On the boundedness of generalized Cesàro operators on Sobolev spaces

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Abstract

For $\beta > 0$ and $p \geq 1$, the generalized Cesàro operator

$$\mathscr{C}_\beta f(t) := \frac{\beta}{t^\beta} \int_0^t (t-s)^{\beta-1} f(s) ds$$

and its companion operator $\mathscr{C}_\beta^*$ defined on Sobolev spaces $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ and $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$ (where $\alpha \geq 0$ is the fractional order of derivation and are embedded in $L^p(\mathbb{R}^+)$ and

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$L^p(\mathbb{R})$ respectively) are studied. We prove that if $p > 1$, then $\mathscr{C}_\beta$ and $\mathscr{C}_\beta^*$ are bounded operators and commute on $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ and $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$. We calculate explicitly their spectra $\sigma(\mathscr{C}_\beta)$ and $\sigma(\mathscr{C}_\beta^*)$ and their operator norms (which depend on p). For $1 < p \le 2$, we prove that $\widehat{\mathscr{C}_\beta(f)} = \mathscr{C}_\beta^*(\widehat{f})$ and $\widehat{\mathscr{C}_\beta^*(f)} = \mathscr{C}_\beta(\widehat{f})$ where $\widehat{f}$ denotes the Fourier transform of a function $f \in L^p(\mathbb{R})$.

Keywords: Cesàro operators, Sobolev spaces, Boundedness.

1 Introduction

Given $1 \le p < \infty$, let $L^p(\mathbb{R}^+)$ be the set of Lebesgue p-integrable functions, that is, f is a measurable function and

$$||f||_p := \left(\int_0^\infty |f(t)|^p dt\right)^{1/p} < \infty.$$

The classical Hardy inequality (see [13, p. 245]) establishes that

$$\left(\int_0^\infty \left|\frac{1}{t}\int_0^t f(s)ds\right|^p dt\right)^{1/p} \le \frac{p}{p-1}||f||_p, \quad f \in L^p(\mathbb{R}^+),$$

for $1 < p < \infty$ and therefore the so-called Cesàro transformation $\mathscr{C}$, defined by

$$\mathscr{C}(f)(t) = \frac{1}{t}\int_0^t f(s)ds, \quad t > 0, \tag{1.1}$$

is a bounded operator on $L^p(\mathbb{R}^+)$ with $||\mathscr{C}|| \le \frac{p}{p-1}$ for $1 < p < \infty$. In fact, it is also known that if $\beta > 0$

$$\left(\int_0^\infty \left|\frac{\beta}{t^\beta}\int_0^t (t-s)^{\beta-1} f(s)ds\right|^p dt\right)^{1/p} \le \frac{\Gamma(\beta+1)\Gamma(1-\frac{1}{p})}{\Gamma(\beta+1-\frac{1}{p})}\|f\|_p, \qquad f \in L^p(\mathbb{R}^+), \tag{1.2}$$

for $1 < p < \infty$ and the constant $\frac{\Gamma(\beta+1)\Gamma(1-\frac{1}{p})}{\Gamma(\beta+1-\frac{1}{p})}$ is optimal in this inequality, see [13, Theorem 329]. A closer (and dual) inequality is the following

$$\left(\int_0^\infty \left|\beta\int_x^\infty \frac{(t-x)^{\beta-1}}{t^\alpha} f(t)dt\right|^p dx\right)^{\frac{1}{p}} \le \frac{\Gamma(\alpha+1)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\alpha+\frac{1}{p}\right)}\|f\|_p. \tag{1.3}$$

Also the constant $\frac{\Gamma(\alpha+1)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\alpha+\frac{1}{p}\right)}$ is optimal in the above inequality ([13, Theorem 329, p.245]).

Note that inequalities (1.2) and (1.3) show that the operators $\mathscr{C}_\beta, \mathscr{C}_\beta^*$ where

$$\mathscr{C}_\beta f(t) := \frac{\beta}{t^\beta}\int_0^t (t-s)^{\beta-1} f(s)ds, \qquad \mathscr{C}_\beta^* f(s) := \beta\int_s^\infty \frac{(t-s)^{\beta-1}}{t^\beta} f(t)dt,$$

define bounded operators on $L^p(\mathbb{R}^+)$, $\mathscr{C}_1 = \mathscr{C}$ and $\mathscr{C}_1^* = \mathscr{C}^*$. By Fubini theorem, the dual operator of $\mathscr{C}_\beta$ on $L^p(\mathbb{R}^+)$ is $\mathscr{C}_\beta^*$ on $L^{p'}(\mathbb{R}^+)$, i.e,

$$\int_0^\infty \mathscr{C}_\beta f(t)g(t)dt = \int_0^\infty f(s)\mathscr{C}_\beta^* g(s)ds, \qquad f \in L^p(\mathbb{R}^+), \quad g \in L^{p'}(\mathbb{R}^+),$$

where $1 < p, p' < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. See other properties about some of these operators in [6, 7, 18].

Recently, A. Arvanitidis and A. Siskakis ([4]) showed that the half-plane versions of Cesàro operators on the Hardy space $\mathscr{H}_p(\mathbb{U})$, defined on $\mathbb{U} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ by

$$C(F)(z) := \frac{1}{z}\int_0^z F(s)ds, \quad C^*(F)(z) := \int_z^\infty \frac{F(s)}{s}ds, \quad F \in H^p(\mathbb{U}), \tag{1.4}$$

define bounded operators on $\mathscr{H}_p(\mathbb{U})$ when $p > 1$. Both operators C and C^* can be obtained as resolvent operators of generators of some appropriate strongly continuous C_0-semigroups on $\mathscr{H}_p(\mathbb{U})$.

Similarly, W. Arendt and B. de Pagter ([3]) studied the Cesàro operator (1.1) defined in an interpolation space E of (L^1, L^∞) on $\mathbb{R}^+$. When $E = L^p(\mathbb{R}^+)$, the authors obtained a representation of $\mathscr{C}$ in terms of an appropriate resolvent operator, see [3, Corollaries 2.2, 4.3].

In [11], Sobolev subspaces $\mathscr{T}_1^{(\alpha)}(t^\alpha)$ and $\mathscr{T}_1^{(\alpha)}(|t|^\alpha)$ (contained in $L^1(\mathbb{R}^+)$ and $L^1(\mathbb{R})$ respectively and where $\alpha \geq 0$ is the fractional order of derivation) were introduced. In fact, these subspaces are sub-algebras for the convolution products given by

$$f * g(t) = \int_0^t f(t-s)g(s)ds, \quad t \geq 0, \tag{1.5}$$

and

$$f * g(t) = \int_{-\infty}^\infty f(t-s)g(s)ds, \qquad t \in \mathbb{R}, \tag{1.6}$$

respectively. These algebras are canonical to define some algebra homomorphisms (defined by integral representations) into $\mathscr{B}(X)$, the set of all linear and bounded operators on a Banach space X. See further details in [11].

Further, in [20] Sobolev subspaces $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ contained in Lebesgue spaces $L^p(\mathbb{R}^+)$ ($p \geq 1$) were introduced and studied in detail. Some remarkable results were proved (see Proposition 2.2 below). In particular, the subspace $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ is a module for the algebra $\mathscr{T}_1^{(\alpha)}(t^\alpha)$ for the convolution product $*$ given by (1.5).

Hence, it is natural to ask to what extent the boundedness property of the operators $\mathscr{C}_\beta$ and $\mathscr{C}_\beta^*$ remain valid in the above described Sobolev spaces.

The main aim of this paper is to study boundedness, representation and spectral properties for the generalized Cesàro operators $\mathscr{C}_\beta$ and $\mathscr{C}_\beta^*$ on Sobolev subspaces of fractional order $\alpha \geq 0$ embedded in $L^p(\mathbb{R}^+)$ and $L^p(\mathbb{R})$ (which are denoted by $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ and $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$ respectively).

The outline of the paper is as follows: In the second section we recall some basic properties of the Sobolev spaces $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ (where $\mathscr{T}_p^{(\alpha)}(t^\alpha) \hookrightarrow L^p(\mathbb{R}^+)$). We also prove new results, see for example Proposition 2.4. The main tool of this section (and in the rest of the paper) is the group of isometries on $\mathscr{T}_p^{(\alpha)}(t^\alpha)$, $(T_{t,p})_{t\in\mathbb{R}}$ given by

$$T_{t,p}f(s) := e^{-\frac{t}{p}} f(e^{-t}s), \qquad f \in \mathscr{T}_p^{(\alpha)}(t^\alpha).$$

In the Theorem 2.5 it is identified its infinitesimal generator and, its spectrum, in Proposition 2.6. We note that this strategy has been pursued by other authors. We mention here [3, 4, 8, 24].

In the third section, we study the generalized Cesàro operators $\mathscr{C}_\beta$ and $\mathscr{C}_\beta^*$ defined on Sobolev spaces $\mathscr{T}_p^{(\alpha)}(t^\alpha)$. We first show that both operators are bounded operators and commute for $p > 1$. In fact, we have

$$||\mathscr{C}_\beta|| = \frac{\Gamma(\beta+1)\Gamma(1/p')}{\Gamma(\beta+1/p')}; \quad ||\mathscr{C}_\beta^*|| = \frac{\Gamma(\beta+1)\Gamma(1/p)}{\Gamma(\beta+1/p)},$$

for $\alpha \geq 0, p > 1, \beta > 0,\ 1/p + 1/p' = 1$. It is remarkable that the composition $\mathscr{C}_\alpha \mathscr{C}_\beta^*$ may be described explicitly involving the Gaussian hypergeometric function ${}_2F_1$ (see Theorem 3.12) as follows:

$$\begin{aligned}(\mathscr{C}_\alpha \mathscr{C}_\beta^*)f(t) \;=\;& \alpha \int_0^t f(r) \frac{1}{t-r} \left(\frac{t-r}{t}\right)^{\alpha+\beta} {}_2F_1(\alpha+\beta, \beta; \beta+1; \frac{r}{t}) dr \\ & + \beta \int_t^\infty f(r) \frac{1}{r-t} \left(\frac{r-t}{t}\right)^{\alpha+\beta} {}_2F_1(\alpha+\beta, \alpha; \alpha+1; \frac{t}{r}) dr,\end{aligned}$$

for $\alpha, \beta > 0$.

Using the description of $\mathscr{C}_\beta$ and $\mathscr{C}_\beta^*$ in terms of the C_0-semigroups (Theorem 3.3 and Theorem 3.7), we are able to determine the spectra, $\sigma(\mathscr{C}_\beta)$ and $\sigma(\mathscr{C}_\beta^*)$ (Theorem 3.5 and 3.9) as:

$$\sigma(\mathscr{C}_\beta) = \Gamma(\beta+1) \overline{\left\{ \frac{\Gamma(\frac{1}{p'} + it)}{\Gamma(\beta + \frac{1}{p'} + it)} \;:\; t \in \mathbb{R} \right\}};$$

and

$$\sigma(\mathscr{C}_\beta^*) = \Gamma(\beta+1) \overline{\left\{ \frac{\Gamma(\frac{1}{p} + it)}{\Gamma(\beta + \frac{1}{p} + it)} \;:\; t \in \mathbb{R} \right\}},$$

where $1/p+1/p'=1$. In particular, the operators $\mathscr{C}_1$ and $\mathscr{C}_1^*$ can be obtained as the resolvent operator of appropriate C_0-semigroups, namely $(T_{t,p})_{t\geq 0}$ and $(T_{-t,p})_{t\geq 0}$, respectively.

We remark that in case $\beta=1$ we obtain:

$$\sigma(\mathscr{C}_1^*)=\left\{w\in\mathbb{C}\ :\ \left|w-\frac{p}{2}\right|=\frac{p}{2}\right\}.$$

This gives a proof of a conjecture posed by F. Móricz on $L^p(\mathbb{R}^+)$ [18, Section 2] and new proofs of some results given in [6, 7].

In Section 4, we introduce and give some basic properties of the Sobolev spaces $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$ (here $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)\hookrightarrow L^p(\mathbb{R})$). We also prove that the space $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$ is a module for the algebra $\mathscr{T}_1^{(\alpha)}(|t|^\alpha)$ and the $*$-convolution product given by (1.6). Moreover, the following interesting inequality holds:

$$|||f*g|||_{\alpha,p}\leq C_{\alpha,p}|||f|||_{\alpha,p}|||g|||_{\alpha,1},\qquad f\in\mathscr{T}_p^{(\alpha)}(|t|^\alpha),\quad g\in\mathscr{T}_1^{(\alpha)}(|t|^\alpha).$$

In Section 5, we study boundedness, representation and spectral properties of generalized Cèsaro operators on $\mathbb{R}$. Again, it is relevant to mention that the C_0-group of isometries on $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$, $(T_{t,p})_{t\in\mathbb{R}}$ given by

$$T_{t,p}f(s):=e^{-\frac{t}{p}}f(e^{-t}s),\qquad f\in\mathscr{T}_p^{(\alpha)}(|t|^\alpha),$$

(Theorem 4.4) is the main tool to prove the main results in this section. The generalized Cesàro operators $\mathscr{C}_\beta$ and $\mathscr{C}_\beta^*$ defined on Sobolev spaces $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$ are described in terms of the C_0-group of isometries $(T_{t,p})_{t\in\mathbb{R}}$. Similar results shown in the case $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ hold in this case, see Theorem 5.2 and 5.3 below.

In the last section we show that $\widehat{\mathscr{C}_\beta(f)}=\mathscr{C}_\beta^*(\widehat{f})$ and $\widehat{\mathscr{C}_\beta^*(f)}=\mathscr{C}_\beta(\widehat{f})$ where $\widehat{f}$ is the Fourier transform of a function $f\in L^p(\mathbb{R})$ and $1<p\leq 2$, see Theorem 6.4. We notice that our studies in this section extends and complement the main result in [19].

2 Composition groups on Sobolev spaces defined on $\mathbb{R}^+$.

Let $\mathscr{D}_+$ be the class of C^∞-functions with compact support on $[0,\infty)$ and $\mathscr{S}_+$ the Schwartz class on $[0,\infty)$. For a function $f\in\mathscr{S}_+$ and $\alpha>0$, the *Weyl fractional integral* of order α, $W_+^{-\alpha}f$, is defined by

$$W_+^{-\alpha}f(t):=\frac{1}{\Gamma(\alpha)}\int_t^\infty(s-t)^{\alpha-1}f(s)ds,\qquad t\in\mathbb{R}^+.$$

The *Weyl fractional derivative* $W_+^\alpha f$ of order α is defined by

$$W_+^\alpha f(t):=(-1)^n\frac{d^n}{dt^n}W_+^{-(n-\alpha)}f(t),\quad t\in\mathbb{R}^+$$

where $n=[\alpha]+1$, and $[\alpha]$ denotes the integer part of α. It is proved that $W_+^{\alpha+\beta}=W_+^{\alpha}(W_+^{\beta})$ for any $\alpha,\beta\in\mathbb{R}$, where $W_+^0=Id$ is the identity operator and $(-1)^nW_+^n=\frac{d^n}{dt^n}$ holds with $n\in\mathbb{N}$, see more details in [16] and [21].

Take $\lambda>0$ and f_λ defined by $f_\lambda(r):=f(\lambda r)$ for $r>0$ and $f\in\mathscr{S}_+$. It is direct to check that

$$W_+^{\alpha}f_\lambda=\lambda^{\alpha}(W_+^{\alpha}f)_\lambda, \qquad f\in\mathscr{S}_+, \tag{2.1}$$

for $\alpha\in\mathbb{R}$.

Now we introduce a family of subspaces $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ which are contained in $L^p(\mathbb{R}^+)$.

Definition 2.1 *For $\alpha>0$ let be the Banach space $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ defined as the completion of the Schwartz class $\mathscr{S}_+$ in the norm*

$$||f||_{\alpha,p}:=\frac{1}{\Gamma(\alpha+1)}\left(\int_0^\infty |W_+^{\alpha}f(t)|^p t^{\alpha p}dt\right)^{\frac{1}{p}}.$$

We understand that $\mathscr{T}_p^{(0)}(t^0)=L^p(\mathbb{R}^+)$ and $||\quad||_{0,p}=||\quad||_p$. The case $p=1$ and $\alpha\in\mathbb{N}$ where introduced in [2] and for $\alpha>0$ in [11].

In the next proposition we collect some results about these family of spaces $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ which we may be found in [20].

Proposition 2.2 *Take $p\ge 1$ and $\beta>\alpha>0$. Then*

(i) $\mathscr{T}_p^{(\beta)}(t^\beta)\hookrightarrow\mathscr{T}_p^{(\alpha)}(t^\alpha)\hookrightarrow L^p(\mathbb{R}^+)$.

(ii) $\mathscr{T}_p^{(\alpha)}(t^\alpha)*\mathscr{T}_1^{(\alpha)}(t^\alpha)\hookrightarrow\mathscr{T}_p^{(\alpha)}(t^\alpha)$ *for $1\le p<\infty$, where*

$$f*g(t)=\int_0^t f(t-s)g(s)ds,\quad t\ge 0,\qquad f\in\mathscr{T}_p^{(\alpha)}(t^\alpha),\quad g\in\mathscr{T}_1^{(\alpha)}(t^\alpha). \tag{2.2}$$

(iii) The operator $D_+^{\alpha}:\mathscr{T}_p^{(\alpha)}(t^\alpha)\to L^p(\mathbb{R}^+)$ defined by

$$f\mapsto D_+^{\alpha}f(t)=\frac{1}{\Gamma(\alpha+1)}t^{\alpha}W_+^{\alpha}f(t),\qquad t\ge 0,\quad f\in\mathscr{T}_p^{(\alpha)}(t^\alpha).$$

is an isometry.

(iv) If $p>1$ and p' satisfies $\frac{1}{p}+\frac{1}{p'}=1$, then the dual of $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ is $\mathscr{T}_{p'}^{(\alpha)}(t^\alpha)$, where the duality is given by

$$\langle f,g\rangle_\alpha=\frac{1}{\Gamma(\alpha+1)^2}\int_0^\infty W_+^{\alpha}f(t)W_+^{\alpha}g(t)t^{2\alpha}dt,$$

for $f\in\mathscr{T}_p^{(\alpha)}(t^\alpha)$, $g\in\mathscr{T}_{p'}^{(\alpha)}(t^\alpha)$.

Note that, in fact,

$$\|f\|_{\alpha,p} = \|D_+^\alpha f\|_p, \qquad \langle f,g\rangle_\alpha = \langle D_+^\alpha f, D_+^\alpha g\rangle_0, \tag{2.3}$$

for $f \in \mathscr{T}_p^{(\alpha)}(t^\alpha)$ and $g \in \mathscr{T}_{p'}^{(\alpha)}(t^\alpha)$ with $\frac{1}{p}+\frac{1}{p'}=1$.

In the next lemma, we consider some functions which belong (or not) to $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ for $p \geq 1$.

Lemma 2.3 *If $\alpha, a > 0$ and $p \geq 1$, then*
(i) $t^\beta \notin \mathscr{T}_p^{(\alpha)}(t^\alpha)$ for $\beta \in \mathbb{C}$.
(ii) $(a+t)^{-\beta} \in \mathscr{T}_p^{(\alpha)}(t^\alpha)$ for $\Re\beta > 1/p$.

Proof. (i) It suffices to note that t^β does not belong to $L^p(\mathbb{R}^+)$.

(ii) For $0 < \Re\gamma < \Re\delta$ and $a > 0$ it is well know that $W_+^{-\gamma}(a+t)^{-\delta} = \frac{\Gamma(\delta-\gamma)}{\Gamma(\delta)}(t+a)^{\gamma-\delta}$, see for example [10, p. 201]. With this formula, it is easy to check that

$$W_+^\alpha (a+t)^{-\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}(t+a)^{-(\alpha+\beta)}.$$

Thus for $f(t) := (a+t)^{-\beta}$ we obtain

$$\begin{aligned} ||f||_{\alpha,p}^p &= \frac{1}{\Gamma(\alpha+1)^p}\int_0^\infty |W_+^\alpha f(t)|^p t^{\alpha p} dt = \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)^p \int_0^\infty \frac{t^{\alpha p}}{|(t+a)^{(\alpha+\beta)p}|} dt \\ &\leq \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)^p \int_0^\infty \frac{1}{(t+a)^{p\Re\beta}} dt < \infty, \end{aligned}$$

and we conclude the proof. ∎

Given $f \in \mathscr{T}_p^{(\alpha)}(t^\alpha)$, as the next result shows, we obtain that the function $f \in C(\mathbb{R}^+)$ for $p, \alpha \geq 1$.

Proposition 2.4 *Take $p, \alpha \geq 1$ and $f \in \mathscr{T}_p^{(\alpha)}(t^\alpha)$. Then $f \in C(\mathbb{R}^+)$, $\lim_{t\to\infty} f(t) = 0$ and*

$$\sup_{t>0} t^p |f(t)| \leq C_{\alpha,p}\|f\|_{\alpha,p}, \qquad f \in \mathscr{T}_p^{(\alpha)}(t^\alpha),$$

where $C_{\alpha,p}$ is independent of f.

Proof. By Proposition 2.2 (i), it is enough to check for $\alpha = 1$. Take $t > s > 0$, and we get that

$$|f(t)-f(s)| \leq \int_s^t |f'(u)|du \leq \frac{1}{s}\int_s^t |f'(u)|u du.$$

For $p=1$, it is clear that f is continuous and for $p>1$, we apply the Hölder inequality to obtain

$$|f(t)-f(s)| \leq \|f\|_{1,p}\,(t-s)^{\frac{1}{p'}}, \qquad \frac{1}{p}+\frac{1}{p'}=1.$$

Then f is continuous in $\mathbb{R}^+$. For $f\in\mathscr{T}_1^{(\alpha)}(t^\alpha)$, we have

$$|f(t)| \leq \int_t^\infty |f'(u)|du \leq \frac{1}{t}\int_t^\infty u|f'(u)|du \leq \frac{C}{t}\|f\|_{1,1} \leq \frac{C}{t}\|f\|_{\alpha,1}, \qquad t>0,$$

and we conclude that $\lim_{t\to\infty} f(t)=0$. Similarly take $f\in\mathscr{T}_p^{(\alpha)}(t^\alpha)$ with $1<p<\infty$. Then we have that

$$|f(t)| \leq \int_t^\infty |f'(u)|du \leq \left(\int_t^\infty u^p|f'(u)|^p du\right)^{\frac{1}{p}} \left(\int_t^\infty \frac{1}{u^{p'}}du\right)^{\frac{1}{p'}} \leq \left(\frac{1}{p' t^{p'-1}}\right)^{\frac{1}{p'}} \|f\|_{1,p}$$

where we conclude that $\sup_{t>0} t^p|f(t)| \leq \left(\frac{1}{p'}\right)^{\frac{1}{p'}} \|f\|_{1,p}$ and the proof is finished. ■

The following is the main result of this section. It will be the key in the study of spectral properties of the generalized Cesàro operators $\mathscr{C}_\beta$ and $\mathscr{C}_\beta^*$ defined on Sobolev spaces.

Theorem 2.5 *For $1\leq p$ and $\alpha\geq 0$, the family of operators $(T_{t,p})_{t\in\mathbb{R}}$ defined by*

$$T_{t,p}f(s) := e^{-\frac{t}{p}} f(e^{-t}s), \qquad f\in\mathscr{T}_p^{(\alpha)}(t^\alpha),$$

is a C_0-group of isometries on $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ whose infinitesimal generator Λ is given by

$$(\Lambda f)(s) := -sf'(s) - \frac{1}{p}f(s)$$

with domain $D(\Lambda)=\mathscr{T}_p^{(\alpha+1)}(t^{\alpha+1})$.

Proof. We check that the operators $(T_{t,p})_{t\in\mathbb{R}}$ are isometries:

$$\begin{aligned}
||T_{t,p}f||_{\alpha,p}^p &= \frac{1}{\Gamma(\alpha+1)^p}\int_0^\infty |W_+^\alpha T_{t,p}f(s)|^p s^{\alpha p} ds = \frac{e^{-t}}{\Gamma(\alpha+1)^p}\int_0^\infty |W_+^\alpha f(e^{-t}s)|^p s^{\alpha p} ds \\
&= \frac{e^{-t}}{\Gamma(\alpha+1)^p}\int_0^\infty e^t |e^{-\alpha t}(W_+^\alpha f)(u)|^p (e^{\alpha t}u^\alpha)^p du = ||f||_{\alpha,p}^p,
\end{aligned}$$

where we have applied the equality (2.1).

Using some known properties for fractional derivative ([21, p. 96]) it can be shown that the family of operatos $(T_{t,p})_{t\in\mathbb{R}}$ are strongly continuous, see similar ideas in [4, Proposition

2.1] and [3, Section 2]. It is straightforward to check that the family $(T_{t,p})_{t\in\mathbb{R}}$ is a group of operators.

On $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ define $\{S_t\}_{t\geq 0}$ by $S_t(f)(s) := f(e^{-t}s)$. Then, an easy computation shows that the generator A of $\{S_t\}_{t\geq 0}$ with domain $\{f \in \mathcal{T}_p^{(\alpha)}(t^\alpha) \ : \ tf' \in \mathcal{T}_p^{(\alpha)}(t^\alpha)\}$ is given by $Af(s) = -sf'(s)$. Therefore, the rescaled semigroup $(T_{t,p})_{t\geq 0}$ has domain $\{f \in \mathcal{T}_p^{(\alpha)}(t^\alpha) \ : \ tf' \in \mathcal{T}_p^{(\alpha)}(t^\alpha)\}$ and his generator is $(\Lambda f)(s) = -sf'(s) - \frac{1}{p}f(s)$. See [9, p. 60] for more details.

Finally, we prove that $D(\Lambda) = \mathcal{T}_p^{(\alpha+1)}(t^{\alpha+1})$. In fact, let $f \in \mathcal{T}_p^{(\alpha+1)}(t^{\alpha+1})$ be given. Since $\mathcal{T}_p^{(\alpha+1)}(t^{\alpha+1}) \hookrightarrow \mathcal{T}_p^{(\alpha)}(t^\alpha)$, we have $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$. From [16, p. 246] it is easy to show that $W_+^\alpha(tf'(t)) = \alpha W_+^\alpha f(t) + tW_+^{\alpha+1}f(t)$. Thus, $tf' \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$ and therefore $f \in D(\Lambda)$. Conversely, if $f \in D(\Lambda)$, then $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$ and $tf' \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$. The same above identity, implies that $t^{\alpha+1}W_+^{\alpha+1}f(t) = t^\alpha W_+^\alpha(tf'(t)) - \alpha t^\alpha W_+^\alpha f(t)$, and therefore $f \in \mathcal{T}_p^{(\alpha+1)}(t^{\alpha+1})$. ■

The proof of the following result is inspired in [4, Proposition 2.3] (see also [1]). We denote by $\sigma(\Lambda)$ the usual spectrum of the operator Λ and by $\sigma_p(\Lambda)$ the point spectrum of the operator Λ.

Proposition 2.6 *For $1 \leq p < \infty$ we have*
(i) $\sigma_p(\Lambda) = \emptyset$;
(ii) $\sigma(\Lambda) = i\mathbb{R}$.

Proof. (i) Let $\lambda \in \mathbb{C}$ and $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$ such that $\Lambda(f) = \lambda f$. Then, f is solution of the differential equation

$$sf'(s) + (\lambda + \frac{1}{p})f(s) = 0.$$

The nonzero solutions to this equation have the form $f(t) = ct^{-(\lambda+1/p)}$ with $c \neq 0$. But by Lemma 2.3, these solutions are not in $\mathcal{T}_p^{(\alpha)}(t^\alpha)$. Therefore $\sigma_p(\Lambda) = \emptyset$.

(ii) Since each $T_{t,p}$ is an invertible isometry its spectrum satisfies

$$\sigma(T_{t,p}) \subseteq \{z \in \mathbb{C} : |z| = 1\}.$$

By the spectral mapping theorem (see Theorem [9, IV.3.6]), we have that

$$e^{t\sigma(\Lambda)} \subseteq \sigma(T_{t,p}).$$

Therefore, if $w \in \sigma(\Lambda)$, then $e^{tw} \in \{z \in \mathbb{C} : |z| = 1\}$. Thus, we obtain that $\sigma(\Lambda) \subseteq i\mathbb{R}$.

Conversely, let $\mu \in i\mathbb{R}$ and assume that $\mu \in \rho(\Lambda)$. Let $\lambda = \mu + \frac{1}{p}$. By Lemma 2.3 the function f defined by $f(t) := (1+t)^{-\lambda-1} \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$. Since $R(\mu, \Lambda)$ is a bounded operator, the function $g(t) := R(\mu, \Lambda)f(t)$ belongs to $\mathcal{T}_p^{(\alpha)}(t^\alpha)$. Therefore, g is solution of equation

$$\lambda g(t) + tg'(t) = f(t).$$

An easy computation shows that the solution of this equation is $G(t) := ct^{-\lambda} + \lambda^{-1}(1+t)^{-\lambda}$, where c is a constant. However, as in Lemma 2.3 one can check that $G \notin \mathscr{T}_p^{(\alpha)}(t^\alpha)$. Therefore, $\mu \in \sigma(\Lambda)$. ∎

Now, consider the negative part $\{T_{-t,p}, t \geq 0\}$ of the group $\{T_{t,p}\}_{t\in\mathbb{R}}$: that is, for $f \in \mathscr{T}_p^{(\alpha)}(t^\alpha)$,

$$T_{-t,p}f(s) = e^{\frac{t}{p}} f(e^t s), \; t \geq 0.$$

Obviously, $\{T_{-t,p}\}_{t\geq 0}$ is a C_0-semigroup on $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ of isometries whose generator is $-\Lambda$.

We finish this section, establishing the relationship between the semigroups $\{T_{t,p}\}_{t\geq 0}$ and $\{T_{-t,p'}\}_{t\geq 0}$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

Proposition 2.7 *The semigroups $\{T_{t,p}\}_{t\geq 0}$ and $\{T_{-t,p'}\}_{t\geq 0}$ are dual operators of each other acting on $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ and $\mathscr{T}_{p'}^{(\alpha)}(t^\alpha)$ with $\frac{1}{p} + \frac{1}{p'} = 1$.*

Proof. This is easily checked by Proposition 2.2 (iv) and (2.1).

∎

3 Generalized Cesàro operators on Sobolev spaces defined on $\mathbb{R}^+$.

For $\beta > 0$ the generalized Cesàro operator on $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ is defined by

$$\mathscr{C}_\beta f(t) := \frac{\beta}{t^\beta} \int_0^t (t-s)^{\beta-1} f(s)ds = \beta \int_0^1 (1-r)^{\beta-1} f(tr)dr, \quad t > 0.$$

Defining the function

$$g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0,$$

we obtain the also equivalent formulation of the generalized Cesàro operator in terms of finite convolution as follows:

$$\mathscr{C}_\beta f(t) := \frac{1}{g_{\beta+1}(t)} \int_0^t g_\beta(t-s) f(s)ds, \quad t > 0.$$

We remark that for certain classes of vector-valued functions f, the asymptotic behavior as $t \to \infty$ of $\mathscr{C}_\beta f(t)$ in the above representation has been studied in [14].

Note that we may calculate $\mathscr{C}_\beta(f)$ for some particular functions:

Example 3.1 (i) Functions g_γ are eigenfunctions of $\mathscr{C}_\beta$ with eigenvalue $\frac{\Gamma(\beta+1)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}$:

$$\mathscr{C}_\beta(g_\gamma)(t) = \frac{\beta}{\Gamma(\gamma)t^{\beta-1}}\int_0^t (t-s)^{\beta-1}s^{\gamma-1}ds = \frac{\Gamma(\beta+1)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}g_\gamma(t), \qquad t>0.$$

(ii) Take $e_\lambda(t) := e^{-\lambda t}$ for $t>0$ and $\lambda\in\mathbb{C}^+$. Then

$$\mathscr{C}_1(e_\lambda)(t) = \frac{1}{\lambda t}(1-e^{-\lambda t}), \qquad \mathscr{C}_2(e_\lambda)(t) = \frac{2}{\lambda t}(e^{-\lambda t}-1+\lambda t), \qquad t>0.$$

Since $\mathscr{C}_1^2(e_\lambda)(t) = \frac{1}{t\lambda}\int_0^t \frac{1-e^{-\lambda s}}{s}ds$ for $t>0$, we conclude that $\mathscr{C}_1^2(e_\lambda)\neq\mathscr{C}_2(e_\lambda)$ and then $\mathscr{C}_1^2\neq\mathscr{C}_2$.

(iii) More generally, take $f_\lambda(t) := E_{\beta,1}(\lambda t^\beta)$ the Mittag-Leffler function, for $t>0$ and $\lambda\in\mathbb{C}^+$. Then

$$\mathscr{C}_\beta(f_\lambda)(t) = \frac{1}{\lambda g_{\beta+1}(t)}(1-f_\lambda(t)), \quad t>0.$$

The relationship between these generalized Cesáro operators and fractional evolution equations of order α can be also observed in [14].

The next lemma shows a key commutativity property.

Lemma 3.2 *Take $\alpha\geq 0$ and $\beta>0$. Then $D_+^\alpha\circ\mathscr{C}_\beta = \mathscr{C}_\beta\circ D_+^\alpha$, i.e.,*

$$D_+^\alpha(\mathscr{C}_\beta(f)) = \mathscr{C}_\beta(D_+^\alpha(f)), \qquad f\in\mathscr{S}_+,$$

where $D_+^\alpha(t) = \frac{1}{\Gamma(\alpha+1)}t^\alpha W_+^\alpha f(t)$ for $f\in\mathscr{S}_+$.

Proof. By the equality (2.1), we have that

$$\begin{aligned}\mathscr{C}_\beta(D_+^\alpha(f))(t) &= \beta\int_0^1(1-r)^{\beta-1}(tr)^\alpha W_+^\alpha f(tr)dr\\ &= t^\alpha W_+^\alpha\left(\beta\int_0^1(1-r)^{\beta-1}f(r)dr\right)(t) = D_+^\alpha(\mathscr{C}_\beta(f))(t)\end{aligned}$$

for $f\in\mathscr{S}_+$ and we conclude the proof. ∎

The first main result in this section is the following theorem.

Theorem 3.3 *The operator $\mathscr{C}_\beta$ is a bounded operator on $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ and*

$$||\mathscr{C}_\beta|| = \frac{\Gamma(\beta+1)\Gamma(1-1/p)}{\Gamma(\beta+1-1/p)},$$

for $\alpha \geq 0$, $p > 1$ and $\beta > 0$. If $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$, then

$$\mathscr{C}_\beta f(t) = \beta \int_0^\infty (1-e^{-r})^{\beta-1} e^{-r(1-1/p)} T_{r,p} f(t) dr, \quad t \geq 0, \tag{3.1}$$

where the semigroup $(T_{r,p})_{t\geq 0}$ is defined in Theorem 2.5.

Proof. Let $\alpha \geq 0$, $\beta > 0$ and $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$ be given. We apply the change of variable $s = te^{-r}$ to get that

$$\mathscr{C}_\beta f(t) := \frac{\beta}{t^\beta} \int_0^t (t-s)^{\beta-1} f(s) ds = \beta \int_0^\infty (1-e^{-r})^{\beta-1} e^{-r} f(te^{-r}) dr,$$

and the equality (3.1) is proved. Observe that by this equality, $\mathscr{C}_\beta$ is well defined and is a bounded operator on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ for $p > 1$. Indeed, we have

$$\begin{aligned} ||\mathscr{C}_\beta f||_{\alpha,p} &\leq \beta \int_0^\infty (1-e^{-r})^{\beta-1} e^{-r(1-1/p)} ||T_r f||_{\alpha,p} dr \\ &= \beta ||f||_{\alpha,p} \int_0^\infty (1-e^{-r})^{\beta-1} e^{-r(1-1/p)} dr = ||f||_{\alpha,p} \frac{\Gamma(\beta+1)\Gamma(1-1/p)}{\Gamma(\beta+1-1/p)}. \end{aligned}$$

To check the exact value of $||\mathscr{C}_\beta||_{\alpha,\beta}$, note that by the Lemma 3.2, the boundedness of $\mathscr{C}_\beta$ on $L^p(\mathbb{R}^+)$ (see the Introduction) and the fact that the operator D_+^α is an isometry (see Proposition 2.2 (iii)), we have

$$\begin{aligned} \|\mathscr{C}_\beta\|_{\alpha,p} &= \sup_{f\neq 0} \frac{\|\mathscr{C}_\beta f\|_{\alpha,p}}{\|f\|_{\alpha,p}} \\ &= \sup_{f\neq 0} \frac{\|D_+^\alpha \circ \mathscr{C}_\beta f\|_p}{\|D_+^\alpha f\|_p} \\ &= \sup_{f\neq 0} \frac{\|\mathscr{C}_\beta \circ D_+^\alpha f\|_p}{\|D_+^\alpha f\|_p} = \sup_{g\neq 0} \frac{\|\mathscr{C}_\beta g\|_p}{\|g\|_p} = \|\mathscr{C}_\beta\|_p. \end{aligned}$$

Finally, we observe that $\|\mathscr{C}_\beta\|_p = \inf\{M > 0 : \|\mathscr{C}_\beta f\|_p \leq M\|f\|_p\} = \frac{\Gamma(\beta+1)\Gamma(1-1/p)}{\Gamma(\beta+1-1/p)}$ because, by (1.2), the constant $\frac{\Gamma(\beta+1)\Gamma(1-1/p)}{\Gamma(\beta+1-1/p)}$ is optimal for the inequality. ∎

Remark 3.4 (i) Recall that the Beta function, also called the Euler integral of the first kind, is defined by:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad x > 0, \quad y > 0,$$

and satisfies the property $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. Hence, the obtained value for the norm of $\mathscr{C}_\beta$ can be rewritten as

$$||\mathscr{C}_\beta|| = \beta B(\beta, 1-1/p), \quad \beta>0, \quad p>1.$$

(ii) In the case $p=1$ we remark that $\mathscr{C}_\beta$ does not take $\mathscr{T}_1^{(\alpha)}(t^\alpha)$ in $\mathscr{T}_1^{(\alpha)}(t^\alpha)$. In fact, from Lemma 2.3 it follows that, for $\beta>0$, $h_\beta(t):=(1+t)^{-(\beta+1)}$ belongs to $\mathscr{T}_1^{(\alpha)}(t^\alpha)$. By [21, Formula 2, p.173] and [17, p. 38], we have

$$\mathscr{C}_\beta h_\beta(t) \;=\; \frac{\beta}{t^\beta}\int_0^t \frac{(t-s)^{\beta-1}}{(1+s)^{\beta+1}}ds = {}_2F_1(1,\beta+1;\beta+1;-t) = (1+t)^{-1},$$

where ${}_2F_1$ denotes the Gaussian hypergeometric function,

$${}_2F_1(a,b;c;z) := \frac{\Gamma(c)}{\Gamma(b)\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}.$$

Since $\mathscr{C}_\beta h_\beta$ does not belong to $L^1(\mathbb{R}^+)$ and $\mathscr{T}_1^{(\alpha)}(t^\alpha)\hookrightarrow L^1(\mathbb{R}^+)$ (see Proposition 2.2 (i)), we obtain $\mathscr{C}_\beta h_\beta \notin \mathscr{T}_1^{(\alpha)}(t^\alpha)$.

(iii) Let $p>1$ be given. Take $\beta=1$ and $f\in\mathscr{T}_p^{(\alpha)}(t^\alpha)$. Then

$$\mathscr{C}_1 f(t) = \int_0^\infty e^{-r(1-1/p)}T_{r,p}f(t)dr = R(\lambda_p,\Lambda)f(t), \quad \lambda_p = 1-1/p>0. \tag{3.2}$$

and by the spectral theorem for resolvent operators (see for example [9, Theorem IV.1.13]) we get that

$$\sigma(\mathscr{C}_1) = \left\{ w\in\mathbb{C} \;:\; \left| w - \frac{p}{2(p-1)}\right| = \frac{p}{2(p-1)}\right\}, \tag{3.3}$$

see [18, Theorem 2] and similar results in [4, Theorem 3.1], and [3, Corollary 2.2]. Here, $R(\cdot,\Lambda)$ denotes the resolvent operator of Λ.

Note that in case $\beta=2$ we obtain

$$\mathscr{C}_2 f(t) = 2\int_0^\infty e^{-r(1-1/p)}(1-e^{-r})T_{r,p}f(t)dr = 2R(\lambda_p,\Lambda)f(t) - 2R(\lambda_p+1,\Lambda)f(t),$$

and, more generally, for $\beta=n+1$,

$$\mathscr{C}_{n+1}f(t) = (n+1)\sum_{k=0}^{n}\binom{n}{k}(-1)^k R(\lambda_p+k,\Lambda)f(t), \quad n\in\mathbb{Z}_+. \tag{3.4}$$

In the next result, we are able to describe $\sigma(\mathscr{C}_\beta)$ for $\beta>0$.

Theorem 3.5 *Let $1 < p < \infty$, and $\mathscr{C}_\beta : \mathscr{T}_p^{(\alpha)}(t^\alpha) \to \mathscr{T}_p^{(\alpha)}(t^\alpha)$ the generalized Cesàro operator. Then*

$$\sigma(\mathscr{C}_\beta) = \beta\overline{B(\beta, 1-1/p+i\mathbb{R})} := \Gamma(\beta+1)\overline{\left\{\frac{\Gamma(1-\frac{1}{p}+it)}{\Gamma(\beta+1-\frac{1}{p}+it)} \ : \ t \in \mathbb{R}\right\}}.$$

Proof. Note that $(T_{t,p})_{t\in\mathbb{R}}$ is an uniformly bounded C_0-group (Theorem 2.5) whose infinitesimal generator is $(\Lambda, D(\Lambda))$ and $\mathscr{C}_\beta = \widehat{f_{\beta,p}}(\Lambda)$, i.e,

$$\mathscr{C}_\beta f = \beta\int_0^\infty (1-e^{-r})^{\beta-1}e^{-r(1-1/p)}T_{r,p}f\,dr = \int_{-\infty}^\infty f_{\beta,p}(r)T_{r,p}f\,dr,$$

where $f_{\beta,p}(r) = \chi_{[0,\infty)}(r)\beta(1-e^{-r})^{\beta-1}e^{-r(1-1/p)}$ for $r \in \mathbb{R}$, see Theorem 3.3. By [22, Theorem 3.1], we obtain

$$\sigma(\mathscr{C}_\beta) = \overline{\widehat{f_{\beta,p}}(\sigma(i\Lambda))}$$

where $\widehat{f_{\beta,p}}$ is the Fourier transform of the function $f_{\beta,p}$. As $\sigma(i\Lambda) = \mathbb{R}$ (see Proposition 2.6 (ii)) and $\widehat{f_{\beta,p}}(t) = \mathscr{L}(f_{\beta,p})(it)$ we use that

$$\mathscr{L}(f_{\beta,p})(z) = \beta\int_0^\infty e^{-zr}(1-e^{-r})^{\beta-1}e^{-r(1-1/p)}dr = \frac{\Gamma(\beta+1)\Gamma(1-\frac{1}{p}+z)}{\Gamma(\beta+1-\frac{1}{p}+z)}, \qquad z \in \overline{\mathbb{C}^+}.$$

to conclude the result. ∎

Remark 3.6 In the case that $n \in \mathbb{N}$, we obtain that

$$\sigma(\mathscr{C}_n) = \left\{\frac{n!p^n}{((n+it)p-1)\dots((1+it)p-1)} \ : \ t \in \mathbb{R}\right\} \cup \{0\},$$

and for $n = 1$

$$\sigma(\mathscr{C}_1) = \left\{\frac{p}{(1+it)p-1} \ : \ t \in \mathbb{R}\right\} \cup \{0\} = \left\{w \in \mathbb{C} \ : \ \left|w - \frac{p}{2(p-1)}\right| = \frac{p}{2(p-1)}\right\}.$$

Now we consider the generalized dual Cesàro operator $\mathscr{C}_\beta^*$ on $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ defined by

$$\mathscr{C}_\beta^* f(t) := \beta\int_t^\infty \frac{(s-t)^{\beta-1}}{s^\beta}f(s)ds = \beta\int_1^\infty \frac{(r-1)^{\beta-1}}{r^\beta}f(tr)dr, \quad t > 0.$$

For $0 < \gamma < 1$, functions g_γ are eigenfunctions of $\mathscr{C}_\beta^*$ with eigenvalue $\frac{\Gamma(\beta+1)\Gamma(1-\gamma)}{\Gamma(\beta-\gamma+1)}$:

$$\mathscr{C}_\beta^*(g_\gamma)(t) = \frac{\beta}{\Gamma(\gamma)}\int_t^\infty \frac{(s-t)^{\beta-1}s^{\gamma-1}}{s^\beta}ds = \frac{\Gamma(\beta+1)\Gamma(1-\gamma)}{\Gamma(\beta-\gamma+1)}g_\gamma(t),$$

for $t > 0$.

Using (2.1), we obtain

$$D_+^\alpha \circ \mathscr{C}_\beta^*(f) = \mathscr{C}_\beta^* \circ D_+^\alpha(f), \qquad f \in \mathscr{S}_+ \tag{3.5}$$

where $D_+^\alpha f(t) = \dfrac{1}{\Gamma(\alpha+1)} t^\alpha W_+^\alpha f(t)$ for $f \in \mathscr{S}_+$ and $t \geq 0$. Hence the proof of the next result follows from duality and Theorem 3.3.

Theorem 3.7 *The operator $\mathscr{C}_\beta^*$ is a bounded operator on $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ and*

$$||\mathscr{C}_\beta^*|| = \frac{\Gamma(\beta+1)\Gamma(1/p)}{\Gamma(\beta+1/p)},$$

for $\alpha \geq 0$, $p > 1$ and $\beta > 0$. The dual operator of $\mathscr{C}_\beta$ on $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ is $\mathscr{C}_\beta^*$ on $\mathscr{T}_{p'}^{(\alpha)}(t^\alpha)$, i.e.

$$\langle \mathscr{C}_\beta f, g\rangle_\alpha = \langle f, \mathscr{C}_\beta^* g\rangle_\alpha, \qquad f \in \mathscr{T}_p^{(\alpha)}(t^\alpha), \quad g \in \mathscr{T}_{p'}^{(\alpha)}(t^\alpha),$$

where $\langle \quad , \quad \rangle_\alpha$ is given in Proposition 2.2 (iv) and $\frac{1}{p} + \frac{1}{p'} = 1$.

If $f \in \mathscr{T}_p^{(\alpha)}(t^\alpha)$, then

$$\mathscr{C}_\beta^* f(t) \;=\; \beta \int_{-\infty}^0 (e^{-r}-1)^{\beta-1} e^{-r(1-1/p-\beta)} T_{r,p} f(t)\,dr, \quad t \geq 0, \tag{3.6}$$

where the C_0-group $(T_{r,p})_{t\in\mathbb{R}}$ is defined in Theorem 2.5.

Remark 3.8 Take $\beta = 1$ and $f \in \mathscr{T}_p^{(\alpha)}(t^\alpha)$. Then

$$\mathscr{C}_1^* f(t) = \int_{-\infty}^0 e^{-\frac{r}{p}} T_{-r,p} f(t)\,drds = R(1/p, -\Lambda) f(t), \qquad t \geq 0.$$

and by the spectral theorem for the resolvent operator, see [9, Theorem IV.1.13], we obtain

$$\sigma(\mathscr{C}_1^*) = \left\{ w \in \mathbb{C} \;:\; \left| w - \frac{p}{2} \right| = \frac{p}{2} \right\}.$$

This gives a proof of a conjecture posed by F. Móricz in [18, Section 2]. See a similar result in [4, Theorem 3.2].

In the following theorem we describe $\sigma(\mathscr{C}_\beta^*)$ for $\beta > 0$. The proof follows from duality and Theorem 3.5.

Theorem 3.9 *Let $\beta > 0$, $1 \le p < \infty$, and $\mathscr{C}_\beta^*: \mathscr{T}_p^{(\alpha)}(t^\alpha) \to \mathscr{T}_p^{(\alpha)}(t^\alpha)$ the generalized dual Cesàro operator. Then*

$$\sigma(\mathscr{C}_\beta^*) = \beta\overline{B(\beta, 1/p + i\mathbb{R})} := \Gamma(\beta+1)\overline{\left\{\frac{\Gamma(\frac{1}{p}+it)}{\Gamma(\beta+\frac{1}{p}+it)} \ : \ t \in \mathbb{R}\right\}}.$$

Remark 3.10 In the case that $n \in \mathbb{N}$, we obtain that

$$\sigma(\mathscr{C}_n^*) \ = \ \left\{\frac{n!p^n}{((n-1)p+1+it)\dots(p+1+it)(1+it)} \ : \ t \in \mathbb{R}\right\} \cup \{0\},$$

and for $n = 1$

$$\sigma(\mathscr{C}_1^*) \ = \ \left\{\frac{p}{1+it} \ : \ t \in \mathbb{R}\right\} \cup \{0\} = \left\{w \in \mathbb{C} \ : \ \left|w - \frac{p}{2}\right| = \frac{p}{2}\right\}.$$

Remark 3.11 In the case that $p = 2$ we have $\sigma(\mathscr{C}_\beta) = \sigma(\mathscr{C}_\beta^*)$ for all $\beta > 0$. Note that in case $p \neq 2$ the spectrum of $\mathscr{C}_\beta$ and $\mathscr{C}_\beta^*$ are dual in the sense that $\sigma(\mathscr{C}_\beta)$, with $\mathscr{C}_\beta$ defined on $\mathscr{T}_p^{(\alpha)}(t^\alpha)$, is identical to $\sigma(\mathscr{C}_\beta^*)$, with $\mathscr{C}_\beta^*$ defined on $\mathscr{T}_{p'}^{(\alpha)}(t^\alpha)$, and where $\frac{1}{p} + \frac{1}{p'} = 1$.

To finish this section we prove the remarkable fact that $\mathscr{C}_\alpha$ and $\mathscr{C}_\beta^*$ commute on $L^p(\mathbb{R}^+)$ (and then on $\mathscr{T}_p^{(\alpha)}(t^\alpha)$). We also give explicitly the value of $\mathscr{C}_\alpha\mathscr{C}_\beta^*$ in terms of the the Gaussian hypergeometric function ${}_2F_1$. This theorem includes [18, Lemma 2] for $\alpha = \beta = 1$.

Theorem 3.12 *Let $\mathscr{C}_\alpha$ and $\mathscr{C}_\beta^*$ the generalized Cesáro operators on $L^p(\mathbb{R}^+)$ for $p > 1$. Then $\mathscr{C}_\alpha\mathscr{C}_\beta^* = \mathscr{C}_\beta^*\mathscr{C}_\alpha$ for $\alpha, \beta > 0$ and*

$$\begin{aligned}(\mathscr{C}_\alpha\mathscr{C}_\beta^*)f(t) \ &= \ \alpha\int_0^t f(r)\frac{1}{t-r}\left(\frac{t-r}{t}\right)^{\alpha+\beta} {}_2F_1(\alpha+\beta, \beta; \beta+1; \frac{r}{t})dr \\ &\quad + \beta\int_t^\infty f(r)\frac{1}{r-t}\left(\frac{r-t}{t}\right)^{\alpha+\beta} {}_2F_1(\alpha+\beta, \alpha; \alpha+1; \frac{t}{r})dr,\end{aligned}$$

in particular

$$\begin{aligned}(\mathscr{C}_1\mathscr{C}_\beta^*)f(t) \ &= \ \mathscr{C}_1 f(t) + \beta\int_t^\infty f(r)\frac{(r-t)^\beta}{r^{\beta+1}} {}_2F_1(\beta+1, 1; 2; \frac{r}{t})dr, \\ (\mathscr{C}_\alpha\mathscr{C}_1^*)f(t) \ &= \ \alpha\int_0^t f(r)\frac{(t-r)^\alpha}{t^{\alpha+1}} {}_2F_1(\alpha+1, 1; 2; \frac{r}{t})dr + \mathscr{C}_1^* f(t), \\ (\mathscr{C}_1\mathscr{C}_1^*)f \ &= \ \mathscr{C}_1 f + \mathscr{C}_1^* f = (\mathscr{C}_1^*\mathscr{C}_1)f,\end{aligned}$$

for $f \in L^p(\mathbb{R}^+)$ and t almost everywhere on $\mathbb{R}^+$.

Proof. By the integral representations (3.1) and (3.6), and since $T_{t,p}$ commutes with $T_{r,p}$ for any $t,r\in\mathbb{R}$, we conclude that $\mathscr{C}_\alpha\mathscr{C}_\beta^*=\mathscr{C}_\beta^*\mathscr{C}_\alpha$ for $\alpha,\beta>0$. Take $f\in L^p(\mathbb{R}^+)$ and we apply the Fubini theorem to get that

$$\begin{aligned}\mathscr{C}_\beta^*\mathscr{C}_\alpha f(t) &= \beta\alpha\int_t^\infty \frac{(x-t)^{\beta-1}}{x^{\beta+\alpha}}\int_0^x (x-r)^{\alpha-1}f(r)drdx\\ &= \beta\alpha\int_0^\infty f(r)\int_{\max\{t,r\}}^\infty \frac{(x-t)^{\beta-1}(x-r)^{\alpha-1}}{x^{\beta+\alpha}}dxdr\end{aligned}$$

for t almost everywhere on $\mathbb{R}^+$. For $0<r<t$, this equality

$$\int_t^\infty \frac{(x-t)^{\beta-1}(x-r)^{\alpha-1}}{x^{\beta+\alpha}}dx=\frac{1}{\beta(t-r)}\left(\frac{t-r}{t}\right)^{\alpha+\beta} {}_2F_1(\alpha+\beta,\beta;\beta+1;\frac{r}{t})$$

holds, see for example [12, p. 314, 3197(1)].

Now take $\alpha=1$. Since

$$(1-z)^a\,{}_2F_1(a,b;c;z)={}_2F_1(a,c-b;c;\frac{z}{z-1})$$

(see for example [17, p.47]), we get that

$$\frac{1}{t-r}\left(\frac{t-r}{t}\right)^{1+\beta} {}_2F_1(1+\beta,\beta;\beta+1;\frac{r}{t})=\frac{1}{t-r}\,{}_2F_1(1+\beta,1;1+\beta;\frac{-r}{t-r})=\frac{1}{t}$$

where we apply that ${}_2F_1(-a,b;b;-z)=(1+z)^a$, ([17, p. 38]). Similarly we prove the case $\beta=1$. ∎

4 Composition groups on Sobolev spaces defined on $\mathbb{R}$.

In this section we introduce the subspaces $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$ which are contained in $L^p(\mathbb{R})$, similarly to $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ are in $L^p(\mathbb{R}^+)$. Let $\mathscr{S}$ be the Schwartz class on $\mathbb{R}$ and we set

$$W_-^{-\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-t)^{\alpha-1}f(t)dt,$$

$$W_-^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{-\infty}^x (x-t)^{n-\alpha-1}f(t)dt,$$

and $W_-^0 f=f$, for $x\in\mathbb{R}$ and a natural number $n>\alpha$. Putting $\tilde{f}(x)=f(-x)$, it is readily seen that $W_+^\alpha f(x)=W_-^\alpha\tilde{f}(-x)$ for all $\alpha\in\mathbb{R}$, $f\in\mathscr{S}$ and $x\in\mathbb{R}$. Equalities $W_-^{\alpha+\beta}=W_-^\alpha W_-^\beta$ and $W_-^n f=f^{(n)}$ hold for each natural number n and $\alpha,\beta\in\mathbb{R}$.

For $f \in \mathscr{S}$, put

$$W_0^\alpha f(t) := \begin{cases} W_-^\alpha f(t), & t < 0, \\ e^{i\pi\alpha} W_+^\alpha f(t), & t > 0. \end{cases}$$

For $\lambda > 0$, we have that $W_0^\alpha(f_\lambda) = \lambda^\alpha (W_0^\alpha f)_\lambda$, where $f_\lambda(t) = f(\lambda t)$ for $t \in \mathbb{R}$.

Definition 4.1 *Let $1 \le p < \infty$. The Banach space $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$ is defined as the completion of the Schwartz class on $\mathbb{R}$ in the norm*

$$|||f|||_{\alpha,p} := \frac{1}{\Gamma(\alpha+1)} \left(\int_{-\infty}^{\infty} (|W_0^\alpha f(t)|\, |t|^\alpha)^p \, dt \right)^{\frac{1}{p}}.$$

Properties similar to those of $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ hold for $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$. The proof of next proposition is similar to the proof of Proposition 2.2 and we skip it.

Proposition 4.2 *Take $p \ge 1$ and $\beta > \alpha > 0$. Then*

(i) $\mathscr{T}_p^{(\beta)}(|t|^\beta) \hookrightarrow \mathscr{T}_p^{(\alpha)}(|t|^\alpha) \hookrightarrow L^p(\mathbb{R})$.

(ii) The operator $D_0^\alpha : \mathscr{T}_p^{(\alpha)}(|t|^\alpha) \to L^p(\mathbb{R})$ defined by

$$f \mapsto D_0^\alpha f(t) := \frac{1}{\Gamma(\alpha+1)} |t|^\alpha W_0^\alpha f(t), \qquad t \in \mathbb{R}, \quad f \in \mathscr{T}_p^{(\alpha)}(|t|^\alpha),$$

is an isometry.

(iii) If $p > 1$ and p' satisfies $\frac{1}{p} + \frac{1}{p'} = 1$, then the dual of $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$ is $\mathscr{T}_{p'}^{(\alpha)}(|t|^\alpha)$, where the duality is given by

$$\langle f, g \rangle_\alpha = \frac{1}{\Gamma(\alpha+1)^2} \int_{-\infty}^{\infty} W_0^\alpha f(t) W_0^\alpha g(t) |t|^{2\alpha} dt,$$

for $f \in \mathscr{T}_p^{(\alpha)}(|t|^\alpha)$, $g \in \mathscr{T}_{p'}^{(\alpha)}(|t|^\alpha)$.

For $p = 1$, the subspace $\mathscr{T}_1^{(\alpha)}(|t|^\alpha)$ was introduced in [11, Definition 1.9]. In fact $\mathscr{T}_1^{(\alpha)}(|t|^\alpha)$ is a subalgebra of $L^1(\mathbb{R})$ for the convolution product

$$f * g(t) = \int_{-\infty}^{\infty} f(t-s) g(s) ds, \qquad t \in \mathbb{R}, \quad f, g \in \mathscr{T}_1^{(\alpha)}(|t|^\alpha), \tag{4.1}$$

see [11, Theorem 1.8] and also [15, Theorem 2] for some more details.

Theorem 4.3 *Let $1 < p < \infty$. The Banach space $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$ is a module for the algebra $\mathscr{T}_1^{(\alpha)}(|t|^\alpha)$ and*

$$|||f*g|||_{\alpha,p} \le C_{\alpha,p}|||f|||_{\alpha,p}|||g|||_{\alpha,1}, \qquad f \in \mathscr{T}_p^{(\alpha)}(|t|^\alpha), \quad g \in \mathscr{T}_1^{(\alpha)}(|t|^\alpha).$$

Proof. Take $f, g \in \mathscr{S}$. We write $f_+ := f\chi_{[0,\infty)}$ and $f_- := f\chi_{(-\infty,0]}$. By considering the decomposition $f*g = (f_+*g_+)+(f_+*g_-)+(f_-*g_+)+(f_-*g_-)$ on $\mathbb{R}$, and we apply [11, Lemma 1.6] and the fact that $f_-*g_- = 0$ on $(0,\infty)$ to obtain that

$$W_+^\alpha(f*g)_+(t) = W_+^\alpha(f_+*g_+)(t) + (W_+^\alpha f_+ * g_-)(t) + (W_+^\alpha g_+ * f_-)(t), \qquad t > 0.$$

Now, first,

$$\|f_+*g_+\|_{\alpha,p} \le C_{\alpha,p}\|f_+\|_{\alpha,p}\|g_+\|_{\alpha,1} \le C_{\alpha,p}|||f|||_{\alpha,p}|||g|||_{\alpha,1}$$

by Proposition 2.2 (ii).

On the other hand, $\mathscr{T}_1^{(\alpha)}(t^\alpha) \subset L^1(\mathbb{R}^+)$, and we apply the Minkowski inequality to get that

$$\begin{aligned}
&\left(\int_0^\infty |W_+^\alpha f_+ * g_-(t)|^p t^{\alpha p} dt\right)^{\frac{1}{p}} \\
\le\ & \left(\int_0^\infty \left(\int_0^\infty |W_+^\alpha f_+(s+t)||g_-(s)| ds\right)^p t^{\alpha p} dt\right)^{\frac{1}{p}} \\
=\ & \int_0^\infty |g_-(s)| \left(\int_0^\infty |W_+^\alpha f_+(t+s)|^p t^{\alpha p} dt\right)^{\frac{1}{p}} ds \\
\le\ & \int_0^\infty |g_-(s)| \left(\int_s^\infty |W_+^\alpha f_+(u)|^p u^{\alpha p} du\right)^{\frac{1}{p}} ds \\
\le\ & \Gamma(\alpha+1)|||g|||_{0,1}||f_+||_{\alpha,p} \le \Gamma(\alpha+1)|||g|||_{\alpha,1}|||f|||_{\alpha,p}.
\end{aligned}$$

As $\mathscr{T}_p^{(\alpha)}(t^\alpha) \subset L^p(\mathbb{R}^+)$ for $p > 1$, and we apply again the Minkowski inequality to obtain that

$$\begin{aligned}
&\left(\int_0^\infty |(W_+^\alpha g_+ * f_-)(t)(t)|^p t^{\alpha p} dt\right)^{\frac{1}{p}} \\
\le\ & \left(\int_0^\infty \left(\int_t^\infty |W_+^\alpha g_+(s)||f_-(t-s)| ds\right)^p t^{\alpha p} dt\right)^{\frac{1}{p}} \\
=\ & \int_0^\infty |W_+^\alpha g_+(s)| \left(\int_0^s |f_-(t-s)|^p t^{\alpha p} dt\right)^{\frac{1}{p}} ds \\
\le\ & |||f|||_{0,p} \int_0^\infty |W_+^\alpha g_+(s)| s^\alpha ds \\
\le\ & \Gamma(\alpha+1)|||f|||_{\alpha,p}\, ||g_+||_{\alpha,1} \\
\le\ & \Gamma(\alpha+1)|||f|||_{\alpha,p}\, |||g|||_{\alpha,1}.
\end{aligned}$$

Combining these estimates obtained, we get

$$\frac{1}{\Gamma(\alpha+1)}\left(\int_0^\infty |W_+^\alpha(f*g)(t)|^p\, t^{\alpha p}dt\right)^{\frac{1}{p}} \leq C|||f|||_{\alpha,p}\,|||g|||_{\alpha,1}.$$

Finally, because $W_-^\alpha(f*g)(t) = W_+^\alpha(\tilde{f}*\tilde{g})(-t)$ if $t<0$ using the inclusion $\mathscr{T}_p^{(\alpha)}(t^\alpha) \subset L^p(\mathbb{R}^+)$ as above for $p \geq 1$, we have that

$$\frac{1}{\Gamma(\alpha+1)}\left(\int_{-\infty}^0 |W_-^\alpha(f*g)(t)|^p\,|t|^{\alpha p}\,dt\right)^{\frac{1}{p}} \leq C|||f|||_{\alpha,p}\,|||g|||_{\alpha,1}.$$

The result follows. ∎

We remark that, as in the case of $\mathscr{T}_p^{(\alpha)}(t^\alpha)$, it is easy to verify that $(T_{t,p})_{t\in\mathbb{R}}$ is a C_0-group of isometries on $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$ as the next theorem shows. The proof runs parallel to the proofs of Theorem 2.5, Proposition 2.6 and Proposition 2.7 and hence we omit it.

Theorem 4.4 *Let $1 \leq p$ and $\alpha \geq 0$. We define the family of operators $(T_{t,p})_{t\in\mathbb{R}}$ by*

$$T_{t,p}f(s) := e^{-\frac{t}{p}}f(e^{-t}s), \qquad f \in \mathscr{T}_p^{(\alpha)}(|t|^\alpha).$$

(i) Then $(T_{t,p})_{t\in\mathbb{R}}$ is a C_0-group of isometries on $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$ whose infinitesimal generator Λ is given by

$$(\Lambda f)(s) := -sf'(s) - \frac{1}{p}f(s)$$

with domain $D(\Lambda) = \mathscr{T}_p^{(\alpha+1)}(|t|^{\alpha+1})$.

(ii) $\sigma_p(\Lambda) = \emptyset$ and $\sigma(\Lambda) = i\mathbb{R}$ (here σ_p denotes the point spectrum).

(iii) The semigroups $(T_{t,p})_{t\geq 0}$ and $(T_{-t,p'})_{t\geq 0}$ are dual operators of each other acting on $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$ and $\mathscr{T}_{p'}^{(\alpha)}(|t|^\alpha)$ with $\frac{1}{p}+\frac{1}{p'} = 1$ for $p > 1$.

5 The generalized Cesàro operators on $\mathbb{R}$.

For $\beta > 0$ we define the generalized Cesàro operator by

$$\mathscr{C}_\beta f(t) := \begin{cases} \dfrac{\beta}{|t|^\beta}\displaystyle\int_t^0 (s-t)^{\beta-1}f(s)ds, & t<0, \\ f(0), & t=0, \\ \dfrac{\beta}{t^\beta}\displaystyle\int_0^t (t-s)^{\beta-1}f(s)ds, & t>0, \end{cases}$$

for $f \in \mathscr{S}$. We are interested in the extension of $\mathscr{C}_\beta$ on $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$. Note that we may write

$$\mathscr{C}_\beta f(t) = \beta \int_0^1 (1-r)^{\beta-1} f(tr)dr, \quad t \in \mathbb{R},\ f \in \mathscr{S}.$$

We use this integral representation to prove the next lemma.

Lemma 5.1 *Take $\alpha \geq 0$ and $\beta > 0$. Then $D_0^\alpha \circ \mathscr{C}_\beta = \mathscr{C}_\beta \circ D_0^\alpha$, i.e.,*

$$D_0^\alpha(\mathscr{C}_\beta(f)) = \mathscr{C}_\beta(D_0^\alpha(f)), \qquad f \in \mathscr{S},$$

where $D_0^\alpha f(t) = \frac{1}{\Gamma(\alpha+1)}|t|^\alpha W_0^\alpha f(t)$ for $f \in \mathscr{S}$.

Proof. Since for $\lambda > 0$, we have that $W_0^\alpha(f_\lambda) = \lambda^\alpha (W_0^\alpha f)_\lambda$, where $f_\lambda(t) = f(\lambda t)$ for $t \in \mathbb{R}$, the proof follows similarly to Lemma 3.2. ∎

Similar results of $\mathscr{C}_\beta$ on $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ hold for $\mathscr{C}_\beta$ on $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$. The proof of next result is analogous to the proof of Theorem 3.3 and Theorem 3.5.

Theorem 5.2 *Let $\alpha \geq 0,\ \beta > 0,\ 1 < p < \infty$ and the generalized Cesàro operator $\mathscr{C}_\beta$ on $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$. Then*

(i) The operator $\mathscr{C}_\beta$ is bounded on $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$ and

$$||\mathscr{C}_\beta|| = \frac{\Gamma(\beta+1)\Gamma(1-1/p)}{\Gamma(\beta+1-1/p)}.$$

(ii) If $f \in \mathscr{T}_p^{(\alpha)}(|t|^\alpha)$, then

$$\mathscr{C}_\beta f(t) = \beta \int_0^\infty (1-e^{-r})^{\beta-1} e^{-r(1-1/p)} T_{r,p} f(t)dr, \quad t \in \mathbb{R},$$

where the C_0-group $(T_{r,p})_{r\in\mathbb{R}}$ is defined in Theorem 4.4.

(iii)

$$\sigma(\mathscr{C}_\beta) = \Gamma(\beta+1)\overline{\left\{\frac{\Gamma(1-\frac{1}{p}+it)}{\Gamma(\beta+1-\frac{1}{p}+it)}\ :\ t \in \mathbb{R}\right\}}.$$

Now we consider the generalized dual Cesàro operator $\mathscr{C}_\beta^*$ defined for $\beta > 0$ by

$$\mathscr{C}_\beta^* f(t) := \begin{cases} \beta \int_{-\infty}^t \frac{(t-s)^{\beta-1}}{|s|^\beta} f(s)ds, & t < 0, \\ 0, & t = 0, \\ \beta \int_t^\infty \frac{(s-t)^{\beta-1}}{s^\beta} f(s)ds, & t > 0, \end{cases}$$

and $D_0^\alpha \circ \mathscr{C}_\beta^*(f) = \mathscr{C}_\beta^* \circ D_0^\alpha(f)$, where $D_0^\alpha f(t) = \frac{1}{\Gamma(\alpha+1)}|t|^\alpha W_0^\alpha f(t)$ for $f \in \mathscr{S}$ and $t \in \mathbb{R}$.

Note that we may write

$$\mathscr{C}_\beta^* f(t) = \beta \int_1^\infty \frac{(s-1)^{\beta-1}}{s^\beta} f(ts)ds,\ t \neq 0,$$

for $f \in \mathscr{S}$. The proof of next result runs parallel to the proof of Theorem 3.7 and 3.9.

Theorem 5.3 *Let $\alpha \geq 0$, $\beta > 0$, $1 \leq p < \infty$ and the generalized dual Cesáro operator $\mathscr{C}_\beta^*$ on $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$. Then*

(i) The operator $\mathscr{C}_\beta^*$ is bounded on $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$ and

$$||\mathscr{C}_\beta^*|| = \frac{\Gamma(\beta+1)\Gamma(1/p)}{\Gamma(\beta+1/p)}.$$

(ii) The dual operator of $\mathscr{C}_\beta$ on $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$ is $\mathscr{C}_\beta^*$ on $\mathscr{T}_{p'}^{(\alpha)}(|t|^\alpha)$, i.e.

$$\langle \mathscr{C}_\beta f, g\rangle_\alpha = \langle f, \mathscr{C}_\beta^* g\rangle_\alpha, \qquad f \in \mathscr{T}_p^{(\alpha)}(|t|^\alpha), \quad g \in \mathscr{T}_{p'}^{(\alpha)}(|t|^\alpha),$$

where $\langle \quad , \quad \rangle_\alpha$ is given in Proposition 4.2 (iii).

(iii) If $f \in \mathscr{T}_p^{(\alpha)}(|t|^\alpha)$, then

$$\mathscr{C}_\beta^* f(t) \;=\; \beta \int_{-\infty}^0 (e^{-r}-1)^{\beta-1} e^{-r(1-1/p-\beta)} T_{r,p} f(t)dr, \quad t \in \mathbb{R}, \qquad (5.1)$$

where the C_0-group $(T_{r,p})_{r\in\mathbb{R}}$ is defined in Theorem 4.4.

(iv)

$$\sigma(\mathscr{C}_\beta^*) = \Gamma(\beta+1)\overline{\left\{ \frac{\Gamma(\frac{1}{p}+it)}{\Gamma(\beta+\frac{1}{p}+it)} \;:\; t \in \mathbb{R} \right\}}.$$

Remark 5.4 Note that for $t = 0$, by the integral representation (5.1)

$$\mathscr{C}_\beta^* f(0) = f(0)\beta \int_0^\infty (1-e^{-r})^{\beta-1} dr = \infty, \quad f \in \mathscr{S}.$$

6 Fourier transform and Cesàro generalized operator

We remind the reader that the Fourier transform of a function f in $L^1(\mathbb{R})$ is defined by

$$\hat{f}(t) := \int_{-\infty}^{\infty} e^{-ixt} f(x)dx, \qquad t \in \mathbb{R}.$$

It is well-known that $\hat{f}$ is continuous on $\mathbb{R}$ and $\hat{f}(t) \to 0$ when $|t| \to \infty$ (the Riemann-Lebesgue lemma). In the case that $f \in L^p(\mathbb{R})$ for some $1 < p \leq 2$, the Fourier transform of f is defined in terms of a limit in the norm of $L^{p'}(\mathbb{R})$ of truncated integrals:

$$\hat{f} := \lim_{R\to\infty} \widehat{f\chi_{(-R,R)}}, \qquad \widehat{f\chi_{(-R,R)}}(t) = \int_{-R}^{R} e^{-ixt} f(x)dx, \qquad t \in \mathbb{R},$$

i.e., $\hat{f} \in L^{p'}(\mathbb{R})$ and $\lim_{R\to\infty} \|\hat{f} - \widehat{f\chi_{(-R,R)}}\|_{p'} = 0$ where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\chi_{(-R,R)}$ is the characteristic function of the interval $(-R,R)$, see for example [25, Vol 2, p.254]. Then the existence of $\hat{f}(t)$ is guaranteed only at almost every t and $\hat{f}$ may be non continuous and the Riemann-Lebesgue lemma could not hold (unlike the case when $f \in L^1(\mathbb{R})$).

In case that $f \in L^p(\mathbb{R})$ for some $2 < p < \infty$, the Fourier transform $\hat{f}$ cannot be defined as an ordinary function although $\hat{f}$ can be defined as a tempered distribution, see for example [23, pp 19-30].

In the next theorem, we consider the Fourier transform on the Sobolev space $\mathscr{T}_p^{(n)}(|t|^n)$.

Theorem 6.1 *Take $1 \leq p \leq 2$ and $n \in \mathbb{N}$. Then $\hat{f} \in \mathscr{T}_{p'}^{(n)}(|t|^n)$ for $f \in \mathscr{T}_p^{(n)}(|t|^n)$ and $\frac{1}{p} + \frac{1}{p'} = 1$.*

Proof. Take $f \in \mathscr{T}_p^{(n)}(|t|^n)$. Since $\mathscr{T}_p^{(n)}(|t|^n) \subset \mathscr{T}_p^{(j)}(|t|^j)$, we have that $x^j f^{(j)} \in L^p(\mathbb{R})$ for $0 \leq j \leq n$. As

$$(it)^n(\hat{f})^{(n)}(t) = \sum_{j=0}^{n} (-1)^n \binom{n}{j} \frac{n!}{j!} \widehat{x^j f^{(j)}}(t), \qquad n \in \mathbb{N}, t \text{ a.e. on } \mathbb{R},$$

(see for example [25]), we conclude that $(it)^n(\hat{f})^{(n)} \in L^{p'}(\mathbb{R})$ and then $\hat{f} \in \mathscr{T}_{p'}^{(n)}(|t|^n)$. ■

In what follows, we show that

$$\widehat{\mathscr{C}_\beta(f)} = \mathscr{C}_\beta^*(\widehat{f}), \qquad \text{and} \qquad \widehat{\mathscr{C}_\beta^*(f)} = \mathscr{C}_\beta(\widehat{f}), \qquad f \in L^p(\mathbb{R}),$$

for $1 < p \leq 2$ (Theorem 6.4). This theorem extends the case $\beta = 1$ formulated in [5] and proved in [19]. Our approach looks like to be new and is based in the integral representations of $\mathscr{C}_\beta(f)$ and $\mathscr{C}_\beta^*(f)$ given in Section 3.

Lemma 6.2 *Let* $1 \le p \le 2$ *and the family of operators* $(T_{t,p})_{t\in\mathbb{R}}$ *defined by* $T_{t,p}(f) := e^{-\frac{t}{p}} f(e^{-t}\cdot)$, *for* $f \in L^p(\mathbb{R})$. *Then*

$$\widehat{T_{t,p}(f)} = T_{-t,p'}(\hat{f}), \qquad f \in L^p(\mathbb{R}), \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

Proof. Consider $1 \le p \le 2$ and $f \in \mathscr{S}$. It is clear that $T_{t,p}(f) \in \mathscr{S}$. Note that

$$\begin{aligned}(\widehat{T_{t,p}(f)})(r) &= e^{\frac{-t}{p}} \int_{-\infty}^{\infty} e^{-irx} f(e^{-t}x)dx = e^{t(1-\frac{1}{p})} \int_{-\infty}^{\infty} e^{-ire^t y} f(y)dy = e^{\frac{t}{p'}} \widehat{f}(e^t r) \\ &= (T_{-t,p'}\widehat{f})(r).\end{aligned}$$

By denseness of $\mathscr{S}$ we conclude the result. ∎

Remark 6.3 Since $\mathscr{T}_p^{(\alpha)}(|t|^\alpha) \hookrightarrow L^p(\mathbb{R})$ (Proposition 4.2 (i)), the equality $\widehat{T_{t,p}(f)} = T_{t,p'}(\hat{f})$ holds for $f \in \mathscr{T}_p^{(\alpha)}(|t|^\alpha)$ for $\alpha \ge 0$ and $1 \le p \le 2$.

Finally, we are ready to prove the main result in this section.

Theorem 6.4 *Let* $\beta > 0$.

(i) If $f \in L^p(\mathbb{R})$ *for some* $1 < p \le 2$, *then* $\widehat{\mathscr{C}_\beta(f)} = \mathscr{C}_\beta^*(\widehat{f})$.

(ii) If $f \in L^p(\mathbb{R})$ *for some* $1 \le p \le 2$, *then* $\widehat{\mathscr{C}_\beta^*(f)} = \mathscr{C}_\beta(\widehat{f})$.

Proof. (i) Take $f \in L^p(\mathbb{R})$ for some $1 < p \le 2$. By Theorem 5.2 (ii) and Lemma 6.2 we have that

$$\begin{aligned}\widehat{\mathscr{C}_\beta(f)}(x) &= \beta \int_0^\infty (1-e^{-r})^{\beta-1} e^{-r(1-1/p)} \widehat{T_{r,p}f}(x)dr \\ &= \beta \int_{-\infty}^0 (e^{-r}-1)^{\beta-1} e^{-r(1/p-\beta)} T_{r,p'}\widehat{f}(x)dr \\ &= \beta \int_{-\infty}^0 (e^{-r}-1)^{\beta-1} e^{-r(1-\frac{1}{p'}-\beta)} T_{r,p'}\widehat{f}(x)dr = \mathscr{C}_\beta^*(\widehat{f})(x)\end{aligned}$$

for almost every x on $\mathbb{R}$ and we use Theorem 5.3 (iii).

(ii) Now take $f \in L^p(\mathbb{R})$ for some $1 \le p \le 2$. By the integral representation (5.1) of $\mathscr{C}_\beta^*$ and Lemma 6.2 we have that

$$\begin{aligned}\widehat{\mathscr{C}_\beta^*(f)}(x) &= \beta \int_{-\infty}^0 (e^{-r}-1)^{\beta-1} e^{-r(1-\frac{1}{p}-\beta)} T_{-r,p'}\widehat{f}(x)dr \\ &= \beta \int_0^\infty (1-e^{-r})^{\beta-1} e^{-\frac{r}{p}} T_{r,p'}\widehat{f}(x)dr \\ &= \beta \int_0^\infty (1-e^{-r})^{\beta-1} e^{-r(1-\frac{1}{p'})} T_{r,p'}\widehat{f}(x)dr = \mathscr{C}_\beta(\widehat{f})(x)\end{aligned}$$

for almost every x on $\mathbb{R}$ and we use the Theorem 5.2 (ii). ■

Remark 6.5 By the Proposition 2.4, we get that $\widehat{\mathscr{C}_\beta(f)}(t) = \mathscr{C}^*_\beta(\widehat{f})(t)$ and $\widehat{\mathscr{C}^*_\beta(f)}(t) = \mathscr{C}_\beta(\widehat{f})(t)$ for $t \neq 0$ and $f \in \mathscr{T}_p^{(\alpha)}(|t|^\alpha)$, $1 < p \leq 2$ and $\alpha \geq 1$.

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