ABSTRACT FRACTIONAL LINEAR PSEUDO–PARABOLIC EQUATIONS IN BANACH SPACES. WELL–POSEDNESS, REGULARITY, AND ASYMPTOTIC BEHAVIOR

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Abstract. In this paper we study the well-posedness, regularity, and asymptotic behavior of the solutions to the abstract pseudo-parabolic equation $\partial_t^\alpha u(t) = Au(t) + B\partial_t^\beta u(t) + f(t)$, where A, B are closed linear operators in a Banach space, and $\partial_t^\gamma u$ denotes the Caputo or Riemann–Liouville fractional derivative of order $\gamma > 0$.

1. Introduction

Consider the prototype pseudo–parabolic equation

$$\partial_t u(x,t) - \varepsilon\Delta\partial_t u(x,t) - \Delta u(x,t) = f(u(x,t)), \qquad (x,t) \in \Omega \times [0,T], \tag{1}$$

along with suitable initial and boundary conditions, where $\Omega \subset \mathbb{R}^n$, $n = 1, 2$, or 3, $\varepsilon > 0$, and ∂_t stands for the time derivative of order one.

The equation (1) arises in several fields of science and engineering. In fact in the earlier work [7] the authors describe how this kind of equations may be used in the study of some materials for which two different temperatures apply (the *conductive* and *thermodynamic* ones). The equation (1) is also related to the analysis of unidirectional propagation of nonlinear, dispersive, long waves [4] where $f(u) = u^p$, $1 < p < +\infty$, and $n = 1, 2$; the aggregation of population [20]; the analysis of nonstationary processes for semi–conductors in presence of sources and a constant homogeneous external electric field [14]; two–phase immiscible flow in porous media with dynamic capillary pressure [1, 2]; electrical conduction in heterogeneous media [3]; or image texture recognition [27].

In the last few years some generalizations of (1) have been studied whose main novelty might be the use of fractional calculus both, in the time and the spatial setting. In fact, in [12] and [24] a fractional Laplacian $(-\Delta)^\alpha$, $\alpha > 0$, replaces the classical one acting both on $u(x,t)$ and some functional of $u(x,t)$ respectively, and the well–posedness and asymptotic behavior of its solutions is studied. In [6, 8, 10, 22, 28] the study is extended to semi–linear pseudo–parabolic equations also involving a fractional Laplacian. In [25, 26] two different powers of the Laplacian acting separately on $u(x,t)$ and $\partial_t u(x,t)$ are considered, and in [16, 21] time fractional derivatives are introduced in the format

$$\partial_t^\alpha u(x,t) + \mu(-\Delta)^{s_1}\partial_t^\alpha u(x,t) + (-\Delta)^{s_2} u(x,t) = f(u(x,t)), \tag{2}$$

where $0 < s_1 \neq s_2 < 1$, $\alpha > 0$, and f stands for a locally Lipchitz function. In [5, 15, 18, 19] second order elliptical operators are considered instead of the Laplacian itself, even within the framework of time fractional derivatives.

We here address the generalization of such a fractional linear pseudo–parabolic problems by considering an abstract approach in the framework of complex Banach spaces and the format

$$\partial_t^\alpha u(t) = Au(t) + B\partial_t^\beta u(t) + f(t), \qquad t > 0, \tag{3}$$

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where A, B stand for two linear operators (might be unbounded) defined in $\mathcal{D}(A), \mathcal{D}(B) \subset X$ respectively, X is a complex Banach space, and $\partial_t^\alpha, \partial_t^\beta$ denote time fractional derivatives of order $\alpha, \beta > 0$ respectively, whose precise definition is discussed below. We keep this notation throughout the paper even if α and β are integers, in that case time derivatives stand for the classical integer derivatives. Convenient initial data for (3) will be also discussed below.

Our first contribution consists of stating conditions on α, β, A, and B for the well–posedness of (3). Moreover, since one of the main issues when time fractional derivatives are involved is the time regularity at $t = 0^+$ we also study the regularity of its solutions as $t \to 0^+$. The present study is then completed with the asymptotic behavior of the solutions as $t \to +\infty$.

The paper organizes as follows. In Section 2 we give the notation, definitions and precise formulation of the problem. Here we introduce a family $E_\gamma(t) : X \to X$, $t \geq 0$, of evolution operators whose Laplace transform $\mathcal{L}(E_\gamma)(z)$, verifies $\mathcal{L}(E_\gamma)(z) = z^\gamma(z^\alpha - A - z^\beta B)^{-1}$. This family allows us to write the solution of (3) as a variation of parameters formula. Section 3 is devoted to the case $A = B$ in (3) where we study the well–posedness, the regularity, and asymptotic behavior of the solutions to (3) in terms of the properties of $E_\gamma(t)$. In Section 4 carry out the same analysis now in the case $A \neq B$, here under suitable but general conditions on the operators A and B.

2. Notation and problem formulation

Let X be a complex Banach space. Recall that a linear operator A is θ–sectorial, $0 < \theta < \pi/2$, if there exist $M > 0$, and $w \in \mathbb{R}$, such that

$$\|(A - \lambda I)^{-1}\| \leq \frac{M}{|\lambda - w|}, \qquad \lambda \notin w + S_\theta = \{w + z : z \in S_\theta\}, \tag{4}$$

where

$$S_\theta := \{z \in \mathbb{C} : |\arg(-z)| < \theta\}, \tag{5}$$

I is the identity operator, and $(A - \lambda I)^{-1}$ stands for the resolvent operator of A defined in their resolvent set $\varrho(A)$ (see [11] Ch. 2 and [17] Ch. 2).

Related to the fractional derivative of order $\alpha \geq 0$ of $g(t)$, $\partial_t^\alpha g(t)$, here we focus on two of the most commonly used in practical instances: The Caputo and the Riemann–Liouville ones. Even though the results shown in the present paper actually coincide for both choices, and there are hardly any differences in the corresponding proofs, some differences arise related to the initial data. For the sake of the convenience of readers recall that the Riemann–Liouville type derivative of order $\alpha \geq 0$, with $n - 1 \leq \alpha < n$, $n \in \mathbb{Z}^+$, and $g \in L^1(0, +\infty)$, reads

$$\partial_t^\alpha g(t) := \partial_t^n(\mathcal{I}_t^{n-\alpha} g(t)), \qquad t \geq 0, \tag{6}$$

where $\mathcal{I}_t^\beta$ stands for the fractional integral of order $\beta > 0$ in the Riemann–Liouville sense, and defines, for $g \in L^1(0, +\infty)$, as the convolution integral

$$\mathcal{I}_t^\beta g(t) := \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s)\, \mathrm{d}s, \qquad t \geq 0. \tag{7}$$

On the other hand, the fractional derivative of $g(t)$ in the Caputo's sense is defined by

$${}_c\partial_t^\alpha g(t) := \mathcal{I}_t^{n-\alpha}(\partial_t^n g(t)), \qquad t \geq 0. \tag{8}$$

In order to simplify the notation and without no confusion we denote $\mathcal{I}_t^\beta = \partial_t^{-\beta}$. See e.g. [23, 13] and references therein for a deeper study on fractional calculus.

Now we are in a position to state the problem which is the main purpose of our study. Let A, B be two linear operators in X, $\mathcal{D}(A), \mathcal{D}(B) \subset X$, and let α, β be two positive constants such that

$$1 \leq \alpha < 2, \quad \text{and} \quad 0 < \beta \leq \alpha. \tag{9}$$

Consider the linear fractional pseudo–parabolic equation

$$\partial_t^\alpha u(t) = Au(t) + B\partial_t^\beta u(t) + f(t), \qquad t > 0, \tag{10}$$

along with some initial conditions. Those initial conditions depend on the definition of fractional derivative one opts for, and this point deserves a short discussion.

Let us consider the definition (6) and take the Laplace transform in (10), in fact for the left–hand side term we have

$$\mathcal{L}(\partial_t^\alpha u)(z) \;\; = \;\; z^\alpha U(z) - z\, \partial_t^{\alpha-2} u(t)\big|_{t\downarrow 0^+} - \partial_t^{\alpha-1} u(t)\big|_{t\downarrow 0^+},$$

where $U(z) := \mathcal{L}(u)(z)$. Analogously, the Laplace transform of the fractional derivative in the right–hand side of (10), taking from apart the operator B, leads to

$$\mathcal{L}(\partial_t^\beta u)(z) \;\; = \;\; z^\beta U(z) - \partial_t^{\beta-1} u(t)\Big|_{t\downarrow 0^+}, \qquad \text{if} \qquad 0 < \beta \le 1,$$

and

$$\mathcal{L}(\partial_t^\beta u)(z) \;\; = \;\; z^\beta U(z) - z\, \partial_t^{\beta-2} u(t)\Big|_{t\downarrow 0^+} - \partial_t^{\beta-1} u(t)\Big|_{t\downarrow 0^+}, \qquad \text{if} \qquad 1 < \beta \le \alpha.$$

In view of the above suitable initial conditions consist of the existence of

$$u_0^{\alpha-2} = \partial_t^{\alpha-2} u(t)\big|_{t\downarrow 0^+}, \quad u_0^{\alpha-1} = \partial_t^{\alpha-1} u(t)\big|_{t\downarrow 0^+} \quad \in \quad X, \tag{11}$$

and the existence of

$$u_0^{\beta-1} = \partial_t^{\beta-1} u(t)\Big|_{t\downarrow 0^+} \in \mathcal{D}(B), \qquad \text{if} \qquad 0 < \beta \le 1, \tag{12}$$

or

$$u_0^{\beta-2} = \partial_t^{\beta-2} u(t)\Big|_{t\downarrow 0^+}, \quad u_0^{\beta-1} = \partial_t^{\beta-1} u(t)\Big|_{t\downarrow 0^+}, \in \mathcal{D}(B), \qquad \text{if} \qquad 1 < \beta \le \alpha. \tag{13}$$

Such a conditions have not by far physical meaning, and in addition lead to solutions that may not be defined at $t = 0$.

On the contrary if one consider the fractional derivatives in Caputo's sense (8), then the Laplace transforms of (10) reads

$$\mathcal{L}({}_c\partial_t^\alpha u)(z) = z^\alpha U(z) - z^{\alpha-1} u(0) - z^{\alpha-2} \partial_t u(0), \tag{14}$$

and taking again from apart the operator B,

$$\mathcal{L}({}_c\partial_t^\beta u)(z) = z^\beta U(z) - z^{\beta-1} u(0), \quad \text{if} \quad 0 < \beta \le 1, \tag{15}$$

or

$$\mathcal{L}({}_c\partial_t^\beta u)(z) = z^\beta U(z) - z^{\beta-1} u(0) - z^{\beta-2} \partial_t u(0), \quad \text{if} \quad 1 < \beta \le \alpha. \tag{16}$$

In this case one may naturally consider the following initial conditions

$$\begin{aligned} u(0) \;\; &= \;\; u_0 + Bu_0, \quad u_0 \in \mathcal{D}(B), \quad \text{and} \quad \partial_t u(0) = u_0^1 \in X, \quad \text{if} \quad 0 < \beta \le 1, \\ &\text{or} \\ u(0) \;\; &= \;\; u_0 + Bu_0, \quad u_0 \in \mathcal{D}(B), \quad \text{and} \quad \partial_t u(0) = u_0^1 + Bu_0^1, \quad u_0^1 \in \mathcal{D}(B), \quad \text{if} \quad 1 < \beta \le \alpha. \end{aligned} \tag{17}$$

Observe that initial conditions have now a precise physical meaning since they are given in terms of u and its first derivative at $t = 0$, and moreover they provide solutions well defined at $t = 0$. Is for that we henceforth adopt the definition (8) of fractional derivative. In fact denote

$$\mathcal{U}_0(z) = z^{\alpha-1} u_0 + z^{\alpha-2} u_0^1 - z^{\beta-1} Bu_0, \quad \text{if} \quad 0 < \beta \le 1, \tag{18}$$

or

$$\mathcal{U}_0(z) = z^{\alpha-1} u_0 + z^{\alpha-2} u_0^1 - z^{\beta-1} Bu_0 - z^{\beta-2} Bu_0^1, \quad \text{if} \quad 1 < \beta \le \alpha, \tag{19}$$

According to (19)–(18) and denoting $F(z) = \mathcal{L}(f)(z)$, the equation (10) may be written in the domain of the Laplace transform as

$$(z^\alpha - A - z^\beta B)U(z) = \mathcal{U}_0(z) + F(z), \tag{20}$$

from where we have, in case of existing the operator $(z^\alpha - B - z^\beta B)^{-1}$

$$U(z) = (z^\alpha - A - z^\beta B)^{-1}(\mathcal{U}_0(z) + F(z)). \tag{21}$$

Therefore, in case of existing the inverse Laplace transform of the operator $(z^\alpha - B - z^\beta B)^{-1}$, we have

$$u(t) = \left(E_{\alpha-1}(t) - E_{\beta-1}(t)B\right) u_0 + E_{\alpha-2}(t)u_0^1 + \int_0^t E_0(t-s)f(s)\,\mathrm{d}s, \quad t>0, \quad \text{if} \quad 0<\beta\le 1. \tag{22}$$

and

$$u(t) = \big(E_{\alpha-1}(t) - E_{\beta-1}(t)B\big)u_0 + \left(E_{\alpha-2}(t) - E_{\beta-2}(t)B\right) u_0^1 + \int_0^t E_0(t-s)f(s)\,\mathrm{d}s, \quad t>0, \quad \text{if} \quad 1<\beta\le \alpha, \tag{23}$$

In (22) and (23) $\{E_\gamma(t)\}_{t\geq 0}$, for $\gamma \le \alpha - 1$, stands for a strongly continuous family of linear and bounded operators $E_\gamma(t) : X \to X$, $t \ge 0$, such that $t \mapsto E_\gamma(t)v$ belongs to $L^1_{loc}([0, +\infty))$, and where in fact, $E_\gamma(t)$ comes given by the inversion Laplace transform formula or Bromwich integral

$$E_\gamma(t) := \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{zt} z^\gamma (z^\alpha - A - z^\beta B)^{-1}\,\mathrm{d}z, \tag{24}$$

for a suitable complex path Γ. The family of operators $\{E_\gamma(t)\}_{t\geq 0}$ might be extended for γ in a larger range of values, however for our purposes it is enough to consider $\gamma \le \alpha - 1$.

If not regularity at all is assumed for u_0 and u_0^1, then (22) and (23) are be adopted as the mild solutions of (10), for $0 < \beta \le 1$ and $1 < \beta \le \alpha$ respectively. Moreover, whether some regularity on the initial data is assumed or not in case satisying (22) and (23) it is said that the problem (10)–(17) is well–posed.

In the following sections we state conditions for the existence of mild solutions for (10), that is for the existence of(24) to be meaningful, in both cases $A = B$ and $A \ne B$. Moreover suitable regularity conditions related to the initial data are stated in both cases in order to get genuine solutions of (10).

Before going to the following sections of the paper let us recall a known result which will be used repeatedly throughout the paper: Let $H(z)$ be a complex function, analytic outside a sector $w+S_\theta$, $0<\theta<\pi/2$, $w\in\mathbb{R}$, and such that there exist $\gamma \in \mathbb{R}$ and $M > 0$ satisfying

$$|H(z)| \le M|z|^{-\gamma}, \qquad z \notin w + S_\theta. \tag{25}$$

Therefore there exists a complex path Γ surrounding $w + S_\theta$, and connecting $-\mathrm{i}\infty$ and $+\mathrm{i}\infty$, such that the inverse Laplace transform writes as

$$h(t) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{zt} H(z)\,\mathrm{d}z, \qquad t>0, \tag{26}$$

and $C > 0$, independent on t, such that

$$|h(t)| \le C t^{\gamma-1}\,\mathrm{e}^{wt}, \qquad t>0. \tag{27}$$

Observe that, if $\gamma > 0$, then $f(t)$ turns out to be locally integrable. However, if $\gamma \le 0$, then those convolutions where $h(t)$ stands for its convolution kernel

$$\int_0^t h(t-s)g(s)\,\mathrm{d}s, \qquad t>0,$$

will be interpreted as the k–th (integer) derivative

$$\partial_t^k \left(\int_0^t \tilde{h}(t-s)g(s)\,\mathrm{d}s \right), \qquad t>0,$$

where $\tilde{h}(t)$ stands for the inverse Laplace transform of $z^{-k}H(z)$, for $\gamma + k > 0$, as long as $g(t)$ is k–times continuously differentiable.

For the sake of the simplicity of presentation, and without lost of generality, from now on we assume that $f(t) = 0$. Besides observe that if $\alpha = \beta = 1$, and $A = B$, then the equation (10) matches the classical linear pseudo–parabolic equations, and if in addition $B = 0$, then (10) matches the classical fractional parabolic equations.

3. Only one operator: $A = B$.

The first part of the paper is devoted to those equations (10) where only one operator is involved. In that way let A be a θ–sectorial operator, $\mathcal{D}(A) \subset X$, $0 < \theta < \pi/2$, and $w \in \mathbb{R}$, and assume that $A = B$.

3.1. **Well–posedness.** The first result we address in this paper concerns the well–posedness of the initial value problem (10)–(17). For the sake of the simplicity of the presentation in this section we assume that $w = 0$. This assumption does not mean a loss of generality since in the case of $w \neq 0$ no relevant differences arise in the final result, and no additional difficulties in the proof.

Theorem 1. *Let A be a linear and θ–sectorial, $0 < \theta < \pi/2$, and α, β positive constants satisfying* (9). *If*

$$\theta < \frac{\pi(2 - \alpha + \beta)}{2}, \tag{28}$$

then the initial value problem (10)–(17) *is well–posed.*

Proof. First of all notice that according to (23) and (22) –(24), now with $A = B$, the proof of the well–posedness consists of the existence of the resolvents

$$\frac{z^\gamma}{1 + z^\beta}\left(\frac{z^\alpha}{1 + z^\beta} - A\right)^{-1}, \quad \gamma \leq \alpha - 1, \tag{29}$$

in a convenient domain, and the convergence of the integral (24) for a suitable complex path Γ. These facts are directly related to the sectorial property of A and in particular to the behavior of $z^\alpha/(1 + z^\beta)$ respect to the sector S_θ associated to A. In this regard note that the left–hand side term $z^\gamma/(1 + z^\beta)$ in (29) does not affect the result, and therefore is avoided hereafter in the proof.

Denote $z = \rho\, \mathrm{e}^{\mathrm{i}\varphi}$, $\rho \geq 0$, and $\pi/2 < \varphi < \pi$. Observe that

$$\arg\left(\frac{z^\alpha}{1 + z^\beta}\right) = \arctan\left(\frac{\sin(\alpha\varphi) + \rho^\beta \sin((\alpha - \beta)\varphi)}{\cos(\alpha\varphi) + \rho^\beta \cos((\alpha - \beta)\varphi)}\right), \tag{30}$$

and asymptotically we have

$$\arg\left(\frac{z^\alpha}{1 + z^\beta}\right) \to (\alpha - \beta)\varphi, \qquad \text{as} \qquad \rho \to +\infty. \tag{31}$$

Henceforth, since $\varphi > \pi/2$, if $\pi(\alpha - \beta)/2 < \pi - \theta$ or equivalently if (28) satisfies, then one can set φ satisfying $\pi(\alpha - \beta)/2 < (\alpha - \beta)\varphi < \pi - \theta$, and $R_0 > 0$, such that $z^\alpha/(1 + z^\beta)$ does not belong to S_θ, for $\rho \geq R_0$.

Now, we are in a position to define a suitable complex Γ for the existence of the evolution operator (24). In fact let φ be belonging to $(\pi/2, \pi)$ such that $\pi(\alpha - \beta)/2 < (\alpha - \beta)\varphi < \pi - \theta$, define $\Gamma = \Gamma_1 \cup \Gamma_2$ where

$$\begin{aligned} \Gamma_1 &:= \{z \in \mathbb{C} : z = \rho\, \mathrm{e}^{\mathrm{i}\varphi}, \rho \geq R_0\}, \\ \Gamma_2 &:= \{z \in \mathbb{C} : z = R_0\, \mathrm{e}^{\mathrm{i}\sigma}, -\varphi \leq \sigma \leq \varphi\}, \end{aligned} \tag{32}$$

positively oriented, that is with increasing imaginary part. The complex path (32) keeps out of S_θ, and the complex integral is certainly convergent. Therefore the representation (24) of the evolution operator $E_\gamma(t)$ is meaningful, as well as the mild solution (23) and (22). □

Notice that if $\alpha - \beta \leq 1$, then there is not restrictions on θ since in that case $\pi(2 - \alpha + \beta)/2 \geq \pi/2$.

3.2. **Regularity.** We here study the regularity of the solution of (10)–(17) as $t \to 0^+$. To this end we first show a result concerning to the behavior of the evolution operator (24) as $t \to 0^+$, and which will be the key to state the regularity and the asymptotic behavior of the solution.

Notice that the value w involved in the sectoriallity of A actually does not affect the regularity of the solution and the corresponding result is shown for the shortness only for $w = 0$. On the contrary, the asymptotic behavior shows differences depending on w and it study in two cases, that is whether $w \geq 0$ or $w < 0$. This is why the following result is stated both for $w \geq 0$, then for $w < 0$.

Theorem 2. *Let α, β be two positive constants satisfying* (9). *Moreover let $\{E_\gamma(t)\}_{t\geq 0}$ be the family of evolution operators defined in* (24), *for $\gamma \leq \alpha - 1$.*

Therefore there exists $C > 0$ independent on t such that, for $t > 0$,

$$\|E_\gamma(t)\| \leq \begin{cases} C\,\mathrm{e}^{wt} t^{\alpha-\gamma-1}, & \text{if} \quad w \geq 0, \\ C \min\left\{ \dfrac{t^{\beta-\gamma-1}}{|w|}, t^{\alpha-\gamma-1} \right\}, & \text{if} \quad w < 0. \end{cases} \tag{33}$$

If in addition $\zeta \in \mathcal{D}(A)$, then there exist an operator $R(t)$ and $C > 0$ so that, for $t > 0$,

$$E_\gamma(t)\zeta = \frac{t^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\zeta + R(t)A\zeta, \tag{34}$$

where

$$\|R(t)\| \leq \begin{cases} C\,\mathrm{e}^{wt} t^{2\alpha-\gamma-\beta-1}, & \text{if} \quad w \geq 0, \\ C \min\left\{ \dfrac{t^{\alpha-\gamma-1}}{|w|}, t^{2\alpha-\gamma-\beta-1} \right\}, & \text{if} \quad w < 0. \end{cases} \tag{35}$$

Proof. First of all, according to the definition of Γ in the proof of the Theorem 1, let Γ_{w_+} be the complex path surrounding the sector $w + S_\theta$ defined by $\Gamma_{w_+} := (w_+ + \Gamma_1) \cup (w_+ + \Gamma_2)$ where $w_+ + \Gamma_j := \{w_+ + z : z \in \Gamma_j\}$, $j = 1, 2$, and $w_+ = \max\{0, w\}$. Assume also that Γ_{w_+} is defined with R_0 large enough. Therefore the evolution operator $E_\gamma(t)$ writes

$$E_\gamma(t) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{w_+}} \mathrm{e}^{zt} \frac{z^\gamma}{1+z^\beta} \left(\frac{z^\alpha}{1+z^\beta} - A \right)^{-1} \mathrm{d}z, \quad t > 0. \tag{36}$$

Notice that, since the integrand could not be longer extended to the left hand side complex plane, the integral is only admitted over Γ_{w_+}. This will implies that in this analysis the exponential growth shown if $w > 0$ has not the counterpart exponential decay if $w < 0$.

According to the sectorial property of A, we have that

$$\left\| \frac{z^\gamma}{1+z^\beta} \left(\frac{z^\alpha}{1+z^\beta} - A \right)^{-1} \right\| \leq \frac{M \left| \dfrac{z^\gamma}{1+z^\beta} \right|}{\left| \dfrac{z^\alpha}{1+z^\beta} - w \right|} \leq \frac{CM}{|z|^{\alpha-\gamma}}, \tag{37}$$

for a $C > 0$ independent on t. This means that $E_\gamma(t)$ stands for a functional whose Laplace transform is bounded by $CM/|z|^{\alpha-\gamma}$, $z \in \Gamma_{w_+}$. Therefore, by (25)–(27) there exists $C > 0$, independent on t, such that

$$\|E_\gamma(t)\| \leq C\,\mathrm{e}^{tw_+} t^{\alpha-\gamma-1}, \quad t > 0.$$

This bound applies for any $w \in \mathbb{R}$, in particular if $w < 0$ the operator A may though merely as with $w = 0$. If $w < 0$ a slightly different analysis must be done. In fact, straightforwardly one has from (37) that

$$\frac{M \left| \dfrac{z^\gamma}{1+z^\beta} \right|}{\left| \dfrac{z^\alpha}{1+z^\beta} - w \right|} \leq M \left| \frac{z^\gamma}{1+z^\beta} \right| \frac{1}{|w|\sin(\theta)} \leq \frac{M/\sin(\theta)}{|w||z|^{\beta-\gamma}},$$

therefore, for $w < 0$ we have also the bound

$$\|E_\gamma(t)\| \le \frac{C}{|w|} t^{\beta-\gamma-1}, \quad t > 0,$$

and the first statement of the theorem follows.

By going a step forward the evolution operator $E_\gamma(t)$ admits the following expression, for $\zeta \in \mathcal{D}(A)$,

$$\begin{aligned} E_\gamma(t)\zeta &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{w_+}} \mathrm{e}^{zt} \frac{z^\gamma}{1+z^\beta} \left(\frac{z^\alpha}{1+z^\beta} - A \right)^{-1} \zeta \, \mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{w_+}} \mathrm{e}^{zt} \frac{1}{z^{\alpha-\gamma}} \frac{z^\alpha}{1+z^\beta} \left(\frac{z^\alpha}{1+z^\beta} - A \right)^{-1} \zeta \, \mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{w_+}} \mathrm{e}^{zt} \frac{1}{z^{\alpha-\gamma}} \left\{ I + \left(\frac{z^\alpha}{1+z^\beta} - A \right)^{-1} A \right\} \zeta \, \mathrm{d}z \\ &= R_0(t)\zeta + R(t)A\zeta, \end{aligned}$$

where

$$R_0(t) := \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{w_+}} \mathrm{e}^{zt} \frac{1}{z^{\alpha-\gamma}} I \, \mathrm{d}z, \quad \text{and} \quad R(t) := \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{w_+}} \mathrm{e}^{zt} \frac{1}{z^{\alpha-\gamma}} \left(\frac{z^\alpha}{1+z^\beta} - A \right)^{-1} \mathrm{d}z. \tag{38}$$

Note that $R_0(t)$ may be written as

$$R_0(t) = \frac{t^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} I, \qquad t > 0, \tag{39}$$

and by the sectorial property of A, and since R_0 is assumed to be large enough, there exists $C > 0$ such that

$$\left\| \frac{1}{z^{\alpha-\gamma}} \left(\frac{z^\alpha}{1+z^\beta} - A \right)^{-1} \right\| \le M \frac{|1+z^\beta|}{|z|^{2\alpha-\gamma}} \le \frac{CM}{|z|^{2\alpha-\gamma-\beta}}, \qquad z \in \Gamma_{w_+}. \tag{40}$$

Therefore $R(t)$ stands for the inverse Laplace transform of a function depending on z bounded by $|z|^{-(2\alpha-\gamma-\beta)}$, for $z \in \Gamma_{w_+}$. Once again, from (25)–(27) it follows that,

$$\|R(t)A\zeta\| \le CM \|A\zeta\| \, \mathrm{e}^{tw_+} t^{2\alpha-\gamma-\beta-1}, \qquad t > 0, \tag{41}$$

for $\zeta \in \mathcal{D}(A)$. Once again, for $w < 0$, the analysis above may be applied here to have

$$\left\| \frac{1}{z^{\alpha-\gamma}} \left(\frac{z^\alpha}{1+z^\beta} - A \right)^{-1} \right\| \le M \frac{\dfrac{1}{|z|^{\alpha-\gamma}}}{\left| \dfrac{z^\alpha}{1+z^\beta} - w \right|} \le \frac{CM/\sin(\theta)}{|w||z|^{\alpha-\gamma}}, \qquad z \in \Gamma_{w_+}, \tag{42}$$

and consequently

$$\|R(t)\| \le \frac{C}{|w|} t^{\alpha-\gamma-1}, \qquad t > 0. \tag{43}$$

In that manner the proof of the theorem concludes. □

In view of (23) and (22) the regularity of the solution is achieved by applying Theorem 2 with some particular values of γ, and suitable regularity conditions for u_0 and u_0^1. All these cases are collected in following corollary. Notice that since the regularity of the solutions is not actually affected by w, for the shortness of the presentation we only show the results for $w = 0$. The results for $w \neq 0$ straightforwardly might achieved.

Corollary 3. *Let α, β be two positive constants satisfying (9), and let $\{E_\gamma(t)\}_{t\geq 0}$ be the family of evolution operators defined in (24), for $\gamma \leq \alpha - 1$, and $w = 0$.*

Therefore,

(1) If $\zeta \in \mathcal{D}(A)$, then we have the following,

$$E_{\alpha-1}(t)\zeta - E_{\beta-1}(t)A\zeta = \zeta + E_{-1}(t)A\zeta, \quad t \geq 0. \tag{44}$$

$$\partial_t \left\{E_{\alpha-1}(t)\zeta - E_{\beta-1}(t)A\zeta\right\}\Big|_{t=0} = 0. \tag{45}$$

$$E_{\alpha-2}(t)\zeta - E_{\beta-2}(t)A\zeta = t\zeta + E_{-2}(t)A\zeta, \quad t \geq 0. \tag{46}$$

$$\partial_t \left\{E_{\alpha-2}(t)\zeta - E_{\beta-2}(t)A\zeta\right\}\Big|_{t=0} = \zeta. \tag{47}$$

$$\left.\begin{array}{rcl} E_{\alpha-2}(t)\zeta &=& t\zeta + R(t)A\zeta, \quad R(t) = O(t^{\alpha-\beta+1}), \\ \partial_t E_{\alpha-2}(t)\zeta &=& \zeta + R(t)A\zeta, \quad R(t) = O(t^{\alpha-\beta}), \end{array}\right\} \quad t \to 0^+. \tag{48}$$

(2) If $\zeta \in \mathcal{D}(A^2)$, then we have the following,

$$E_{\alpha-1}(t)\zeta - E_{\beta-1}(t)A\zeta = \zeta + \frac{t^\alpha}{\Gamma(\alpha+1)}A\zeta + R(t)A^2\zeta, \quad t \geq 0, \qquad R(t) = O(t^{2\alpha-\beta}),\ t \to 0^+. \tag{49}$$

$$E_{\alpha-2}(t)\zeta - E_{\beta-2}(t)A\zeta = t\zeta + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}A\zeta + R(t)A^2\zeta, \quad t \geq 0, \qquad R(t) = O(t^{2\alpha-\beta+1}),\ t \to 0^+. \tag{50}$$

$$E_{\alpha-2}(t)\zeta = t\zeta + R_1(t)A\zeta + R_2(t)A^2\zeta, \qquad R_1(t) = O(t^{\alpha-\beta+1}), \quad R_2(t) = O(t^{2(\alpha-\beta)+1}),\ t \to 0^+. \tag{51}$$

Proof. In order to prove (44) recall that the operators $E_\gamma(t)$ admit the integral representation (24) along a suitable complex path Γ, in fact we adopt again the path $\Gamma = \Gamma_0$ according the notation in the proof of the Theorem 2, here again for $R_0 > 0$ large enough. Therefore we have

$$\begin{array}{rl} & E_{\alpha-1}(t)\zeta - E_{\beta-1}(t)A\zeta \\ = & \displaystyle\frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{zt}\frac{z^{\alpha-1}}{1+z^\beta}\left(\frac{z^\alpha}{1+z^\beta} - A\right)^{-1}\zeta\,\mathrm{d}z - \frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{zt}\frac{z^{\beta-1}}{1+z^\beta}\left(\frac{z^\alpha}{1+z^\beta} - A\right)^{-1}A\zeta\,\mathrm{d}z \\ = & \displaystyle\zeta + \frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{zt}\frac{1}{z}\left(\frac{z^\alpha}{1+z^\beta} - A\right)^{-1}A\zeta\,\mathrm{d}z - \frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{zt}\frac{z^{\beta-1}}{1+z^\beta}\left(\frac{z^\alpha}{1+z^\beta} - A\right)^{-1}A\zeta\,\mathrm{d}z \\ = & \displaystyle\zeta + \frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{zt}\left(\frac{1}{z} - \frac{z^{\beta-1}}{1+z^\beta}\right)\left(\frac{z^\alpha}{1+z^\beta} - A\right)^{-1}A\zeta\,\mathrm{d}z \\ = & \displaystyle\zeta + \frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{zt}\frac{z^{-1}}{1+z^\beta}\left(\frac{z^\alpha}{1+z^\beta} - A\right)^{-1}A\zeta\,\mathrm{d}z \\ = & \zeta + E_{-1}(t)A\zeta\,\mathrm{d}z. \end{array}$$

So the statement (44) follows, and in a similar manner the proof of (46) follows as well.

On the other hand, once observed that

$$\partial_t\big\{E_{\alpha-1}(t)\zeta - E_{\beta-1}(t)A\zeta\big\} = \frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{zt}\frac{1}{1+z^\beta}\left(\frac{z^\alpha}{1+z^\beta} - A\right)^{-1}A\zeta\,\mathrm{d}z = E_0(t)A\zeta\,\mathrm{d}z, \quad t \geq 0,$$

the Theorem 2, now with $\gamma = 0$, leads to

$$\|E_0(t)\| \leq Ct^{\alpha-1}, \quad t > 0,$$

and accordingly to (45). Likewise the proof of (47) is done.

The proof of (49) in based on the fact that, according to the Theorem 2 and (44), if $\zeta \in \mathcal{D}(A^2)$ the operator $E_{-1}(t)$ admits the following expression

$$E_{-1}(t)\zeta = \frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{zt}\frac{1}{z^{\alpha+1}}\frac{z^\alpha}{1+z^\beta}\left(\frac{z^\alpha}{1+z^\beta}-A\right)^{-1}A\zeta\,\mathrm{d}z = \frac{t^\alpha}{\Gamma(\alpha+1)}A\zeta + R(t)A^2\zeta,$$

where

$$R(t) = \frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{zt}\frac{1}{z^{\alpha+1}}\left(\frac{z^\alpha}{1+z^\beta}-A\right)^{-1}\mathrm{d}z, \quad t>0.$$

Here, we have that

$$\left\|\frac{1}{z^{\alpha+1}}\left(\frac{z^\alpha}{1+z^\beta}-A\right)^{-1}\right\| \le \frac{CM}{|z|^{2\alpha-\beta+1}}, \quad z\in\Gamma, \tag{52}$$

and therefore

$$\|R(t)\| \le Ct^{2\alpha-\beta}, \quad t>0. \tag{53}$$

In that manner the proof of (49) concludes. The proof of (50) follows the same steps as with $E_{-1}(t)$, now $E_{-2}(t)$.

Finally proofs of (48) and (51) follow similar steps and by the shortness of the paper are omitted. □

Theorem 4. *Let α,β be two positive constants satisfying* (9)*. Moreover let $u(t)$ be the mild solution* (23) *of the initial value problem* (10)–(17)*, for $0<\beta\le 1$, and $1<\beta\le\alpha$ respectively. If $u_0, u_1^0 \in \mathcal{D}(A)$, then $u(t)$ is a genuine solution of* (10)–(17) *such that*

$$u(0) = u_0, \quad \partial_t u(0) = u_0^1,$$

and satisfies that,

(1) For $1<\beta\le\alpha$,

$$u(t) = u_0 + tu_0^1 + E_{-1}(t)Au_0 + E_{-2}(t)Au_0^1, \quad t>0,$$

and if moreover $u_0, u_0^1 \in \mathcal{D}(A^2)$, then

$$u(t) = u_0 + tu_0^1 + \frac{t^\alpha}{\Gamma(\alpha+1)}Au_0 + \frac{t^{\alpha+1}}{\Gamma(\alpha+1)}Au_0^1 + R_1(t)A^2u_0 + R_2(t)A^2u_0^1,$$

where there exists $C>0$, independent on t, such that

$$\|R_1(t)\| \le Ct^{2\alpha-\beta}, \quad \text{and} \quad \|R_2(t)\| \le Ct^{2\alpha-\beta+1}, \quad t>0.$$

(2) For $0<\beta\le 1$, there exists $C>0$ such that

$$u(t) = u_0 + tu_0^1 + E_{-1}(t)Au_0 + R(t)Au_0^1, \quad \text{where} \quad \|R(t)\| \le Ct^{\alpha-\beta+1}, \quad t>0,$$

and if moreover $u_0, u_0^1 \in \mathcal{D}(A^2)$, then

$$u(t) = u_0 + tu_0^1 + \frac{t^\alpha}{\Gamma(\alpha+1)}Au_0 + R_1(t)Au_0^1 + R_2(t)A^2u_0 + R_3(t)A^2u_0^1,$$

where

$$\|R_1(t)\| \le Ct^{\alpha-\beta+1}, \quad \|R_2(t)\| \le Ct^{2\alpha-\beta}, \quad \text{and} \quad \|R_3(t)\| \le Ct^{2(\alpha-\beta)+1}, \quad t>0.$$

The proof of the Theorem 4 is a straightforward consequence of the Corollary 3.

3.3. **Asymptotic behavior.** In this section we show the behavior of the solution of (10)–(17) as $t \to +\infty$. In this section the coefficient w plays an important role, henceforth we here consider any $w \in \mathbb{R}$ instead of merely $w = 0$.

Theorem 5. *Let α, β be two positive constants satisfying* (9)*, $u_0, u_1^0 \in \mathcal{D}(A)$, and $u(t)$ the solution of the initial value problem* (10)–(17)*.*

Therefore there exists $C > 0$ such that

(1) If $1 < \beta \leq \alpha$, then

$$\|u(t)\| \leq \begin{cases} C\,\mathrm{e}^{wt} t^{\alpha-\beta+1}, & w \geq 0, \\ \dfrac{Ct}{|w|}, & w < 0, \end{cases} \qquad \text{as} \quad t \to +\infty. \tag{54}$$

(2) If $0 < \beta \leq 1$, then

$$\|u(t)\| \leq \begin{cases} C\,\mathrm{e}^{wt} t^{\max\{\alpha-\beta,1\}}, & w \geq 0, \\ \dfrac{Ct^{\max\{\beta-\alpha+1,0\}}}{|w|}, & w < 0, \end{cases} \qquad \text{as} \quad t \to +\infty. \tag{55}$$

Proof. Let us consider the expressions (22), and (23) of the solution $u(t)$ of (10)–(17), for $0 < \beta \leq 1$, and $1 < \beta \leq \alpha$ respectively.

First of all notice that one might consider the expressions of $u(t)$ provided by the Theorem 4 instead of (23) and (22), however no more accurate bounds can be achieved. Therefore consider two cases, $1 < \beta \leq \alpha$ and $0 < \beta \leq 1$ as follows:

1. Let β be a positive constant such that $1 < \beta \leq \alpha$. According to (23) and (33), for $w \geq 0$, we have

$$E_{\alpha-1}(t) \leq C\,\mathrm{e}^{wt}, \quad E_{\beta-1}(t) \leq C\,\mathrm{e}^{wt} t^{\alpha-\beta}, \quad E_{\alpha-2}(t) \leq C\,\mathrm{e}^{wt} t, \quad E_{\beta-2}(t) \leq C\,\mathrm{e}^{wt} t^{\alpha-\beta+1}, \tag{56}$$

for $t > 0$. The first statement of (54) then follows. On the other hand, if $w < 0$, then and according again to (33) we have

$$E_{\alpha-1}(t) \leq \frac{C}{|w| t^{\alpha-\beta}}, \quad E_{\beta-1}(t) \leq \frac{C}{|w|}, \quad E_{\alpha-2}(t) \leq \frac{Ct^{\beta-\alpha+1}}{|w|}, \quad E_{\beta-2}(t) \leq \frac{Ct}{|w|}, \tag{57}$$

for $t > 0$. Since $\beta - \alpha + 1 \leq 1$ the second statement of (54) follows as well.

2. Let β be a positive constant such that $0 < \beta \leq 1$. In this case we only have to take into account the first third terms in (56), so that the dominant terms are

$$E_{\beta-1}(t) \leq C\,\mathrm{e}^{wt} t^{\alpha-\beta}, \quad E_{\alpha-2}(t) \leq C\,\mathrm{e}^{wt} t, \qquad t > 0.$$

Therefore we have the first statement of (55). On the same manner if if $w < 0$, then the last term in (57) does nort affect the bound, and according one more time to (33) we have that the dominant terms are

$$E_{\beta-1}(t) \leq \frac{C}{|w|}, \quad E_{\alpha-2}(t) \leq \frac{Ct^{\beta-\alpha+1}}{|w|}, \qquad t > 0.$$

So, since $\beta - \alpha + 1 \geq 0$ is not always satisfies, the proof of the second statement of (55) follows, and the proof of the theorem concludes. □

The Theorem 5 deserves some comment, in particular note that if $B = 0$, $\beta = 0$, and $u_0^1 = 0$, that is if one has the classical parabolic fractional integral equation $u(t) = u_0 + \partial_t^{-\alpha} Au(t)$, then the asymptotic behavior shown above perfectly matches the one provided in [9].

4. Two operators: $A \neq B$

Despite the results shown in this section turn out to be fairly similar to the ones in the case $A = B$, two different operators are now involved and the proofs are slightly different. This is why the proofs below are shown.

4.1. **Well–posedness.** In this section we consider two different linear operators A and B, $\mathcal{D}(A), \mathcal{D}(B) \subset X$.

Let us recall a definition that will prove useful hereafter: Given two linear operators $A, B : X \to X$, the operator A is called B–bounded if $\mathcal{D}(A) \subseteq \mathcal{D}(B)$, and there exists $b > 0$ such that

$$\|A\zeta\| \leq b\|B\zeta\|, \qquad \zeta \in \mathcal{D}(A). \tag{58}$$

In that case b is so–called the B–bound of A, if $b := \inf\{a > 0 : \|A\zeta\| \leq a\|B\zeta\|, \zeta \in \mathcal{D}(A)\}$.

Once again assume also here that $w = 0$ since as in Section 3 no relevant differences arise if $w \neq 0$. Now we have the following result.

Theorem 6. *Assume that A and B commute. Let B be a linear θ_B–sectorial operator, $0 \leq \theta_B < \pi/2$, such that*

$$\theta_B < \frac{\pi(2 - \alpha + \beta)}{2}, \tag{59}$$

M_B is the associated sectorial bound, and let $A : \mathcal{D}(A) \subset X \to X$ be a linear B–bounded operator with B–bound $b > 0$. Moreover let α, β be positive constants satisfying (9)*. Then the problem* (10)–(17) *is well–posed.*

Proof. Similarly to the proof of Theorem 1, the term z^γ in (24) does not affect the result, therefore we concentrate in the term $(z^\alpha - A - z^\beta B)^{-1}$.

First of all observe that the operator in (20) may be written as

$$(z^\alpha - A - z^\beta B)^{-1} = z^{-\beta}(I - z^{-\beta}A(z^{\alpha-\beta} - B)^{-1})^{-1}(z^{\alpha-\beta} - B)^{-1}. \tag{60}$$

Now the proof consists of the existence of the resolvent $(z^{\alpha-\beta} - B)^{-1}$ and the operator $(I - A(z^\alpha - z^\beta B)^{-1})^{-1}$ in a convenient domain, and then existence of a complex path Γ to be the integral (24) convergent.

Since the argument of $\arg(z^{\alpha-\beta}) = (\alpha - \beta)\arg(z)$ the condition on θ is straightforward by following the same step as in the Theorem 1. Moreover, the complex path Γ surrounding S_{θ_B} defined in that Theorem may be used here as well. Having in mind all these facts there holds

$$\|(z^\alpha - z^\beta B)^{-1}\| = \frac{1}{|z|^\beta}\|(z^{\alpha-\beta} - B)^{-1}\| \leq \frac{M_B}{|z|^\alpha}, \quad z \notin S_{\theta_B}. \tag{61}$$

As A and B commute we have that if $x \in \mathcal{D}(A)$, then $(z^\alpha - z^\beta B)^{-1}x \in \mathcal{D}(A)$. Therefore by the B–boundness of A we have, for $x \in \mathcal{D}(A)$, and $z \notin S_{\theta_B}$, that

$$\|A(z^\alpha - z^\beta B)^{-1}x\| \leq b\|B(z^\alpha - z^\beta B)^{-1}x\| \leq \frac{b(1 + M_B)}{|z|^\beta}\|x\|. \tag{62}$$

Let R_0 be a positive constant large enough, in fact so that $R_0 > (b(1 + M_B))^{1/\beta}$, and set $z \notin S_{\theta_B}$. In that case $\|A(z^\alpha - z^\beta B)^{-1}\| < 1$ and in view of (60) we have

$$\begin{aligned}
\|(z^\alpha - A - z^\beta B)^{-1}\| &= \left\| \left(\sum_{j=0}^{+\infty} (A(z^\alpha - z^\beta B)^{-1})^j \right) (z^\alpha - z^\beta B)^{-1} \right\| \\
&\leq \sum_{j=0}^{+\infty} \left(\frac{b(1 + M_B)}{|z|^\beta} \right)^j \frac{M_B}{|z|^\alpha} \\
&\leq \frac{1}{1 - \dfrac{b(1 + M_B)}{|z|^\beta}} \frac{M_B}{|z|^\alpha} \\
&= \frac{M_B}{R_0^\beta - b(1 + M_B)} \frac{1}{|z|^{\alpha-\beta}}.
\end{aligned}$$

Therefore the operator $(z^\alpha - A - z^\beta B)^{-1}$ is bounded.

Accordingly, since $\beta \le \alpha$, Γ keeps out of S_{θ_B} (with R_0 large enough), and the operator $(z^\alpha - A - z^\beta B)^{-1}$ is bounded, for $z \in \Gamma$ and then the expression (24) of the evolution operator $E_\gamma(t)$ is meaningful. Consequently the mild solutions (22) exists, that is the problem (10)–(17) is well–posed, and the proof concludes. $\square$

4.2. **Regularity.** Let A, B be two linear operators such that B is θ_B–sectorial, $0 < \theta_B < \pi/2$, with sectorial bound $M_B > 0$, and A of type B–bounded with B–bound $b > 0$.

As in the Section 3.2 we first show a result concerning to the behavior of the evolution operator (24) as $t \to 0^+$.

The regularity of the genuine solution then follows, depending again on the regularity of the initial data.

Theorem 7. *Let α, β be two positive constants satisfying* (9). *Moreover let $\{E_\gamma(t)\}_{t\ge 0}$ be the family of evolution operators defined in* (24), *for $\gamma \le \alpha - 1$.*

Therefore, there exists $C > 0$, independent on t, such that, for $t > 0$,

$$\|E_\gamma(t)\| \le \begin{cases} C\,\mathrm{e}^{wt} t^{\alpha-\gamma-1}, & \text{if} \quad w \ge 0, \\ C \min\left\{ \dfrac{t^{\beta-\gamma-1}}{|w|}, t^{\alpha-\gamma-1} \right\}, & \text{if} \quad w < 0. \end{cases} \tag{63}$$

If $\zeta \in \mathcal{D}(A)$, then there exist an operator $R(t)$ and $C > 0$ so that,

$$E_\gamma(t)\zeta = \frac{t^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\zeta + E_{\gamma-\alpha}(t)A\zeta + E_{\gamma-\alpha+\beta}(t)B\zeta, \quad t > 0. \tag{64}$$

And if $\zeta \in \mathcal{D}(B)$, but $\zeta \notin \mathcal{D}(A)$, then

$$E_\gamma(t)\zeta = R(t)\zeta + E_{\gamma-\alpha+\beta}(t)B\zeta, \quad t > 0, \tag{65}$$

where

$$\|R(t)\| \le C\,\mathrm{e}^{w_+ t} t^{\alpha-\gamma-1}, \quad t > 0. \tag{66}$$

Proof. Let Γ_{w_+} be once again the complex path surrounding the sector $w + S_\theta$ defined by $\Gamma_{w_+} := (w_+ + \Gamma_1) \cup (w_+ + \Gamma_2)$ where $w_+ + \Gamma_j := \{w_+ + z : z \in \Gamma_j\}$, $w_+ = \max\{0, w_B\}$, and Γ_j is defined according to that in the Theorem 1, for $j = 1, 2$. Assume again that Γ_{w_+} is defined with R_0 large enough. Therefore the evolution operator $E_\gamma(t)$ writes

$$E_\gamma(t) = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_{w_+}} \mathrm{e}^{zt} z^\gamma (z^\alpha - A - z^\beta B)^{-1}\,\mathrm{d}z, \quad t > 0. \tag{67}$$

As in Theorem 6 let us write the evolution operator $E_\gamma(t)$ as

$$E_\gamma(t) = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_{w_+}} \mathrm{e}^{zt} z^\gamma (I - A(z^\alpha - z^\beta B)^{-1})^{-1} (z^\alpha - z^\beta B)^{-1}\,\mathrm{d}z, \quad t > 0.$$

On the one hand,

$$\|(z^\alpha - z^\beta B)^{-1}\| = \frac{1}{|z|^\beta} \|(z^{\alpha-\beta} - B)^{-1}\| \le \frac{M_B}{|z^\alpha - z^\beta w|}, \quad z \notin w + S_{\theta_B}. \tag{68}$$

Since A and B commute, we have that if $x \in \mathcal{D}(A)$, then $(z^\alpha - z^\beta B)^{-1}x \in \mathcal{D}(A)$. Therefore if R_0 is large enough, in fact $R_0 > (b(1+2M_B))^{1/\beta}$, so that

$$\frac{|z|^{\alpha-\beta}}{|z^{\alpha-\beta} - w|} \le 2, \quad z \in \Gamma_{w_+},$$

then according to (68) we have, for $x \in \mathcal{D}(A)$, that

$$\|A(z^\alpha - z^\beta B)^{-1}x\| \le b\|B(z^\alpha - z^\beta B)^{-1}x\| \le \frac{b(1+2M_B)}{|z|^\beta}\|x\|, \quad z \notin w + S_{\theta_B}. \tag{69}$$

Therefore, following the ideas of the proof of Theorem 6, if $z \in \Gamma_{w_+}$, then

$$\|z^\gamma(z^\alpha - A - z^\beta B)^{-1}\| \;=\; \left\| z^\gamma \left(\sum_{j=0}^{+\infty} (A(z^\alpha - z^\beta B)^{-1})^j\right)(z^\alpha - z^\beta B)^{-1}\right\| \le \frac{M_B}{|z|^\beta - b(1+2M_B)} \frac{|z|^{\gamma+\beta}}{|z^\alpha - z^\beta w|}.$$

Once again, for $z \in \Gamma_{w_+}$, there exists $C > 0$ such that

$$\frac{M_B}{|z|^\beta - b(1+2M_B)} \frac{|z|^{\gamma+\beta}}{|z^\alpha - z^\beta w|} \le \frac{C}{|z|^{\alpha-\gamma}},$$

that is the Laplace transform of $E_\gamma(t)$ is bounded by $C/|z|^{\alpha-\gamma}$, $z \in \Gamma_{w_+}$. Accordingly there exists $C > 0$ such that

$$\|E_\gamma(t)\| \le C\, \mathrm{e}^{w_+ t} t^{\alpha-\gamma-1}, \quad t > 0.$$

However, if $w < 0$, then we have a slightly different bound,

$$|z^{\alpha-\beta} - w| \ge |w| \sin(\theta), \quad z \in \Gamma_{w_+},$$

therefore there exists $C > 0$ such that

$$\frac{M_B}{|z|^\beta - b(1+M_B)} \frac{|z|^{\gamma+\beta}}{|z^\alpha - z^\beta w|} \le \frac{M_B}{|z|^\beta - b(1+M_B)} \frac{|z|^{\gamma}}{|w|\sin(\theta)} \le \frac{C}{|w||z|^{\beta-\gamma}},$$

and (63) then follows.

Assume that $\zeta \in \mathcal{D}(A)$. Therefore we have

$$\begin{aligned} E_\gamma(t)\zeta &= \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_{w_+}} \mathrm{e}^{zt} z^{\gamma-\alpha} \left(I + (A + z^\beta B)(z^\alpha - A - z^\beta B)^{-1}\right)\zeta \,\mathrm{d}z \\ &= \frac{t^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\zeta + \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_{w_+}} \mathrm{e}^{zt} z^{\gamma-\alpha} (z^\alpha - A - z^\beta B)^{-1} A\zeta \,\mathrm{d}z \\ &\quad + \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_{w_+}} \mathrm{e}^{zt} z^{\gamma-\alpha+\beta} (z^\alpha - A - z^\beta B)^{-1} B\zeta \,\mathrm{d}z, \end{aligned}$$

and (64) follows as well.

If $\zeta \in \mathcal{D}(B)$ but $\zeta \notin \mathcal{D}(A)$, then the last term in (64) remains, and only the first ones change. In particular that term writes as

$$\begin{aligned} R(t) &= \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_{w_+}} \mathrm{e}^{zt} z^{\gamma-\alpha} (I + A(z^\alpha - A - z^\beta B)^{-1})\zeta \,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_{w_+}} \mathrm{e}^{zt} z^{\gamma-\alpha} (z^\alpha - z^\beta B)(z^\alpha - A - z^\beta B)^{-1}\zeta \,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_{w_+}} \mathrm{e}^{zt} z^{\gamma-\alpha} (I - A(z^\alpha - z^\beta B)^{-1})^{-1}\zeta \,\mathrm{d}z. \end{aligned}$$

Repeating again the arguments, straightforwardly follows that the operator $(I - A(z^\alpha - z^\beta B)^{-1})^{-1}$, for $z \in \Gamma_{w_+}$, is merely bounded. Moreover since there exits $C > 0$ such that the Laplace transform of $R(t)$ is bounded by $C|z|^{\gamma-\alpha}$, we have

$$\|R(t)\| \le C\, \mathrm{e}^{w_+ t} t^{\alpha-\gamma-1}, \quad t > 0.$$

The case $w < 0$ does not allow to achieve different bounds, therefore the proof of the theorem ends. ☐

The following corollary collects those particular cases of the Theorem 7 required for the regularity of the solution of (10) according the regularity of the initial data.

Corollary 8. *Let α, β be two positive constants satisfying* (9)*, and let $\{E_\gamma(t)\}_{t\ge 0}$ be the family of evolution operators defined in* (24)*, for $\gamma \le \alpha - 1$. Therefore,* *(1) If $\zeta \in \mathcal{D}(A)$, then we have the following,*

$$E_{\alpha-1}(t)\zeta - E_{\beta-1}B\zeta = \zeta + E_{-1}(t)A\zeta, \quad t>0. \tag{70}$$

$$E_{\alpha-2}(t)\zeta - E_{\beta-2}B\zeta = t\zeta + E_{-2}(t)A\zeta, \quad t>0. \tag{71}$$

$$\partial_t \left\{E_{\alpha-1}(t)\zeta - E_{\beta-1}B\zeta\right\}_{t=0} = 0. \tag{72}$$

$$\partial_t \left\{E_{\alpha-2}(t)\zeta - E_{\beta-2}B\zeta\right\}_{t=0} = \zeta. \tag{73}$$

$$\left.\begin{array}{rcl} E_{\alpha-2}(t)\zeta &=& t\zeta + E_{-2}(t)A\zeta + E_{\beta-2}(t)B\zeta, \\ \partial_t E_{\alpha-2}(t)\zeta &=& E_{-1}(t)A\zeta + E_{\beta-1}(t)B\zeta. \end{array}\right\} \qquad t>0. \tag{74}$$

(2) If $\zeta \in \mathcal{D}(A^2)$, then we have the following,

$$E_{\alpha-1}(t)\zeta - E_{\beta-1}B\zeta = \zeta + \frac{t^{\alpha}}{\Gamma(\alpha+1)}A\zeta + E_{-\alpha-1}(t)A^2\zeta + E_{\beta-\alpha-1}(t)BA\zeta, \quad t>0. \tag{75}$$

$$E_{\alpha-2}(t)\zeta - E_{\beta-2}B\zeta = t\zeta + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}A\zeta + E_{-\alpha-2}(t)A^2\zeta + E_{\beta-\alpha-2}(t)BA\zeta, \quad t>0. \tag{76}$$

If in addition $B\zeta \in \mathcal{D}(A)$, then

$$\begin{aligned} E_{\alpha-2}(t)\zeta &= t\zeta + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}A^2\zeta + \frac{t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}B\zeta \\ &\quad + E_{-\alpha-2}(t)A^2\zeta + E_{\beta-\alpha-2}(t)(BA+AB)\zeta + E_{2\beta-\alpha-2}(t)B^2\zeta. \end{aligned} \tag{77}$$

The case $\zeta \in \mathcal{D}(B)\backslash\mathcal{D}(A)$ may be straightforwardly derived but for the shortness of the paper is omitted.

Proof. First of all consider the representation (24)-(32), of the operators $E_\gamma(t)$, for $R_0>0$ large enough. Secondly notice that if we apply directly the Theorem 7 some key cancelations are not revealed, therefore we make use in this proof of the expression of the evolution operators.

In particular if $\zeta \in \mathcal{D}(A)$, then we have that

$$\begin{aligned} E_{\alpha-1}(t)\zeta &= \frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{tz}\frac{1}{z}(I + (A+z^\beta B)(z^\alpha - A - z^\beta A)^{-1})\zeta \,\mathrm{d}z \\ &= \zeta + \frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{tz}\frac{1}{z}(z^\alpha - A - z^\beta A)^{-1})A\zeta\,\mathrm{d}z + \frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{tz}z^{\beta-1}(z^\alpha - A - z^\beta A)^{-1})B\zeta\,\mathrm{d}z \\ &= \zeta + E_{-1}(t)A\zeta + E_{\beta-1}(t)B\zeta, \end{aligned}$$

and (70) follows. In the same manner, the proof of (71) may be done, and the proof of (72)–(74) follows easily from (70)–(71) and (64) by repeating the same arguments.

Since $\zeta \in \mathcal{D}(A^2)$, and $\mathcal{D}(A) \subset \mathcal{D}(B)$, we have that $A^2\zeta$ and $BA\zeta$ are meaningful. Therefore the proof of (75) follows this steps

$$\begin{aligned} E_{\alpha-1}(t)\zeta &= \zeta + \frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{tz}\frac{1}{z^{\alpha+1}}\left(I + (A+z^\beta B)(z^\alpha - A - z^\beta B)^{-1}\right)A\zeta\,\mathrm{d}z \\ &\quad + \frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{tz}\frac{1}{z^{\alpha-\beta+1}}\left(I + (A+z^\beta B)(z^\alpha - A - z^\beta B)^{-1}\right)B\zeta\,\mathrm{d}z \\ &= \zeta + \frac{t^\alpha}{\Gamma(\alpha+1)}A\zeta + \frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}B\zeta \\ &\qquad + E_{-\alpha-1}(t)A^2\zeta + E_{\alpha-\beta-1}(t)(BA+AB)\zeta + E_{2\beta-\alpha-1}(t)B^2\zeta. \end{aligned}$$

Analogously,

$$\begin{aligned} E_{\beta-1}(t)B\zeta &= \frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{e}^{tz}\frac{1}{z^{\alpha-\beta+1}}\left(I+(A+z^\beta B)(z^\alpha - A - z^\beta B)^{-1}\right)B\zeta\,\mathrm{d}z \\ &= \frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}B\zeta + E_{\beta-\alpha-1}(t)AB\zeta + E_{2\beta-\alpha-1}(t)B^2\zeta. \end{aligned}$$

By subtracting both expressions the statement follows.

The proof of (76) and (77) straightforwardly follow the same steps, and so the proof ends. □

The proof of the next result follows from Corollary 8.

Theorem 9. *Let α, β be two positive constants satisfying* (9). *Moreover let $u(t)$ be the mild solution* (23) *and* (23) *of the initial value problem* (10)–(17)*, for $1 < \beta \leq \alpha$ and $0 < \beta \leq 1$, respectively. If $u_0, u_1^0 \in \mathcal{D}(A)$, then $u(t)$ is a genuine solution of* (10)–(17) *such that*

$$u(0) = u_0, \quad \partial_t u(0) = u_0^1,$$

and satisfies that,

(1) For $1 < \beta \leq \alpha$,

$$u(t) = u_0 + tu_0^1 + E_{-1}(t)Au_0 + E_{-2}(t)Au_0^1, \quad t > 0,$$

and if moreover $u_0, u_0^1 \in \mathcal{D}(A^2)$, then

$$\begin{aligned} u(t) &= u_0 + tu_0^1 + \frac{t^\alpha}{\Gamma(\alpha+1)}Au_0 + \frac{t^{\alpha+1}}{\Gamma(\alpha+1)}Au_0^1 \\ &\quad + E_{-\alpha-1}(t)A^2u_0 + E_{\beta-\alpha-1}(t)BAu_0 + E_{-\alpha-2}(t)A^2u_0^1 + E_{\beta-\alpha-2}(t)BAu_0^1, \quad t > 0. \end{aligned}$$

(2) For $0 < \beta \leq 1$,

$$u(t) = u_0 + tu_0^1 + E_{-1}(t)Au_0 + E_{-2}(t)Au_0^1 + E_{\beta-2}(t)Bu_0^1,$$

and if moreover $u_0, u_0^1 \in \mathcal{D}(A^2)$, and $Bu_0^1 \in \mathcal{D}(A)$, then for any $t > 0$, we have

$$\begin{aligned} u(t) &= u_0 + tu_0^1 + \frac{t^\alpha}{\Gamma(\alpha+1)}Au_0 + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}Au_0^1 + \frac{t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}Bu_0^1 \\ &\quad + E_{-\alpha-1}(t)A^2u_0 + E_{\beta-\alpha-1}(t)BAu_0 + E_{-\alpha-2}(t)A^2u_0^1 \\ &\quad + E_{-\beta-\alpha-2}(t)(BA+AB)u_0^1 + E_{2\beta-\alpha-2}B^2u_0^1. \end{aligned}$$

4.3. **Asymptotic behavior.** The asymptotic behavior in the case of $A \neq B$, under the assumptions stated at the beginning of the section for both, perfectly fits the one in the case $A = B$, that is there is hardly any difference, both if $u_0, u_0^1 \in \mathcal{D}(A)$, and even if $u_0, u_0^1 \in \mathcal{D}(B)/\mathcal{D}(A)$. Therefore the Theorem 5 is perfectly valid here, the proof straightforwardly follows, and this is why both are omitted.

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6. Declarations

6.1. **Conflict of interest.** The authors have no conflicts of interest to declare.

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