of those in the proof of Theorem 4, and therefore, we just present a sketch of that.

Proof. Once again, let Γ be the path defined in Section 3. Then, we can write

$$u'(t) = E'(t)\xi = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{zt} h(z)(h(z) - A)^{-1} \xi \, \mathrm{d}z, \qquad t > 0.$$

On the other hand, sectorial property (A1) of A, now with $\omega = 0$, and the change of variable $\nu = tz$ leads to

$$\|u'(t)\| \leq \frac{M}{2\pi} \int_\Gamma |\mathrm{e}^{tz}| \, \mathrm{d}z = \frac{M}{2\pi t} \int_{\Gamma^*} |\mathrm{e}^{\nu}| \, \mathrm{d}\nu, \qquad t > 0,$$

where the complex path Γ^* results from the change of variable $\nu = tz$. The boundness of the last integral is straightforward, and proceeding similarly for $\omega < 0$ the proof follows. □

The proof of the next Corollary follows similar steps as the proof of Theorem 4. We omit the details.

Corollary 6. *Let $u(t)$ be the solution of (10) satisfying (H1)–(H3), and (A1). If $u_0 \in D(A)$, then $u'' \in \mathcal{F}_{2-m_1}$.*

5. Continuous solution: Asymptotic behavior

In this section we study the asymptotic behavior in norm of the solution of (10) as t goes to $+\infty$ through the associated evolution operator $E(t)$. We remark that the study of the constant case, that is, $\alpha(t)$ =const. can be found in [13].

Theorem 7. *Let $E(t)$ be the evolution operator (15) associated to the problem (10) satisfying (H1)–(H3), and (A1). Then, there exists a constant $C > 0$ independent of t such that*

$$\|E(t)\| \leq \frac{CM}{1 + |\omega| t^{m_1}}, \qquad \textit{as} \quad t \to +\infty. \tag{28}$$

Notice that, in view of Theorem 7, the asymptotic behavior of the solution $u(t)$ of (10) is independent of the initial value, i.e. this holds for u_0 merely belonging to X.

Proof. Let $E(t)$ be the evolution operator (15)

$$E(t) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{zt} \frac{h(z)}{z} \left(h(z)I - A\right)^{-1} \mathrm{d}z, \qquad t > 0,$$

where Γ is a suitable path connecting $-\mathrm{i}\infty$ and $+\mathrm{i}\infty$ positively oriented. In fact for the convenience of the proof we choose again Γ as the union of Γ_1^{1/m_2} and Γ_2^{1/m_2} defined in Section 3, and again with a depending on time as $1/t^{m_2}$.

In Section 3 we have proven that $E(t)$ is bounded, for $t>0$, and now we get a finer bound to prove it. To this end, we write

$$E(t)=\sum_{j=1}^{2} I_j(t), \quad t>0, \quad \text{where} \quad I_j(t)=\frac{1}{2\pi \mathrm{i}}\int_{\Gamma_j^{1/m_2}} \mathrm{e}^{zt}\frac{h(z)}{z}\left(h(z)I-A\right)^{-1}\,\mathrm{d}z,$$

for $j=1,2$. Consider $t>0$ large enough, and study separately $I_1(t)$ and $I_2(t)$. Along this proof $C>0$ will denote a generic constant independent of t.

First part. Consider $I_1(t)$, and $z\in\Gamma_1^{1/m_2}$. By the definition of Γ_1^{1/m_2}

$$z=\left(\frac{1}{t^{m_2}}+\rho \mathrm{e}^{\mathrm{i}(\pi-\theta)}\right)^{1/m_2}, \quad \text{for some} \quad 0\le\rho\le\rho_0,$$

where ρ_0 was computed in (16).

Notice that $|\mathrm{e}^{zt}|\le \mathrm{e}$, for $z\in\Gamma_1^{1/m_2}$. Therefore, if $\omega=0$, then there exists $C>0$ such that parametrizing Γ_1^{1/m_2} we have

(29)
$$\|I_1(t)\|\le\frac{CM\mathrm{e}}{2\pi}\int_{\Gamma_1^{1/m_2}}\frac{1}{|z|}|\,\mathrm{d}z|\le\frac{CM\mathrm{e}\,t^{m_2}}{\pi\sin(\theta)}\int_0^{\rho_0}\mathrm{d}\rho=\frac{CM\mathrm{e}}{\pi\sin(\theta)\cos(\theta)\left(1-\frac{\tan(\theta)}{\tan(\pi-\varphi)}\right)},$$

and therefore, $I_1(t)$ is bounded, for $t>0$.

If $\omega<0$, then

$$\|I_1(t)\|\le\frac{M}{2\pi}\int_{\Gamma_1^{1/m_2}}\frac{|h(z)|\,|\mathrm{e}^{zt}|}{|z|\,|h(z)-\omega|}|\,\mathrm{d}z|.$$

For $z\in\Gamma_1^{1/m_2}$ there exists $C>0$ such that $|z|\le C/t$, and by (22)

$$|h(z)|=|z|^{g_R(z)}\mathrm{e}^{-\mathrm{Arg}(z)g_I(z)}\le C_M\left(\frac{C}{t}\right)^{g_R(z)}\le\frac{CC_M}{t^{m_1}}. \tag{30}$$

Moreover, there exists $C>0$ such that

$$|z|\ge\left(\frac{1}{t^{m_2}}\sin(\theta)\right)^{1/m_2} \quad\Longrightarrow\quad \frac{1}{|z|}\le Ct. \tag{31}$$

Also, we have

$$\frac{1}{|h(z)-\omega|}\le\frac{1}{\sin(\theta)}\frac{t^{m_2}}{1+|\omega|t^{m_2}}, \quad \text{for all} \quad z\in\Gamma_1^{1/m_2}. \tag{32}$$

Finally, since

$$\int_{\Gamma_1^{1/m_2}}|\,\mathrm{d}z|=\mathrm{long}(\Gamma_1^{1/m_2})\le C\left(\frac{1/t^{m_2}}{\cos(\theta)\left(1-\frac{\tan(\theta)}{\tan(\pi-\varphi)}\right)}\right)^{1/m_2}\le\frac{C}{t}, \tag{33}$$

for some $C>0$, bounds (30)–(33) lead us to

$$\|I_1(t)\|\le\frac{CM}{|\omega|t^{m_1}}, \qquad \text{as} \quad t\to+\infty,$$

and in virtue of boundness of $E(t)$ we have

$$\|I_1(t)\|\le\frac{CM}{1+|\omega|t^{m_1}}, \qquad \text{as} \quad t\to+\infty. \tag{34}$$

Second part. Consider now $I_2(t)$, and $z \in \Gamma_2^{1/m_2}$. By the definition of Γ_2^{1/m_2} $z = \left(\rho \mathrm{e}^{\mathrm{i}\varphi}\right)^{1/m_2}$ for some $\rho \geq \rho_1$. First of all observe that the choice of Γ, and more precisely the choice of φ, implies an exponential decay of $|\mathrm{e}^{zt}|$ over Γ_2^{1/m_2}.

In this case we have to split the analysis into two parts: $|z| \leq R$, and $|z| \geq R$, for $R > 0$ large enough. Denote $\Gamma_{2,l}^{1/m_2} = \Gamma_2^{1/m_2} \cap \{z \in \mathbb{C} : |z| \leq R\}$, and $\Gamma_{2,u}^{1/m_2} = \Gamma_2^{1/m_2} \cap \{z \in \mathbb{C} : |z| \geq R\}$.

Consider first $\omega = 0$, then

$$\begin{aligned} \|I_2(t)\| &\leq \frac{M}{2\pi} \int_{\Gamma_2^{1/m_2}} \left| \frac{\mathrm{e}^{zt}}{z} \right| |\,\mathrm{d}z| = \frac{M}{2\pi} \left(\int_{\Gamma_{2,l}^{1/m_2}} \left| \frac{\mathrm{e}^{zt}}{z} \right| |\,\mathrm{d}z| + \int_{\Gamma_{2,u}^{1/m_2}} \left| \frac{\mathrm{e}^{zt}}{z} \right| |\,\mathrm{d}z| \right) \\ &= \frac{M}{2\pi} (I_{2,l}(t) + I_{2,u}(t)). \end{aligned}$$

On the one hand, parametrizing $\Gamma_{2,l}^{1/m_2}$ along the interval $\rho_1 \leq \rho \leq R^{m_2}$, we have

$$\begin{aligned} I_{2,l}(t) &= \frac{1}{m_2} \int_{\rho_1}^{R^{m_2}} \frac{\exp(\cos(\varphi/m_2) t \rho^{1/m_2}) \left| \frac{1}{m_2} \rho^{1/m_2 - 1} \mathrm{e}^{\mathrm{i}\varphi/m_2} \right|}{\left| (\rho \mathrm{e}^{\mathrm{i}\varphi})^{1/m_2} \right|} \,\mathrm{d}\rho \\ &= \frac{1}{m_2} \int_{\rho_1}^{R^{m_2}} \frac{\exp(\cos(\varphi/m_2) t \rho^{1/m_2})}{\rho} \,\mathrm{d}\rho \\ &\leq \int_{t\rho_1^{1/m_2}}^{tR} \frac{\exp(\cos(\varphi/m_2)\mu)}{\mu} \,\mathrm{d}\mu \qquad (t\rho^{1/m_2} = \mu) \\ &\leq \int_{t\rho_1^{1/m_2}}^{+\infty} \frac{\exp(\cos(\varphi/m_2)\mu)}{\mu} \,\mathrm{d}\mu \leq C. \end{aligned} \tag{35}$$

Notice that bound C in (35) does not depend on t because the integral lower limit $t\rho_1^{1/m_2}$ does not.

Moreover, parametrizing $\Gamma_{2,u}^{1/m_2}$ along the interval $\rho \geq R^{m_2}$ we have

$$\begin{aligned} I_{2,u}(t) &\leq \frac{1}{m_2} \int_{R^{m_2}}^{+\infty} \frac{\exp(\cos(\varphi/m_2) t \rho^{1/m_2})}{\rho} \,\mathrm{d}\rho \\ &= \int_{tR}^{+\infty} \frac{\exp(\cos(\varphi/m_2)\mu)}{\mu} \,\mathrm{d}\rho \qquad (t\rho^{1/m_2} = \mu) \\ &\leq \frac{\exp(\cos(\varphi/m_2) tR)}{-\cos(\varphi/m_2) tR}. \end{aligned} \tag{36}$$

Observe that, for $\omega = 0$, (36) shows an exponential decay, more than needed to state the estimate (28) for $\omega = 0$. Therefore the bounds (35) and (36) lead to the statement of theorem.

Now, we consider $\omega < 0$. As in the case $\omega = 0$, we can write

$$\begin{aligned} \|I_2(t)\| &\leq \frac{M}{2\pi} \left(\int_{\Gamma_{2,l}^{1/m_2}} \frac{|h(z)|\,|\mathrm{e}^{zt}|}{|z|\,|h(z) - \omega|} |\,\mathrm{d}z| + \int_{\Gamma_{2,u}^{1/m_2}} \frac{|h(z)|\,|\mathrm{e}^{zt}|}{|z|\,|h(z) - \omega|} |\,\mathrm{d}z| \right) \\ &= \frac{M}{2\pi} (I_{2,l}(t) + I_{2,u}(t)). \end{aligned}$$

In this point, it is straightforward to show that, for $z \in \Gamma_2^{1/m_2}$,

$$|h(z) - \omega| \geq |\omega| \sin(\theta), \tag{37}$$

and again

$$|h(z)| = |z|^{g_R(z)} \mathrm{e}^{-\mathrm{Arg}(z) g_I(z)} \leq C_M \rho^{g_R(z)/m_2}. \tag{38}$$

Therefore, with some abuse of the notation, we have

$$\begin{aligned}
I_{2,l}(t) &\leq \frac{C}{m_2|\omega|\sin(\theta)} \int_{\rho_1}^{R^{m_2}} \rho^{g_R(z)/m_2-1} \exp(\cos(\varphi/m_2)t\rho^{1/m_2})\,\mathrm{d}\rho \\
&\leq \frac{C}{|\omega|\sin(\theta)} \int_0^{+\infty} \mu^{g_R(z)-m_2} \exp(\cos(\varphi/m_2)t\mu)\mu^{m_2-1}\,\mathrm{d}\mu \quad (\rho^{1/m_2} = \mu) \\
&\leq \frac{C}{|\omega|\sin(\theta)} \int_0^1 \mu^{m_1-1} \exp(\cos(\varphi/m_2)t\mu)\,\mathrm{d}\mu \\
&\quad + \frac{1}{|\omega|\sin(\theta)} \int_1^{+\infty} \mu^{m_2-1} \exp(\cos(\varphi/m_2)t\mu)\,\mathrm{d}\mu \\
&\leq \frac{C}{|\omega|\sin(\theta)} \left(\frac{1}{(\cos(\varphi/m_2))^{m_1} t^{m_1}} + \frac{1}{(\cos(\varphi/m_2))^{m_2} t^{m_2}} \right).
\end{aligned}$$

Since we are assuming that $t \gg 0$, we conclude that there exists $C > 0$ such that

$$I_{2,l}(t) \leq \frac{C}{|\omega| t^{m_1}}. \tag{39}$$

Finally, admitting again some abuse of notation, the parametrization of Γ_2^{1/m_2} and (37) lead to

$$\begin{aligned}
I_{2,u}(t) &\leq \int_{R^{m_2}}^{+\infty} \exp(\cos(\varphi/m_2)t\rho^{1/m_2}) \frac{\rho^{g_R(z)/m_2}}{\rho^{1/m_2}} \frac{1}{|h(z)-\omega|} \frac{\rho^{1/m_2-1}}{m_2}\,\mathrm{d}\rho \\
&\leq \frac{1}{|\omega|\sin(\theta) m_2} \int_{R^{m_2}}^{+\infty} \exp(\cos(\varphi/m_2)t\rho^{1/m_2})\,\mathrm{d}\rho \\
&= \frac{1}{|\omega|\sin(\theta) t^{m_2}} \int_{tR}^{+\infty} \exp(\cos(\varphi/m_2)\mu)\mu^{m_2-1}\,\mathrm{d}\mu \qquad (t\rho^{1/m_2} = \mu) \\
&= \frac{\Gamma(m_2-1)}{|\omega|\sin(\theta) t^{m_2} \cos(\varphi/m_2)^{m_2}}.
\end{aligned} \tag{40}$$

Notice that a different bound, finer than (40), could be achieved, however the asymptotic behavior for $\omega < 0$ is restricted by (34), therefore the statement of theorem is satisfied and the proof concludes. □

Remark 8. *By the uniqueness of the Laplace transform, the evolution operator $E(t)$ defined in (15) satisfies the equation*

$$E(t)\xi = \xi + \int_0^t k(t-s) A E(s)\xi\,\mathrm{d}s \tag{41}$$

for all $\xi \in X$. This means that the family of bounded operators $\{E(t)\}_{t\geq 0}$ is a resolvent family. *These families of operators were introduced by Da Prato and Ianelli in* [41, Definition 1]*, as an extension of the notion of C_0-semigroups to solve integro–differential equations. For general kernels k in (41) and under the* 1*-regularity of k (see Definition in* [42, Chapter 1, Section 3]*) it was proved that* $\lim_{t\to+\infty} \|E(t)\| = 0$, *see* [29]. *However, the results in* [29] *show that if k is* 1*-regular, then $\|E(t)\| \leq \frac{C}{t}$ whereas the Theorem 7 above provides a better description of the behavior of $\|E(t)\|$ as $t \to +\infty$.*

6. Discrete solution: Definition and asymptotic behavior

The time discretization of (10) has been addressed in the literature by several means, numerical inverse Laplace transform [30], collocation methods [7], Adomian decomposition methods [19, 20], among others.

In this work we focus on the convolution quadrature based methods whose convergence and stability have been deeply studied [15, 16], even within the wide framework described in this work for (10), and not only but also in the more general context of Volterra equations [8, 9].

6.1. **Convolution quadratures.** Let $g : (0, +\infty) \to X$ be a function belonging to $L^1((0, +\infty), X)$, $N \in \mathbb{N}$, $T > 0$, $\tau = T/N$, and denote $t_n = n\tau$, for $0 \le n \le N$.

In this section we briefly recall how convolution quadratures formulate [32], in fact

$$\int_0^{t_n} k(t_n - s) g(s)\,\mathrm{d}s \approx \sum_{j=0}^{n} k_{n-j} g_j, \qquad 0 \le n \le N,$$

for certain weights k_n defined below, and where $g_n = g(t_n)$, for $0 \le n \le N$.

First of all if $k : (0, +\infty) \to \mathbb{R}$ is a convolution kernel admitting Laplace transform K (e.g. the ones we are considering in this paper) the inversion formula for the Laplace transform ensures the existence of a complex path Γ connecting $-\mathrm{i}\infty$ and $+\mathrm{i}\infty$ positively oriented, such that

$$\begin{aligned} \int_0^{t_n} k(t_n - s) g(s)\,\mathrm{d}s &= \int_0^{t_n} \left(\frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{\lambda(t_n - s)} K(\lambda)\,\mathrm{d}\lambda \right) g(s)\,\mathrm{d}s \\ &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma K(\lambda) \left(\int_0^{t_n} \mathrm{e}^{\lambda(t_n - s)} g(s)\,\mathrm{d}s \right) \mathrm{d}\lambda \\ &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma K(\lambda) y_\lambda(t_n)\,\mathrm{d}\lambda, \end{aligned}$$

where $y_\lambda(t)$ is the solution of the initial value problem

$$y'(t) = \lambda y(t) + g(t), \quad t > 0, \quad \text{with} \qquad y(0) = 0. \tag{42}$$

Now, we set the numerical solution $\{y_n\}_{n\ge 0}$ of (42), provided by a multistep linear method of m steps

$$\sum_{j=0}^{m} \alpha_j y_{n+j-m} = \tau \sum_{j=0}^{m} \beta_j (\lambda y_{n+j-m} + g_{n+j-m}), \tag{43}$$

where y_n stands for the approximation to $y(t_n)$, for $n \ge 0$.

These methods admit a compact formulation in terms of generating functions. In fact, if

$$P(\xi) = \alpha_0 \xi^m + \alpha_1 \xi^{m-1} + \ldots + \alpha_m \xi^0, \quad Q(\xi) = \beta_0 \xi^m + \beta_1 \xi^{m-1} + \ldots + \beta_m \xi^0,$$

$$Y(\xi) = \sum_{j=0}^{+\infty} y_j \xi^j \quad \text{and} \quad G(\xi) = \sum_{j=0}^{+\infty} g_j \xi^j,$$

then, the numerical method (43) can be compactly written

$$P(\xi) Y(\xi) = \tau Q(\xi) (\lambda Y(\xi) + G(\xi)),$$

or equivalently

$$Y(\xi) = \left(\frac{\sigma(\xi)}{\tau} - \lambda\right)^{-1} G(\xi), \tag{44}$$

where $\sigma(\xi)$ stands for the characteristic function corresponding to the multistep linear method, i.e. $\sigma(\xi) = P(\xi)/Q(\xi)$. Note that all formal series are valid for $|\xi| \leq r$, $0 < r < 1$. Therefore, if $[\cdot]_n$ denotes de $n-$th term of the formal series inside the brackets, then we have

$$\int_0^{t_n} k(t_n - s)g(s)\,\mathrm{d}s \approx \left[\frac{1}{2\pi\mathrm{i}}\int_\Gamma K(\lambda)\left(\frac{\sigma(\xi)}{\tau} - \lambda\right)^{-1} G(\xi)\,\mathrm{d}\lambda\right]_n = [L(\xi)G(\xi)]_n,$$

where $L(\xi)$ stands for the integral term

$$L(\xi) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma K(\lambda)\left(\frac{\sigma(\xi)}{\tau} - \lambda\right)^{-1} \mathrm{d}\lambda.$$

Moreover, by Cauchy's formula it is straightforward that

$$L(\xi) = K\left(\frac{\sigma(\xi)}{\tau}\right). \tag{45}$$

Therefore, (42), (44), and (45) lead to the formulation in terms of formal series of a multistep linear method based convolution quadrature, for the discretization of (10),

$$U(\xi) = \frac{\xi}{1-\xi}u_0 + K\left(\frac{\sigma(\xi)}{\tau}\right) AU(\xi), \tag{46}$$

where $U(\xi) = \sum_{j=0}^{+\infty} u_j \xi^j$, and u_j stands for the approximation to the analytic solution $u(t)$ at time level t_j. Therefore, according the notation of Section 3 we can write

$$\begin{aligned} U(\xi) &= \frac{\xi}{1-\xi}\left(I - K\left(\frac{\sigma(\xi)}{\tau}\right) A\right)^{-1} u_0 \\ &= \frac{\xi}{1-\xi}\cdot h\left(\frac{\sigma(\xi)}{\tau}\right)\left(h\left(\frac{\sigma(\xi)}{\tau}\right) I - A\right)^{-1} u_0. \end{aligned} \tag{47}$$

Notice that the function

$$\frac{\xi}{1-\xi}\cdot h\left(\frac{\sigma(\xi)}{\tau}\right)\left(h\left(\frac{\sigma(\xi)}{\tau}\right) I - A\right)^{-1},$$

is holomorphic in $|\xi| \leq r$ (even if $\omega > 0$, in that case for $\tau > 0$ small enough). The Cauchy formula along the complex path $S(\nu) = r\mathrm{e}^{\nu\mathrm{i}}$, for $-\pi < \nu \leq \pi$, allows us to write

$$u_n = D_n u_0, \tag{48}$$

where

$$D_n := \frac{1}{2\pi\mathrm{i}}\int_S \frac{1}{(1-\xi)\xi^n}\cdot h\left(\frac{\sigma(\xi)}{\tau}\right)\left(h\left(\frac{\sigma(\xi)}{\tau}\right) I - A\right)^{-1} \mathrm{d}z. \tag{49}$$

Now, applying these ideas to (10), we choose as multistep linear method the backward Euler method whose characteristic function reads $\sigma(\xi) = 1 - \xi$. In that case, the change of variable $z = (1-\xi)/\tau$, and a convenient path deformation lead to the following expression of the discrete evolution operator D_n

$$D_n = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{h(z)}{z}\, r_n(\tau z)\, (h(z)I - A)^{-1}\, \mathrm{d}z, \tag{50}$$

where $r_n(z) = 1/(1-z)^n$, $n \geq 1$.

6.2. **Asymptotic behavior.** The convergence, and stability of the method (50), and results related to the representation of the numerical solution have been already studied in [8, 9, 16]. In this section we extend the asymptotic behavior of the discrete solution studied for $\alpha(t)$ =const. [13, 28] to $\alpha(t)$ varying in time.

Theorem 9. *Let $\alpha(t) : (0,+\infty) \to (1,2)$ a function belonging to $L^1(0,+\infty)$, satisfying (H1)–(H3), and let A be an operator satisfying Assumption (A1). Assume that $\omega \leq 0$. Then there exists a constant $C > 0$ independent of n and τ such that the numerical solution (48) and (50) satisfies*

$$\|u_n\| \leq \frac{CM}{1+|\omega| t_n^{m_1}}, \qquad \textit{as} \qquad n \to +\infty.$$

Proof. First of all, we notice that, in virtue of the representation given in [8, 16] where the backward Euler based convolution quadrature is also considered, and under Assumption (A1), the discrete evolution operator (50) admits the following representation in terms of the continuous evolution operator (15)

$$D_n = \int_0^{+\infty} E(s) \rho_n(s)\, \mathrm{d}s, \quad n \geq 1, \tag{51}$$

where $\rho_n(t)$ is the measure given by

$$\rho_n(t) := \frac{\mathrm{e}^{-t/\tau}}{\tau(n-1)} \left(\frac{t}{\tau}\right)^{n-1}, \quad n \geq 0 \quad \left(\text{notice that} \quad \int_0^{+\infty} \rho_n(s)\, \mathrm{d}s = 1\right).$$

Therefore, it is straightforward that the numerical solution inherits some properties of the continuous one through this representation. In fact, since $E(t)$ is bounded as we stated in Section 3, the discrete evolution operator D_n is bounded as well, and therefore the numerical solution is bounded independently of the regularity of u_0.

On the other hand, consider the representation (50) of D_n with the complex path Γ defined in Section 3. Therefore we can write

$$D_n = \sum_{j=1}^{2} I_j^n, \ \text{ where } \ I_j^n := \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_j^{1/m_2}} \frac{h(z)}{z}\, r_n(\tau z)\, (h(z)I - A)^{-1}\, \mathrm{d}z, \quad j = 1, 2.$$

Now we prove that both integrals, I_1^n and I_2^n, satisfy the statement of the Theorem. Now and hereafter we assume that τ is small enough in each instance of the proof.

Since the proof follows the ideas of Theorem 7, we only present the key points of the proof.

First part. Consider first I_1^n, and $z \in \Gamma_1^{1/m_2}$. Therefore

$$z = \left(\frac{1}{t_n^{m_2}} + \rho \mathrm{e}^{\mathrm{i}(\pi-\theta)}\right)^{1/m_2}, \quad \text{for some} \quad 0 \leq \rho \leq \rho_0.$$

One straightforwardly has that $|1-\tau z| \geq 1-1/n$, for $z \in \Gamma_1^{1/m_2}$ and n large enough, and therefore

$$|r_n(\tau z)| = \frac{1}{|1-\tau z|^n} \leq C, \qquad \text{for} \quad z \in \Gamma_1^{1/m_2}. \tag{52}$$

If $\omega = 0$, then there exists $C > 0$ such that, as in (29)

$$\|I_1^n\| \leq \frac{CM}{2\pi} \int_{\Gamma_1^{1/m_2}} \frac{1}{|z|} |\,\mathrm{d}z| \leq \frac{CMt_n^{m_2}}{\pi \sin(\theta)} \int_0^{\rho_0} \mathrm{d}\rho = \frac{CM}{\pi \sin(\theta)\cos(\theta)\left(1 - \frac{\tan(\theta)}{\tan(\pi-\varphi)}\right)},$$

and the boundness of I_1^n is proven.

If $\omega < 0$, then applying (30)–(33) there exists $C > 0$ such that

$$\|I_1^n\| \leq \frac{M}{2\pi} \int_{\Gamma_1^{1/m_2}} \frac{|h(z)|\,|r_n(\tau z)|}{|z|\,|h(z)-\omega|} |\,\mathrm{d}z| \leq \frac{CM}{1+|\omega| t_n^{m_1}}, \quad n \geq 1, \quad (n \text{ large enough}).$$

Second part. Consider now I_2^n, and $z \in \Gamma_2^{1/m_2}$. Denote as in Theorem 7, $\Gamma_{2,l}^{1/m_2}$, and $\Gamma_{2,u}^{1/m_2}$.

First, we observe that, by virtue of the choice of Γ, and more precisely of φ, there exists $\eta > 0$ such that, for $\xi = |\xi|\mathrm{e}^{\mathrm{i}\varphi}$, there holds $1/|1-\xi| \leq \mathrm{e}^{\eta\cos(\varphi)|\xi|}$, and hence we easily have

$$|r_n(\tau z)| = \left|\frac{1}{(1-\tau z)^n}\right| \leq \exp\left(\eta \cos(\varphi/m_2)\rho^{1/m_2} t_n\right), \tag{53}$$

for $n \geq 1$ and $\rho \geq \rho_1$. In this point it is important to notice that by virtue of the choice of φ, $\left|\exp\left(\eta\cos(\varphi/m_2)\rho^{1/m_2}t_n\right)\right|$ decays exponentially over Γ_2^{1/m_2}, as $t \to +\infty$.

If $\omega = 0$, then

$$\begin{aligned}\|I_2^n\| &\leq \frac{M}{2\pi} \int_{\Gamma_2^{1/m_2}} \left|\frac{r_n(\tau z)}{z}\right| |\,\mathrm{d}z| \\ &= \frac{M}{2\pi}\left(\int_{\Gamma_{2,l}^{1/m_2}} \left|\frac{r_n(\tau z)}{z}\right| |\,\mathrm{d}z| + \int_{\Gamma_{2,u}^{1/m_2}} \left|\frac{r_n(\tau z)}{z}\right| |\,\mathrm{d}z|\right) = \frac{M}{2\pi}(I_{2,l}^n + I_{2,u}^n).\end{aligned}$$

On the one hand, parametrizing $\Gamma_{2,l}^{1/m_2}$ along the interval $\rho_1 \leq \rho \leq R^{m_2}$, regarding (53), and following the steps of the estimation in (35), we obtain the boundness of $I_{2,l}^n$, i.e. there exists $C > 0$ such that

$$I_{2,l}^n \leq C. \tag{54}$$

Moreover, parametrizing $\Gamma_{2,u}^{1/m_2}$ along the interval $\rho \geq R^{m_2}$ we have

$$\begin{aligned} I_{2,u}^n &\leq \int_{R^{m_2}}^{+\infty} \frac{\left|\frac{1}{m_2}\rho^{1/m_2-1}\mathrm{e}^{\mathrm{i}\varphi/m_2}\right|}{\left|1-\tau(\rho\mathrm{e}^{\mathrm{i}\varphi})^{1/m_2}\right|^n \left|(\rho\mathrm{e}^{\mathrm{i}\varphi})^{1/m_2}\right|}\,\mathrm{d}\rho \\ &\leq \frac{1}{m_2}\int_{R^{m_2}}^{+\infty} \frac{1}{\rho\left|\tau(\rho\mathrm{e}^{\mathrm{i}\varphi})^{1/m_2}\right|^n}\,\mathrm{d}\rho = \frac{1}{(\tau R)^n}, \end{aligned} \tag{55}$$

which is bounded if R is large enough, in fact if $\tau R \geq C$ for some $C > 1$.

The bounds (54) and (55) lead to the statement of theorem for $\omega = 0$.

Now, we set $\omega < 0$. As in the case $\omega = 0$, we can estimate $\|I_2^n\|$ as

$$\begin{aligned}\|I_2^n\| &\leq \frac{M}{2\pi}\left(\int_{\Gamma_{2,l}^{1/m_2}} \frac{|h(z)|\,|r_n(\tau z)|}{|z|\,|h(z)-\omega|}|\,\mathrm{d}z| + \int_{\Gamma_{2,u}^{1/m_2}} \frac{|h(z)|\,|r_n(\tau z)|}{|z|\,|h(z)-\omega|}|\,\mathrm{d}z|\right)\\ &= \frac{M}{2\pi}(I_{2,l}^n + I_{2,u}^n).\end{aligned}$$

In view of (37) and (38) it is easy to show that there exists $C > 0$, for $z \in \Gamma_2^{1/m_2}$,

$$I_{2,l} \leq \frac{C}{|\omega| t_n^{m_1}} \qquad \text{as} \qquad n \to +\infty. \tag{56}$$

Finally, and admitting again some abuse of the notation, we have

$$\begin{aligned}I_{2,u}^n &\leq \int_{R^{m_2}}^{+\infty} \frac{\rho^{g_R(z)/m_2}}{\rho^{1/m_2}} \frac{1}{\left|1-\tau\left(\rho \mathrm{e}^{\mathrm{i}\varphi}\right)^{1/m_2}\right|^n} \frac{1}{|h(z)-\omega|} \frac{\rho^{1/m_2-1}}{m_2}\,\mathrm{d}\rho\\ &\leq \frac{C}{|\omega|\sin(\theta) m_2} \int_{R^{m_2}}^{+\infty} \frac{1}{\left(\tau \rho^{1/m_2}\right)^n}\,\mathrm{d}\rho\\ &= \frac{C}{|\omega|\sin(\theta)\tau^{m_2}} \int_{\tau R}^{+\infty} \frac{1}{\mu^{n-m_2+1}}\,\mathrm{d}\mu \qquad (\mu = \tau\rho^{1/m_2})\\ &= \frac{C R^{m_2}}{|\omega|\sin(\theta)(n-m_2)(\tau R)^n}.\end{aligned}$$

Assuming again that $\tau R \geq C > 1$, the bound (56) is valid for $I_{2,u}^n$ as well, and the proof concludes. □

Remark 10. *Note that the numerical solution inherits the asymptotic behavior of the continuous one, in fact in case $\omega = 0$ the numerical solution is merely bounded, and if $\omega < 0$ the numerical solution decays as $1/t_n^{m_1}$, for any $u_0 \in X$ (i.e. not necessarily belonging to $D(A)$),*

7. Numerical experiments

We devote this section to illustrate numerically some of theoretical aspects discussed in previous sections.

7.1. **On the different definitions of fractional integrals of variable order.**

In this sub–section we consider the three definitions we have discussed in Section 2 for the fractional integral equations of order varying in time, recall

Def. 1: $\partial^{-\alpha(t)} g(t) := \displaystyle\int_0^t \frac{(t-s)^{\alpha(t-s)-1}}{\Gamma(\alpha(t-s))} g(s)\,\mathrm{d}s, \quad t > 0.$

Def. 2: $\partial^{-\alpha(t)} g(t) := \displaystyle\frac{1}{\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(s)-1} g(s)\,\mathrm{d}s, \quad t > 0.$

Def. 3: $\partial^{-\alpha(t)} g(t) := \displaystyle\int_0^t k(t-s) g(s)\,\mathrm{d}s, \quad t > 0, \quad \text{where} \quad k(t) := (\mathcal{L}^{-1}K)(t),$

and $\quad K(z) := \dfrac{1}{z^{z\tilde{\alpha}(z)}}.$

We also consider several choices for the viscosity function $\alpha(t)$:

$$\alpha_1(t) = 0.4\sin(t) + 1.5, \quad (\tilde{\alpha}(z) = 0.4/(z^2+1) + 1.5/z). \tag{57}$$

$$\alpha_2(t) = 0.4\cos(t) + 1.5 \quad (\tilde{\alpha}(z) = 0.4z/(z^2+1) + 1.5/z). \tag{58}$$

$$\alpha_3(t) = 0.1\mathrm{e}^{-t} + 1 \quad (\tilde{\alpha}(z) = 0.1/(z+1) + 1/z). \tag{59}$$

In Figures 2–5 we show the results of the fractional integration with variable order according the definitions Def. 1 – Def. 3, with the viscosity functions (57)–(59), and applied to the function $g(t) = t^2$. The numerical results in Figures 2–5 has been obtained by means of the trapezoidal rule for Def. 1 and Def. 2, and the backward Euler based convolution quadrature [16] for Def. 3. For all of them the final time is $T = 10$, the number of time steps is $N = 1000$, and therefore the time step size is $h = 0.01$.

Figures 2–5 are organized as follows. For each figure, the first row shows the results of integrating $g(t)$ according the three definitions Def. 1 – Def. 3, and the second row shows the differences between the results reached with these definitions. Finally, Figure 2 shows the results of integrating with integer and constant order $\alpha(t) = 1$, and Figures 3–5 show the result of using the viscosity functions $\alpha_1(t)$, $\alpha_2(t)$, and $\alpha_3(t)$ respectively.

Regarding Figures 2–5 we must point out several facts:

- The integer integration ($\alpha(t) = 1$, Figure 2) matches perfectly with the expected results. To be more precise, since trapezoidal rule is exact for polynomials up to degree 2 ($g(t) = t^2$) Def. 1 and Def. 2 coincide, in other words the difference between them vanishes (first column second row, Figure 2). On the other hand, the differences between the results provided by Def. 1 and Def. 2, and the results provided by Def. 3 are within the order of the backward Euler convolution quadrature (see Th. 2.2 [34] with $p = 1$, $\beta = 3$, and $\eta = 1$), i.e. differences are $O(1)$ (second row, first and second column Figure 2).
- The qualitative behavior of the integration differs depending on the definition but also on the viscosity function $\alpha(t)$, it keeps clearer if Def. 2 and Def. 3 are compared. In fact, observe that profile provided by Def. 1 seems to be less affected by the choice of $\alpha(t)$ (first column, first row Figures 3–5), however the profiles provided by Def. 2 and Def. 3 are more oscillatory for $\alpha_1(t)$, that for $\alpha_2(t)$ and $\alpha_3(t)$ (second and third column, first row Figures 3–4).

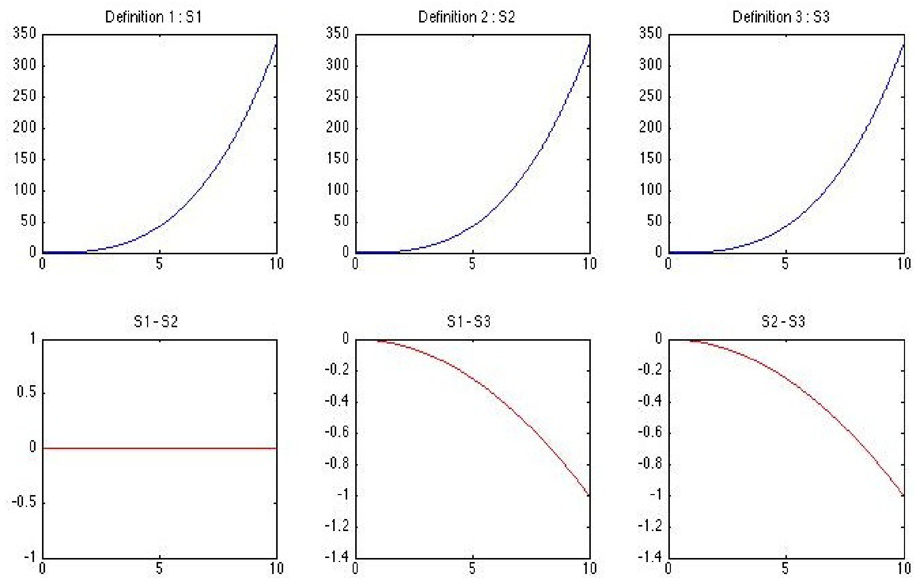

FIGURE 2. Integer integration of order 1 of $g(t) = t^2$.

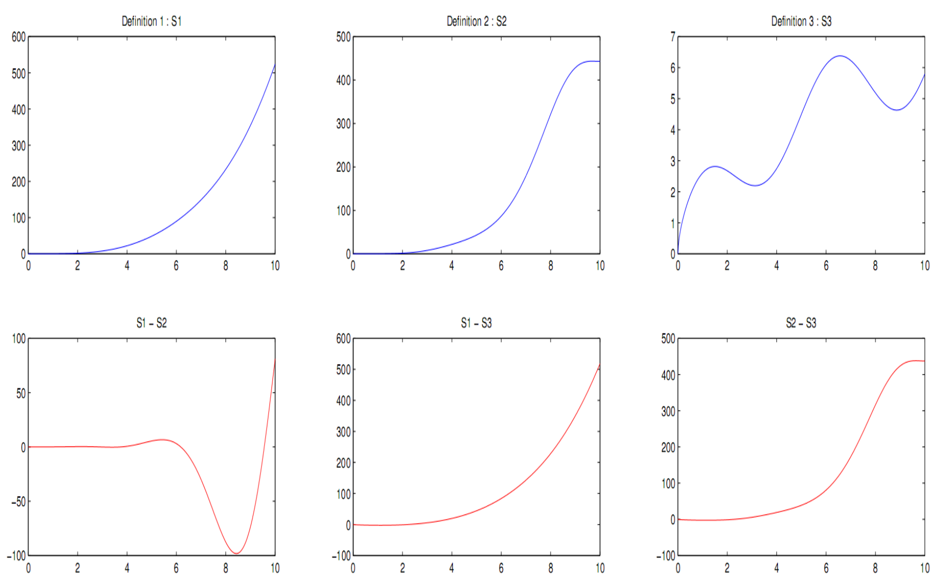

FIGURE 3. Fractional integration of $g(t) = t^2$. with $\alpha_1(t) = 0.4\sin(t) + 3/2$.

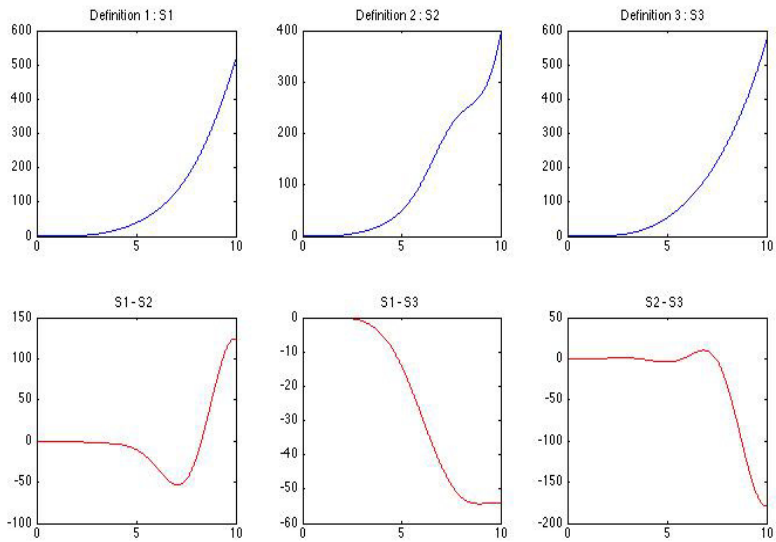

FIGURE 4. Fractional integration of $g(t) = t^2$. with $\alpha_2(t) = 0.4\cos(t) + 3/2$.

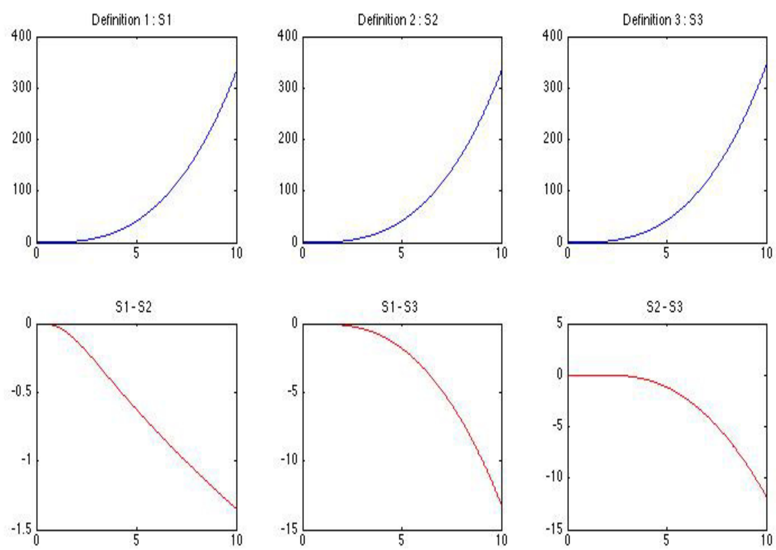

FIGURE 5. Fractional integration of $g(t) = t^2$. with $\alpha_3(t) = 0.1\mathrm{e}^{-t} + 1$.

7.2. **On the initial and boundary value problem.** Finally, we devote this sub–section to illustrate numerically the behavior of the solution of (7)–(8) when a non scalar equation is considered.

In fact, consider the integral initial–boundary value problem

$$u(x,t) = u_0(x) + \int_0^t k(t-s)(\Delta u)(x,s)\,\mathrm{d}s, \quad 0 \le t \le T, \quad x \in [a,b], \tag{60}$$

where Δ stands for the one dimensional Laplacian in an interval $[a,b]$ with homogeneous Dirichlet/Newmann boundary conditions, u_0 stands for the initial data, and k is a given convolution kernel defined as in (8). To illustrate numerically our results we consider here the viscosity functions $\alpha_1(t)$, $\alpha_3(t)$, (57) and (59) respectively, and to compare to the constant order fractional equations we also consider $\alpha_0(t) = \alpha$ constant. Notice that (60) fits in the abstract framework stated for (10).

Since differences in the behavior of 2D models cannot be accurately observed in a printed version i.e. merely by showing some selected frames, we restrict our attention to 1D examples. In the context of 2D models we refer the readers e.g. to [14] where image processing problems have been addressed by considering piecewise constant viscosity functions, always within the framework of (10).

First of all we discretize the operator Δ over an uniform spatial mesh of size $h > 0$, $x_m = a + mh$ with $h = (b-a)/M$. To this end we set a second order difference scheme whose formulation leads to the system of integral equations

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t k(t-s)(\Delta_h \mathbf{u})(s)\,\mathrm{d}s, \quad 0 \le t \le T, \tag{61}$$

where the vector $\mathbf{u}_0$ stands for the restriction of u_0 to the spatial mesh, and Δ_h stands for the three–diagonal matrix corresponding to the mentioned difference scheme for the Laplacian, including homogeneous Dirichlet/Newmann discrete boundary conditions.

Notice that the sectorial property of Δ is inherited by Δ_h, therefore the semi–discrete problem (61) fits in sectorial framework of (10) as well.

The time discretization is carried out by means of the backward Euler based convolution quadrature method described in Section 6.1. The fully discrete problem now reads

$$\mathbf{u}_n = \mathbf{u}_0 + \sum_{j=1}^{n} q_{n-j} \Delta_h \mathbf{u}_j, \quad 1 \le n \le N, \tag{62}$$

where the vector $\mathbf{u}_n$ of size $(M+1)^2 \times 1$ represents the approximation to the analytical solution $u(\cdot, t_n)$ restricted to the spatial mesh, with $t_n = n\tau$, for $0 \le n \le N$ and the step size $\tau = T/N > 0$. Let us highlight that each convolution weight q_j, for $j \ge 0$, is the j-*th* coefficient of the expansion provided when evaluating the Laplace transform of k in $\sigma(\xi)/\tau$ according (45), and all of them have been computed by means of Fast Fourier Transform techniques [33].

Since the issues related to the convergence and stability have been precisely studied in [8, 9], in this section we will focus on showing the different behaviors of the solutions depending on the choice of $\alpha(t)$. In fact in Figure 6 we consider three initial data (first column) on the spatial intervals $[0,1]$, $[0,\pi]$ and $[0,\sqrt{\pi}]$ respectively, and homogeneous Dirichlet boundary conditions. The time discretization is carried out with the final time $T = 5$, and $N = 500$, and the spatial mesh is $x_m = mh$ with $h = (b-a)/M$ and $M = 200$, on the respective intervals, and for each initial

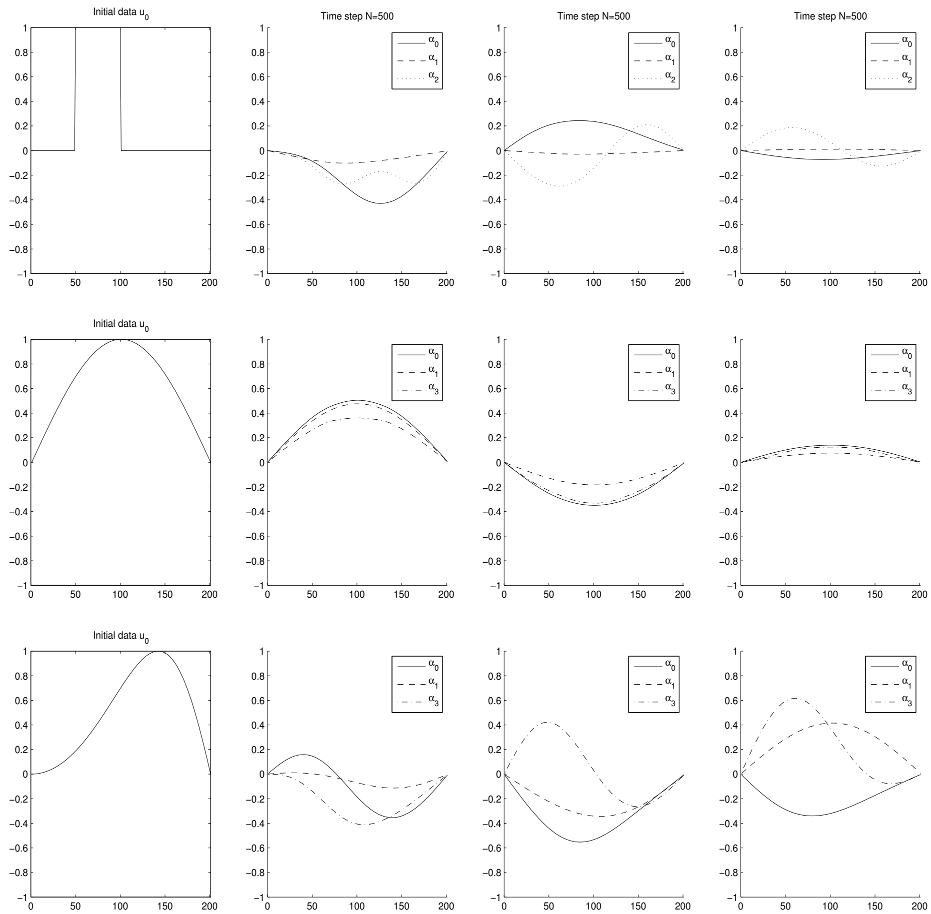

FIGURE 6. $\alpha_0(t) = 1.9, \alpha_1(t) = 0.4\sin(t) + 1.5$, and $\alpha_3(t) = 0.9\mathrm{e}^{-t} + 1$. Boundary conditions: Dirichlet homogeneous.

data. The results shown in columns 2–4 of Figure 6 stand for the evolution of $u(x,t)$ at time levels $n = 100, 200$, and 500, for each initial data. All results are compared with the results achieved with constant order of integration, in fact we have considered $\alpha_0(t) = \alpha_0$ with $\alpha_0 = 1.9$.

In Figure 7 we repeat the first experiment of Figures 6, but changing the boundary conditions now with homogeneous Newmann boundary conditions.

In view of experiments in Figures 6 and 7, we highlight the following:

- In all cases the solutions decay, as t tends to $+\infty$, even for $\alpha_3(t)$ (last row, Figures 6) whose decay turns out to be slower and cannot be clearly observed for $T = 5$. However in this case, for longer time e.g. $T = 20$, it can be numerically observed that the solution decays to 0 as well.
- The behavior of solutions strongly depend on the viscosity functions $\alpha(t)$, in fact the oscillatory behavior, or the decay as $t \to +\infty$.

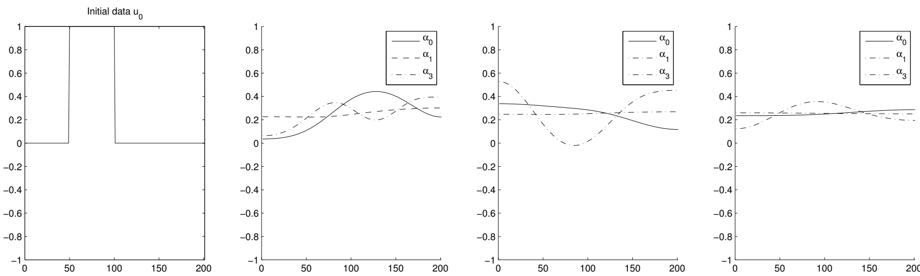

FIGURE 7. $\alpha_0(t) = 1.9, \alpha_1(t) = 0.4\sin(t) + 1.5$, and $\alpha_3(t) = 0.9e^{-t} + 1$. Boundary conditions: Newmann homogeneous.

Conclusions. In this paper we state conditions rather undemanding for the well–posedness of fractional integral equations of variable order in time, with special regard to the conditions for viscosity functions beyond the conditions stated in the literature for general Volterra equations.

The numerical results confirmed that different choices of the viscosity function and/or the choice of the definition of fractional integral lead to solutions with very different profiles.

Finally, if a source term $f(t)$ is introduced in the equation (10), and since most parts of the proofs are carried out in terms of the continuous and discrete evolution operators, then no additional and relevant difficulties are expected to extend most of results to the non homogeneous problem.

Acknowledgments. The first author has been supported by Spanish Ministerio de Economía y Competitividad under the Research Grant MTM2014-54710-P, and the second author has been supported by Fondecyt Grant 11130619.

References

[1] A. Akgul, M. Inc, and D. Baleanu. On solutions of variable–order fractional differential equations. *Int. J. Optim. Control. Theor. Appl.* **7** (2017), 112–116.

[2] A. Almeida and S. Samko. Fractional and hypersingular operators in variable exponent spaces on metrics measure spaces. *Mediterr. J. Math.* **6** (2009), 215–232.

[3] T. Atanackovic and S. Pilipovic. Hamilton's principle with variable order fractional derivatives. *Fract. Cal. Appl. Anal.* **14** (2011), 94–109.

[4] A. H. Bhrawy and M. A. Zaky. Numerical simulation for two–dimensional variable–order fractional nonlinear cable equation. *Nonlinear Dyn.* **80** (2015), 101–116.

[5] A. H. Bhrawy and M. A. Zaky. Highly accurate numerical schemes for multi–dimensional space variable–order fractional Schrödinger equations, *Comp. Math. Appl.* **73** (2017), 1100–1117.

[6] A. H. Bhrawy and M. A. Zaky. An improved collocation method for multi–dimensional space–time variable–order fractional Schrödinger equation. *Appl. Numer. Math.* **111** (2017), 197–218.

[7] H. Brunner. *Collocation Methods for Volterra Integral and Related Functional Equations,* vol. 15, Cambridge University Press, New York. P. G. Ciarlet, A. Iserles, R. V. Kohn, and M. H. Wright (eds.), 2004.

[8] M. P. Calvo, E. Cuesta, and C. Palencia. *Backward Euler method as a positivity preserving method for abstract integral equations of convolution type.* Proceedings in 2nd IFAC Workshop on Fractional Differentiation and its Applications–FDA '06, 2006, pp. 530–534.

[9] M. P. Calvo, E. Cuesta, and C. Palencia. Runge–Kutta convolution quadrature methods for well–posed equations with memory. *Numer. Math.* **107** (2007), no. 4, 589–614.

[10] W. Chen, J. Zhang, and J. Zhang. A variable–order time-fractional derivative model for chloride ions sub–diffusion in concrete structures. *Fract. Cal. Appl. Anal.* **16** (2013), 76–92.
[11] Y. Chen, L. Liu, D. Liu, and D. Boutat. Numerical study of a class of variable order nonlinear fractional differential equation in terms of Bernstein polynomials. *Eng. Physics and Math.* (2016), (in Press).
[12] C. F. M. Coimbra. Mechanics with variable–order differential operators. *Ann. Phys.* **12** (2003), no. 11-12, 692–703.
[13] E. Cuesta. Asymptotic behaviour of the solutions of fractional integro–differential equations and some time discretizations. *Discrete Contin. Dyn. Syst., Proceedings of the 6th AIMS International Conference, suppl.* (2007), 277–285.
[14] E. Cuesta, A. Durán, and M. Kirane. *On evolutionary integral models for image restoration.* Developments in Medical Image Processing and Computational Vision, Springer International Publishing, 2015, pp. 241–260.
[15] E. Cuesta, Ch. Lubich, and C. Palencia. Convolution quadrature time discretizations of fractional diffusion–wave equation. *Math. Comput.* **75** (2006), no. 254, 673–696.
[16] E. Cuesta and C. Palencia. A numerical method for an integro–differential equation with memory in Banach spaces: Qualitative properties. *SIAM J. Numer. Anal.* **41** (2003), no. 4, 1232–1241.
[17] M. Dalir. Applications of fractional calculus. *Appl. Math. Sci.* **4** (2010), no. 21, 1021–1032.
[18] G. Diaz and C. F. M. Coimbra. Nonlinear dynamics and control of a variable order oscillator with applications to the Van der Pol equation. *Nonlinear Dyn.* **56** (2009), no. 1-2, 145–157.
[19] J.-S. Duan, T. Chaolu, R. Rach, and L. Lu. The adomian decomposition method with convergence acceleration techniques for nonlinear fractional differential equations. *Comp. Math. App.* **66** (2013), 728–736.
[20] J. S. Duan, R. Rach, D. Baleanu, and A. M. Wazwaz. A review of the Adomian decomposition method and its applications to fractional differential equations. *Commun. Frac. Calc.* **3** (2012), 73–99.
[21] D. Henry. *Geometric Theory of Semilinear Parabolic Equations.* LNM 840, Springer, Berlin, 1981.
[22] R. Hilfer (ed.). *Applications of Fractional Calculus in Physics.* World Scientific Publishing Co. Pte. Ltd., 2000.
[23] D. Ingman and J. Suzdalnitsky. Control of damping oscillations by fractional differential operator with time–dependent order. *Comput. Methods Appl. Mech. Eng.* **193** (2004), 5585–5595.
[24] Y. Jia, M. Xu, and Y. Lin. A numerical solution for variable order fractional functional differential equation. *Applied Math. Letters* **64** (2017), 125–130.
[25] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations.* North–Holand Mathematics Studies 204, Elsevier B. V., 2006.
[26] X. Li, H. Li, and B. Wu. A new numerical method for variable order fractional functional differential equations. *Applied Math. Letters* **68** (2017), 80–86.
[27] Z. Liu and S.-Da Zeng. Maximum principles for multi–term space–time variable–order fractional diffusion equations and their applications. *Fract. Calc. Appl. Anal.* **19** (2016), no. 1, 188–211.
[28] C. Lizama. The Poisson distribution, abstract fractional difference equations, and stability. *Proc. Amer. Math. Soc.* **145** (2017), no. 9, 3809–3827.
[29] C. Lizama and V. Vergara. Uniform stability of resolvent families. *Proc. Amer. Math. Soc.* **132** (2004), no. 1, 175–181.
[30] M. López-Fernández and C. Palencia. On the numerical inversion of the Laplace transform of certain holomorphic mappings. *App. Numer. Math.* **51** (2005), 289–303.
[31] C. F. Lorenzo and T. T. Hartley. Variable order and distributed order fractional operators. *Nonlinear Dyn.* **29** (2002), no. 1-4, 57–98.
[32] Ch. Lubich. Convolution quadrature and discretized operational calculus, I. *Numer Math.* **52** (1988), 129–145.
[33] Ch. Lubich. Convolution quadrature and discretized operational calculus, II. *Numer. Math.* **52** (1988), 413–425.
[34] Ch. Lubich. Convolution quadrature revisited. *BIT Numer. Math.* **44** (2004), 503–514.
[35] R. L. Magin (ed.). *Fractional Processes and Fractional–Order Signal Processing. Techniques and Applications.* Springer–Verlag London, 2012.

[36] W. Malesza, D. Sierociuk, and M. Macias. Solution of fractional variable order differential equation. *2015 American Control Conference Proceedings* (2015).

[37] A. B. Malinowska, D. F. M. Torres, and T. Odzijewicz. *Advanced Methods in the Fractional Calculus of Variations.* Springerbriefs in Applied Sciences and Technology, Springer International Publishing AG Switzerland, 2015.

[38] V. Anh P. Zhuang, F. Liu and I. Turner. Numerical methods for the for the variable–order fractional advection–diffusion equation with a nonlinear source term. *SIAM J. Numer. Anal.* **47** (2009), 1760–1781.

[39] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer-Verlag, New York, 1983.

[40] H. T. C. Pedro, M. H. Kobayashi, J. M. C. Pereira, and C. F. M. Coimbra. Variable order modeling of diffusive–convectie effects on the oscillatory flow past a sphere. *J. Vib. Control* **14** (2008), no. 9-10, 1569–1672.

[41] G. Da Prato and M. Iannelli. Linear integrodifferential equations in Banach spaces. *Rend. Sem. Mat. Univ. Padova* **62** (1980), 207–219.

[42] J. Prüss. *Evolutionary Integral Equations and Applications.* Monographs in Mathematics, vol. 87, Birkhäuser Verlag, 1993.

[43] L. E. S. Ramirez and C. F. M. Coimbra. On the selection and meaning of variable order operators for dynamics modeling. *Int. J. Differ. Equ.* (2010), 16 pages.

[44] L. E. S. Ramirez and C. F. M. Coimbra. On the variable order dynamics of the nonlinear wake caused by a sediment particle. *Phys. D.* **240** (2011), no. 13, 1111–1118.

[45] A. Raziminia, A. F. Dizaji, and V. J. Majd. Solution existence for non-autonomous variable-order fractional differential equations. *Math. and Comput. Model.* **55** (2012), 1106–1117.

[46] B. Ross. Fractional integration operator of variable order in the Hölder spaces $\mathrm{H}^{\lambda(x)}$. *Internat. J. Math. & Math. Sci.* **18** (1995), no. 4, 777–788.

[47] S. G. Samko. Fractional integration and differentiation with variable order. *Anal. Mathematica* **21** (1995), 213–236.

[48] S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional Integrals and Derivatives.* Gordon and Breach Science Publishers, 1993.

[49] G. Scarpi. Sulla posibilitá di un modello reologico intermedio di tipo evolutivo. *Rend. Sc. Fis. Mat. e Nat.* **LII** (1972), 570–575.

[50] Q. Zhou, J. Gao, Z. Wang, and Kexue Li. Adaptive variable time fractional anisotropic diffusion filtering for seismic data noise attenuation. *IEEE Trans. Geosci. Remote Sens.* **54** (2016), no. 4, 1905–1917.

Department of Applied Mathematics, E.T.S.I. of Telecomunication, Campus Miguel Delibes, University of Valladolid 47011, Spain.
E-mail address: eduardo@mat.uva.es

Universidad de Talca, Instituto de Matemática y Física, Casilla 747, Talca-Chile.
E-mail address: rponce@inst-mat.utalca.cl