ALMOST SECTORIAL OPERATORS IN FRACTIONAL SUPERDIFFUSION EQUATIONS

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Abstract. In this paper the resolvent family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}\subset \mathcal{L}(X,Y)$ generated by an almost sectorial operator A, where $\alpha,\beta>0$, X,Y are complex Banach spaces and its Laplace transform satisfies $\hat{S}_{\alpha,\beta}(z)=z^{\alpha-\beta}(z^\alpha-A)^{-1}$ is studied. This family of operators allows to write the solution to an abstract initial value problem of time fractional type of order $1<\alpha<2$ as a variation of constants formula. Estimates of the norm $\|S_{\alpha,\beta}(t)\|$, as well as the continuity and compactness of $S_{\alpha,\beta}(t)$, for $t>0$, are shown. Moreover, the Hölder regularity of its solutions is also studied.

1. Introduction

Sectorial operators have been widely studied during the last four decades because in many differential equations in applied mathematics the differential operators in the linear part are one of those operators. The resolvent of a sectorial operator A satisfies the estimate $\|(z-A)^{-1}\|\leq M|z|^{-1}$ for $z\in\mathbb{C}\setminus\Sigma_\omega$ (see below for the definition of Σ_ω). Many elliptic differential operators with homogeneous boundary conditions are sectorial when they are considered in L^p-spaces or in spaces of continuous functions. For example, if Ω is a bounded subset in $\mathbb{R}^d$, $X:=C(\overline{\Omega})$ denotes the space of all continuous functions defined in $\overline{\Omega}$ and the operator A, defined by $Au:=\Delta u$, is the realization of the second order operator in X with domain $D(A)=\{u\in X:\Delta u\in X, \partial u/\partial\nu=0\}$, where $\partial u/\partial\nu$ denotes the normal derivative at the boundary of Ω, then A is a sectorial operator in X, [22, Chapter 1]. However, this elliptic operator in a more regular functions space, such as the spaces of Hölder continuous functions, may be not sectorial. In fact, if we consider A defined by $Au:=\Delta u$, with domain $D(A)=\{u\in C^{2+\beta}([0,\pi]):u(0)=u(\pi)=0\}$, where $0<\beta\leq 1$ and $C^{2+\beta}([0,\pi])$ denotes the Hölder space of all twice continuously differentiable functions u in $[0,\pi]$ such that Δu belongs to the Hölder space $C^\beta([0,\pi])$, then A is not sectorial [22, Example 3.1.33]. However, in this last case, the operator A satisfies the estimate $\|(z-A)^{-1}\|\leq M|z|^\gamma$, for all $z\in\mathbb{C}\setminus\Sigma_\omega$ and some $-1<\gamma<0$ (instead of $\gamma=-1$ as in the case of sectorial operators). Operators A satisfying this last inequality are known as *almost sectorial operators.*

On the other hand, the theory of fractional differential equations of sub and super diffusion type has been a topic of great interest in the last two decades. In particular the problem of the existence of solutions (and its regularity) to the problem

$$\partial_t^\alpha u(t)=Au(t)+f(t),\quad t\geq 0,\quad u(0)=x, \tag{1}$$

where A is a closed linear operator defined in a Banach space X, $x\in X$, f is a suitable vector-valued function (linear or non-linear), $0<\alpha<1$, and $\partial_t^\alpha u$ denotes the Caputo time–fractional derivative of u, has been widely studied over the last years. See for instance [1, 5, 6, 12, 15, 16, 17, 25] for the general theory of abstract fractional in time evolution problems, some practical applications e.g. in [23], in particular the fractional Navier–Stokes equation [11], or for a suitable background in almost sectorial operators in [29], or [32] for semilinear problems. If A is a generator of an $(\alpha,1)$–resolvent family (see below for its definition),

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then the solution to (1) is given in terms of a variation of constants formula as

$$u(t) = S_{\alpha,1}(t)x + \int_0^t S_{\alpha,\alpha}(t-s)f(s)\,\mathrm{d}s, \qquad t>0, \tag{2}$$

where, for $\alpha, \beta > 0$, $S_{\alpha,\beta}(t)$ is defined by

$$S_{\alpha,\beta}(t) := \frac{1}{2\pi i}\int_\Gamma \mathrm{e}^{zt} z^{\alpha-\beta}(z^\alpha - A)^{-1}\,\mathrm{d}z, \qquad t\geq 0, \tag{3}$$

and Γ is a suitable complex path defined within the domain of the resolvent operator $(z^\alpha - A)^{-1}$. See for instance [20].

Recently, the tools on functional calculus for almost sectorial operators have been used in [29] to study (1). More concretely, if $0<\alpha<1$ and A is an almost sectorial operators, then the resolvent families $\{S_{\alpha,1}(t)\}_{t>0}$, $\{S_{\alpha,2}(t)\}_{t>0}$, and $\{S_{\alpha,\alpha}(t)\}_{t>0}$ are continuous and compact in $\mathcal{L}(X)$. Moreover, there exist C_s and C_p positive constants depending on α and γ such that the following estimates hold

$$\|S_{\alpha,1}(t)\| \leq C_s t^{-\alpha(1+\gamma)} \quad \text{and} \quad \|S_{\alpha,\alpha}(t)\| \leq C_p t^{-\alpha(1+\gamma)}, \quad t>0.$$

As a consequence of these results, the authors study properties of the solutions to some linear abstract fractional differential equations in Banach spaces. However, we notice that these results can not be used or extended directly to study the same problem in case of $1<\alpha<2$. Therefore, the problem of the existence of solutions to the fractional initial value problem

$$\begin{cases} \partial_t^\alpha u(t) &=& Au(t)+f(t), \quad t\in[0,T],\\ u(0) &=& x,\\ u'(0) &=& y, \end{cases} \tag{4}$$

where $x,y\in X$, $1<\alpha<2$, and A is an almost sectorial operator becomes a natural one.

We notice that fractional differential equations in the form of (1) and (4) for $0<\alpha<2$, with A being a sectorial operator have been widely studied in the last decades, see for instance [3, 9, 7, 8, 10, 18, 33] and the references therein. However, the case in which $1<\alpha<2$ and A is an almost sectorial operator remains as an open problem.

From the uniqueness of the Laplace transform, it is easy to see that the solution to (4) is given by

$$u(t) = S_{\alpha,1}(t)x + S_{\alpha,2}(t)y + \int_0^t S_{\alpha,\alpha}(t-s)f(s)\,\mathrm{d}s, \quad t\in[0,T],$$

where $S_{\alpha,\beta}(t)$ given by (3) provides an important tool to study its properties in the case where A is an almost sectorial operator.

Other approaches to fractional order Cauchy problems of type (4) have been considered in the literature, e.g. with other fractional derivative/integral definitions like Hilfer one (see [26, 27, 28]).

In this paper, we consider, to the best of our knowledge, by the first time the properties of the resolvent families $\{S_{\alpha,1}(t)\}_{t>0}$, $\{S_{\alpha,2}(t)\}_{t>0}$, and $\{S_{\alpha,\alpha}(t)\}_{t>0}$, for $1<\alpha<2$, where A is an almost sectorial operator in a complex Banach space X. In fact, we study:

(1) Some estimates of the norms $\|S_{\alpha,\beta}(t)\|$, $\|AS_{\alpha,\beta}(t)\|$ for different values of $1\leq\beta\leq 2$. We notice that all the estimates provided in this paper are given in terms of computable constants, which are a key tool to find, for example, a posteriori error estimates for the time discretizations of linear and non-linear fractional differential equations, see for instance [7, 9, 8, 10].
(2) The continuity and compactness of the linear mapping $t\mapsto S_{\alpha,\beta}(t)$, for $t>0$. Here, we prove that this map is norm continuous and give a characterization (in terms of the resolvent operator $(z^\alpha - A)^{-1}$) that ensures that the function $t\mapsto S_{\alpha,\beta}(t)$ is compact for $t>0$. We notice that this criteria has great importance to study of existence of mild solutions to (1) and (4), because some fixed points arguments can be applied to solve it, see for instance [13, 19, 21].

(3) The Hölder regularity of the solutions to (4) for a given Hölder continuous function $f \in C^{\alpha_1}_{\alpha_2}((0,T];X)$, for $0 < \alpha_1 \le \alpha_2 < 1$. We notice here that the Hölder regularity can be used to study, for example, the existence and uniqueness of non-linear version of problem (4) or a posteriori error estimations for its time discretization, see for instance [10].

The paper is organized as follows. Section 2 provides the Preliminaries. Section 3 is devoted to study the properties of the resolvent family $\{S_{\alpha,\beta}(t)\}_{t\ge 0}$. Here we find estimates for the norm of $S_{\alpha,\beta}(t)$ and we prove its continuity, for $t > 0$. In Section 4 we study the Hölder regularity of the solution to the fractional Cauchy problem (4), and finally we study the compactness of the resolvent family in Section 5.

2. Preliminaries and notation

In this section, we give the preliminaries and the notation will be used throughout the paper. First of all let $X \equiv (X, \|\cdot\|_X)$ be a Banach space, which for the sake of the simplicity, and if not confusing, we denote now and hereafter merely by X, and the associated norm simply by $\|\cdot\|$. Therefore given two complex Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the Banach space of all linear and bounded operators from X into Y. If $X = Y$, then we write $\mathcal{L}(X,X) = \mathcal{L}(X)$.

Definition 1. [24] *Let $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$. By $\Theta^{\gamma}_{\omega}(X)$ we denote the family of all linear closed operators $A : D(A) \subset X \to X$ which satisfy*

(1) $\sigma(A) \subset \Sigma_\omega := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| \le \omega\} \cup \{0\}$, *and*
(2) *for every $\omega < \mu < \pi$, there exists a constant C_μ such that*

$$\|(z-A)^{-1}\| \le C_\mu |z|^{\gamma}, \qquad z \in \mathbb{C} \setminus \Sigma_\mu.$$

A linear operator A will be called almost sectorial on X *if $A \in \Theta^{\gamma}_{\omega}(X)$ (see [24])*

We remark that if A is almost sectorial, then it is not possible to conclude that A is the generator of a C_0-semigroup. Moreover, it is well known that $0 \in \rho(A)$ and therefore, A is an injective operator. Examples of sectorial, almost sectorial, and almost sectorial operators which are not sectorial, and their applications can be found in [22, Chapter 2], [24] and [31, Chapter 2].

Recall that a family of operators $\{S(t)\}_{t\ge0} \subset \mathcal{L}(X)$ is *exponentially bounded* if there exist real numbers $M > 0$ and $\omega_0 \in \mathbb{R}$ such that

$$\|S(t)\| \le M e^{\omega_0 t}, \quad t \ge 0.$$

Definition 2. [2] *Let $1 \le \alpha, \beta \le 2$, X a complex Banach space, and A be a closed linear operator with domain $D(A) \subset X$. The operator A is called the generator of an* (α,β)–resolvent family *if there exist $\omega_0 \ge 0$ and a strongly continuous function $S_{\alpha,\beta} : \mathbb{R}_+ \to \mathcal{L}(X)$ such that $\{z^\alpha : z \in \mathbb{C}, \operatorname{Re} z > \omega_0\} \subset \rho(A)$, and*

$$z^{\alpha-\beta}(z^\alpha - A)^{-1}x = \int_0^{+\infty} e^{-zt} S_{\alpha,\beta}(t)x \, dt,$$

for $\operatorname{Re} z > \omega_0$, and $x \in X$. The family $\{S_{\alpha,\beta}(t)\}_{t\ge0}$ is called the (α,β)–resolvent family generated by A.

Now, for $\beta > 0$, g_β defines the function $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}$, for $t > 0$, where $\Gamma(\cdot)$ here stands for the Gamma function. It is easy to see that, for $\alpha, \beta > 0$, $(g_\alpha * g_\beta)(t) = g_{\alpha+\beta}(t)$, where $*$ denotes the usual finite convolution, that is, $(f * g)(t) := \int_0^t f(t-s)g(s)\,ds$. Moreover, if an operator A with domain $D(A)$ is the infinitesimal generator of a resolvent family $\{S_{\alpha,\beta}(t)\}_{t\ge0}$, then, for $x \in D(A)$, we have

$$Ax = \lim_{t\to0^+} \frac{S_{\alpha,\beta}(t)x - g_\beta(t)x}{g_{\alpha+\beta}(t)}.$$

For example, if $\alpha = \beta = 1$, then $S_{1,1}(t)$ corresponds to a C_0–semigroup, if $\alpha = 2, \beta = 1$, then $S_{2,1}(t)$ is a cosine family, and if $\alpha = \beta = 2$, then $S_{2,2}(t)$ is a sine family. See [4] for further details.

For $\alpha>0$, let $m=\lceil\alpha\rceil$ be the smallest integer m greater than or equal to α. The *Caputo fractional derivative of order* α of a m–times differentiable function $f:\mathbb{R}_+\to X$ is defined by

$$\partial_t^\alpha f(t):=\int_0^t g_{m-\alpha}(t-s)f^{(m)}(s)\,\mathrm{d}s.$$

For a given Banach space Y, and $0<\alpha_1<1$, the space $C^{\alpha_1}([0,T];Y)$ denotes the set of all bounded α_1–Hölder continuous functions $g:[0,T]\to Y$, endowed with the norm

$$\|g\|_{C^{\alpha_1}([0,T];Y)}:=\sup_{0\le t\le T}\|g(t)\|_Y+[[g]]_{C^{\alpha_1}([0,T];Y)},$$

where $[[g]]_{C^{\alpha_1}([0,T];Y)}$ denotes the semi–norm

$$[[g]]_{C^{\alpha_1}([0,T];Y)}:=\sup_{0\le s<t\le T}\frac{\|g(t)-g(s)\|_Y}{(t-s)^{\alpha_1}}.$$

Moreover, if $0<\alpha_1\le\alpha_2<1$, then we define the space $C^{\alpha_1}_{\alpha_2}((0,T];Y)$ as the set of all bounded functions $g:(0,T]\to Y$ such that $t\mapsto t^{\alpha_2-\alpha_1}g(t)$ is α_1-Hölder continuous in $(0,T]$ endowed with the norm

$$\|g\|_{C^{\alpha_1}_{\alpha_2}((0,T];Y)}:=\sup_{0<t\le T}\|g(t)\|_Y+[[g]]_{C^{\alpha_1}_{\alpha_2}((0,T];Y)},$$

where $[[g]]_{C^{\alpha_1}_{\alpha_2}((0,T];Y)}$ denotes the semi–norm

$$[[g]]_{C^{\alpha_1}_{\alpha_2}((0,T];Y)}:=\sup_{0\le s<t\le T}\frac{s^{\alpha_2}\|g(t)-g(s)\|_Y}{(t-s)^{\alpha_1}}.$$

For a given $0\le\vartheta\le1$, and an almost sectorial operator A, we denote by X^ϑ the domain of the fractional power $\vartheta>0$ of A, that is $X^\vartheta:=D(A^\vartheta)$ endowed with the norm $\|x\|_\vartheta=\|A^\vartheta x\|$. In particular $X^1=D(A)$ and $X^0=X$. The following result gives a moment inequality for almost sectorial operators.

Throughout the paper we will make use over and over of a type of complex path which has always the same structure. Let us fix its notation once for all as follows: Let r be positive, $0<\theta<\pi$, and $\Gamma_{r,\theta}=\Gamma^1_{r,\theta}\cup\Gamma^2_{r,\theta}\cup\Gamma^3_{r,\theta}$ where

$$\left\{\begin{array}{lllll}\Gamma^1_{r,\theta}: & \gamma^1_{r,\theta}(\rho) & = & \rho\mathrm{e}^{\mathrm{i}\theta}, & \rho\ge r,\\ \Gamma^2_{r,\theta}: & \gamma^2_{r,\theta}(\varphi) & = & r\mathrm{e}^{\mathrm{i}\varphi}, & -\theta\le\varphi\le\theta,\\ \Gamma^3_{r,\theta}: & \gamma^3_{r,\theta}(\rho) & = & \rho\mathrm{e}^{-\mathrm{i}\theta}, & \rho\ge r.\end{array}\right. \tag{5}$$

Proposition 3 (Moment inequality)**.** *Let $A\in\Theta^\gamma_\omega(X)$ and $0<\varepsilon<1$ such that $\gamma+\varepsilon<0$. Then, there exists a constant $k>0$, depending on C_μ,γ and ε, such that*

$$\|A^\varepsilon x\|\le k\|Ax\|^{1+\gamma+\varepsilon}\|x\|^{-(\gamma+\varepsilon)},\quad x\in D(A). \tag{6}$$

Proof. For $x\in D(A)\subset X^\varepsilon$ we have (see [24], Th 2.5)

$$A^\varepsilon x=\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{r,\theta}}z^\varepsilon(z-A)^{-1}x\,\mathrm{d}z,$$

where $\Gamma_{r,\theta}$ is defined according to (5), with $\omega<\theta<\mu$ and $r>0$ is small enough. Alternatively there satisfies

$$A^\varepsilon x=A^{\varepsilon-1}(Ax)=\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{r,\theta}}z^{\varepsilon-1}(z-A)^{-1}Ax\,\mathrm{d}z.$$

Now consider $R\ge r>0$, and the complex paths

$$\Gamma_1:=\{z\in\mathbb{C}:z\in\Gamma_{r,\theta},|z|\le R\}\cup\{z\in\mathbb{C}:z=R\mathrm{e}^{\mathrm{i}\phi},-\theta<\phi\le\theta\},$$

and

$$\Gamma_2:=\{z\in\mathbb{C}:z\in\Gamma_{r,\theta},|z|\ge R\}\cup\{z\in\mathbb{C}:z=R\mathrm{e}^{\mathrm{i}\phi},-\theta<\phi\le\theta\},$$

both of them with $\mathrm{Re}(z)$ going from $-\infty$ to $+\infty$. Straightforwardly it follows that

$$A^{\varepsilon}x = I_1 + I_2,$$

where,

$$I_j := \frac{1}{2\pi \mathrm{i}}\int_{\Gamma_j} z^{\varepsilon-1}(z-A)^{-1}Ax\,\mathrm{d}z, \qquad j=1,2.$$

On the one hand, re–writing I_1, and applying the Cauchy's Theorem,

$$I_1 = \frac{1}{2\pi \mathrm{i}}\int_{\Gamma_1}\big(z^{\varepsilon}(z-A)^{-1} - z^{\varepsilon-1}\big)x\,\mathrm{d}z = \frac{1}{2\pi \mathrm{i}}\int_{\Gamma_1} z^{\varepsilon}(z-A)^{-1}x\,\mathrm{d}z,$$

Since $r>0$ may be taken as small as one needs, if in I_1 we take the limit $r\to 0^+$ and apply the boundness of the resolvent of A, then there exists $C>0$ such that

$$\|I_1\| \le \frac{1}{2\pi}\int_{\Gamma_1} |z|^{\varepsilon+\gamma}|\,\mathrm{d}z|\,\|x\| \le CR^{\varepsilon+\gamma+1}\|x\|.$$

Notice that C stands for a computable constant.

On the other hand, the parametrization of Γ_2 and the boundness of the resolvent of A, lead us to the bound

$$\|I_2\| \le \frac{1}{2\pi}\int_{\Gamma_2} |z|^{\varepsilon+\gamma-1}|\,\mathrm{d}z|\|Ax\| \le CR^{\varepsilon+\gamma}\|Ax\|,$$

where $C>0$ is a computable constant as well. Therefore

$$\|A^{\varepsilon}x\| \le C\big(R^{\varepsilon+\gamma+1}\|x\| + R^{\varepsilon+\gamma}\|Ax\|\big).$$

The choice $R=\|Ax\|/\|x\|$ gives rise to the statement of the Proposition and the proof concludes. □

The proof of the next Lemma follows as in [10, Lemma 2].

Lemma 4. *Let $\delta\ge 0$, $\alpha\pi/2<\phi<\pi$, and $1<\alpha<2$. Therefore*

$$\int_{\Gamma_{1/t,\phi}} \left|\frac{\mathrm{e}^{zt}}{z^{\delta}}\right| |\,\mathrm{d}z| \le C_0 t^{\delta-1}, \quad t>0,$$

where

$$C_0 := \left(C_\alpha + \frac{2\mathrm{e}^{\cos(\phi/\alpha)}}{-\cos(\phi/\alpha)}\right) \quad \text{and} \quad C_\alpha := \frac{1}{\alpha}\int_{-\phi}^{\phi} e^{\cos(\psi/\alpha)}\,\mathrm{d}\psi.$$

3. Estimates and continuity of the resolvent family

In this Section we provide estimates of the norm of the resolvent families $S_{\alpha,\beta}(t)$ and $AS_{\alpha,\beta}(t)$, for $1<\alpha<2$, and different values of $\beta\ge 0$. Moreover, we study the continuity of $S_{\alpha,\beta}(t)$. Throughout this section A will be an operator in $\Theta_\omega^\gamma(X)$ with $-1<\gamma<0$ and $0<\omega<\pi/2$.

Moreover, from now on, the complex path $\Gamma_{1/t,\phi}$, $t>0$, defined in the previous section will be taken with $\pi/2<\phi^\alpha<\pi$.

Theorem 5. *Let $0<\vartheta<1$ and $A\in\Theta_\omega^\gamma(X)$. Suppose that $1\le\beta\le 2$, $1+\gamma-\vartheta<0$, and $\beta-\alpha(2+2\gamma-\vartheta)>0$. If $x\in X^\vartheta$, then*

$$\|S_{\alpha,\beta}(t)x\| \le \frac{C_0}{2\pi}t^{\beta-1}\|x\| + C_1(t^{\alpha\gamma(\gamma+1-\vartheta)+\beta-1} + t^{-\alpha(2+2\gamma-\vartheta)+\beta-1})\|A^\vartheta x\|, \quad t\ge 0,$$

where $C_1 := kC_0(C_\mu+1)^{2+\gamma-\vartheta}C_\mu^{-(\gamma+1-\vartheta)}/2\pi>0$.

Proof. Let $x \in X^\vartheta$. Therefore as $z^\alpha(z^\alpha - A)^{-1} = I + A(z^\alpha - A)^{-1}$ we have $z^{\alpha-\beta}(z^\alpha - A)^{-1} = \frac{1}{z^\beta}(I + A(z^\alpha - A)^{-1})$. Hence, for $x \in X$, and $\Gamma_{1/t,\phi} = \Gamma_1 \cup \Gamma_2$ defined according to Proposition 3 (where $R > 1/t$), we have

$$\begin{aligned} S_{\alpha,\beta}(t)x &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{1/t,\phi}} \frac{e^{zt}}{z^\beta} x \, \mathrm{d}z + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{1/t,\phi}} \frac{e^{zt}}{z^\beta} A(z^\alpha - A)^{-1} x \, \mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{1/t,\phi}} \frac{e^{zt}}{z^\beta} x \, \mathrm{d}z + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{1/t,\phi}} \frac{e^{zt}}{z^\beta} A^{1-\vartheta}(z^\alpha - A)^{-1} A^\vartheta x \, \mathrm{d}z. \end{aligned}$$

On the one hand, by Lemma 4, the first integral can be estimated as

$$\left\| \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{1/t,\phi}} \frac{e^{zt}}{z^\beta} x \, \mathrm{d}z \right\| \le \frac{1}{2\pi} \int_{\Gamma_{1/t,\phi}} \frac{|e^{zt}|}{|z|^\beta} \|x\| | \, \mathrm{d}z| \le \frac{C_0}{2\pi} t^{\beta-1} \|x\|.$$

On the other hand, the second integral makes use of the facts that $(z^\alpha - A)^{-1}x$ belongs to $D(A)$ and $A(z^\alpha - A)^{-1} = z^\alpha(z^\alpha - A)^{-1} - I$, whether $z \in \Gamma_{1/t,\phi}$. Therefore we have (by the moment inequality (6) with $\varepsilon = 1 - \vartheta$) that, for $y \in X$,

$$\begin{aligned} \|A^{1-\vartheta}(z^\alpha - A)^{-1} y\| &\le k \|A(z^\alpha - A)^{-1} y\|^{1+\gamma+(1-\vartheta)} \|(z^\alpha - A)^{-1} y\|^{-(\gamma+(1-\vartheta))} \\ &\le k \Big((|z|^{\alpha(\gamma+1)} C_\mu + 1) \|y\| \Big)^{2+\gamma-\vartheta} \Big(C_\mu |z|^{\alpha\gamma} \|y\| \Big)^{-(\gamma+1-\vartheta)} \\ &= k \Big(|z|^{\alpha(\gamma+1)} C_\mu + 1 \Big)^{2+\gamma-\vartheta} C_\mu^{-(\gamma+1-\vartheta)} |z|^{-\alpha\gamma(\gamma+1-\vartheta)} \|y\|. \end{aligned} \tag{7}$$

Therefore,

$$\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{1/t,\phi}} \frac{e^{zt}}{z^\beta} A^{1-\vartheta}(z^\alpha - A)^{-1} A^\vartheta x \, \mathrm{d}z = \sum_{j=1,2} \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_j} \frac{e^{zt}}{z^\beta} A^{1-\vartheta}(z^\alpha - A)^{-1} A^\vartheta x \, \mathrm{d}z.$$

Firstly, if $z \in \Gamma_1$ and $y \in X$, then according to (7),

$$\|A^{1-\vartheta}(z^\alpha - A)^{-1} y\| \le C |z|^{-\alpha\gamma(\gamma+1-\vartheta)} \|y\|,$$

where $C = k(C_\mu + 1)^{2+\gamma-\vartheta} C_\mu^{-(\gamma+1-\vartheta)} > 0$. Therefore, by Lemma 4, we have

$$\begin{aligned} \left\| \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} \frac{e^{zt}}{z^\beta} A^{1-\vartheta}(z^\alpha - A)^{-1} A^\vartheta x \, \mathrm{d}z \right\| &\le \frac{1}{2\pi} \int_{\Gamma_1} \frac{|e^{zt}|}{|z|^\beta} \|A^{1-\vartheta}(z^\alpha - A)^{-1} A^\vartheta x\| | \, \mathrm{d}z| \\ &\le \frac{C}{2\pi} \int_{\Gamma_1} \frac{|e^{zt}|}{|z|^{\alpha\gamma(\gamma+1-\vartheta)+\beta}} | \, \mathrm{d}z| \|A^\vartheta x\| \\ &\le \frac{CC_0}{2\pi} t^{\alpha\gamma(\gamma+1-\vartheta)+\beta-1} \|A^\vartheta x\|. \end{aligned}$$

Now, if $z \in \Gamma_2$ and $y \in X$, and again according to (7)

$$\begin{aligned} \|A^{1-\vartheta}(z^\alpha - A)^{-1} y\| &\le k(C_\mu + 1)^{2+\gamma-\vartheta} |z|^{\alpha(\gamma+1)(2+\gamma-\vartheta)} C_\mu^{-(\gamma+1-\vartheta)} |z|^{-\alpha\gamma(\gamma+1-\vartheta)} \|y\| \\ &= C |z|^{\alpha(2+2\gamma-\vartheta)} \|y\|, \end{aligned}$$

where $C > 0$ stands for the positive constant defined above. Thus, by Lemma 4, we have

$$\begin{aligned} \left\| \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_2} \frac{e^{zt}}{z^\beta} A^{1-\vartheta}(z^\alpha - A)^{-1} A^\vartheta x \, \mathrm{d}z \right\| &\le \frac{1}{2\pi} \int_{\Gamma_2} \frac{|e^{zt}|}{|z|^\beta} \|A^{1-\vartheta}(z^\alpha - A)^{-1} A^\vartheta x\| | \, \mathrm{d}z| \\ &\le \frac{C}{2\pi} \int_{\Gamma_2} \frac{|e^{zt}|}{|z|^{\beta-\alpha(2+2\gamma-\vartheta)}} | \, \mathrm{d}z| \|A^\vartheta x\| \\ &\le \frac{CC_0}{2\pi} t^{\beta-\alpha(2+2\gamma-\vartheta)-1} \|A^\vartheta x\|. \end{aligned}$$

This finishes the proof. □

Remark 6. *From the Proof of Theorem 5, if $x \in X^{\vartheta}$, then*

(1) *If $z \in \Gamma_1$, then $\|A^{1-\vartheta}(z^{\alpha}-A)^{-1}A^{\vartheta}x\| \leq \frac{C_1}{C_0}|z|^{-\alpha\gamma(\gamma+1-\vartheta)}\|A^{\vartheta}x\|$.*

(2) *If $z \in \Gamma_2$, then $\|A^{1-\vartheta}(z^{\alpha}-A)^{-1}A^{\vartheta}x\| \leq \frac{C_1}{C_0}|z|^{\alpha(2+2\gamma-\vartheta)}\|A^{\vartheta}x\|$.*

From now on, wherever we split the path Γ into $\Gamma_1 \cup \Gamma_2$, we will assume that $R > 1/t$.

Remark 7. *If $x \in X^{\vartheta}$, in particular since $D(A) \subset X^{\theta}$ if $x \in D(A)$, then we have by the closed graph Theorem applied to the identity operator $I : X^{\vartheta} \mapsto X$ that $\|x\| \leq \|A^{\vartheta}x\|$. Therefore, by Theorem 5 it satisfies that, for $0 \leq t \leq T$,*

$$\begin{aligned}
&\|S_{\alpha,\beta}(t)\|_{\mathcal{L}(X^{\vartheta},X)} \\
&= \sup\{\|S_{\alpha,\beta}(t)x\| : x \in X^{\vartheta}, \|x\|_{\vartheta} \leq 1\} \\
&\leq \sup\left\{\frac{C_0}{2\pi}t^{\beta-1}\|A^{\vartheta}x\| + C_1(t^{\alpha\gamma(\gamma+1-\vartheta)+\beta-1} + t^{-\alpha(2+2\gamma-\vartheta)+\beta-1})\|A^{\vartheta}x\| : x \in X^{\vartheta}, \|x\|_{\vartheta} \leq 1\right\} \\
&\leq \frac{C_0}{2\pi}T^{\beta-1} + C_1(T^{\alpha\gamma(\gamma+1-\vartheta)+\beta-1} + T^{-\alpha(2+2\gamma-\vartheta)+\beta-1}) \\
&\leq \max\left\{\frac{C_0}{2\pi}, C_1\right\} T^{\beta-1}(1 + T^{\alpha\gamma(\gamma+1-\vartheta)} + T^{-\alpha(2+2\gamma-\vartheta)}).
\end{aligned}$$

Theorem 8. *Let $0 < \vartheta < 1$, and $A \in \Theta^{\gamma}_{\omega}(X)$. Suppose that $\alpha\gamma(\gamma+1-\vartheta)-1 > 0$, and $-\alpha(2+2\gamma-\vartheta)-1 > 0$, and denote $C_2 := C_1/2\pi$ where C_1 stands for the constant defined in Theorem 5. Therefore*

(1) *If $x \in X^{\vartheta}$, then*
$$\|AS_{\alpha,\beta}(t)x\| \leq C_2\left(t^{\beta-\alpha+\alpha\gamma(\gamma+1-\vartheta)-1} + t^{\beta-\alpha-\alpha(2+2\gamma-\vartheta)-1}\right)\|A^{\vartheta}x\|, \quad t > 0.$$
for $1 \leq \beta \leq 2$.

(2) *If in addition $x \in X^{\vartheta+1}$, then*
$$\|AS_{\alpha,1}(t)x\| \leq \frac{C_0}{2\pi}\|Ax\| + C_2(t^{\alpha\gamma(\gamma+1-\vartheta)} + t^{-\alpha(2+2\gamma-\vartheta)})\|A^{\vartheta+1}x\|, \quad t > 0.$$

Proof. Firstly, for $x \in X^{\vartheta}$, we can write
$$AS_{\alpha,\beta}(t)x = \sum_{j=1,2}\frac{1}{2\pi \mathrm{i}}\int_{\Gamma_j} e^{zt}z^{\alpha-\beta}A^{1-\vartheta}(z^{\alpha}-A)^{-1}A^{\vartheta}x\,\mathrm{d}z,$$
where the complex path $\Gamma_{1/t,\phi} = \Gamma_1 \cup \Gamma_2$ is defined throughout this proof as in the proof of Theorem 5, where $R > 1/t$.

By Remark 6,
$$\|AS_{\alpha,\beta}(t)x\| \leq \frac{C_1}{2\pi C_0}\int_{\Gamma_1}|e^{zt}||z|^{\alpha-\beta-\alpha\gamma(\gamma+1-\vartheta)}|\,\mathrm{d}z|\|A^{\vartheta}x\| + \frac{C_1}{2\pi C_0}\int_{\Gamma_2}|e^{zt}||z|^{\alpha-\beta+\alpha(2+2\gamma-\vartheta)}|\,\mathrm{d}z|\|A^{\vartheta}x\|.$$

Since $2+2\gamma-\vartheta < 0$, we have $(\gamma+1-\vartheta)+(\gamma+1) < 0$ and therefore $(\gamma+1)-\vartheta < -(\gamma+1) < 0$, that is, $(\gamma+1) < \vartheta$. As $-1 < \gamma < 0$, and $1 < \alpha < 2$, it satisfies $\alpha-\beta-\alpha\gamma(\gamma+1-\vartheta) < 0$, and $\alpha-\beta+\alpha(2+2\gamma-\vartheta) < 0$, for $1 \leq \beta \leq 2$. Consequently, by Theorem 5
$$\|AS_{\alpha,\beta}(t)x\| \leq \frac{C_1}{2\pi}\left(t^{\beta-\alpha+\alpha\gamma(\gamma+1-\vartheta)-1} + t^{\beta-\alpha-\alpha(2+2\gamma-\vartheta)-1}\right)\|A^{\vartheta}x\|, \quad t > 0,$$
which stands for the first statement of the theorem.

Secondly we consider $x \in X^{\vartheta+1}$. Since $z^{\alpha-1}(z^{\alpha}-A)^{-1} = z^{-1}A(z^{\alpha}-A)^{-1} - z^{-1}I$ we have that
$$AS_{\alpha,1}(t)x = \sum_{j=1,2}\frac{1}{2\pi \mathrm{i}}\int_{\Gamma_j}\frac{e^{zt}}{z}A^{1-\vartheta}(z^{\alpha}-A)^{-1}A^{\vartheta+1}x\,\mathrm{d}z - \frac{1}{2\pi \mathrm{i}}\int_{\Gamma_{1/t,\phi}}\frac{e^{zt}}{z}Ax\,\mathrm{d}z.$$

Therefore by Remark 6 we have

$$\begin{aligned}\|AS_{\alpha,1}(t)x\| &\leq \frac{C_1}{2\pi C_0}\int_{\Gamma_1}|e^{zt}||z|^{-1-\alpha\gamma(\gamma+1-\vartheta)}|\,\mathrm{d}z|\|A^{\vartheta+1}x\| + \frac{C_1}{2\pi C_0}\int_{\Gamma_2}|e^{zt}||z|^{-1+\alpha(2+2\gamma-\vartheta)}|\,\mathrm{d}z|\|A^{\vartheta+1}x\| \\ &\quad + \frac{1}{2\pi}\int_{\Gamma_{1/t,\phi}}\frac{|e^{zt}|}{|z|}\|Ax\||\,\mathrm{d}z|.\end{aligned}$$

Since $-\alpha\gamma(\gamma+1-\vartheta)-1<-2$, and $\alpha(2+2\gamma-\vartheta)<0$, we have by Lemma 4 that

$$\|AS_{\alpha,1}(t)x\| \leq \frac{C_1}{2\pi}t^{\alpha\gamma(\gamma+1-\vartheta)}\|A^{\vartheta+1}x\| + \frac{C_1}{2\pi}t^{-\alpha(2+2\gamma-\vartheta)}\|A^{\vartheta+1}x\| + \frac{C_0}{2\pi}\|Ax\|,$$

and the proof concludes. □

Remark 9. *Let us highlight a fact which is particularly interesting since this appears more than once in the sections below. If $\beta=\alpha$, and $x\in X^\vartheta$, then by the first statement of Theorem 8*

$$\begin{aligned}\|AS_{\alpha,\alpha}(t)\|_{\mathcal{L}(X^\vartheta,X)} &= \sup\{\|AS_{\alpha,\alpha}(t)x\| : x\in X^\vartheta, \|x\|_\vartheta\leq 1\} \\ &\leq \sup\left\{C_2(t^{\alpha\gamma(\gamma+1-\vartheta)-1}+t^{-\alpha(2+2\gamma-\vartheta)-1})\|A^\vartheta x\| : x\in X^\vartheta, \|x\|_\vartheta\leq 1\right\} \\ &\leq C_2(T^{\alpha\gamma(\gamma+1-\vartheta)-1}+T^{-\alpha(2+2\gamma-\vartheta)-1}).\end{aligned}$$

Next we show the continuity of the resolvent family.

Theorem 10. *Let $0<\vartheta<1$ and $A\in\Theta_\omega^\gamma(X)$. Suppose that $2+2\gamma-\vartheta<0$. If $1\leq\beta\leq 2$, then the map $t\mapsto S_{\alpha,\beta}(t)$ is continuous in $\mathcal{L}(X^\vartheta,X)$, for $t>0$.*

Proof. Let $x\in X^\vartheta$, $0<s<t$, and $1\leq\beta\leq 2$. We may write

$$\begin{aligned}(S_{\alpha,\beta}(t)-S_{\alpha,\beta}(s))x &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{1/t,\phi}}(\mathrm{e}^{zt}-\mathrm{e}^{zs})z^{\alpha-\beta}(z^\alpha-A)^{-1}x\,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{1/t,\phi}}\mathrm{e}^{zs}\frac{\mathrm{e}^{z(t-s)}-1}{z}z^{\alpha-\beta+1}(z^\alpha-A)^{-1}x\,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{1/t,\phi}}\mathrm{e}^{zs}\frac{\mathrm{e}^{z(t-s)}-1}{z}z^{-\beta+1}\Big(I+A(z^\alpha-A)^{-1}\Big)x\,\mathrm{d}z \\ &= I_1+I_2,\end{aligned}$$

where

$$I_1 := \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{1/t,\phi}}\mathrm{e}^{zs}\frac{\mathrm{e}^{z(t-s)}-1}{z}\frac{1}{z^{\beta-1}}x\,\mathrm{d}z,\quad I_2 := \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{1/t,\phi}}\mathrm{e}^{zs}\frac{\mathrm{e}^{z(t-s)}-1}{z}\frac{1}{z^{\beta-1}}A^{1-\vartheta}(z^\alpha-A)^{-1}A^\vartheta x\,\mathrm{d}z.$$

On the one hand there exists $C>0$ (in particular $C=\max_{z\in\Gamma_{1/t,\phi}}\left\{\frac{|\mathrm{e}^{z(t-s)}-1|}{(t-s)|z|}\right\}$) such that

$$\frac{|\mathrm{e}^{z(t-s)}-1|}{|z|}\leq C(t-s),\quad z\in\Gamma_{1/t,\phi}.$$

Therefore by Lemma 4, straightforwardly follows that

$$\|I_1\|\leq\frac{CC_0(t-s)s^{\beta-2}\|x\|}{2\pi},\qquad 0<s<t,$$

and that $I_1\to 0$, as s tends to t.

On the other hand by the bound above and Remark 6 we have

$$\|I_2\|\leq\frac{CC_1(t-s)}{2\pi C_0}\int_{\Gamma_{1/t,\phi}}|\mathrm{e}^{zs}|\left\{\frac{1}{|z|^{\beta-1+\alpha\gamma(\gamma+1-\theta)}}+\frac{1}{|z|^{\beta-1-\alpha(2\gamma+2-\theta)}}\right\}|\,\mathrm{d}z|\|A^\vartheta x\|,$$

and since $\beta-1+\alpha\gamma(\gamma+1-\theta)>0$, and $\beta-1-\alpha(2\gamma+2-\theta)>0$, by Lemma 4 again, there satisfies

$$\|I_2\| \le \frac{CC_1(t-s)}{2\pi}(s^{\beta-2+\alpha\gamma(\gamma+1-\theta)}+s^{\beta-2-\alpha(2\gamma+2-\theta)})\|A^\vartheta x\|, \quad 0<s<t.$$

In view of the above, if s tends to t, then $I_2\to 0$, and the proof concludes. □

4. Hölder regularity

In this Section we study the regularity of the solutions to (4) in the Hölder continuity sense where $A\in\Theta^\gamma_\omega(X)$. First of all, recall that the solution to Problem (4) can be written as

$$u(t) \;=\; S_{\alpha,1}(t)x+S_{\alpha,2}(t)y+\int_0^t S_{\alpha,\alpha}(t-s)f(s)\,\mathrm{d}s.$$

In fact in this Section we estimate $\|u\|_{C^{\alpha_1}_{\alpha_2}((0,T];D(A))}$ which consists of estimating the terms involved in the that norm, that is,

$$\|u\|_{C^{\alpha_1}_{\alpha_2}((0,T];D(A))}=\sup_{0<t\le T}\|u(t)\|_{D(A)}+[[u]]_{C^{\alpha_1}_{\alpha_2}((0,T];D(A))},$$

where

$$\sup_{0<t\le T}\|u(t)\|_{D(A)}=\sup_{0<t\le T}\|u(t)\|+\sup_{0<t\le T}\|Au(t)\|,$$

and

$$[[u]]_{C^{\alpha_1}_{\alpha_2}((0,T];D(A))}=\sup_{0\le s<t\le T}\frac{s^{\alpha_2}\|u(t)-u(s)\|}{(t-s)^{\alpha_1}}+\sup_{0\le s<t\le T}\frac{s^{\alpha_2}\|Au(t)-Au(s)\|}{(t-s)^{\alpha_1}}.$$

The propositions below are devoted to show estimates for each of these terms. Although most the results below can be stated in a more general framework of values of β, we here focus our attention in those required by (8).

Notice also that within this Section we assume that $f\in C^{\alpha_1}_{\alpha_2}((0,T],X^\vartheta)$, for $0<\alpha_1\le\alpha_2<1$, Hence all constants involved in the bounds below will also implicitly depend on α_1 and α_2, even though they will not explicitly appear in the notation. However, all the constants are in fact, computable.

Moreover, we state once for all the following assumptions which will be required from now on in all results below, although they are not explicitly mentioned in the statement of results. In fact, assume that

$$\alpha\gamma(\gamma+1-\vartheta)-1>0, \quad \text{and} \quad -\alpha(2+2\gamma-\vartheta)-1>0.$$

Before starting with the results and proofs of this Section, and since this will appear repeatedly we assume now and hereafter that the complex path $\Gamma_{1/t,\phi}=\Gamma_1\cup\Gamma_2$ is defined as in the proof of Theorem 5.

Proposition 11. *Let $0<\vartheta<1$ and $A\in\Theta^\gamma_\omega(X)$. If $x,y\in X^\vartheta$, then there exist constants $K_1,K_2,K_3>0$ depending on α,γ,ϑ and T such that*

$$\sup_{0<t\le T}\|u(t)\|\le K_1\|x\|_\vartheta+K_2\|y\|_\vartheta+K_3\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T],X^\vartheta)}. \tag{8}$$

Proof. As $x\in X^\vartheta$, $\alpha\gamma(\gamma+1-\vartheta)>0$, and $-\alpha(2+2\gamma-\vartheta)>1>0$, by Theorem 5 with $\beta=1$, we have

$$\|S_{\alpha,1}(t)x\| \;\le\; \frac{C_0}{2\pi}\|x\|+C_1(T^{\alpha\gamma(\gamma+1-\vartheta)}+T^{-\alpha(2+2\gamma-\vartheta)})\|A^\vartheta x\|.$$

Once again since $X^\vartheta\subset X$, we have $\|x\|\le\|A^\vartheta x\|$, for $x\in X^\vartheta$, and we get

$$\|S_{\alpha,1}(t)x\|\le K_1\|A^\vartheta x\|, \quad 0\le t\le T,$$

where

$$K_1:=\max\left\{\frac{C_0}{2\pi},C_1(T^{\alpha\gamma(\gamma+1-\vartheta)}+T^{-\alpha(2+2\gamma-\vartheta)})\right\}.$$

Similarly, as $y\in X^\vartheta$, by Theorem 5 with $\beta=2$, we have

$$\|S_{\alpha,2}(t)y\| \;\le\; \frac{C_0}{2\pi}T\|y\|+C_1(T^{1+\alpha\gamma(\gamma+1-\vartheta)}+T^{1-\alpha(2+2\gamma-\vartheta)})\|A^\vartheta y\|\le K_2\|A^\vartheta y\|, \quad 0\le t\le T,$$

where

$$K_2 := \max\left\{\frac{C_0}{2\pi}T, C_1(T^{1+\alpha\gamma(\gamma+1-\vartheta)} + T^{1-\alpha(2+2\gamma-\vartheta)})\right\}.$$

On the one hand, by Remark 7, we have (with $\beta = \alpha$)

$$\begin{aligned}
\left\|\int_0^t S_{\alpha,\alpha}(t-s)(f(s)-f(t))\,\mathrm{d}s\right\| &\leq \int_0^t \|S_{\alpha,\alpha}(t-s)\|_{\mathcal{L}(X^\vartheta,X)}\|f(s)-f(t)\|_\vartheta\,\mathrm{d}s \\
&\leq \max\left\{\frac{C_0}{2\pi}, C_1\right\} T^{\alpha-1}(1+T^{\alpha\gamma(\gamma+1-\vartheta)}+T^{-\alpha(2+2\gamma-\vartheta)})\int_0^t \frac{s^{\alpha_2}\|f(s)-f(t)\|_\vartheta}{(t-s)^{\alpha_1}}\frac{(t-s)^{\alpha_1}}{s^{\alpha_2}}\,\mathrm{d}s \\
&\leq \max\left\{\frac{C_0}{2\pi}, C_1\right\} T^{\alpha+\alpha_1-\alpha_2}(1+T^{\alpha\gamma(\gamma+1-\vartheta)}+T^{-\alpha(2+2\gamma-\vartheta)})\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)}B(\alpha_1+1,1-\alpha_2),
\end{aligned}$$

where $B(\cdot,\cdot)$ stands for the Beta function. Similarly,

$$\begin{aligned}
\left\|\int_0^t S_{\alpha,\alpha}(s)f(t)\,\mathrm{d}s\right\| &\leq \int_0^t \|S_{\alpha,\alpha}(s)\|_{\mathcal{L}(X^\vartheta,X)}\|f(t)\|_\vartheta\,\mathrm{d}s \\
&\leq \max\left\{\frac{C_0}{2\pi}, C_1\right\} T^{\alpha-1}(1+T^{\alpha\gamma(\gamma+1-\vartheta)}+T^{-\alpha(2+2\gamma-\vartheta)})\int_0^t \sup_{0<t\leq T}\|f(t)\|_\theta\,\mathrm{d}s \\
&\leq \max\left\{\frac{C_0}{2\pi}, C_1\right\} T^{\alpha}(1+T^{\alpha\gamma(\gamma+1-\vartheta)}+T^{-\alpha(2+2\gamma-\vartheta)})\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)}.
\end{aligned}$$

We conclude that

$$\sup_{0<t\leq T}\|u(t)\| \leq K_1\|x\|_{X^\vartheta} + K_2\|y\|_{X^\vartheta} + K_3\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)},$$

where $K_3 := \max\left\{\frac{C_0}{2\pi}, C_1\right\}\left(T^{\alpha+\alpha_1-\alpha_2}B(\alpha_1+1,1-\alpha_2)+T^\alpha\right)\left(1+T^{\alpha\gamma(\gamma+1-\vartheta)}+T^{-\alpha(2+2\gamma-\vartheta)}\right)$. □

Proposition 12. *Let $0<\vartheta<1$ and $A \in \Theta^\gamma_\omega(X)$. If $x \in X^{\theta+1}, y \in X^\vartheta$, then there exist constants $K_4, K_5, K_6 > 0$ depending on $\alpha, \gamma, \vartheta$ and T such that*

$$\sup_{0<t\leq T}\|Au(t)\| \leq K_4\|x\|_{X^{\vartheta+1}} + K_5\|y\|_{X^\vartheta} + K_6\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)}.$$

Proof. As $x \in X^{\vartheta+1} \subset D(A)$ and $\alpha\gamma(\gamma+1-\vartheta) > 0$ and $-\alpha(2+2\gamma-\vartheta) > 0$, by the second statement of Theorem 8 we have

$$\|AS_{\alpha,1}(t)x\| \leq \frac{C_0}{2\pi}\|Ax\| + C_2(T^{\alpha\gamma(\gamma+1-\vartheta)}+T^{-\alpha(2+2\gamma-\vartheta)})\|A^{\vartheta+1}x\|.$$

From hypotheses of the theorem it follows that $2-\alpha+\alpha\gamma(\gamma+1-\vartheta)-1>0$, and $2-\alpha-\alpha(2+2\gamma-\vartheta)-1>0$, and along with the first statement of Theorem 8, now with $\beta = 2$, we have

$$\|AS_{\alpha,2}(t)y\| \leq C_2(T^{2-\alpha+\alpha\gamma(\gamma+1-\vartheta)-1}+T^{2-\alpha-\alpha(2+2\gamma-\vartheta)-1})\|A^\vartheta y\|.$$

Again by Theorem 8, now with $\beta = \alpha$, we get

$$\begin{aligned}
\left\|\int_0^t AS_{\alpha,\alpha}(t-s)(f(s)-f(t))\,\mathrm{d}s\right\| &\leq \int_0^t \|AS_{\alpha,\alpha}(t-s)\|_{\mathcal{L}(X^\vartheta,X)}\|f(s)-f(t)\|_\vartheta\,\mathrm{d}s \\
&\leq C_2\int_0^t \left((t-s)^{\alpha\gamma(\gamma+1-\vartheta)+\alpha_1-1}+(t-s)^{-\alpha(2+2\gamma-\vartheta)+\alpha_1-1}\right)s^{(1-\alpha_2)-1} \\
&\quad \cdot\frac{s^{\alpha_2}\|f(t)-f(s)\|_\vartheta}{(t-s)^{\alpha_1}}\,\mathrm{d}s.
\end{aligned}$$

Notice that

$$\begin{aligned}\int_0^t \Big((t-s)^{\alpha\gamma(\gamma+1-\vartheta)+\alpha_1-1}+(t-s)^{-\alpha(2+2\gamma-\vartheta)+\alpha_1-1}\Big)s^{(1-\alpha_2)-1}\,\mathrm{d}s \\ = t^{\alpha\gamma(\gamma+1-\vartheta)+\alpha_1-\alpha_2}B(\alpha\gamma(\gamma+1-\vartheta)+\alpha_1,1-\alpha_2) \\ +t^{-\alpha(2+2\gamma-\vartheta)+\alpha_1-\alpha_2}B(-\alpha(2+2\gamma-\vartheta)+\alpha_1,1-\alpha_2),\end{aligned}$$

where $B(\cdot,\cdot)$ stands once again for the Beta function. If we denote

$$C_3:=C_2\Big(T^{\alpha\gamma(\gamma+1-\vartheta)+\alpha_1-\alpha_2}B(\alpha\gamma(\gamma+1-\vartheta)+\alpha_1,1-\alpha_2)+T^{-\alpha(2+2\gamma-\vartheta)+\alpha_1-\alpha_2}B(-\alpha(2+2\gamma-\vartheta)+\alpha_1,1-\alpha_2)\Big),$$

we obtain

$$\left\|\int_0^t AS_{\alpha,\alpha}(t-s)[f(s)-f(t)]\,\mathrm{d}s\right\|\leq C_3\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)}.$$

Since $f\in C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)$ we have that $f(t)\in X^\vartheta$, for $t\in(0,T]$, and as $\alpha\gamma(\gamma+1-\vartheta)>0,-\alpha(2+2\gamma-\vartheta)>0$ by Theorem 8 we get

$$\begin{aligned}\left\|\int_0^t AS_{\alpha,\alpha}(s)f(t)\,\mathrm{d}s\right\| &\leq \int_0^t\|AS_{\alpha,\alpha}(s)\|_{\mathcal{L}(X^\vartheta,X)}\|f(t)\|_\vartheta\,\mathrm{d}s \\ &\leq C_2\int_0^t(s^{\alpha\gamma(\gamma+1-\vartheta)-1}+s^{-\alpha(2+2\gamma-\vartheta)-1})\,\mathrm{d}s\|A^\vartheta f(t)\| \\ &\leq C_4(T^{\alpha\gamma(\gamma+1-\vartheta)}+T^{-\alpha(2+2\gamma-\vartheta)})\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)},\end{aligned}$$

where $C_4:=C_2\max\left\{\frac{1}{\alpha\gamma(\gamma+1-\vartheta)},\frac{1}{-\alpha(2+2\gamma-\vartheta)}\right\}$.

As $\|Ax\|\leq\|A^{1+\vartheta}x\|$, for $x\in X^{\vartheta+1}$, straightforwardly follows that

$$\sup_{0<t\leq T}\|Au(t)\| \leq K_4\|A^{1+\vartheta}x\|+K_5\|A^\vartheta y\|+K_6\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)},$$

where

$$\begin{aligned}K_4 &:= \frac{C_0}{2\pi}+C_2(T^{\alpha\gamma(\gamma+1-\vartheta)}+T^{-\alpha(2+2\gamma-\vartheta)}),\\ K_5 &:= C_2(T^{1-\alpha+\alpha\gamma(\gamma+1-\vartheta)}+T^{1-\alpha-\alpha(2+2\gamma-\vartheta)}),\\ K_6 &:= \max\Big\{C_3,C_4(T^{\alpha\gamma(\gamma+1-\vartheta)}+T^{-\alpha(2+2\gamma-\vartheta)})\Big\}.\end{aligned}$$

□

Next, we notice that if u is the solution to Problem (4), then

$$\begin{aligned}u(t)-u(s) &= (S_{\alpha,1}(t)-S_{\alpha,1}(s))x+(S_{\alpha,2}(t)-S_{\alpha,2}(s))y+\int_0^t S_{\alpha,\alpha}(t-r)f(r)\,\mathrm{d}r-\int_0^s S_{\alpha,\alpha}(t-r)f(r)\,\mathrm{d}r\\ &= (S_{\alpha,1}(t)-S_{\alpha,1}(s))x+(S_{\alpha,2}(t)-S_{\alpha,2}(s))y+\int_0^s S_{\alpha,\alpha}(r)(f(t-r)-f(s-r))\,\mathrm{d}r+\\ &\quad\int_s^t S_{\alpha,\alpha}(r)f(t-r)\,\mathrm{d}r.\end{aligned}$$

In the following propositions, we estimate the terms involved in $[[u]]_{C^{\alpha_1}_{\alpha_2}((0,T];D(A))}$ according to the expression of $u(t)-u(s)$ above.

Proposition 13. *Let $0<\vartheta<1$ and $A\in\Theta^\gamma_\omega(X)$. If $x\in X^\vartheta$, then there exists a constant $K_7>0$ depending on α,γ,ϑ and T, such that*

$$\sup_{0\leq s<t\leq T}\frac{s^{\alpha_2}\|(S_{\alpha,1}(t)-S_{\alpha,1}(s))x\|}{(t-s)^{\alpha_1}}\leq K_7\|x\|_{X^\vartheta}.$$

Proof. Let $x \in X^{\vartheta}$, and $0 \le s < t \le T$. Therefore

$$\begin{aligned}(S_{\alpha,1}(t)-S_{\alpha,1}(s))x &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{1/t,\phi}} \mathrm{e}^{zt} z^{\alpha-1}(z^{\alpha}-A)^{-1}x\,\mathrm{d}z - \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{1/t,\phi}} \mathrm{e}^{zs} z^{\alpha-1}(z^{\alpha}-A)^{-1}x\,\mathrm{d}z\\ &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{1/t,\phi}} (\mathrm{e}^{zt}-\mathrm{e}^{zs}) z^{\alpha-1}(z^{\alpha}-A)^{-1}x\,\mathrm{d}z\\ &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{1/t,\phi}} \mathrm{e}^{zs}(\mathrm{e}^{z(t-s)}-1) z^{\alpha-1}(z^{\alpha}-A)^{-1}x\,\mathrm{d}z.\end{aligned}$$

On the one hand

$$z^{\alpha-1}(z^{\alpha}-A)^{-1}x = \frac{1}{z}\Big(I + A(z^{\alpha}-A)^{-1}\Big)x = \frac{1}{z}\Big(I + A^{1-\vartheta}(z^{\alpha}-A)^{-1}A^{\vartheta}\Big)x.$$

On the other hand, by holomorphy matters

$$\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{1/t,\phi}} \mathrm{e}^{zs}(\mathrm{e}^{z(t-s)}-1)\frac{1}{z}\,\mathrm{d}z = 0.$$

Therefore since $|1-\mathrm{e}^{z(t-s)}| \le C|z|(t-s)$, for $z \in \Gamma_{1/t,\phi}$, $C := \max_{z\in\Gamma_{1/t,\phi}}\left\{\frac{|1-\mathrm{e}^{z(t-s)}|}{|z|(t-s)}\right\}$, by Remark 6 we have

$$\begin{aligned}\|(S_{\alpha,1}(t)-S_{\alpha,1}(s))x\| &\le \frac{C_1(t-s)}{2\pi C_0}\left(\int_{\Gamma_1} |\mathrm{e}^{zs}||z|^{-\alpha\gamma(\gamma+1-\vartheta)}|\,\mathrm{d}z| + \int_{\Gamma_2} |\mathrm{e}^{zs}||z|^{\alpha(2\gamma+2-\vartheta)}|\,\mathrm{d}z|\right)\|A^{\vartheta}x\|\\ &\le \frac{CC_1(t-s)}{2\pi}\left(s^{\alpha\gamma(\gamma+1-\vartheta)-1} + s^{-\alpha(2\gamma+2-\vartheta)-1}\right)\|A^{\vartheta}x\|.\end{aligned}$$

In view of the above we straightforwardly have

$$\begin{aligned}\frac{s^{\alpha_2}\|(S_{\alpha,1}(t)-S_{\alpha,1}(s))x\|}{(t-s)^{\alpha_1}} &\le \frac{CC_1}{2\pi}\frac{s^{\alpha_2}}{(t-s)^{\alpha_1-1}}\left(s^{\alpha\gamma(\gamma+1-\vartheta)-1} + s^{-\alpha(2\gamma+2-\vartheta)-1}\right)\|A^{\vartheta}x\|\\ &\le \frac{CC_1}{2\pi}\left(T^{\alpha\gamma(\gamma+1-\vartheta)+\alpha_2-\alpha_1} + T^{-\alpha(2\gamma+2-\vartheta)+\alpha_2-\alpha_1}\right)\|A^{\vartheta}x\|,\\ &\le K_7\|x\|_{\vartheta},\end{aligned}$$

where

$$K_7 := \frac{CC_1}{2\pi}\left(T^{\alpha\gamma(\gamma+1-\vartheta)+\alpha_2-\alpha_1} + T^{-\alpha(2\gamma+2-\vartheta)+\alpha_2-\alpha_1}\right),$$

which concludes the proof. □

The following proposition straightforwardly follows merely assuming some additional regularity on the data. The proof is therefore omitted.

Proposition 14. *Let $0 < \vartheta < 1$ and $A \in \Theta_{\omega}^{\gamma}(X)$. If $x \in X^{\vartheta+1}$, then there exists a constant $K_8 > 0$ depending on $\alpha, \gamma, \vartheta$ and T, such that*

$$\sup_{0\le s<t\le T}\frac{s^{\alpha_2}\|(AS_{\alpha,1}(t)-AS_{\alpha,1}(s))x\|}{(t-s)^{\alpha_1}} \le K_8\|x\|_{X^{\vartheta+1}}.$$

Proposition 15. *Let $0 < \vartheta < 1$ and $A \in \Theta_{\omega}^{\gamma}(X)$. If $x \in X^{\vartheta}$, then there exists a constant $K_9 > 0$ depending on $\alpha, \gamma, \vartheta$ and T, such that*

$$\sup_{0\le s<t\le T}\frac{s^{\alpha_2}\|(S_{\alpha,2}(t)-S_{\alpha,2}(s))x\|}{(t-s)^{\alpha_1}} \le K_9\|x\|_{X^{\vartheta}}.$$

Proof. Let $x \in X^{\vartheta}$, and $0 < s < t$. Then we have

$$\begin{aligned}(S_{\alpha,2}(t) - S_{\alpha,2}(s))x &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{1/t,\phi}} (\mathrm{e}^{zt} - \mathrm{e}^{zs}) z^{\alpha-2} (z^{\alpha} - A)^{-1} x \, \mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{1/t,\phi}} \mathrm{e}^{zs} \left(\frac{\mathrm{e}^{z(t-s)} - 1}{z} \right) \frac{1}{z} \Big(I + A(z^{\alpha} - A)^{-1} \Big) x \, \mathrm{d}z \\ &= I_1 + I_2,\end{aligned}$$

where

$$\begin{aligned} I_1 &:= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{1/t,\phi}} \mathrm{e}^{zs} \left(\frac{\mathrm{e}^{z(t-s)} - 1}{z} \right) \frac{1}{z} x \, \mathrm{d}z, \\ I_2 &:= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{1/t,\phi}} \mathrm{e}^{zs} \left(\frac{\mathrm{e}^{z(t-s)} - 1}{z} \right) \frac{1}{z} A^{1-\vartheta} (z^{\alpha} - A)^{-1} A^{\vartheta} x \, \mathrm{d}z.\end{aligned}$$

On the one hand, if C is the bound of $\frac{|1-\mathrm{e}^{z(t-s)}|}{|z|(t-s)}$ reached above, then

$$\|I_1\| \le \frac{CC_0(t-s)}{2\pi} \|x\| \le \frac{CC_0(t-s)}{2\pi} \|x\|_{X^{\vartheta}}.$$

Moreover, by Remark 6, and as $1 + \alpha\gamma(\gamma + 1 - \vartheta) > 0$ and $1 - \alpha(2\gamma + 1 - \vartheta) > 0$ by Lemma 4 we also have

$$\begin{aligned}\|I_2\| &\le \frac{CC_1(t-s)}{2\pi C_0} \left\{ \int_{\Gamma_1} |\mathrm{e}^{zs}| \frac{1}{|z|^{1+\alpha\gamma(\gamma+1-\vartheta)}} |\, \mathrm{d}z| + \int_{\Gamma_2} |\mathrm{e}^{zs}| \frac{1}{|z|^{1-\alpha(2\gamma+1-\vartheta)}} |\, \mathrm{d}z| \right\} \|x\|_{X^{\vartheta}} \\ &\le \frac{CC_1(t-s)}{2\pi} (s^{\alpha\gamma(\gamma+1-\vartheta)} + s^{1-\alpha(2\gamma+1-\vartheta)}) \|x\|_{X^{\vartheta}}.\end{aligned}$$

Therefore, with the same bound for $s^{\alpha_2}(t-s)^{1-\alpha_1}$ as in Proposition 13, there satisfies

$$\begin{aligned}\frac{s^{\alpha_2} \|(S_{\alpha,2}(t) - S_{\alpha,2}(s))x\|}{(t-s)^{\alpha_1}} &\le \left\{ \frac{CC_0(t-s)}{2\pi} + \frac{CC_1(t-s)}{2\pi} (s^{\alpha\gamma(\gamma+1-\vartheta)} + s^{-\alpha(2\gamma+1-\vartheta)}) \right\} \frac{s^{\alpha_2}}{(t-s)^{\alpha_1}} \|x\|_{X^{\vartheta}} \\ &\le K_8 \|x\|_{X^{\vartheta}},\end{aligned}$$

where

$$K_9 := \frac{C}{2\pi} \left\{ C_0 T^{1+\alpha_2-\alpha_1} + C_1 (T^{\alpha\gamma(\gamma+1-\vartheta)+1+\alpha_2-\alpha_1} + T^{-\alpha(2\gamma+1-\vartheta)+1+\alpha_2-\alpha_1}) \right\}.$$

□

The proof of the next theorem does not differ so much from the previous one, so we only present some guidelines of it.

Proposition 16. *Let $0 < \vartheta < 1$ and $A \in \Theta^{\gamma}_{\omega}(X)$. If $x \in X^{1+\vartheta}$, then there exists a constant $K_{10} > 0$ depending on $\alpha, \gamma, \vartheta$ and T, such that*

$$\sup_{0 \le s < t \le T} \frac{s^{\alpha_2} \|(AS_{\alpha,2}(t) - AS_{\alpha,2}(s))x\|}{(t-s)^{\alpha_1}} \le K_{10} \|x\|_{X^{1+\vartheta}}.$$

Proof. Similarly to the proof of Proposition 15 we have

$$\begin{aligned}&(AS_{\alpha,2}(t) - AS_{\alpha,2}(s))x \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{1/t,\phi}} (\mathrm{e}^{zt} - \mathrm{e}^{zs}) z^{\alpha-2} A (z^{\alpha} - A)^{-1} x \, \mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{1/t,\phi}} \mathrm{e}^{zs} \left(\frac{\mathrm{e}^{z(t-s)} - 1}{z} \right) \frac{1}{z} \left\{ Ax + A^{1-\vartheta} (z^{\alpha} - A)^{-1} A^{1+\vartheta} x \right\} \mathrm{d}z.\end{aligned}$$

Therefore, by Remark 6 again

$$\begin{aligned}
&\|AS_{\alpha,2}(t) - AS_{\alpha,2}(s)x\| \\
\leq\quad & \frac{C(t-s)}{2\pi}\Bigg(\int_{\Gamma_{1/t,\phi}} \frac{|e^{zs}|}{|z|}|\,dz|\|Ax\| \\
&+\frac{C_1}{C_0}\left\{\int_{\Gamma_1} |e^{zs}||z|^{-\alpha\gamma(1+\gamma-\vartheta)-1}\,dz + \int_{\Gamma_2} |e^{zs}||z|^{\alpha(2+2\gamma-\vartheta)-1}|\,dz|\right\}\left\|A^{1+\vartheta}x\right\|\Bigg) \\
\leq\quad & \frac{C(t-s)}{2\pi}\left(C_0\|Ax\| + C_1\left\{s^{\alpha\gamma(1+\gamma-\vartheta)} + s^{-\alpha(2+2\gamma-\vartheta)}\right\}\left\|A^{1+\vartheta}x\right\|\right).
\end{aligned}$$

The proof concludes noticing that $\|Ax\| \leq \|x\|_{1+\vartheta}$, and that

$$\frac{s^{\alpha_2}\|(AS_{\alpha,2}(t) - AS_{\alpha,2}(s))x\|}{(t-s)^{\alpha_1}} \leq K_{12}\|x\|_{X^{1+\vartheta}},$$

where

$$K_{10} := \frac{C}{2\pi}\left\{C_0 T^{1+\alpha_2-\alpha_1} + C_1(T^{1+\alpha_2-\alpha_1+\alpha\gamma(1+\gamma-\vartheta)} + T^{1+\alpha_2-\alpha_1-\alpha(2+2\gamma-\vartheta)})\right\}.$$

□

Proposition 17. *Let $0 < \vartheta < 1$ and $A \in \Theta_\omega^\gamma(X)$. There exists a constant $K_{11} > 0$ depending on $\alpha, \gamma, \vartheta$ and T, such that*

$$\sup_{0\leq s<t\leq T} \frac{s^{\alpha_2}\left\|\int_0^s S_{\alpha,\alpha}(r)(f(t-r) - f(s-r))\,dr\right\|}{(t-s)^{\alpha_1}} \leq K_{11}\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)}.$$

Proof. First of all notice that the following bounds which are useful in the present proof,

$$\|f(s)\| \leq \|A^\vartheta f(s)\| \leq \|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)}, \quad 0 < s \leq T,$$

and,

$$\|f(t) - f(r)\| \leq \|A^\vartheta(f(t) - f(s))\| \leq \frac{(t-s)^{\alpha_1}}{s^{\alpha_2}}\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)}, \quad 0 \leq s < t \leq T.$$

Secondly applying Remark 7 with $\beta = \alpha$ we have

$$\begin{aligned}
\left\|\int_0^s S_{\alpha,\alpha}(r)(f(t-r) - f(s-r))\,dr\right\| \quad &\leq \quad \sup_{0\leq r\leq s}\left\{\|A^\vartheta(f(t-r) - f(s-r))\|\right\}\int_0^s \|S_{\alpha,\alpha}(r)\|_{\mathcal{L}(X^\vartheta,X)}\,dr \\
&\leq K_{11}\frac{(t-s)^{\alpha_1}}{s^{\alpha_2}}\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)},
\end{aligned}$$

where

$$K_{11} := \max\left\{\frac{C_0}{2\pi}, C_1\right\}\left(\frac{T^\alpha}{\alpha} + \frac{T^{\alpha+\alpha\gamma(1+\gamma-\vartheta)}}{\alpha+\alpha\gamma(1+\gamma-\vartheta)} + \frac{T^{\alpha-\alpha\gamma(2+2\gamma-\vartheta)}}{\alpha-\alpha\gamma(2+2\gamma-\vartheta)}\right).$$

Last bound straightforwardly leads to the statement of the Proposition and the proof concludes. □

Proposition 18. *Let $0 < \vartheta < 1$ and $A \in \Theta_\omega^\gamma(X)$. There exists a constant $K_{12} > 0$ depending on $\alpha, \gamma, \vartheta$ and T, such that*

$$\sup_{0\leq s<t\leq T} \frac{s^{\alpha_2}\left\|\int_s^t S_{\alpha,\alpha}(r)f(t-r)\,dr\right\|}{(t-s)^{\alpha_1}} \leq K_{12}\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)}.$$

Proof. On the one hand, by Theorem 5 with $\beta = \alpha$ we have, for $0 < s < t \le T$,

$$\begin{aligned}
&\left\| \int_s^t S_{\alpha,\alpha}(r) f(t-r)\,\mathrm{d}r \right\| \\
&\le (t-s) \sup_{s \le r \le t} \{ \| S_{\alpha,\alpha}(r) f(t-r) \| \} \\
&\le \frac{C_0 (t-s)}{2\pi} \sup_{s \le r \le t} \left\{ (t-r)^{\alpha-1} \| f(t-r) \| \right\} \\
&\quad + (t-s) C_1 \sup_{s \le r \le t} \left\{ \left((t-r)^{\alpha + \alpha\gamma(\gamma+1-\vartheta) - 1} + (t-r)^{\alpha - \alpha(2+2\gamma-\vartheta) - 1} \right) \| A^{\vartheta} f(t-r) \| \right\} \\
&\le (t-s) K_{12}^{(1)} \| f \|_{C^{\alpha_1}_{\alpha_2}((0,T];X^{\vartheta})},
\end{aligned}$$

where

$$K_{12}^{(1)} := \frac{C_0 T^{\alpha-1}}{2\pi} + C_1 \left(T^{\alpha + \alpha\gamma(\gamma+1-\vartheta) - 1} + T^{\alpha - \alpha(2+2\gamma-\vartheta)} \right).$$

Therefore

$$\frac{s^{\alpha_2} \left\| \int_s^t S_{\alpha,\alpha}(r) f(t-r)\,\mathrm{d}r \right\|}{(t-s)^{\alpha_1}} \le s^{\alpha_2} (t-s)^{1-\alpha_1} K_{12}^{(1)} \| f \|_{C^{\alpha_1}_{\alpha_2}((0,T];X^{\vartheta})} \le K_{12} \| f \|_{C^{\alpha_1}_{\alpha_2}((0,T];X^{\vartheta})},$$

where

$$K_{12} := T^{1+\alpha_2-\alpha_1} K_{12}^{(1)},$$

which concludes the proof □

Proposition 19. *Let $0 < \vartheta < 1$ and $A \in \Theta^{\gamma}_{\omega}(X)$. If $f \in C^{\alpha_1}_{\alpha_2}((0,T];X^{\vartheta})$, then there exists a constant $K_{13} > 0$ depending on γ, ϑ and T, such that*

$$\sup_{0 \le s < t \le T} \frac{s^{\alpha_2} \left\| \int_s^t A S_{\alpha,\alpha}(r) (f(t-r) - f(s-r))\,\mathrm{d}r \right\|}{(t-s)^{\alpha_1}} \le K_{13} \| f \|_{C^{\alpha_1}_{\alpha_2}((0,T];X^{\vartheta})}.$$

Proof. Using again the notation $\Gamma_{1/t,\phi} = \Gamma_1 \cup \Gamma_2$ as in Theorem 5 we can write

$$A S_{\alpha,\alpha}(r)(f(t-r) - f(s-r)) = \sum_{j=1,2} \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_j} \mathrm{e}^{zr} A^{1-\vartheta} (z^{\alpha} - A)^{-1} A^{\vartheta} (f(t-r) - f(s-r))\,\mathrm{d}z.$$

By hypothesis, Remark 6 and Lemma 4 we get

$$\begin{aligned}
\| A S_{\alpha,\alpha}(r)(f(t-r) - f(s-r)) \| &\le \frac{C_1}{2\pi C_0} \int_{\Gamma_1} |\mathrm{e}^{zr}| |z|^{-\alpha\gamma(\gamma+1-\vartheta)} \| A^{\vartheta} (f(t-r) - f(s-r)) \| |\,\mathrm{d}z| + \\
&\quad \frac{C_1}{2\pi C_0} \int_{\Gamma_2} |\mathrm{e}^{zr}| |z|^{\alpha(2+2\gamma-\vartheta)} \| A^{\vartheta} (f(t-r) - f(s-r)) \| |\,\mathrm{d}z| \\
&\le \frac{C_1}{2\pi} (r^{\alpha\gamma(\gamma+1-\vartheta)-1} + r^{-\alpha(2+2\gamma-\vartheta)-1}) \| A^{\vartheta} (f(t-r) - f(s-r)) \|.
\end{aligned}$$

Now, as long as

$$\| A^{\vartheta} (f(t-r) - f(s-r)) \| = \frac{\| A^{\vartheta} (f(t-r) - f(s-r)) \| (s-r)^{\alpha_2}}{(t-s)^{\alpha_1}} \frac{(t-s)^{\alpha_1}}{(s-r)^{\alpha_2}} \le \| f \|_{C^{\alpha_1}_{\alpha_2}((0,T];X^{\vartheta})} \frac{(t-s)^{\alpha_1}}{(s-r)^{\alpha_2}},$$

we have

$$\begin{aligned}
&\int_0^s \| A S_{\alpha,\alpha}(r)(f(t-r) - f(s-r)) \|\,\mathrm{d}r \\
&\le \frac{C_1 (t-s)^{\alpha_1}}{2\pi} \int_0^s (r^{\alpha\gamma(\gamma+1-\vartheta)-1} + r^{-\alpha(2+2\gamma-\vartheta)-1}) (s-r)^{-\alpha_2}\,\mathrm{d}r \| f \|_{C^{\alpha_1}_{\alpha_2}((0,T];X^{\vartheta})}.
\end{aligned}$$

Now, by the definition of the Beta function it follows

$$\begin{aligned}&\int_0^s (r^{\alpha\gamma(\gamma+1-\vartheta)-1}+r^{-\alpha(2+2\gamma-\vartheta)-1})(s-r)^{-\alpha_2}\,\mathrm{d}r\\ &= s^{\alpha\gamma(\gamma+1-\vartheta)-1+1-\alpha_2}B(\alpha\gamma(\gamma+1-\vartheta),1-\alpha_2)+s^{-\alpha(2+2\gamma-\vartheta)-1+1-\alpha_2}B(-\alpha(2+2\gamma-\vartheta),1-\alpha_2).\end{aligned}$$

Therefore,

$$\begin{aligned}&\left\|\frac{s^{\alpha_2}}{(t-s)^{\alpha_1}}\int_0^s AS_{\alpha,\alpha}(r)(f(t-r)-f(s-r))\,\mathrm{d}r\right\|\\ &\le \frac{C_1}{2\pi}(s^{\alpha\gamma(\gamma+1-\vartheta)}+s^{-\alpha(2+2\gamma-\vartheta)})\\ &\quad\cdot\max\{B(\alpha\gamma(\gamma+1-\vartheta),1-\alpha_2),B(-\alpha(2+2\gamma-\vartheta),1-\alpha_2)\}\,\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)}\\ &\le K_{13}\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)},\end{aligned}$$

where

$$K_{13}:=\frac{C_1}{2\pi}\left(T^{\alpha\gamma(\gamma+1-\vartheta)}+T^{-\alpha(2+2\gamma-\vartheta)}\right)\max\{B(\alpha\gamma(\gamma+1-\vartheta),1-\alpha_2),B(-\alpha(2+2\gamma-\vartheta),1-\alpha_2)\}.$$

□

Proposition 20. *Let $0<\vartheta<1$ and $A\in\Theta^\gamma_\omega(X)$. If $f\in C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)$, then there exists a constant $K_{14}>0$ depending on α,γ,ϑ and T, such that*

$$\sup_{0\le s<t\le T}\frac{s^{\alpha_2}\left\|\int_s^t AS_{\alpha,\alpha}(r)f(t-r)\,\mathrm{d}r\right\|}{(t-s)^{\alpha_1}}\le K_{14}\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)}.$$

Proof. As in the Proof of Proposition 19 we can write

$$AS_{\alpha,\alpha}(r)f(t-r)=\sum_{j=1,2}\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_j}\mathrm{e}^{zr}A^{1-\vartheta}(z^\alpha-A)^{-1}A^\vartheta f(t-r)\,\mathrm{d}z$$

where $\Gamma_{1/t,\phi}$ splits again as in Theorem 5.

Since $\|A^\vartheta f(t-r)\|\le\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)}$, we obtain by Remark 6 and Lemma 4 that

$$\begin{aligned}\|AS_{\alpha,\alpha}(r)f(t-r)\| &\le \frac{C_1}{2\pi C_0}\left(\int_{\Gamma^a_t}|\mathrm{e}^{zr}||z|^{-\alpha\gamma(\gamma+1-\vartheta)}|\,\mathrm{d}z|+\int_{\Gamma^b_t}|\mathrm{e}^{zr}||z|^{\alpha(2+2\gamma-\vartheta)}|\,\mathrm{d}z|\right)\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)}\\ &\le \frac{C_1}{2\pi}\left(r^{\alpha\gamma(\gamma+1-\vartheta)-1}+r^{-\alpha(2+2\gamma-\vartheta)-1}\right)\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)}\\ &\le \frac{C_1}{2\pi}\left(T^{\alpha\gamma(\gamma+1-\vartheta)-1}+T^{-\alpha(2+2\gamma-\vartheta)-1}\right)\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{s^{\alpha_2}\left\|\int_s^t AS_{\alpha,\alpha}(r)f(t-r)\,\mathrm{d}r\right\|}{(t-s)^{\alpha_1}}\\ &\le \frac{C_1 s^{\alpha_2}(t-s)^{1-\alpha_1}}{2\pi}\left(T^{\alpha\gamma(\gamma+1-\vartheta)-1}+T^{-\alpha(2+2\gamma-\vartheta)-1}\right)\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)}\\ &\le K_{14}\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^\vartheta)},\end{aligned}$$

where

$$K_{14}:=\frac{C_1}{2\pi}T^{1+\alpha_2-\alpha_1}(T^{\alpha\gamma(\gamma+1-\vartheta)-1}+T^{-\alpha(2+2\gamma-\vartheta)-1}).$$

□

The next theorem is the main results of this Section and gives a Hölder regularity result of the solution u to Problem (4).

Theorem 21. *Let $0<\vartheta<1$ and $A\in\Theta_\omega^\gamma(X)$. If $x\in X^{\theta+1}, y\in X^\vartheta$, and $f\in C_{\alpha_2}^{\alpha_1}((0,T];X^\vartheta)$, then there exist constants $D_1,D_2,D_3>0$ depending on α,γ,ϑ, and T such that the solution u to Problem (4) verifies*

$$\|u\|_{C_{\alpha_2}^{\alpha_1}((0,T];D(A))}\leq D_1\|x\|_{X^{\vartheta+1}}+D_2\|y\|_{X^\vartheta}+D_3\|f\|_{C_{\alpha_2}^{\alpha_1}((0,T];X^\vartheta)}.$$

Proof. Recall that the solution to Problem (4) writes,

$$u(t)=S_{\alpha,1}(t)x+S_{\alpha,2}(t)y+\int_0^t S_{\alpha,\alpha}(r)f(t-r)\,\mathrm{d}r,\qquad 0\leq t\leq T.$$

Now, by definition

$$\|u\|_{C_{\alpha_2}^{\alpha_1}((0,T];D(A))}=\sup_{0<t\leq T}\|u(t)\|_{D(A)}+[[u]]_{C_{\alpha_2}^{\alpha_1}((0,T];D(A))},$$

where

$$\sup_{0<t\leq T}\|u(t)\|_{D(A)}=\sup_{0<t\leq T}\|u(t)\|+\sup_{0<t\leq T}\|Au(t)\|,$$

and

$$[[u]]_{C_{\alpha_2}^{\alpha_1}((0,T];D(A))}=\sup_{0\leq s<t\leq T}\frac{s^{\alpha_2}\|u(t)-u(s)\|}{(t-s)^{\alpha_1}}+\sup_{0\leq s<t\leq T}\frac{s^{\alpha_2}\|Au(t)-Au(s)\|}{(t-s)^{\alpha_1}}.$$

Let $f\in C_{\alpha_2}^{\alpha_1}((0,T];X^\vartheta)$. By Proposition 11 we have

$$\sup_{0<t\leq T}\|u(t)\|\leq K_1\|x\|_{X^\vartheta}+K_2\|y\|_{X^\vartheta}+K_3\|f\|_{C_{\alpha_2}^{\alpha_1}((0,T],X^\vartheta)},$$

for $x,y\in X^\vartheta$, and by Proposition 12

$$\sup_{0<t\leq T}\|Au(t)\|\leq K_4\|x\|_{X^{1+\vartheta}}+K_5\|y\|_{X^\vartheta}+K_6\|f\|_{C_{\alpha_2}^{\alpha_1}((0,T];X^\vartheta)},$$

for $x\in X^{\vartheta+1}, y\in X^\vartheta$. As $X^{\vartheta+1}\subset X^\vartheta$ and $\|x\|_{X^\vartheta}\leq\|x\|_{X^{\vartheta+1}}$ we obtain

$$\sup_{0<t\leq T}\|u(t)\|_{D(A)}\leq(K_1+K_4)\|x\|_{X^{\vartheta+1}}+(K_2+K_5)\|y\|_{X^\vartheta}+(K_3+K_6)\|f\|_{C_{\alpha_2}^{\alpha_1}((0,T],X^\vartheta)}. \tag{9}$$

On the other hand, as

$$\begin{aligned}u(t)-u(s) &= (S_{\alpha,1}(t)-S_{\alpha,1}(s))x+(S_{\alpha,2}(t)-S_{\alpha,2}(s))y+\int_0^s S_{\alpha,\alpha}(r)(f(t-r)-f(s-r))\,\mathrm{d}r\\ &\quad+\int_s^t S_{\alpha,\alpha}(r)f(t-r)\,\mathrm{d}r,\end{aligned}$$

we have for every single term that

$$\sup_{0\leq s<t\leq T}\frac{s^{\alpha_2}\|(S_{\alpha,1}(t)-S_{\alpha,1}(s))x\|}{(t-s)^{\alpha_1}}\leq K_7\|x\|_{X^{\vartheta+1}},\qquad \sup_{0\leq s<t\leq T}\frac{s^{\alpha_2}\|(S_{\alpha,2}(t)-S_{\alpha,2}(s))y\|}{(t-s)^{\alpha_1}}\leq K_9\|y\|_{X^\vartheta},$$

$$\sup_{0\leq s<t\leq T}\frac{s^{\alpha_2}\left\|\int_0^s S_{\alpha,\alpha}(r)(f(t-r)-f(s-r))\,\mathrm{d}r\right\|}{(t-s)^{\alpha_1}}\leq K_{11}\|f\|_{C_{\alpha_2}^{\alpha_1}((0,T];X^\vartheta)},$$

and

$$\sup_{0\leq s<t\leq T}\frac{s^{\alpha_2}\left\|\int_s^t S_{\alpha,\alpha}(r)f(t-r)\,\mathrm{d}r\right\|}{(t-s)^{\alpha_1}}\leq K_{12}\|f\|_{C_{\alpha_2}^{\alpha_1}((0,T];X^\vartheta)}.$$

Therefore,

$$\sup_{0\leq s<t\leq T}\frac{s^{\alpha_2}\|u(t)-u(s)\|}{(t-s)^{\alpha_1}}\leq K_7\|x\|_{X^{\vartheta+1}}+K_9\|y\|_{X^\vartheta}+(K_{11}+K_{12})\|f\|_{C_{\alpha_2}^{\alpha_1}((0,T];X^\vartheta)}. \tag{10}$$

Since

$$\begin{aligned} A(u(t)-u(s)) &= A(S_{\alpha,1}(t)-S_{\alpha,1}(s))x + A(S_{\alpha,2}(t)-S_{\alpha,2}(s))y + \int_0^s AS_{\alpha,\alpha}(r)(f(t-r)-f(s-r)\,\mathrm{d}r \\ &\quad + \int_s^t AS_{\alpha,\alpha}(r)f(t-r)\,\mathrm{d}r, \end{aligned}$$

we have for every single term

$$\sup_{0\le s<t\le T}\frac{s^{\alpha_2}\|(AS_{\alpha,1}(t)-AS_{\alpha,1}(s))x\|}{(t-s)^{\alpha_1}} \le K_8\|x\|_{X^{\vartheta+1}}, \quad \sup_{0\le s<t\le T}\frac{s^{\alpha_2}\|(AS_{\alpha,2}(t)-AS_{\alpha,2}(s))y\|}{(t-s)^{\alpha_1}} \le K_{10}\|y\|_{X^{\vartheta}},$$

$$\sup_{0\le s<t\le T}\frac{s^{\alpha_2}\left\|\int_s^t AS_{\alpha,\alpha}(r)(f(t-r)-f(s-r))\,\mathrm{d}r\right\|}{(t-s)^{\alpha_1}} \le K_{13}\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^{\vartheta})},$$

and

$$\sup_{0\le s<t\le T}\frac{s^{\alpha_2}\left\|\int_s^t AS_{\alpha,\alpha}(r)f(t-r)\,\mathrm{d}r\right\|}{(t-s)^{\alpha_1}} \le K_{14}\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^{\vartheta})}.$$

This implies that

$$\sup_{0\le s<t\le T}\frac{s^{\alpha_2}\|Au(t)-Au(s)\|}{(t-s)^{\alpha_1}} \le K_8\|x\|_{X^{\vartheta+1}} + K_{10}\|y\|_{X^{\vartheta}} + (K_{13}+K_{14})\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T];X^{\vartheta})}. \tag{11}$$

From the estimates (9),(10) and (11) we conclude that

$$\begin{aligned} \|u\|_{C^{\alpha_1}_{\alpha_2}((0,T];D(A))} &\le (K_1+K_4+K_7+K_8)\|x\|_{X^{\vartheta+1}} + (K_2+K_5+K_9+K_{10})\|y\|_{X^{\vartheta}} \\ &\quad + (K_3+K_6+K_{11}+K_{12}+K_{13}+K_{14})\|f\|_{C^{\alpha_1}_{\alpha_2}((0,T],X^{\vartheta})}. \end{aligned}$$

If one takes $D_1 := (K_1+K_4+K_7+K_8)$, $D_2 := (K_2+K_5+K_9+K_{10})$, and $D_3 := (K_3+K_6+K_{11}+K_{12}+K_{13}+K_{14})$, then the proof finishes. □

5. Compactness of the resolvent family

In the present Section we afford the compactness of the resolvent family $\{S_{\alpha,\beta}(t)\}_{t\ge 0}$, for $\alpha,\beta>0$, defined in Section 1. For technical reasons we study separately both cases, $1<\beta\le 2$, and $\beta=1$.

Theorem 22. *Let $0<\vartheta<1$, $A\in\Theta^{\gamma}_{\omega}(X)$, and γ,θ, so that $-1<\gamma<0$, $0<\vartheta<1$, and $2+2\gamma<\vartheta$. If $1<\beta\le 2$ then the following assertions are equivalent*

i) *$S_{\alpha,\beta}(t)$ is a compact operator in $\mathcal{L}(X^{\vartheta},X)$, for $t>0$.*
ii) *$(z-A)^{-1}$ is a compact operator, for $z\in\mathbb{C}$, $\operatorname{Re} z>\omega^{1/\alpha}$.*

Proof. $(i)\Rightarrow(ii)$ Suppose that $t\mapsto S_{\alpha,\beta}(t)$ is compact in $\mathcal{L}(X^{\vartheta},X)$, for $t>0$, and $1<\beta\le 2$. For any $z\in\mathbb{C}$, $\operatorname{Re} z>\omega^{1/\alpha}$, we have

$$z^{\alpha-\beta}(z^{\alpha}-A)^{-1} = \int_0^{+\infty} e^{-zt}S_{\alpha,\beta}(t)\,\mathrm{d}t.$$

By Theorem 10, the integral in the right–hand side exists in the sense of Bochner because $t\mapsto S_{\alpha,\beta}(t)$ is continuous in $\mathcal{L}(X^{\vartheta},X)$. By [30, Corollary 2.3] we have that $(z^{\alpha}-A)^{-1}$ is a compact operator.

$(ii)\Rightarrow(i)$ Let $t>0$. For $1<\beta<2$, recall we may write

$$S_{\alpha,\beta}(t) = (g_{\beta-1}*S_{\alpha,1})(t), \quad t>0,$$

in $\mathcal{L}(X^{\vartheta},X)$. Therefore,

$$\frac{1}{2\pi i}\int_{\Gamma} \mathrm{e}^{zt}z^{\alpha-\beta}(z^{\alpha}-A)^{-1}\,\mathrm{d}z = S_{\alpha,\beta}(t), \quad t>0,$$

where $\Gamma = \{\omega + is : s \in \mathbb{R}\}$ is nothing but a complex path with increasing imaginary part, and by [30, Corollary 2.3] $S_{\alpha,\beta}(t)$ is compact in $\mathcal{L}(X^\vartheta, X)$.

Now, we take $\beta = 2$. In $\mathcal{L}(X^\vartheta, X)$ we have again

$$S_{\alpha,2}(t) = (g_1 * S_{\alpha,1})(t), \quad t > 0,$$

and by [14, Proposition 2.1], we conclude that $S_{\alpha,2}(t)$, $t > 0$, is compact as well. □

Theorem 23. *Let $0 < \vartheta < 1$, $A \in \Theta_\omega^\gamma(X)$, and γ, ϑ so that $-1 < \gamma < 0$, $0 < \vartheta < 1$, $2 + 2\gamma - \vartheta < 0$, and $\alpha\gamma(\gamma + 1 - \vartheta) - 1 > 0$. Therefore the following assertions are equivalent*

i) *$S_{\alpha,1}(t)$ is a compact operator in $\mathcal{L}(X^\vartheta, X)$, for $t > 0$.*
ii) *$(z - A)^{-1}$ is a compact operator, for $z \in \mathbb{C}$, $\operatorname{Re} z > \omega^{1/\alpha}$.*

Proof. $(i) \Rightarrow (ii)$ Suppose that $t \mapsto S_{\alpha,1}(t)$ is compact in $\mathcal{L}(X^\vartheta, X)$, for $t > 0$. For $\operatorname{Re} z > \omega^{1/\alpha}$ we have

$$z^{\alpha-1}(z^\alpha - A)^{-1} = \int_0^{+\infty} \mathrm{e}^{-zt} S_{\alpha,1}(t)\,\mathrm{d}t.$$

Since by Theorem 10, the map $t \mapsto S_{\alpha,1}(t)$ is continuous in $\mathcal{L}(X^\vartheta, X)$, the integral in the right–hand side is well defined in the sense of Bochner, and by [30, Corollary 2.3] we have that $(z^\alpha - A)^{-1}$ is a compact operator.

$(ii) \Rightarrow (i)$ Conversely, let $0 < s < t$, and $x \in X^\vartheta$. Therefore

$$\begin{aligned}(S_{\alpha,1}(t) - S_{\alpha,1}(s))x &= \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_{1/t,\phi}} \frac{\mathrm{e}^{zt} - \mathrm{e}^{sz}}{z} z^\alpha (z^\alpha - A)^{-1} x\,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_{1/t,\phi}} \mathrm{e}^{sz} \frac{\mathrm{e}^{z(t-s)} - 1}{z} (I + A(z^\alpha - A)^{-1}) x\,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_{1/t,\phi}} \mathrm{e}^{sz} \frac{\mathrm{e}^{z(t-s)} - 1}{z} x\,\mathrm{d}z + \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_{1/t,\phi}} \mathrm{e}^{sz} \frac{\mathrm{e}^{z(t-s)} - 1}{z} A^{1-\vartheta}(z^\alpha - A)^{-1}) A^\vartheta x\,\mathrm{d}z.\end{aligned}$$

As the first integral in the last equality turns out to be zero since the singularity in there stands for a removable singularity, we have that

$$(S_{\alpha,1}(t) - S_{\alpha,1}(s))x = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_{1/t,\phi}} \mathrm{e}^{sz} \frac{\mathrm{e}^{z(t-s)} - 1}{z} A^{1-\vartheta}(z^\alpha - A)^{-1}) A^\vartheta x\,\mathrm{d}z.$$

On the other hand, as we have noticed before, there exists $C > 0$ (precisely detailed above) such that $|\mathrm{e}^{z(t-s)} - 1)|/|z|(t-s) \le C$, for $z \in \Gamma_{1/t,\phi}$. Therefore, by Lemma 4, and Remarks 6, according to the notation $\Gamma_{1/t,\phi} = \Gamma_1 \cup \Gamma_2$ in Theorem 5, we have

$$\begin{aligned}&\|(S_{\alpha,1}(t) - S_{\alpha,1}(s))x\| \\ &\le \frac{C(t-s)}{2\pi} \int_{\Gamma_{1/t,\phi}} |\mathrm{e}^{sz}| \|A^{1-\vartheta}(z^\alpha - A)^{-1}) A^\vartheta x\| |\,\mathrm{d}z| \\ &\le \frac{C_1 C(t-s)}{2\pi C_0} \left\{ \int_{\Gamma_1} |\mathrm{e}^{sz}||z|^{-\alpha\gamma(1+\gamma-\vartheta)} |\,\mathrm{d}z| + \int_{\Gamma_2} |\mathrm{e}^{sz}||z|^{\alpha(2+2\gamma-\vartheta)} |\,\mathrm{d}z| \right\} \|A^\vartheta x\| \\ &\le \frac{C_1 C(t-s)}{2\pi} \left\{ s^{1+\alpha\gamma(1+\gamma-\vartheta)} + s^{1-\alpha(2+2\gamma-\vartheta)} \right\} \|A^\vartheta x\| \\ &\le \frac{C_1 C(t-s)}{2\pi} \left\{ T^{1+\alpha\gamma(1+\gamma-\vartheta)} + T^{1-\alpha(2+2\gamma-\vartheta)} \right\} \|A^\vartheta x\|\end{aligned}$$

Thus, if t tends to s, then the last inequality implies that $\|(S_{\alpha,1}(t) - S_{\alpha,1}(s))x\| \to 0$. That is, $S_{\alpha,1}(t)$ is continuous in $\mathcal{L}(X^\vartheta, X)$, and by [30, Corollary 2.3] we conclude that $S_{\alpha,1}(t)$ is compact in $\mathcal{L}(X^\vartheta, X)$, for $t > 0$.

□

6. Conclusions and future works

In the present paper the authors focused on abstract evolution equations of fractional type in the framework of almost sectorial operators. Actually fractional equations belong to a very large class of evolution equations wich are known as Volterra type equations or more generally equations with memory.

In a natural manner the extension of the present study to general Volterra equations where the convolution kernel is not restricted to a fractional type one but to more general kernels is expected to be addressed in the near future. But not only, as far as semilinear equations are highly interesting in practical instances, the study of this kind of equations in a semilinear format are also expected to be considered in the short term.

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