ON C^{α}-HÖLDER CLASSICAL SOLUTIONS FOR NON-AUTONOMOUS NEUTRAL DIFFERENTIAL EQUATIONS: THE NONLINEAR CASE

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Abstract. In this paper we continue our developments in [17] on the existence of classical solutions for abstract neutral differential equations. In the current work we extend the results in [17] for nonlinear neutral differential equations. Some applications involving nonlinear partial neutral differential equations are presented.

1. Introduction

In this paper we discuss the existence of classical solutions for some models of neutral differential equations which can be described in the abstract form

$$\frac{d}{dt}\left[u(t)+g(t,u_t)\right] = A(t)u(t)+f(t,u_t), \qquad t\in[0,a], \tag{1.1}$$

$$u_0 = \varphi\in U\subset\mathcal{B}=C([-p,0];\mathcal{D}), \ \ (p>0), \tag{1.2}$$

where $(A(t))_{t\in[0,a]}$ is a family of sectorial operators defined on a common domain $D(A)\subset X$, each operator $A(t)$ is the generator of an analytic semigroup of bounded linear operators on a Banach space $(X,\|\cdot\|)$, $\mathcal{D}$ denotes the space $D(A)$ endowed with the norm $\|\cdot\|_{\mathcal{D}}$, $U\subset\mathcal{B}$ is open, $\varphi\in U$, the history $u_t:[-p,0]\to X$ $(u_t(\theta)=x(t+\theta))$ belongs to $\mathcal{B}=C([-p,0];\mathcal{D})$ and $f,g:[0,a]\times U\to X$ are continuous functions.

The literature on abstract neutral differential equations of the form (1.1)-(1.2) is extensive and includes topics on the existence and qualitative properties of solutions, see [2, 3, 4, 5, 9, 11, 16, 17, 18, 19, 20, 22, 27] and the references therein. In the cited works some restrictive conditions are used in order to guarantee the integrability of the function $s\to AT(t-s)g(s,u_s)$ which appears in the variation of constant formulae used to define the concept of mild solution of (1.1)-(1.2). We can present these conditions in the following unified manner.

> **($\mathbf{H_g}$) There exists a Banach space $(\mathbf{Y},\|\cdot\|_{\mathbf{Y}})$ continuously included in X such that $\mathbf{g}\in\mathbf{C}([\mathbf{0},\mathbf{a}]\times\mathcal{B};\mathbf{Y})$ and the operator function $\mathbf{t}\to\mathbf{AT}(\cdot)$ belongs to $\mathbf{L^1}([\mathbf{0},\mathbf{b}];\mathcal{L}(\mathbf{Y},\mathbf{X}))$, where $\mathcal{L}(\mathbf{Y},\mathbf{X})$ denotes the space of bounded linear operator from Y into X endowed with the norm of operators.**

The condition ($\mathbf{H_g}$) is an elegant mathematical condition which can be verified in some situations. However, due to the nature of the operator A, this assumption is a severe restriction. To illustrate this fact, we note that in Datko [9], for example, $X=C_{UC}([0,\infty),\mathbb{R})$ (the space of bounded uniformly continuous functions from $(-\infty,0]$ into $\mathbb{R}$ endowed with the uniform norm), $A(t)=A$ for all t, and $A:D(A)\subset X\to X$ is the operator given by $Ax=x'$ with domain $D(A)=\{x\in X: x'\in X\}$. In this case, the function $g(\cdot)$ has an unusual regularizing property which transforms a continuous function into a C^1 function. A similar situation is observed in [1, 18, 19, 20].

In connection to the above, we note that in [2, 3, 4, 5] it is assumed that the set $\{AT(t):t\in(0,a]\}$ is bounded in the space of bounded linear operator on X which correspond to the case $Y=X$.

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However, as pointed out in [20], this condition is valid if and only if the operator A is a bounded linear operator, so this restricts the applications to ordinary differential equations.

On the other hand, in [16] is studied the existence of solutions for a neutral problem of the form

$$\frac{d}{dt}[x(t) + g(t, x(t - r_1))] = Ax(t) + f(t, x_t), \qquad (r_1 < r) \tag{1.3}$$

$$x_0 = \varphi \in C([-r, 0]; X), \tag{1.4}$$

without using the condition $\mathbf{H_g}$. We note that the approach in [16] is completely different to that used in this current paper and in the cited literature. The results in [16] are proved by assuming that the function $t \to g(t, \varphi(t - r_1))$ is smooth enough on $[0, r_1]$ when the function $g(\cdot)$ and the initial condition φ are smooth in an appropriate sense which permit the study of the existence of solutions for the associated integral equation

$$\begin{aligned} u(t) &= T(t)[\varphi(0) + g(0, \varphi(-r_1))] - g(t, \varphi(t - r_1)) \\ &\quad - \int_0^t AT(t - s)g(s, \varphi(s - r_1))ds + \int_0^t T(t - s)f(s, u_s)ds, \quad t \in [0, r_1]. \end{aligned}$$

Motivated by the above comments, in [17] we introduced a new approach to study abstract neutral differential equations based on maximal regularity type techniques for abstract parabolic problems. Using this new approach, in [17] we discussed the existence of classical solutions for abstract neutral problems of the form

$$\frac{d}{dt}[u(t) + g(t, u_t)] = A(t)u(t) + f(t, u_t).$$

The results in [17] are proved assuming that the functions $f(\cdot)$, $g(\cdot)$ are α-Hölder in an appropriated sense and $f(t, \cdot)$, $g(t, \cdot)$ are bounded linear operator for all $t \in [0, a]$. The assumption on $f(t, \cdot)$, $g(t, \cdot)$ is used to guarantee that functions of the form $t \to f(t, u_t) - f(t, v_t)$, $t \to g(t, u_t) - g(t, v_t)$ are α-Hölder continuous when the involved functions $u(\cdot), v(\cdot)$ are α-Hölder and this enables us to use the contraction mapping principle in spaces of α-Hölder continuous functions. The conditions on $f(t, \cdot)$, $g(t, \cdot)$ are useful and satisfied in several situations. However, they restrict the applications to linear type neutral differential equations. This relevant fact is the main motivation of the current work. In this paper, by assuming that $f(\cdot)$, $g(\cdot)$ are smooth enough on an open neighborhood of the initial condition φ, we study the local existence of classical solutions for neutral problems of the form

$$\frac{d}{dt}[u(t) + g(t, u_t)] = A(t)u(t) + f(t, u_t), \tag{1.5}$$

$$\frac{d}{dt}\left[u(t) + \int_{-p}^t B(t, \tau)u(\tau)d\tau\right] = A(t)u(t) + f(t, u_t), \tag{1.6}$$

which were treated earlier, for the linear type case, in the paper [17].

To finish our comments on the associated literature, is convenient to include a remark on the paper [15] where is introduced and studied a new type of neutral differential equation described in the form

$$u'(t) = Au(t) + f(t, u_t, u_t'), \qquad u_0 = \varphi \in \mathcal{B}, \tag{1.7}$$

which has a slight similarity with the problem and the technical approach in this current work. Concerning the relations between [15] and our current paper, we remark that now we study "nonlinear neutral" problems with applications to nonlinear partial differential equations whereas the results in [15] are only applicable to neutral type linear equations.

Abstract neutral differential equations arise in many situations. The abstract problem (1.1)-(1.2) arises, for example, in the theory of heat conduction in fading memory material. In the classical theory of heat conduction, it is assumed that the internal energy and the heat flux depends linearly on the temperature u and on its gradient ∇u. Under these conditions, the classical heat equation describes sufficiently well the evolution of the temperature in different types of materials. However, this description is not satisfactory in materials with fading memory. In the theory developed in [13, 26], the internal energy and the heat flux are described as functionals of u and u_x. The next system, see [6, 7, 8, 24], has been frequently used to describe this phenomena,

$$\begin{aligned}\frac{d}{dt}\left[u(t,x)+\int_{-\infty}^{t}k_1(t-s)u(s,x)ds\right] &= c\triangle u(t,x)+\int_{-\infty}^{t}k_2(t-s)\triangle u(s,x)ds,\\ u(t,x) &= 0, \qquad x\in\partial\Omega.\end{aligned}$$

In this system, $\Omega\subset\mathbb{R}^n$ is open, bounded and has smooth boundary, $(t,x)\in[0,\infty)\times\Omega$, $u(t,x)$ represents the temperature in x at the time t, c is a physical constant and $k_i:\mathbb{R}\to\mathbb{R}$, $i=1,2$, are the internal energy and the heat flux relaxation respectively. By assuming the solution u is known on $(-\infty,0]$ and $k_2\equiv 0$, we can transform this system into the abstract form (1.1)-(1.2).

The literature on ordinary neutral differential equations in the theory of population dynamics is extensive, see for example, [10, 12, 22, 23]. If in some of these works we consider the spatial diffusion phenomena, which arises in the natural tendency of biological populations to migrate from high population density regions to regions with minor density, then it is possible to obtain partial neutral differential systems of the form

$$\frac{d}{dt}\left[u(t,\xi)+g(t,u(t-p_1,\xi))\right]=\triangle u(t,\xi)+f(t,u(t-p_1,\xi)),$$

which can be described in the abstract form (1.1).

For additional references and new developments in abstract neutral differential equations we also cite [15, 16] and the references therein.

This paper has four sections. In the next section we introduce some notations, definitions and results used in this work. In Section 3 we study the existence of classical solutions for the neutral problems (1.5)-(1.6). In the last section some applications to nonlinear partial neutral differential equations are presented.

2. Preliminaries

In this section we introduce some notations, definitions and technical results used in this paper. Let $(Z,\|\cdot\|_Z)$ and $(W,\|\cdot\|_W)$ be Banach spaces. In this paper, $\mathcal{L}(Z,W)$ denotes the space of bounded linear operators from Z into W endowed with the norm of operators denoted by $\|\cdot\|_{\mathcal{L}(Z,W)}$ and we write $\mathcal{L}(Z)$ and $\|\cdot\|_{\mathcal{L}(Z)}$ when $Z=W$. In addition, $B_l(z,Z)=\{x\in Z:\|x-z\|_Z\leq l\}$ and for a given function $h(\cdot)$ we use the notation $d_ih(\cdot)$ for the i-th partial derivative.

As usual, $C([b,c];Z)$ is the space of continuous functions from $[b,c]$ into Z endowed with the uniform norm denoted by $\|\cdot\|_{C(I;Z)}$. The space $C^\gamma([b,c];Z)$, $\gamma\in(0,1)$, is formed by all the functions $\xi\in C([b,c];Z)$ such that $[\xi]_{C^\gamma([b,c];Z)}=\sup_{t,s\in[b,c],t\neq s}\frac{\|\xi(s)-\xi(t)\|_Z}{|t-s|^\gamma}<\infty$, provided with the norm $\|\xi\|_{C^\gamma([b,c];Z)}=\|\xi\|_{C([b,c];Z)}+[\xi]_{C^\gamma([b,c];Z)}$.

Concerning the operator family $(A(t))_{t\in[0,a]}$, we adopt all the notations and assumptions considered in [17]. In particular, $A=A(0)$ and $\mathcal{D}$ denotes the domain of A endowed with the graph norm $\|x\|_{\mathcal{D}}=\|Ax\|$. In addition, we suppose that $0\in\rho(A)$, each operator $A(t)$ is the infinitesimal generator of an analytic semigroup on X, the domain of $A(t)$ is independent of t and the operator function $t\to A(t)$ belongs to $C^\alpha([0,a];\mathcal{L}(\mathcal{D};X))$ for some $\alpha\in(0,1)$. In this work, $(T(t))_{t\geq 0}$ represents the semigroup generated by A and C_i, $i\in\mathbb{N}$, are positive constants such that $\|A^iT(t)\|\leq C_it^{-i}$ for every $t>0$. The notation $(X,\mathcal{D})_{\eta,\infty}$, $\eta\in(0,1)$, stands for the space

$$(X,\mathcal{D})_{\eta,\infty}=\{x\in X:[x]_{\eta,\infty}=\sup_{t\in(0,1)}\|t^{1-\eta}AT(t)x\|<\infty\},$$

endowed with the norm $\|x\|_{\eta,\infty}=[x]_{\eta,\infty}+\|x\|$. In addition, $C^k_{\eta,\infty}$, $k\in\mathbb{N}$, are real numbers such that $s^{1-\eta}\|A^kT(s)\|_{\mathcal{L}((X,\mathcal{D})_{\eta,\infty},X)}\leq C^k_{\eta,\infty}$ for all $s\in(0,a]$ and every $k\in\mathbb{N}$.

We also need to include some remarks on the abstract Cauchy problem

$$\frac{d}{dt}(x(t)+\xi_1(t))=Ax(t)+\xi_2(t),\qquad t\in[0,a],\qquad x(0)=x_0. \tag{2.1}$$

We note that the mild solution of (2.1) on $[0,b]$, $0<b\le a$, is the function given by

$$u(t)=T(t)(x_0+\xi_1(0))-\xi_1(t)-\int_0^t AT(t-s)\xi_1(s)ds+\int_0^t T(t-s)\xi_2(s)ds,\quad \forall\, t\in[0,b].$$

In addition, a function $u\in C([0,b];X)$ is said to be a classical solution of (2.1) on $[0,b]$ if $u+\xi\in C^1([0,b];X)$, $u\in C([0,b];\mathcal{D})$ and $u(\cdot)$ satisfies (2.1) on $[0,b]$.

Lemma 2.1. [17, Lemma 2.1] *Assume $\xi_1\in C^\alpha([0,b],\mathcal{D})$, $\xi_2\in C^\alpha([0,b],X)$, $x\in\mathcal{D}$ and let $u:[0,b]\to X$ be the mild solution of (2.1) on $[0,b]$. If $Ax+\xi_2(0)\in(X,\mathcal{D})_{\alpha,\infty}$, then $u(\cdot)$ is a classical solution of (2.1), $u\in C^\alpha([0,b],\mathcal{D})$, the function $u+\xi_1$ is differentiable, $\frac{d}{dt}(u+\xi_1)\in C^\alpha([0,b],X)$ and*

$$[u]_{C^\alpha([0,b],\mathcal{D})}\le\Lambda([\xi_1]_{C^\alpha([0,b],\mathcal{D})}+[\xi_2]_{C^\alpha([0,b],X)})+\frac{C^1_{\alpha,\infty}}{\alpha}\parallel Ax+\xi_2(0)\parallel_{\alpha,\infty},$$

where $\Lambda=\frac{2C_1}{\alpha}+3C_0+2+\frac{C_2}{\alpha(1-\alpha)}$.

To prove our results we introduce the $\mathbf{H}^\alpha_{\mathbf{Z},\mathbf{W}}$ condition.

($\mathbf{H}^\alpha_{\mathbf{Z},\mathbf{W}}$) Let $V\subset Z$ be an open set, $z\in V$, $\alpha\in(0,1)$, and $H\in C([0,a]\times V;W)$. We say that $H(\cdot)$ satisfies the $\mathbf{H}^\alpha_{\mathbf{Z},\mathbf{W}}$ condition at $z\in V$ if $d_2H\in C([0,a]\times V;W)$ and there is a function $L_H\in C([0,a]\times[0,\infty);(0,\infty))$ such that

$$\parallel H(t,x)-H(s,x)\parallel_W+\parallel d_2H(t,x)-d_2H(s,x)\parallel_{\mathcal{L}(Z,W)}\le L_H(b,r)\mid t-s\mid^\alpha,$$
$$\parallel d_2H(s,x_1)-d_2H(s,x_2))\parallel_{\mathcal{L}(Z,W)}\le L_H(b,r)\parallel x_1-x_2\parallel_Z,$$

for all $0\le s<t\le b\le a$, $x_1,x_2\in B_r(z,Z)$ and each $r>0$ such that $B_r(z,Z)\subset V$.

The following result is proved in [21] and we include the proof for completeness. In this result, we use the notation $H_w(\cdot)=H(\cdot,w(\cdot))$ for appropriate functions $H(\cdot)$ and $w(\cdot)$.

Lemma 2.2. *Assume $H\in C([0,a]\times V;W)$ satisfies the $\mathbf{H}^\alpha_{\mathbf{Z},\mathbf{W}}$ condition at $z\in V$, $B_r(z,Z)\subset V$, $0\le b\le a$ and $u,v\in C^\alpha([0,b];B_r(z,Z))$ are such that $u(0)=v(0)=z$. Then $H_u\in C^\alpha([0,b];W)$ and*

$$\begin{aligned}&[H_u]_{C^\alpha([0,b];W)}\\ &\quad\le\ L_H(b,r)+\left(\parallel d_2H(\cdot,z)\parallel_{C([0,b];\mathcal{L}(Z,W))}+L_H(b,r)b^\alpha[u]_{C^\alpha([0,b];Z)}\right)[u]_{C^\alpha([0,b];Z)},\end{aligned}\tag{2.2}$$

$$\begin{aligned}&[H_u-H_v]_{C^\alpha([0,b];W)}\\ &\quad\le\ \left(b^\alpha L_H(b,r)(1+2B(u,v))+\parallel d_2H(\cdot,z)\parallel_{C([0,b];\mathcal{L}(Z,W))}\right)[u-v]_{C^\alpha([0,b];Z)},\end{aligned}\tag{2.3}$$

where $B(u,v)=[u]_{C^\alpha([0,b];Z)}+[v]_{C^\alpha([0,b];Z)}$.

Proof: First we prove that $H_u\in C^\alpha([0,b];W)$. For $0<s\le t\le b$ we see that

$$\begin{aligned}
&\parallel H_u(t)-H_u(s)\parallel_W\le\parallel H(t,u(t))-H(s,u(t))\parallel_W+\parallel H(s,u(t))-H(s,u(s))\parallel_W\\
&\le\ \parallel H(t,u(t))-H(s,u(t))\parallel_W+\int_0^1\parallel\frac{\partial}{\partial\xi}H(s,\xi u(t)+(1-\xi)u(s))\parallel_W d\xi\\
&\le\ L_H(b,r)(t-s)^\alpha+\int_0^1\parallel d_2H(s,\xi u(t)+(1-\xi)u(s))\parallel_{\mathcal{L}(Z,W)}\parallel u(t)-u(s)\parallel_Z d\xi\\
&\le\ L_H(b,r)(t-s)^\alpha+[u]_{C^\alpha([0,b];Z)}(t-s)^\alpha\int_0^1\parallel d_2H(s,z)\parallel_{\mathcal{L}(Z;W)}d\xi\\
&\quad+[u]_{C^\alpha([0,b];Z)}(t-s)^\alpha\int_0^1\parallel d_2H(s,\xi u(t)+(1-\xi)u(s))-d_2H(s,z)\parallel_{\mathcal{L}(Z,W)}d\xi\\
&\le\ L_H(b,r)(t-s)^\alpha+[u]_{C^\alpha([0,b];Z)}(t-s)^\alpha\parallel d_2H(\cdot,z)\parallel_{C([0,b];\mathcal{L}(Z,W))}\\
&\quad+[u]_{C^\alpha([0,b];Z)}(t-s)^\alpha\int_0^1 L_H(b,r)(\xi\parallel u(t)-z\parallel_Z+(1-\xi)\parallel u(s)-z\parallel_Z)d\xi\\
&\le\ L_H(b,r)(t-s)^\alpha+\parallel d_2H(\cdot,z)\parallel_{C([0,b];\mathcal{L}(Z,W))}[u]_{C^\alpha([0,b];Z)})(t-s)^\alpha\\
&\quad+[u]_{C^\alpha([0,b];Z)}(t-s)^\alpha L_H(b,r)[u]_{C^\alpha([0,b];Z)}b^\alpha,
\end{aligned}$$

which implies that $H_u \in C^\alpha([0,b];W)$ and establishes the first inequality.

To prove the second inequality, for $0 < s \leq t \leq b$ we note that

$$\begin{aligned}
&H_u(t) - H_v(t) - (H_u(s) - H_u(s)) \\
&\quad = H(t,u(t)) - H(t,v(t)) - (H(s,u(t)) - H(s,v(t)) \\
&\qquad +H(s,u(t)) - H(s,v(t)) - (H(s,u(s)) - H(s,v(s))) \\
&\quad = \int_0^1 \frac{\partial}{\partial\xi}[H(t,\xi u(t) + (1-\xi)v(t)) - H(s,\xi u(t) + (1-\xi)v(t))]d\xi \\
&\qquad + \int_0^1 \frac{\partial}{\partial\xi}[H(s,\xi u(t) + (1-\xi)v(t)) - H(s,\xi u(s) + (1-\xi)v(s))]d\xi.
\end{aligned}$$

From this equality and using that $w = u - v \in C^\alpha([0,b];Z)$ and $w(0) = 0$, we get

$$\begin{aligned}
&\| H(t,u(t)) - H(t,v(t)) - (H(s,u(s)) - H(s,v(s)) \|_Z \\
&\leq \int_0^1 \| d_2H(t,\xi u(t) + (1-\xi)v(t)) - d_2H(s,\xi u(t) + (1-\xi)v(t)) \|_{\mathcal{L}(Z,W)} \| w(t) \|_Z \, d\xi \\
&\quad + \int_0^1 \| (d_2H(s,\xi u(t) + (1-\xi)v(t))w(t) - d_2H(s,\xi u(s) + (1-\xi)v(s))w(s) \|_Z \, d\xi \\
&\leq \int_0^1 \| d_2H(t,\xi u(t) + (1-\xi)v(t)) - d_2H(s,\xi u(t) + (1-\xi)v(t)) \|_{\mathcal{L}(Z,W)} \| w(t) \|_Z \, d\xi \\
&\quad + \int_0^1 \| [d_2H(s,\xi u(t) + (1-\xi)v(t)) - d_2H(s,\xi u(s) + (1-\xi)v(s))]w(t) \|_Z \, d\xi \\
&\quad + \int_0^1 \| [d_2H(s,\xi u(s) + (1-\xi)v(s)) - d_2H(s,z)](w(t) - w(s)) \|_Z \, d\xi \\
&\quad + \int_0^1 \| d_2H(s,z)(w(t) - w(s)) \|_W \, d\xi \\
&\leq L_H(b,r)(t-s)^\alpha [w]_{C^\alpha([0,b];Z)} b^\alpha \\
&\quad + L_H(b,r) \int_0^1 (\xi \| u(t) - u(s) \|_Z + (1-\xi) \| v(t) - v(s) \|_Z) \| w(t) \|_Z \, d\xi \\
&\quad + L_H(b,r) \int_0^1 (\xi \| u(s) - z \|_Z + (1-\xi) \| v(s) - z \|_Z) \| w(t) - w(s) \|_Z \, d\xi \\
&\quad + [w]_{C^\alpha([0,b];Z)} (t-s)^\alpha \| d_2H(\cdot,z) \|_{C([0,b];\mathcal{L}(Z,W))} \\
&\leq L_H(b,r)(t-s)^\alpha [w]_{C^\alpha([0,b];Z)} b^\alpha + L_H(b,r)B(u,v)(t-s)^\alpha [w]_{C^\alpha([0,b];Z)} t^\alpha \\
&\quad + L_H(b,r)(s^\alpha [u]_{C^\alpha([0,b];Z)} + s^\alpha [v]_{C^\alpha([0,b];Z)})[w]_{C^\alpha([0,b];Z)} (t-s)^\alpha \\
&\quad + [w]_{C^\alpha([0,b];Z)} (t-s)^\alpha \| d_2H(s,z) \|_{\mathcal{L}(Z,W)},
\end{aligned}$$

which shows that $H_u - H_v \in C^\alpha([0,b];W)$ and

$$\begin{aligned}
&[H_u - H_v]_{C^\alpha([0,b];W)} \\
&\quad \leq \left(b^\alpha L_H(b,r)(1 + 2B(u,v)) + \| d_2H(\cdot,z) \|_{C([0,b];\mathcal{L}(Z,W))} \right) [u-v]_{C^\alpha([0,b];Z)}. \quad \blacksquare
\end{aligned}$$

Remark 2.1. In the remainder of this work, Λ is the constant introduced in Lemma 2.1 and $\widetilde{\varphi} : [-p,a] \to \mathcal{B}$ is the function given by $\widetilde{\varphi}(t) = \varphi(t)$ for $t \in [-p,0]$ and $\widetilde{\varphi}(t) = T(t)\varphi(0)$ for $t \in [0,a]$. In addition, for a function $u \in C([-p,b];\mathcal{D})$, $0 < b \leq a$, we use the notations P_u and $L_u(t)$ for the functions $L_u : [0,b] \to X$ and $P_u : [0,b] \to \mathcal{B}$ given by $P_u(t) = u_t$ and $L_u(t) = (A(t) - A)u(t)$.

3. Existence of classical solutions

In this section we discuss the existence of solutions for the equations (1.5) and (1.6).

3.1. **Classical solutions for a general class of neutral system.** In this section we consider the problem of the existence of classical solutions for a neutral system of the form

$$\frac{d}{dt}(x(t)+g(t,x_t)) = A(t)x(t)+f(t,x_t), \qquad t\in[0,a], \tag{3.1}$$

$$x_0 = \varphi\in U\subset \mathcal{B}=C([-p,0],\mathcal{D}). \tag{3.2}$$

To begin we note the following concept of solution.

Definition 3.1. *A function $u\in C([-p,b];X)$ is called a classical solution of (3.1)-(3.2) on $[0,b]$ if $u(\cdot)+g(\cdot,P_u(\cdot))\in C^1([0,b];X)$, $u\in C([0,b];\mathcal{D})$, $u_0=\varphi$ and $u(\cdot)$ satisfies (3.1) on $[0,b]$.*

To establish the results of this section, from [17] we include the following lemma.

Lemma 3.3. *Let $u,w\in C([-p,b];\mathcal{D})$ and assume $w=0$. Then $L_u\in C^\alpha([0,b],\mathcal{D})$ and*

$$[L_u]_{C^\alpha([0,b];X)} \le [A]_{C^\alpha([0,b];\mathcal{L}(\mathcal{D},X))}(\| u\|_{C([0,b],\mathcal{D})} + b^\alpha[u]_{C^\alpha([0,b],\mathcal{D})}), \tag{3.3}$$

$$[L_w]_{C^\alpha([0,b];X)} \le 2b^\alpha[A]_{C^\alpha([0,b];\mathcal{L}(\mathcal{D},X))}[w]_{C^\alpha([0,b],\mathcal{D})}). \tag{3.4}$$

Remark 3.2. It is convenient to include some additional notations. For $b\in(0,b]$ and $R>0$ we use the notations

$$\Theta_{\rm A}({\rm b,R})=[{\rm A}]_{C^\alpha([0,b];\mathcal{L}(\mathcal{D},X))}(2{\rm b}^\alpha{\rm R}+\|\varphi(0)\|_{\mathcal{D}}) \text{ and } \Theta_{\rm A}({\rm b})=2{\rm b}^\alpha[{\rm A}]_{C^\alpha([0,b];\mathcal{L}(\mathcal{D},X))}.$$

We note that if u,w are the functions in Lemma 3.3, $u(0)=\varphi(0)$, $[u]_{C^\alpha([0,b];\mathcal{D})}\le R$ and $[w]_{C^\alpha([0,b];\mathcal{D})}\le R$, then $[L_u]_{C^\alpha([0,b];X)}\le\Theta_A(b,R)$ and $[L_w]_{C^\alpha([0,b];X)}\le\Theta_A(b)$.

We can establish now our first result on the existence of a classical solution for (3.1)-(3.2).

Theorem 3.1. *Assume that $g(\cdot)$ satisfies the $\mathbf{H}^\alpha_{\mathcal{B},\mathcal{D}}$ condition at $\varphi\in U$ and $f(\cdot)$ satisfies the $\mathbf{H}^\alpha_{\mathcal{B},\mathbf{X}}$ condition at $\varphi\in U$. Suppose, $P_{\tilde\varphi}\in C^\alpha([0,b];\mathcal{B})$, $\varphi\in C^\alpha([-p,0];\mathcal{D})$, $\{A\varphi(0),f(0,\varphi)\}\subset(X,\mathcal{D})_{\alpha,\infty}$ and*

$$2\Lambda(L_f(0,0)+L_g(0,0)+\| d_2f(0,\varphi)\|_{\mathcal{L}(\mathcal{B},X)}+\| d_2g(0,\varphi)\|_{\mathcal{L}(\mathcal{B},\mathcal{D})})<1, \tag{3.5}$$

where $L_g(\cdot),L_f(\cdot)$ are the functions associated to $g(\cdot),f(\cdot)$ via the $\mathbf{H}^\alpha_{\mathcal{B},\mathcal{D}}$ and the $\mathbf{H}^\alpha_{\mathcal{B},\mathbf{X}}$ conditions. Then there exists a unique classical solution $u\in C^\alpha([-p,b];\mathcal{D})$ of the problem (3.1)-(3.2) on $[0,b]$ for some $0<b\le a$.

Proof: Let $r>0$ be such that $B_r(\varphi,\mathcal{B})\subset U$. By using that the functions $L_g(\cdot),L_f(\cdot)$ are continuous, from condition (3.5) we select $R>r>0$ and $b\in(0,a]$ such that

$$\begin{aligned}
b^\alpha(\Lambda(\Theta_f(r,R,b)+\Theta_g(r,R,b)+\Theta_A(b,R))+\frac{C_{\alpha,\infty}}{\alpha}\| f(0,\varphi)\|_{\alpha,\infty}+[P_{\tilde\varphi}]_{C^\alpha([0,b];\mathcal{B})}) &\le r,\\
2\Lambda(\Theta_f(r,R,b)+\Theta_g(r,R,b)+\Theta_A(b,R))+[P_{\tilde\varphi}]_{C^\alpha([0,b];\mathcal{B})}&\\
+\frac{C_{\alpha,\infty}}{\alpha}(\| f(0,\varphi)\|_{\alpha,\infty}+\| A\varphi(0)+f(0,\varphi)\|_{\alpha,\infty}) &\le R,\\
2(b^\alpha+1)\Lambda((L_f(b,r)+L_g(b,r))(1+4Rb^\alpha)+\Theta(b)) &< 1,
\end{aligned}$$

where $\Theta_f(b)=\| d_2f(\cdot,\varphi)\|_{C([0,b];\mathcal{L}(\mathcal{B},X))}$, $\Theta_g(b)=\| d_2g(\cdot,\varphi)\|_{C([0,b];\mathcal{L}(\mathcal{B},\mathcal{D}))}$, $\Theta(b)=\Theta_f(b)+\Theta_g(b)+\Theta_A(b)$, $\Theta_h(r,R,b)=L_h(b,r)+(L_h(b,r)b^\alpha R+\Theta_h(b))R$ for $h\in\{f,g\}$, and $\Theta_A(b)$, $\Theta_A(b,R)$ are the numbers introduced in Remark 3.2.

Let $Y=C^\alpha([-p,b];\mathcal{D})$ and $\mathcal{Y}(b,r,R)$ be the space

$$\mathcal{Y}(b,r,R)=\{u\in Y; u_0=\varphi,\ P_u\in C^\alpha([0,b];\mathcal{B}),\ [P_u]_{C^\alpha([0,b];\mathcal{B})}\le R,\ \| P_u-\varphi\|_{C([0,b];\mathcal{B})}\le r\},$$

endowed with the metric $d(u,v)=\| P_u-P_v\|_{C^\alpha([0,b];\mathcal{B})}$. On the space $\mathcal{Y}(b,r,R)$ we define the map $\Gamma:\mathcal{Y}(b,r,R)\to C([-p,b];X)$ by $(\Gamma u)_0=\varphi$ and

$$\begin{aligned}
\Gamma u(t) &= T(t)(\varphi(0)+g(0,\varphi))-g_u(t)-\int_0^t AT(t-s)g_u(s)ds\\
&\quad+\int_0^t T(t-s)[f_u(s)+L_u(s)]ds, \qquad \text{for } t\in[0,b],
\end{aligned}$$

where $g_u(t)=g(t,u_t)$, $f_u(t)=f(t,u_t)$ and $L_u(t)=(A(t)-A)u(t)$.

By noting that $g_u\in C([0,b];\mathcal{D})$, it is easy to see that $\Gamma u\in C([-p,b];X)$. Next, we prove that Γ is a contraction on $\mathcal{Y}(b,r,R)$. In the remainder of the proof, we assume $u,v\in\mathcal{Y}(b,r,R)$ and we use the notation $\Gamma^1 u$ for the function $\Gamma^1 u=\Gamma u-\tilde{\varphi}$.

Step 1. The map Γ has values in Y and $\parallel P_u-\varphi\parallel_{C([0,b];\mathcal{B})}\leq r$.

From the definition of Y, the properties of the functions $g(\cdot), f(\cdot)$ and Lemma 2.2 we have that $g_u\in C^\alpha([0,b];\mathcal{D})$, $f_u\in C^\alpha([0,b];X)$ and

$$\begin{aligned}
&[f_u]_{C^\alpha([0,b];X)}\\
&\quad\leq\ L_f(b,r)+(L_f(b,r)b^\alpha[P_u]_{C^\alpha([0,b];\mathcal{B})}+\parallel d_2f(\cdot,\varphi)\parallel_{C([0,b];\mathcal{L}(\mathcal{B},X))})[P_u]_{C^\alpha([0,b];\mathcal{B})},\\
&[g_u]_{C^\alpha([0,b];\mathcal{D})}\\
&\quad\leq\ L_g(b,r)+(L_g(b,r)b^\alpha[P_u]_{C^\alpha([0,b];\mathcal{B})}+\parallel d_2g(\cdot,\varphi)\parallel_{C([0,b];\mathcal{L}(\mathcal{B},\mathcal{D}))})[P_u]_{C^\alpha([0,b];\mathcal{B})}.
\end{aligned}$$

From the above and Remark 3.2 we obtain that

$$[L_u]_{C^\alpha([0,b];X)}\leq\Theta_A(b,R),\ [f_u]_{C^\alpha([0,b];X)}\leq\Theta_f(r,R,b)\ \text{and}\ [g_u]_{C^\alpha([0,b];\mathcal{D})}\leq\Theta_g(r,R,b).\tag{3.6}$$

On the other hand, from Lemma 2.1 we see that $\Gamma u_{|[0,b]}\in C^\alpha([0,b];\mathcal{D})$ and

$$\begin{aligned}
[(\Gamma u)_{|[0,b]}]_{C^\alpha([0,b];\mathcal{D})}\ &\leq\ \Lambda([f_u]_{C^\alpha([0,b];X)}+[g_u]_{C^\alpha([0,b];\mathcal{D})}+[L_u]_{C^\alpha([0,b];X)})\\
&\quad+\frac{C_{\alpha,\infty}}{\alpha}\parallel A\varphi(0)+f(0,\varphi)\parallel_{\alpha,\infty},
\end{aligned}$$

from which we obtain that

$$\begin{aligned}
[(\Gamma u)_{|[0,b]}]_{C^\alpha([0,b];\mathcal{D})}\ &\leq\ \Lambda(\Theta_f(r,R,b)+\Theta_g(r,R,b)+\Theta_A(b,R))\\
&\quad+\frac{C_{\alpha,\infty}}{\alpha}\parallel A\varphi(0)+f(0,\varphi)\parallel_{\alpha,\infty}.
\end{aligned}\tag{3.7}$$

In addition, by noting that $(\Gamma^1 u)_{|[-p,0]}$ is the mild solution of (2.1) with $\xi_1=g_u$, $\xi_2=f_u+L_u$ and $x_0=0$ and using Lemma 2.1, for $t\in[-p,0]$ and $h>0$ such that $t+h>0$ we get

$$\begin{aligned}
&\parallel\Gamma u(t+h)-\Gamma u(t)\parallel_{\mathcal{D}}=\parallel\Gamma u(t+h)-\varphi(t)\parallel_{\mathcal{D}}\\
&\quad\leq\ \parallel\Gamma u(t+h)-T(t+h)\varphi(0)\parallel_{\mathcal{D}}+\parallel T(t+h)\varphi(0)-\varphi(0)\parallel_{\mathcal{D}}+\parallel\varphi(0)-\varphi(t)\parallel_{\mathcal{D}}\\
&\quad\leq\ (t+h)^\alpha(\Lambda([f_u]_{C^\alpha([0,b];X)}+[g_u]_{C^\alpha([0,b];\mathcal{D})}+[L_u]_{C^\alpha([0,b];X)})+\frac{C_{\alpha,\infty}}{\alpha}\parallel f(0,\varphi)\parallel_{\alpha,\infty})\\
&\qquad+\int_0^{t+h}\parallel T(s)A\varphi(0)\parallel ds+(-t)^\alpha[\varphi]_{C^\alpha([0,b];\mathcal{D})}\\
&\quad\leq\ h^\alpha(\Lambda([f_u]_{C^\alpha([0,b];X)}+[g_u]_{C^\alpha([0,b];\mathcal{D})}+[L_u]_{C^\alpha([0,b];X)})+\frac{C_{\alpha,\infty}}{\alpha}\parallel f(0,\varphi)\parallel_{\alpha,\infty})\\
&\qquad+(t+h)^\alpha b^{1-\alpha}C_0\parallel A\varphi(0)\parallel+h^\alpha[\varphi]_{C^\alpha([0,b];\mathcal{D})},
\end{aligned}$$

and hence,

$$\begin{aligned}
&\parallel\Gamma u(t+h)-\Gamma u(t)\parallel_{\mathcal{D}}\\
&\qquad\leq\ h^\alpha(\Lambda([f_u]_{C^\alpha([0,b];X)}+[g_u]_{C^\alpha([0,b];\mathcal{D})}+[L_u]_{C^\alpha([0,b];X)})+\frac{C_{\alpha,\infty}}{\alpha}\parallel f(0,\varphi)\parallel_{\alpha,\infty})\\
&\qquad\quad+h^\alpha(b^{1-\alpha}C_0\parallel A\varphi(0)\parallel+[\varphi]_{C^\alpha([0,b];\mathcal{D})}),\qquad\text{for } t\in[-p,0], h>0, t+h>0.
\end{aligned}$$

Using this inequality, (3.6), (3.7) and the fact that $\varphi\in C^\alpha([-p,0];\mathcal{D})$ we infer that

$$\begin{aligned}
[\Gamma u]_{C^\alpha([-p,b];\mathcal{D})}\ &\leq\ \Lambda(\Theta_f(r,R,b)+\Theta_g(r,R,b)+\Theta_A(b,R))\\
&\quad+\frac{C_{\alpha,\infty}}{\alpha}(\parallel f(0,\varphi)\parallel_{\alpha,\infty}+\parallel A\varphi(0)+f(0,\varphi)\parallel_{\alpha,\infty})\\
&\quad+b^{1-\alpha}C_0\parallel A\varphi(0)\parallel+[\varphi]_{C^\alpha([0,b];\mathcal{D})},
\end{aligned}$$

which shows that $\Gamma u\in C^\alpha([-p,b];\mathcal{D})$ and $\Gamma u\in Y$.

To prove that $\| P_u - \varphi \|_{C([0,b];\mathcal{B})} \leq r$, it is convenient to estimate $\| \Gamma u(t) - \widetilde{\varphi}(t) \|_{\mathcal{D}}$. From the inequalities in (3.6) and Lemma 2.1, we note that for $t \in [0,b]$

$$\begin{aligned}\| \Gamma u(t) - \widetilde{\varphi}(t) \|_{\mathcal{D}} &= \| \Gamma^1 u(t) \|_{\mathcal{D}} \\ &\leq t^\alpha (\Lambda([f_u]_{C^\alpha([0,b];X)} + [g_u]_{C^\alpha([0,b];\mathcal{D})} + [L_u]_{C^\alpha([0,b];X)}) + \frac{C_{\alpha,\infty}}{\alpha} \| f(0,\varphi) \|_{\alpha,\infty}) \\ &\leq t^\alpha (\Lambda(\Theta_f(r,R,b) + \Theta_g(r,R,b) + \Theta_A(b,R)) + \frac{C_{\alpha,\infty}}{\alpha} \| f(0,\varphi) \|_{\alpha,\infty}).\end{aligned}$$

Using this estimate, for $t \in [0,b]$ we get

$$\begin{aligned}&\| P_{\Gamma u}(t) - \varphi \|_{\mathcal{B}} \leq \| P_{\Gamma u}(t) - P_{\widetilde{\varphi}}(t) \|_{\mathcal{B}} + \| P_{\widetilde{\varphi}}(t) - \varphi \|_{\mathcal{B}} \\ &\leq \sup_{s\in[0,t]} \| \Gamma u(s) - \widetilde{\varphi}(s) \|_{\mathcal{B}} + t^\alpha [P_{\widetilde{\varphi}}]_{C^\alpha([0,b];\mathcal{B})} \\ &\leq t^\alpha (\Lambda(\Theta_f(r,R,b) + \Theta_g(r,R,b) + \Theta_A(b,R)) + \frac{C_{\alpha,\infty}}{\alpha} \| f(0,\varphi) \|_{\alpha,\infty}) + t^\alpha [P_{\widetilde{\varphi}}]_{C^\alpha([0,b];\mathcal{B})}, \qquad (3.8)\end{aligned}$$

which shows (see the choice of b) that $\| P_{\Gamma u}(t) - \varphi \|_{\mathcal{B}} \leq r$ for all $t \in [0,b]$.

Step 2. $P_{\Gamma u} \in C^\alpha([0,b];\mathcal{B})$ and $[P_{\Gamma u}]_{C^\alpha([0,b];\mathcal{B})} \leq R$.

From (3.7), (3.8) and Lemma 2.1, for $t \in [0,b)$ and $h > 0$ with $t+h \in [0,b]$ we get

$$\begin{aligned}&\| P_{\Gamma u}(t+h) - P_{\Gamma u}(t) \|_{\mathcal{B}} \\ &\leq \| P_{\Gamma u}(h) - \varphi \|_{\mathcal{B}} + \sup_{s\in[0,t]} \| \Gamma u(s+h) - \Gamma u(s) \| \\ &\leq h^\alpha (\Lambda(\Theta_f(r,R,b) + \Theta_g(r,R,b) + \Theta_A(b,R)) + \frac{C_{\alpha,\infty}}{\alpha} \| f(0,\varphi) \|_{\alpha,\infty} + [P_{\widetilde{\varphi}}]_{C^\alpha([0,b];\mathcal{B})}) \\ &\quad + h^\alpha (\Lambda(\Theta_f(r,R,b) + \Theta_g(r,R,b) + \Theta_A(b,R)) + \frac{C_{\alpha,\infty}}{\alpha} \| A\varphi(0) + f(0,\varphi) \|_{\alpha,\infty}), \\ &\leq 2h^\alpha (\Lambda(\Theta_f(r,R,b) + \Theta_g(r,R,b) + \Theta_A(b,R)) + h^\alpha [P_{\widetilde{\varphi}}]_{C^\alpha([0,b];\mathcal{B}_X)} \\ &\quad + h^\alpha \frac{C_{\alpha,\infty}}{\alpha} (\| f(0,\varphi) \|_{\alpha,\infty} + \| A\varphi(0) + f(0,\varphi) \|_{\alpha,\infty}), \qquad (3.9)\end{aligned}$$

which shows that $P_{\Gamma u} \in C^\alpha([0,b];\mathcal{B})$ and $[P_{\Gamma u}]_{C^\alpha([0,b];\mathcal{B})} \leq R$.

Step 3. The map Γ is a contraction on $\mathcal{Y}(b,r,R)$.

From Step 1 and Step 2 it follows that Γ has values in $\mathcal{Y}(b,r,R)$. To prove that Γ is a contraction, we estimate $[P_{\Gamma u} - P_{\Gamma v}]_{C^\alpha([0,b];\mathcal{B})}$.

By noting that $\Gamma^1 u - \Gamma^1 v$ is the mild solution of the problem (2.1) with $\xi_1 = g_u - g_v$, $\xi_2 = f_u + L_u - (f_v + L_v)$ and $x_0 = 0$ and using Lemma 2.1, for $t \in [0,b)$ and $h > 0$ such that $t+h \in [0,b]$ we see that

$$\begin{aligned}&\| P_{\Gamma u}(t+h) - P_{\Gamma v}(t+h) - (P_{\Gamma u}(t) - P_{\Gamma v}(t)) \|_{\mathcal{B}} \\ &\leq \| P_{\Gamma u}(h) - P_{\Gamma v}(h) \|_{\mathcal{B}} + \sup_{s\in[0,t]} \| \Gamma u(s+h) - \Gamma v(s+h) - (\Gamma u(s) - \Gamma v(s)) \|_{\mathcal{D}} \\ &\leq \sup_{s\in[0,h]} \| \Gamma u(s) - \Gamma v(s) - (\Gamma u(0) - \Gamma v(0)) \|_{\mathcal{D}} + h^\alpha [(\Gamma u)_{|[0,b]} - (\Gamma v)_{|[0,b]}]_{C^\alpha([0,b];\mathcal{D})} \\ &\leq 2h^\alpha [(\Gamma u)_{|[0,b]} - (\Gamma v)_{|[0,b]}]_{C^\alpha([0,b];\mathcal{D})} \\ &\leq 2h^\alpha \Lambda([f_u - f_v]_{C^\alpha([0,b];X)} + [g_u - g_v]_{C^\alpha([0,b];\mathcal{D})} + [L_{u-v}]_{C^\alpha([0,b];X)})\end{aligned}$$

which implies, via Lemma 2.2 and Lemma 3.3 (see the inequalities (2.3) and (3.4) respectively), that

$$\begin{aligned}&[P_{\Gamma u} - P_{\Gamma v}]_{C^\alpha([0,b];\mathcal{B})} \\ &\leq 2\Lambda(L_f(b,r)(1 + 2B(P_u,P_v)b^\alpha) + \| d_2 f(\cdot,\varphi) \|_{C([0,b];\mathcal{L}(\mathcal{B}))}[P_u - P_v]_{C^\alpha([0,b];\mathcal{B})}) \\ &\quad + 2\Lambda(L_g(b,r)(1 + 2B(P_u,P_v)b^\alpha) + \| d_2 g(\cdot,\varphi) \|_{C([0,b];\mathcal{L}(\mathcal{B}))}[P_u - P_v]_{C^\alpha([0,b];\mathcal{B})}) \\ &\quad + 2b^\alpha [A]_{C^\alpha([0,b];\mathcal{L}(\mathcal{D},X))} d(u,v),\end{aligned}$$

and

$$[P_{\Gamma u} - P_{\Gamma v}]_{C^\alpha([0,b];\mathcal{B})} \leq 2\Lambda((L_f(b,r) + L_g(b,r))(1 + 4Rb^\alpha) + \Theta(b)) d(u,v).$$

Moreover, by noting that $P_{\Gamma u}(0) = P_{\Gamma v}(0)$ it follows that

$$\| P_{\Gamma u} - P_{\Gamma v} \|_{C^\alpha([0,b];\mathcal{B})} \leq 2(b^\alpha + 1)\Lambda((L_f(b,r) + L_g(b,r))(1 + 4Rb^\alpha) + \Theta(b))d(u,v),$$

which proves that Γ is a contraction on $\mathcal{Y}(b,r,R)$.

Finally, from the contraction mapping principe we conclude that there exists a unique classical solution $u \in C^\alpha([0,b];\mathcal{D})$ of (3.1)-(3.2) on $[0,b]$. This completes the proof. ■

In connection with the results in [17] we have the following corollary.

Corollary 3.1. *Assume that* $g \in C^\alpha([0,a];\mathcal{L}(\mathcal{B};\mathcal{D}))$, $f \in C^\alpha([0,a];\mathcal{L}(\mathcal{B};X))$, *the function* $P_{\widetilde{\varphi}}$ *belongs to* $C^\alpha([0,a];\mathcal{B})$, $\varphi \in C^\alpha([-p,0];\mathcal{D})$, $\{A\varphi(0), f(0)\varphi\} \subset (X,\mathcal{D})_{\alpha,\infty}$ *and*

$$4\Lambda(\| f(0) \|_{\mathcal{L}(\mathcal{B};X)} + \| g(0) \|_{\mathcal{L}(\mathcal{B};\mathcal{D})}) < 1.$$

Then there exists a unique classical solution $u \in C^\alpha([-p,b];\mathcal{D})$ *of (3.1)-(3.2) on* $[0,b]$ *for some* $0 < b \leq a$.

Proof: The assertion follows directly from Theorem 3.1. We only note that the functions $g(\cdot)$, $f(\cdot)$ satisfies the $\mathbf{H}^\alpha_{\mathcal{B},\mathcal{D}}$ and the $\mathbf{H}^\alpha_{\mathcal{B},\mathbf{X}}$ condition with $L_g(t,r) = \| g \|_{C([0,t];\mathcal{L}(\mathcal{B},\mathcal{D}))} (1+r)$ and $L_f(t,r) = \| f \|_{C([0,t];\mathcal{L}(\mathcal{B},X))} (1+r)$. ■

In the next corollary we establish the existence of a classical solution for the neutral problem

$$\frac{d}{dt}\left[x(t) + G(t, x(t-p_1))\right] = A(t)x(t) + F(t, x(t-p_2)), \quad t \in [0,a], \tag{3.10}$$

$$x_0 = \varphi, \tag{3.11}$$

where $0 < p_1, p_2 \leq p$, $F, G \in C^1([0,a] \times V; X)$ and $V \subseteq \mathcal{D}$ is a open neighborhood of $\varphi(0)$.

Corollary 3.2. *Assume that* $G(\cdot), F(\cdot)$ *satisfies the* $\mathbf{H}^\alpha_{\mathcal{D},\mathcal{D}}$ *and the* $\mathbf{H}^\alpha_{\mathcal{D},\mathbf{X}}$ *conditions at* $\varphi(0) \in V$, $P_{\widetilde{\varphi}} \in C^\alpha([0,b];\mathcal{B})$, $\{A\varphi(0), F(0,\varphi(-p_2))\} \subset (X,\mathcal{D})_{\alpha,\infty}$ *and*

$$\Lambda(L_F(0,0) + L_G(0,0) + \| d_2F(0,\varphi(-p_2)) \|_{\mathcal{L}(\mathcal{D},X)} + \| d_2G(0,\varphi(-p_1)) \|_{\mathcal{L}(\mathcal{D},\mathcal{D})}) < 1.$$

Then there exists a unique classical solution $u \in C^\alpha([-p,b];\mathcal{D})$ *of (3.10)-(3.11) on* $[0,b]$ *for some* $0 < b \leq a$.

Proof: The results follows from Theorem 3.1 by defining the function $f(\cdot), g(\cdot)$ by $f(t,\psi)(\xi) = F(t,\psi(-p_2))$ and $g(t,\psi)(\xi) = G(t,\psi(-p_1))$. ■

3.2. **On integro-differential abstract neutral equations.** In this section we extend the results in [17, Section 2.2]. We study the existence of solutions for a class of "nonlinear" integro-differential neutral equations of the form

$$\frac{d}{dt}\left[u(t) + \int_{-p}^{t} B(t,\tau)u(\tau)d\tau\right] = A(t)u(t) + f(t,u_t), \quad t \in [0,a], \ (p \geq a), \tag{3.12}$$

$$u_0 = \varphi. \tag{3.13}$$

Definition 3.2. *A function* $u \in C([-p,b];X)$ *is said to be a classical solution of (3.12)-(3.13) on* $[0,b]$ *if the function* $t \to \left[u(t) + \int_{-p}^{t} B(t,\tau)u(\tau)d\tau\right]$ *belongs to* $C^1([0,b];X)$, $u \in C([0,b];\mathcal{D})$, $u_0 = \varphi$ *and* $u(\cdot)$ *satisfies (3.12) on* $[0,b]$.

To prove our following results, from [17] we note the following conditions.

($\mathbf{H_1}$) For all $t \geq s$, $B(t,s) \in \mathcal{L}(\mathcal{D})$, $B(t,\cdot) \in L^{\frac{1}{(1-\alpha)}}([0,t],\mathcal{L}(\mathcal{D}))$,

$$\Theta_B(b) = \sup_{t\in[0,b]} (\int_0^t \| B(t,\sigma) \|^{\frac{1}{1-\alpha}}_{\mathcal{L}(\mathcal{D})} d\sigma)^{(1-\alpha)} < \infty,$$

for all $b < a$, and there is $L_B \in L^1([0,a];\mathbb{R}^+)$ such that

$$\| B(t,t-\tau) - B(s,s-\tau) \|_{\mathcal{L}(\mathcal{D})} \leq L_B(\tau) \mid t-s \mid^\alpha, \quad t \geq s, \ \tau \in [0,s].$$

($\mathbf{H_2}$) For all $t \geq s$, $B(t,s) \in \mathcal{L}(\mathcal{D})$, $B(t,\cdot) \in L^{\frac{1}{(1-\alpha)}}([0,t],\mathcal{L}(\mathcal{D}))$, $\Theta_B(b)$ (see condition $\mathbf{H_1}$) is finite for all $0 < b \leq a$ and there is a integrable function $L_B \in L^1([0,a];\mathbb{R}^+)$ such that $\| B(t,\tau) - B(s,\tau) \|_{\mathcal{L}(\mathcal{D})} \leq L_B(\tau) \mid t-s \mid^{\alpha}$ for all $a \geq t \geq s \geq 0$ and every $\tau \in [0,s]$.

Remark 3.3. In the remainder of this section, $\Phi(\varphi) : [0,a] \to X$ is the function given by $\Phi(\varphi)(t) = \int_{-p}^{0} B(t,\tau)\varphi(\tau)d\tau$. In addition, for $u \in ([-p,b];\mathcal{D})$, $0 < b \leq a$, we use the notation S_u for the function $S_u : [0,b] \to X$ given by $S_u(t) = \int_0^t B(t,\tau)u(\tau)d\tau$.

The proof of Lemma 3.4 follows with minor modifications from the proof of [17, Lemma 2.4].

Lemma 3.4. *Assume $u \in C^{\alpha}([0,b],\mathcal{D})$ and the condition $\mathbf{H_1}$ is satisfied. Then S_u belongs to $C^{\alpha}([0,b],\mathcal{D})$, $\| S_u \|_{C([0,b],\mathcal{D})} \leq \| u \|_{C([0,b],\mathcal{D})} \Theta_B(b)b^{\alpha}$ and*

$$[S_u]_{C^{\alpha}([0,b],\mathcal{D}))} \leq (\| L_B \|_{L^1([0,b])} + \Theta_B(b)) \| u \|_{C([0,b];\mathcal{D})} + \Theta_B(b)b^{\alpha}[u]_{C^{\alpha}([0,b],\mathcal{D})}.$$

In addition, if $u(0) = 0$ then $\| S_u \|_{C([0,b],\mathcal{D})} \leq [u]_{C^{\alpha}([0,b],\mathcal{D})}\Theta_B(b)b^{2\alpha}$ and

$$[S_u]_{C^{\alpha}([0,b],\mathcal{D}))} \leq (\| L_B \|_{L^1([0,b])} + 2\Theta_B(b))b^{\alpha}[u]_{C^{\alpha}([0,b],\mathcal{D})}.$$

Remark 3.4. In the remainder of this section, for positive number b, R we use the notations

$$\begin{aligned} \Theta_{1,B}(b,R) &:= (\| L_B \|_{L^1([0,b])} + \Theta_B(b))(Rb^{\alpha} + \| \varphi(0) \|) + \Theta_B(b)b^{\alpha}R, \\ \Theta_{2,B}(b) &:= (\| L_B \|_{L^1([0,b])} + 2\Theta_B(b))b^{\alpha}. \end{aligned}$$

If $u(\cdot)$ is the function in Lemma 3.4 and $[u]_{C^{\alpha}([0,b];\mathcal{D})} \leq R$, then $[S_u]_{C^{\alpha}([0,b],\mathcal{D}))} \leq \Theta_{1,B}(b,R)$ when $u(0) = \varphi(0)$ and $[S_u]_{C^{\alpha}([0,b],\mathcal{D}))} \leq \Theta_{2,B}(b)[u]_{C^{\alpha}([0,b],\mathcal{D})}$ when $u(0) = 0$.

The proof of the next theorem follows an argument similar at that in the proof of Theorem 3.1. We include a sketch of the proof for completeness.

Theorem 3.2. *Assume the condition $\mathbf{H_1}$ is satisfied and $f(\cdot)$ satisfies the $\mathbf{H}^{\alpha}_{\mathcal{B},\mathbf{X}}$ condition at $\varphi \in U$. Suppose $P_{\widetilde{\varphi}} \in C^{\alpha}([0,b];\mathcal{B})$, $\varphi \in C^{\alpha}([-p,0];\mathcal{D})$, $\{A\varphi(0), f(0,\varphi)\} \subset (X,\mathcal{D})_{\alpha,\infty}$, $\Phi(\varphi) \in C^{\alpha}([0,b];\mathcal{D})$ and*

$$2\Lambda(L_f(0,0) + \| d_2 f(0,\varphi) \|_{\mathcal{L}(\mathcal{B},X)}) < 1, \tag{3.14}$$

where $L_f(\cdot)$ is the function associated to $f(\cdot)$ via the $\mathbf{H}^{\alpha}_{\mathcal{B},\mathbf{X}}$ condition. Then there exists a unique classical solution $u \in C^{\alpha}([0,b],\mathcal{D})$ of the neutral problem (3.12)-(3.13) for some $0 < b \leq a$.

Proof: Let $\Theta_A(s)$, $\Theta_A(s,l)$, $\Theta_{1,B}(s,l)$ and $\Theta_{2,B}(s)$ be defined as in Remark 3.2 and Remark 3.4. Let $r > 0$ be such that $B_r(\varphi,\mathcal{B}) \subset U$. By noting that $\Theta_{2,B}(s) + \Theta_A(s) \to 0$ as $s \to 0$, from condition (3.14) we select $0 < b \leq a$ and $R > r$ such that

$$\begin{aligned} & b^{\alpha}\Lambda(\Theta_f(r,R,b) + \Theta_{1,B}(b,R) + \Theta_A(b,R)) \\ & + b^{\alpha}(\Lambda[\Phi(\varphi)]_{C^{\alpha}([0,b];\mathcal{D})} + \frac{C_{\alpha,\infty}}{\alpha} \| f(0,\varphi) \|_{\alpha,\infty} + [P_{\widetilde{\varphi}}]_{C^{\alpha}([0,b];\mathcal{B}_X)}) \leq r, \\ & 2\Lambda(\Theta_f(r,R,b) + \Theta_{1,B}(b,R) + \Theta_A(b,R)) + 2\Lambda[\Phi(\varphi)]_{C^{\alpha}([0,b];\mathcal{D})} + [P_{\widetilde{\varphi}}]_{C^{\alpha}([0,b];\mathcal{B}_X)} \\ & + \frac{C_{\alpha,\infty}}{\alpha}(\| f(0,\varphi) \|_{\alpha,\infty} + \| A\varphi(0) + f(0,\varphi) \|_{\alpha,\infty}) \leq R, \\ & 2(b^{\alpha}+1)\Lambda\,(L_f(b,r)(1+4Rb^{\alpha}) + \Theta_f(b) + \Theta_{2,B}(b) + \Theta_A(b)) < 1, \end{aligned}$$

where $\Theta_f(r,R,b) = L_f(r,b) + (L_f(r,b)b^{\alpha}R + \Theta_f(b))R$ and $\Theta_f(b) = \| d_2 f(\cdot,\varphi) \|_{C([0,b];\mathcal{L}(\mathcal{B},X))}$.

Let $Y, \mathcal{Y}(b,r,R)$ be the spaces introduced in the proof of Theorem 3.1 and $\Gamma : \mathcal{Y}(b,r,R) \to C([-p,b];X)$ the map given by $(\Gamma u)_0 = \varphi$ and

$$\begin{aligned} \Gamma u(t) &= T(t)\varphi(0) - [S_u(t) + \Phi(\varphi)(t)] - \int_0^t AT(t-s)[S_u(s) + \Phi(\varphi)(s)]ds \\ &\quad + \int_0^t T(t-s)[f_u(s) + L_u(s)]ds, \quad t \in [0,b], \end{aligned}$$

where g_u, f_u and L_u are defined in the proof of Theorem 3.1. Next we prove that Γ is a contraction on $\mathcal{Y}(b,r,R)$. In the sequel we assume $u, v \in \mathcal{Y}(b,r,R)$.

Step 1. The map Γ has values in $\mathcal{Y}(b,r,R)$.

From the assumptions, Lemma 2.2, Lemma 3.3 and Lemma 3.4 we have that the functions AS_u, $A\Phi(\varphi)$, f_u and L_u belong to $C^\alpha([0,b];X)$ which implies via Lemma 2.1 that $\Gamma u_{|_{[0,b]}}$ belongs to $C^\alpha([0,b];\mathcal{D})$ and Γu is a well defined continuous function. Moreover, from these results we also obtain that

$$[S_u]_{C^\alpha([0,b];\mathcal{D})} \leq (\| L_B \|_{L^1([0,b])} + \Theta_B(b)) \| u \|_{C([0,b];\mathcal{D})} + \Theta_B(b) b^\alpha [u]_{C^\alpha([0,b],\mathcal{D})}$$

$$[L_u]_{C^\alpha([0,b];X)} \leq [A]_{C^\alpha([0,b];\mathcal{L}(\mathcal{D},X))}(\| u \|_{C([0,b],\mathcal{D})} + b^\alpha [u]_{C^\alpha([0,b],\mathcal{D})})$$

$$[f_u]_{C^\alpha([0,b];X)} \leq L_f(b,r) + (L_f(b,r) b^\alpha [P_u]_{C^\alpha([0,b];\mathcal{B})} + \| d_2 f(\cdot,\varphi) \|_{C([0,b];\mathcal{L}(\mathcal{B},X))})[P_u]_{C^\alpha([0,b];\mathcal{B})},$$

and

$$[L_u]_{C^\alpha([0,b];X)} \leq \Theta_A(b,R),\ [f_u]_{C^\alpha([0,b];X)} \leq \Theta_f(r,R,b),\ [S_u]_{C^\alpha([0,b];\mathcal{D})} \leq \Theta_{1,B}(b,R). \tag{3.15}$$

From the above estimates and Lemma 2.1 we obtain that $\Gamma u_{|_{[0,b]}} \in C^\alpha([0,b];\mathcal{D})$ and

$$\begin{aligned} [\Gamma u]_{C^\alpha([0,b];\mathcal{D})} \leq\ & \Lambda(\Theta_f(r,R,b) + \Theta_{1,B}(b,R) + \Theta_A(b,R)) \\ & + \Lambda[\Phi(\varphi)]_{C^\alpha([0,b];\mathcal{D})} + \frac{C_{\alpha,\infty}}{\alpha} \| A\varphi(0) + f(0,\varphi) \|_{\alpha,\infty} . \end{aligned} \tag{3.16}$$

Moreover, arguing as in the proof of Theorem 3.1, for $t \in [-p,0]$ and $h > 0$ such that $t+h > 0$ we obtain that

$$\begin{aligned} \| \Gamma u(t+h) - \Gamma u(t) \|_{\mathcal{D}} \leq\ & h^\alpha (\Lambda([f_u]_{C^\alpha([0,b];X)} + [S_u]_{C^\alpha([0,b];\mathcal{D})} + [L_u]_{C^\alpha([0,b];X)})) \\ & h^\alpha (\Lambda[\Phi(\varphi)]_{C^\alpha([0,b];\mathcal{D})} + \frac{C_{\alpha,\infty}}{\alpha} \| f(0,\varphi) \|_{\alpha,\infty}) \\ & + h^\alpha (b^{1-\alpha} C_0 \| A\varphi(0) \| + [\varphi]_{C^\alpha([0,b];\mathcal{D})}). \end{aligned} \tag{3.17}$$

From the above estimates and the fact that $\varphi \in C^\alpha([-p,0];\mathcal{D})$ we have that

$$\begin{aligned} [\Gamma u]_{C^\alpha([-p,b];\mathcal{D})} \leq\ & \Lambda(\Theta_f(r,R,b) + \Theta_{1,B}(b,R) + \Theta_A(b,R)) + \Lambda[\Phi(\varphi)]_{C^\alpha([0,b];\mathcal{D})} \\ & + \frac{C_{\alpha,\infty}}{\alpha}(\| f(0,\varphi) \|_{\alpha,\infty} + \| A\varphi(0) + f(0,\varphi) \|_{\alpha,\infty}) \\ & + b^{1-\alpha} C_0 \| A\varphi(0) \| + [\varphi]_{C^\alpha([0,b];\mathcal{D})}, \end{aligned} \tag{3.18}$$

which shows that $\Gamma u \in Y$. Moreover, from the estimates (3.8) and (3.9) we infer that

$$\begin{aligned} \| P_{\Gamma u}(t) - \varphi \|_{\mathcal{B}} \leq\ & t^\alpha (\Lambda(\Theta_f(r,R,b) + \Theta_{1,B}(b,R) + \Theta_A(b,R))) + t^\alpha \Lambda[\Phi(\varphi)]_{C^\alpha([0,b];\mathcal{D})} \\ & + t^\alpha (\frac{C_{\alpha,\infty}}{\alpha} \| f(0,\varphi) \|_{\alpha,\infty} + [P_{\tilde{\varphi}}]_{C^\alpha([0,b];\mathcal{B})}), \\ [P_{\Gamma u}]_{C^\alpha([0,b];\mathcal{B})} \leq\ & 2\Lambda(\Theta_f(r,R,b) + \Theta_{1,B}(b,R) + \Theta_A(b,R)) + 2\Lambda[\Phi(\varphi)]_{C^\alpha([0,b];\mathcal{D})} \\ & + [P_{\tilde{\varphi}}]_{C^\alpha([0,b];\mathcal{B}_X)} + \frac{C_{\alpha,\infty}}{\alpha}(\| f(0,\varphi) \|_{\alpha,\infty} + \| A\varphi(0) + f(0,\varphi) \|_{\alpha,\infty}), \end{aligned} \tag{3.19}$$

which implies that $\| P_u - \varphi \|_{C([0,b];\mathcal{B})} \leq r$, $[P_{\Gamma u} \|]_{C^\alpha([0,b];\mathcal{B})} \leq R$ and Γ is a $\mathcal{Y}(b,r,R)$-valued function.

Step 2. Γ is a contraction on $\mathcal{Y}(b,r,R)$.

From Lemma 2.1, Lemma 2.2, Lemma 3.3, Lemma 3.4 and proceedings as in the proof of Theorem 3.1 step 3, it is easy to infer that

$$\begin{aligned} [P_{\Gamma u} - P_{\Gamma v}]_{C^\alpha([0,b];\mathcal{B})} \leq\ & 2\Lambda([f_u - f_v]_{C^\alpha([0,b];X)} + [S_{u-v}]_{C^\alpha([0,b];\mathcal{D})} + [L_{u-v}]_{C^\alpha([0,b];X)}) \\ \leq\ & 2\Lambda\left(L_f(b,r)(1+4Rb^\alpha) + \Theta_f(b) + \Theta_{2,B}(b) + \Theta_A(b)\right) d(u,v), \end{aligned}$$

and

$$\| P_{\Gamma u} - P_{\Gamma v} \|_{C^\alpha([0,b];\mathcal{B})} \leq 2(b^\alpha + 1)\Lambda\left(L_f(b,r)(1+4Rb^\alpha) + \Theta_f(b) + \Theta_{2,B}(b) + \Theta_A(b)\right) d(u,v),$$

which proves (see the choice of b) that Γ is a contraction on $\mathcal{Y}(b,r,R)$.

Finally, from the contraction mapping principe we infer that there exists a unique classical solution $u \in C^\alpha([-p,b];\mathcal{D})$ of (3.12)-(3.13). This completes the proof. ■

In the next proposition we study the existence of solutions for (3.12)-(3.13) by assuming that the derivative $\Phi'(\varphi)$ of $\Phi(\varphi)$ belongs to $C^{\alpha}([0,a],X)$.

Proposition 3.1. *Assume the condition* $\mathbf{H_1}$ *is satisfied and* $f(\cdot)$ *satisfies the* $\mathbf{H}^{\alpha}_{\mathcal{B},\mathbf{X}}$ *condition at* $\varphi \in U \subset \mathcal{B}$. *Suppose* $P_{\widetilde{\varphi}} \in C^{\alpha}([0,b];\mathcal{B})$, $\varphi \in C^{\alpha}([-p,0];\mathcal{D})$, *the function* $\Phi(\varphi)$ *is continuously differentiable,* $\Phi'(\varphi) \in C^{\alpha}([0,a];X)$, $\{A\varphi(0), f(0,\varphi) - \Phi'(\varphi)(0)\} \subset (X,\mathcal{D})_{\alpha,\infty}$ *and* $2\Lambda(L_f(0,0)+ \| d_2 f(0,\varphi) \|_{\mathcal{L}(\mathcal{B},X)}) < 1$. *Then there exists a unique classical solution* $u \in C^{\alpha}([0,b],\mathcal{D})$ *of (3.12)-(3.13) for some* $0 < b \leq a$.

Proof: The proof follows from the proof of Theorem 3.1. For the sake of clarity, we include some details. We adopt all the notations in the cited proof.

Let $R > r > 0$ and $0 < b \leq a$ such that $B_r(\varphi,\mathcal{B}) \subset U$ and

$$\begin{aligned}
& b^{\alpha}\Lambda(\Theta_f(r,R,b) + \Theta_{1,B}(b,R) + \Theta_A(b,R)) \\
+b^{\alpha}(\Lambda[\Phi'(\varphi)]_{C^{\alpha}([0,b];X)} + \frac{C_{\alpha,\infty}}{\alpha} \| f(0,\varphi) - \Phi'(\varphi)(0) \|_{\alpha,\infty} + [P_{\widetilde{\varphi}}]_{C^{\alpha}([0,b];\mathcal{B}_X)}) &\leq r, \\
2\Lambda(\Theta_f(r,R,b) + \Theta_{1,B}(b,R) + \Theta_A(b,R)) + 2\Lambda[\Phi'(\varphi)]_{C^{\alpha}([0,b];X)} + [P_{\widetilde{\varphi}}]_{C^{\alpha}([0,b];\mathcal{B}_X)} & \\
+\frac{C_{\alpha,\infty}}{\alpha}(\| f(0,\varphi) - \Phi'(\varphi)(0) \|_{\alpha,\infty} + \| A\varphi(0) + f(0,\varphi) - \Phi'(\varphi)(0) \|) &\leq R, \\
2(b^{\alpha}+1)\Lambda\left(L_f(b,r)(1+Rb^{\alpha}) + \Theta_f(b) + \Theta_{2,B}(b) + \Theta_A(b)\right) &< 1.
\end{aligned}$$

Let $Y, \mathcal{Y}(b,r,R)$ be defined as in the proof of Theorem 3.1 and $\Gamma : \mathcal{Y}(b,r,R) \to C([-p,b];X)$ be the map given by $(\Gamma u)_0 = \varphi$ and

$$\begin{aligned}
\Gamma u(t) &= T(t)\varphi(0) - S_u(t) - \int_0^t AT(t-s)S_u(s)ds \\
&\quad + \int_0^t T(t-s)[F_u(s) - \Phi'(\varphi)(s) + L_u(s)]ds, \qquad \text{for } t \in [0,b].
\end{aligned}$$

From the choice of b and arguing as in the proof of Theorem 3.2 (replacing $\Phi(\varphi)$ by zero and f by $f - \Phi'(\varphi)$) we can prove that Γ is a contraction on $\mathcal{Y}(b,r,R)$ and there exists a classical solution $u \in C^{\alpha}([0,b],\mathcal{D})$ of the problem (3.12)-(3.13). ■

By using the condition $\mathbf{H_2}$ we also can establish the existence of a classical solution for (3.12)-(3.13). We omit the proof of the following results.

Lemma 3.5. [17, Lemma 2.5] *If the condition* $\mathbf{H_2}$ *is valid and* $u \in C^{\alpha}([0,b],\mathcal{D})$, *then* $S_u \in C^{\alpha}([0,b],\mathcal{D})$, $\| S_u \|_{C([0,b],\mathcal{D})} \leq \Theta_B(b)b^{\alpha} \| u \|_{C([0,b],\mathcal{D})}$ *and* $[S_u]_{C^{\alpha}([0,b],\mathcal{D}))} \leq (\| L_B \|_{L^1([0,b])} + \Theta_B(b)) \| u \|_{C([0,b];\mathcal{D})}$. *In addition, if* $u(0) = 0$ *then* $\| S_u \|_{C([0,b],\mathcal{D})} \leq \Theta_B(b)b^{2\alpha}[u]_{C^{\alpha}([0,b],\mathcal{D})}$ *and* $[S_u]_{C^{\alpha}([0,b],\mathcal{D}))} \leq (\| L_B \|_{L^1([0,b])} + \Theta_B(b))b^{\alpha}[u]_{C^{\alpha}([0,b],\mathcal{D})}$.

Theorem 3.3. *Assume the condition* $\mathbf{H_2}$ *is satisfied and* $f(\cdot)$ *satisfies the* $\mathbf{H}^{\alpha}_{\mathcal{B},\mathbf{X}}$ *condition at* $\varphi \in U$. *Suppose* $P_{\widetilde{\varphi}} \in C^{\alpha}([0,b];\mathcal{B})$, $\varphi \in C^{\alpha}([-p,0];\mathcal{D})$, $\{A\varphi(0), f(0,\varphi)\} \subset (X,\mathcal{D})_{\alpha,\infty}$, $\Phi(\varphi) \in C^{\alpha}([0,b];\mathcal{D})$ *and* $2\Lambda(L_f(0,0)+ \| d_2 f(0,\varphi) \|_{\mathcal{L}(\mathcal{B},X)}) < 1$. *Then there exists a unique classical solution* $u \in C^{\alpha}([0,b],\mathcal{D})$ *of the neutral problem (3.12)-(3.13) for some* $0 < b \leq a$.

Proposition 3.2. *Assume the condition in Proposition 3.1 are satisfied with* $\mathbf{H_2}$ *replacing the condition* $\mathbf{H_1}$. *Then there exists a unique classical solution* $u \in C^{\alpha}([0,b],\mathcal{D})$ *of (3.12)-(3.13) for some* $0 < b \leq a$.

4. Applications

In this section we present some applications of our abstract results. To begin, we include some technicalities on the Laplacian operator. In the remainder of this section $X = C([0,\pi])$ and $A : \mathcal{D} \subset X \to X$ is the operator defined by $Ax = x''$ on $\mathcal{D} = \{x \in C^2([0,\pi]) : x(0) = x(\pi) = 0\}$. The operator A is the generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X (which is not a C_0-semigroup). We adopt all the notations used in Section 3.

We consider the integro-differential neutral problem

$$\frac{\partial}{\partial t}\left[u(t,\xi)+\int_{t-p}^{t}\alpha_1(t)\nu_1(u(s,\xi))ds\right] = \gamma(t)\frac{\partial^2 u}{\partial \xi^2}(t,\xi)+\int_{t-p}^{t}\alpha_2(t)\nu_2(u(s,\xi))ds, \tag{4.1}$$

$$u(t,0) = u(t,\pi)=0, \tag{4.2}$$

$$u(\theta,\xi) = \varphi(\theta,\xi), \quad \theta\in[-p,0],\ \xi\in[0,\pi], \tag{4.3}$$

for $(t,\xi)\in[0,a]\times[0,\pi]$, where $\nu_i\in C^1(\mathbb{R};\mathbb{R})$, $\alpha_i\in C^\alpha([0,a],\mathbb{R})$ for some $\alpha\in(0,1)$, $\gamma\in C^\alpha([0,a],\mathbb{R}^+)$ and $\gamma(0)=1$.

To treat this problem we define the functions $A(t):\mathcal{D}\subset X\to X$, $f,g:[0,a]\times\mathcal{B}\to X$ by $A(t)u=\gamma(t)u''$, $g(t,\psi)(\xi)=\int_{-p}^{0}\alpha_1(t)\nu_1(\psi(s,\xi))ds$ and $f(t,\psi)(\xi)=\int_{-p}^{0}\alpha_2(t)\nu_2(\psi(s,\xi))ds$. Under the assumptions we have that $f(\cdot)$, $g(\cdot)$ are C^1 functions and

$$\begin{aligned} d_2g(t,\psi)\phi(\xi) &= \int_{-p}^{0}\alpha_1(t)\nu_1'(\psi(s,\xi))\phi(s,\xi)ds,\\ d_2f(t,\psi)\phi(\xi) &= \int_{-p}^{0}\alpha_2(t)\nu_2'(\psi(s,\xi))\phi(s,\xi)ds.\end{aligned}$$

In addition, next we assume that $\nu_1(\cdot)$ is of class C^3, $v_1(0)=0$ and there is a $L\in C(\mathbb{R};\mathbb{R}^+)$ such that $\mid\nu_i(x)\mid+\mid\nu_i'(x)\mid\le L(r)\mid x\mid$ and $\mid\nu_i'(x)-\nu_i'(y)\mid\le L(r)\mid x-y\mid$ for $i=1,2$ every $x,y\in B_r(0,\mathbb{R})$ and all $r>0$. Moreover, to simplify and for sake of brevity, we also assume that $\parallel\alpha_i\parallel_{C^\alpha([0,b];\mathbb{R})}\to 0$ as $b\to 0$ for $i=1,2$. Under the above conditions, the functions f,g satisfies the conditions in Theorem 3.1, $d_2f(0,\varphi)=0$, $d_2g(0,\varphi)=0$ and the functions $L_f(\cdot)$, $L_g(\cdot)$ (see the statement of Theorem 3.1) are such that $L_f(0,r)=L_g(0,r)=0$ for $r\ge 0$,

In the next result, which is a consequence of Theorem 3.1, we say that $u\in C([-p,b];X)$, $b>0$, is a classical solution of (4.1)-(4.3) on $[0,b]$ if $u(\cdot)$ is a classical solution of the associated abstract neutral problem (3.1)-(3.2) on $[0,b]$. Next, $\tilde\varphi$ is defined as in Section 3.

Proposition 4.3. *Assume $\varphi\in C^\alpha([-p,0],\mathcal{D})$, the function $P_{\tilde\varphi}$ belongs to $C^\alpha([0,b];\mathcal{B})$ and*

$$\{A\varphi(0,\cdot),\int_{-p}^{0}\alpha_2(0)\nu_2(\varphi(s,\cdot))ds\}\subset(X,\mathcal{D})_{\alpha,\infty}.$$

Then there exists a unique classical solution $u\in C^\alpha([0,b],\mathcal{D})$ of (4.1)-(4.3) on $[0,b]$ for some $b\in(0,a]$.

In the next example we discuss briefly the existence of solutions for the problem

$$\frac{\partial}{\partial t}\left[u(t,\xi)+G(t,\int_{t-p}^{t}u(s,\xi)ds)\right] = \frac{\partial^2 u}{\partial \xi^2}(t,\xi)+F(t,\int_{t-p}^{t}u(s,\xi)ds), \tag{4.4}$$

$$u(t,0) = u(t,\pi)=0, \tag{4.5}$$

$$u(\theta,\xi) = \varphi(\theta,\xi), \quad \theta\in[-p,0],\ \xi\in[0,\pi], \tag{4.6}$$

for $(t,\xi)\in[0,a]\times[0,\pi]$, where $F,G:[0,a]\times\mathbb{R}\to\mathbb{R}$ are smooth functions.

To study this problem we assume $G\in C^3([0,a]\times\mathbb{R};\mathbb{R})$, $G(t,0)=0$, $F\in C^1([0,a]\times\mathbb{R};\mathbb{R})$ and there is a continuous function $L\in C(\mathbb{R}\times\mathbb{R};\mathbb{R}^+)$ such that

$$\mid\frac{\partial^i G}{\partial x^i}(t,x)-\frac{\partial^i G}{\partial x^i}(s,y)\mid+\mid\frac{\partial^j F}{\partial x^j}(t,x)-\frac{\partial^j F}{\partial x^j}(s,y)\mid \le L(r)(\mid t-s\mid+\mid x-y\mid),$$

for $t\in[0,a]$, $x,y\in B_r(0,\mathbb{R})$, $i=0,\ldots,3$ and $j=0,1$. In addition and for simplification, we suppose $\frac{\partial G}{\partial}(0,\cdot)=\frac{\partial F}{\partial}(0,\cdot)=G(0,\cdot)=F(0,\cdot)=0$.

Under the above conditions, the functions $f : [0, a] \times \mathcal{B} \to X$, $g : [0, a] \times \mathcal{B} \to X$ defined by $f(t, \psi)(\xi) = F(t, \int_{-p}^{0} \psi(s, \xi) ds)$ and $g(t, \psi)(\xi) = G(t, \int_{-p}^{0} \psi(s, \xi) ds)$ are C^1 functions and

$$\begin{aligned} d_2 f(t, \psi_1)\psi_2(\xi) &= d_2 F(t, \int_{-p}^{0} \psi_1(s, \xi) ds) \int_{-p}^{0} \psi_2(s, \xi) ds, \\ d_2 g(t, \psi_1)\psi_2(\xi) &= d_2 G(t, \int_{-p}^{0} \psi_1(s, \xi) ds) \int_{-p}^{0} \psi_2(s, \xi) ds. \end{aligned}$$

From the above, $d_2 g(0, \cdot) = d_2 f(0, \cdot) = 0$ and the functions $g(\cdot)$, $f(\cdot)$ satisfies the $\mathbf{H}^{\alpha}_{\mathcal{B},\mathcal{D}}$ and the $\mathbf{H}^{\alpha}_{\mathcal{B},\mathbf{X}}$ conditions. Moreover, the numbers $L_f(b, r)$, $L_g(b, r)$ depend on $L(r)b^{1-\alpha}$, so that, $L_f(b, r) \to 0$ and $L_g(b, r) \to 0$ as $b \to 0$. From Theorem 3.1 we have the following result.

Proposition 4.4. *Assume $\varphi \in C^{\alpha}([-p, 0], \mathcal{D})$, $P_{\widetilde{\varphi}} \in C^{\alpha}([0, b]; \mathcal{B})$ and $A\varphi(0, \cdot) \in (X, D)_{\alpha,\infty}$. Then there exists a classical solution of (4.4)-(4.6) in $C^{\alpha}([0, b], \mathcal{D})$ for some $b \in (0, a]$.*

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