NORM CONTINUITY FOR STRONGLY CONTINUOUS FAMILIES OF OPERATORS

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ABSTRACT. One-parameter strongly continuous families $\{R(t)\}_{t\geq 0}$ of bounded operators, defined on a Banach space, are useful instruments in the study of wide classes of abstract evolution equations. In this paper we show conditions which secure the uniform continuity (or norm continuity) of (a, k)-regularized resolvent families $R(t)$ for $t \geq 0$. We prove that on certain Banach spaces (e.g. $L^\infty(S, \Sigma, \mu)$) each exponentially bounded (a, k)-regularized resolvent family is in fact uniformly continuous for $t \geq 0$. We also characterize families $R(t)$ such that $R(t) - k(t)I$ is a compact operator for all $t > 0$. Finally, we prove that in Hilbert spaces the uniform continuity of $R(t)$ for $t > 0$ (also called immediate norm continuity) is equivalent to the decay to zero of $\hat{k}(\lambda)(I - \hat{a}(\lambda)A)^{-1}$ along some imaginary axis. Our results widely generalize known properties for strongly continuous semigroups and cosine families of bounded operators.

1. INTRODUCTION

The property of uniform continuity (or norm-continuity) for one-parameter families of bounded operators is a topic of increasing interest in recent research, mainly because their important role in the exploration of useful criteria for the existence of solutions to nonlinear partial differential equations, when they are modeled as an abstract evolution equation on vector-valued spaces of functions, see e.g. the monographs [2] and [27]. The applications of uniform continuity are usually found in the use of fixed point arguments that try to avoid hypothesis of compactness on the data of the problem, but where this hypothesis need to be replaced by some better behavior on the family of bounded operators dealing with the well posedness of the associated abstract linear problem. See e.g. [1, Remark 3.4], [5, Theorem 3.4], [11, Theorem 4.1] [30, Theorems 4.1 and 5.3] and [33] to cite a few references. Note that uniform continuity also plays a crucial role in investigating the stability of solutions to abstract Volterra equations [6, Theorem 2.9] and abstract Cauchy problems.

Recently, Zhenbin Fan [11] derives characterizations of compactness for families of bounded operators associated to a class of semilinear fractional Cauchy problem. In the searching of this characterizations, one of the difficult points is that they requires the uniform continuity of the family under consideration [11, Theorem 3.6 and Theorem 3.7]. As remarked by Fan, the main difficulty is the non existence of practical criteria that can assure uniform continuity of the given family of bounded operators.

The main goal of this paper is to give a complete answer to the question raised by Fan and other authors (see e.g. [30, p.208, item (ii)]). We give an exhaustive study of uniform continuity not only for families of bounded operators associated to fractional Cauchy problems, but also for a very wide class of families of bounded operators $\{R(t)\}_{t\geq 0}$, namely, the class of (a, k)-regularized resolvent families [22]. We recall that this notion generalizes the theories

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of C_0-semigroups [2, Section 3.1], α-times-integrated semigroups [2, Section 3.2], convoluted semigroups [8], cosine functions [2, Section 3.14], n-times integrated cosine families [3], resolvent families [27] and α -resolvent families [4], among others. For example, if $a(t) = k(t) = 1$ for all $t \geq 0$ and if $a(t) = t, k(t) = 1$, then we obtain the well-known cases of strongly continuous semigroups and cosine operator functions, respectively. If $a(t) = 1$ and $k(t) = t^n/n!$ then $R(t)$ is an n-integrated semigroup. Taking $a \in L^1_{loc}(\mathbb{R}_+)$ and $k(t) = 1$ for all $t \geq 0$ we have that $R(t)$ is a resolvent family, which are the central object to study in the theory of abstract Volterra equations [27]. Finally, if $a(t) = t^{\alpha-1}/\Gamma(\alpha)$ $(\alpha > 0)$ and $k(t) = 1$ for all $t \geq 0$, then $R(t)$ corresponds to a α- resolvent family. For an up to date overwiew we refer the reader to [23, Section 2] and references therein.

Previous studies on uniform continuity for families of bounded operators have been done mainly in the case of C_0-semigroups [18]. K. Latrach, Paoli and Simonett [15], [16] have studied the problem for diverse perspectives. See also [20] and [21] for an study in case of resolvent families associated to Volterra equations. In the case of cosine and sine families of bounded operators, first studies are due to Travis and Webb [28, Proposition 4.1], [29, Proposition 2.4]. See also [17]. More recently, in [13] the authors prove that when the semigroup generated by the linear part of some linear neutral partial functional differential equations in L^p-spaces is norm continuous, then the solution semigroup associated to the neutral system is eventually norm continuous.

It is well known that if a C_0-semigroup $\{T(t)\}_{t\geq0}$ of type (M, ω) defined on a Banach space X generated by an operator A is continuous in the operator norm for all $t > 0$, then $\|(s + i\tau - A)^{-1}\| \to 0$ as $|\tau| \to \infty$ for all $s > \omega$. See [10]. The converse is also true in Hilbert spaces H (see [10, 12, 32]) but it may fail in general Banach spaces. See for example [26]. We also note the recent paper [7] where the authors construct an example where the semigroup is nowhere continuous in operator norm but the resolvent tends to 0 along $2 + i\tau$ almost logarithmically. However, the question of find a similar characterization for (a, k) regularized resolvent families under reasonable conditions on the kernels a and k remains as an open problem.

One of the main issues of this paper, is that we are able to solve this problem assuming that a and k are 2-regular (see Section 2 for definitions) and certain behavior of $\hat{k}(\lambda)$ along the imaginary axis. More precisely, we prove that the following assertions are equivalent:

(i) $\{R(t)\}_{t\geq0}$ is continuous in $\mathcal{B}(H)$ for $t > 0$,

(ii) $\displaystyle\lim_{|\tau|\to\infty} \|\hat{k}(s + i\tau)\left(I - \hat{a}(s + i\tau)A\right)^{-1}\| = 0$ for some $s > \omega$.

This paper is organized as follows: Section 2 is devoted to preliminaries, recalling the definition of (a, k)-regularized resolvent families and their main properties related with the contents of this article. We also recall in this section, the notion of Grothendieck space and the Dunford-Pettis property. A notion of regularity on the kernels will be also useful as well as an important result due to Lotz [24] that is the key to establish one of our main results in the forthcoming sections. Section 3 shows a characterization of uniformly continuity ($t \geq 0$) of (a, k)-regularized resolvent families (Theorem 3.2) which is not surprising at all, but that is absent in the current literature on the subject. The main novelty here are the hypothesis needed in the kernels a and k where one of them need to be positive, for instance. In Section 4, we give a somewhat surprising result. We show in Theorem 4.1 that a strongly continuous (a, k)-regularized resolvent family on a class of Banach spaces containing all L^∞-spaces is necessarily uniformly continuous ($t \geq 0$). This result generalize a known theorem in the case of C_0-semigroups due to Lotz [24]. Then, we

remark an interesting corollary: The result is also true in the case of certain families of bounded operators (called α-resolvent families) that play a central and decisive role in the development of qualitative properties for solutions to fractional partial differential equations. In Section 5, we characterize those (a, k)-regularized resolvent families which have the property of being near $k(t)$ times the identity (i.e., $R(t) - k(t)I$ is compact for some positive value of t). First results on such property are due to Cuthbert [9], Henríquez [14], Lutz [25] and the first author of this paper [20]. It turns out that this property is equivalent to the compactness of the generator. This equivalence is proved in Theorem 5.1. Finally, Section 6 is concerned with an important characterization of uniform continuity ($t > 0$) in case of Hilbert spaces. See Theorem 6.7. This characterization constitutes a remarkable and non trivial extension of previous results (see [21] and [32]) and will be useful in establish practical criteria on the e.g. compactness of (a, k)-regularized families of operators in general, and their specialization in different cases of interest, in particular. An example of this assertion is given in Corollary 6.8 and further applications are indicated in Remark 6.9.

2. Preliminaries

Let X be a Banach space, A be a closed linear operator with domain $D(A)$ densely defined on X.

Definition 2.1. *Let $k \in C(\mathbb{R}_+), k \neq 0$ and $a \in L^1_{loc}(\mathbb{R}_+), a \neq 0$ be given. A strongly continuous family $\{R(t)\}_{t \geq 0} \subset \mathcal{B}(X)$ is called (a, k)-regularized resolvent family on X having A as a generator if the following properties hold:*

(i) $R(0) = k(0)I$;

(ii) $R(t)x \in D(A)$ and $R(t)Ax = AR(t)x$ for all $x \in D(A)$ and $t \geq 0$;

(iii) $R(t)x = k(t)x + \displaystyle\int_0^t a(t-s)AR(s)x \, ds, \ t \geq 0, x \in D(A).$

It is well-known that if an (a, k)-regularized resolvent family exists, then it is unique [22]. Let $\{R(t)\}_{t \geq 0}$ be an (a, k)-regularized resolvent family with generator A such that

$$(2.1) \qquad \|R(t)\| \leq Mk(t), \quad t \geq 0,$$

for some constant $M > 0$. Then, under certain hypothesis on the kernels a and k (see [23, Section 2] and references therein) we have

$$D(A) = \{x \in X \ : \ \lim_{t \to 0^+} \frac{R(t)x - k(t)x}{(a * k)(t)} \text{ exists }\},$$

and

$$(2.2) \qquad Ax = \lim_{t \to 0^+} \frac{R(t)x - k(t)x}{(a * k)(t)}.$$

Here we denote $(a * k)(t) := \int_0^t k(t - s)a(s) \, ds$ the finite convolution between a and k. We note that there is a one-to-one correspondence between (a, k)-regularized resolvent families and their generators.

We say that $\{R(t)\}_{t \geq 0}$ is exponentially bounded (or, of type (M, ω)) if there exists constants $M \geq 0$ and $\omega \in \mathbb{R}$ such that

$$\|R(t)\| \leq Me^{\omega t} \quad \text{for all} \ \ t \geq 0.$$

For exponentially bounded (a, k)-regularized resolvent families is well known the following characterization.

Theorem 2.2. [19] *Let X be a Banach space. The following assertions are equivalent:*

(i) *A is the generator of an (a, k)-regularized resolvent family of type (M, ω);*

(ii) *For all $\lambda > \omega$, the resolvent set $\rho(A)$ contains the set $\{\frac{1}{\hat{a}(\lambda)} : \lambda > \omega\}$ and*

$$\hat{k}(\lambda)\Big(I - \hat{a}(\lambda)A\Big)^{-1}x = \int_0^\infty e^{-\lambda t}R(t)x dt, \quad x \in X, \quad \lambda > \omega.$$

Here, without loss of generality, we are assuming that a and k are Laplace transformable for $\lambda > \omega$.

We recall that a Banach space X is called a *Grothendieck space* if every weak* convergent sequence in X' converges weakly, where X' denotes the dual space of X.

Definition 2.3. *A Banach space X is said to have the Dunford-Pettis property if for all sequence $\{x_n\}_{n\geq 0}$ in X such that $x_n \to 0$ weakly in X and $x'_n \to 0$ in X' then $\langle x_n, x'_n \rangle \to 0$.*

For example, the spaces $L^\infty(X, \Omega, \mu)$ where (X, Ω, μ) is a positive measure space, and $C(X)$ (where X is a compact $\sigma-$Stonian space) are examples of Grothendieck spaces with the Dunford-Pettis property. Recall that X is Stonian if the closure of every open set is open, and it is σ-Stonian if the closure of every open F_σ-set is open. On the other hand, a Banach space E is *injective* if for every Banach space X and every subspace Y of X, each operator $T : Y \to E$ admits an extension $\tilde{T} : X \to E$. Every injective Banach space is a Grothendieck space with the Dunford-Pettis property. Finally, a reflexive space does not have the Dunford-Pettis property, unless the space is finite dimensional.

Definition 2.4. [27] *Let $a \in L^1_{loc}(\mathbb{R}_+)$ be Laplace transformable and $n \in \mathbb{N}$. The kernel $a(t)$ is called n-regular if there is a constant $c > 0$ such that*

$$|\lambda^m \widehat{a}^{(m)}(\lambda)| \leq c|\widehat{a}(\lambda)|,$$

for all $Re(\lambda) > 0$ and $0 \leq m \leq n$.

We finally recall the following result due to Lotz [24].

Theorem 2.5. [24, Theorem 10] *Let E be a Grothendieck space with the Dunford-Pettis property and let $(T_n) \subset \mathcal{B}(E)$ with $\lim_{n\to\infty} \|T_m(T_n - I)\| \to 0$ for all $m \in \mathbb{N}$. If (T_n) tends to the identity in the strong operator topology, then $\lim_{n\to\infty} \|T_n - I\| \to 0$. If, in addition, $\lim_{n\to\infty} \|(T_n - I)T_m\| \to 0$ for every $m \in \mathbb{N}$, in particular, if all operators T_n commute, then suffices to assume that (T_n) converges to the identity in the weak operator topology.*

3. UNIFORMLY CONTINUOUS (a, k)-REGULARIZED RESOLVENT FAMILIES

Let X be a complex Banach space. We recall that a strongly continuous family of bounded and linear operators $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be uniformly continuous if

(3.1) $$\|S(t) - S(s)\|_{\mathcal{B}(X)} \to 0 \quad \text{as} \quad t \to s,$$

holds for all $s \geq 0$. This property is also called norm-continuity for some authors [7], [10] [12], [26], but also it sometimes refers to the case that (3.1) holds for all $s > 0$. To distinguish between both cases, some authors say that $\{S(t)\}_{t\geq 0}$ is immediate norm continuous when refers to the continuity of $\{S(t)\}_{t\geq 0}$ in the uniform operator topology for $s > 0$.

Our first result is probably not surprising, but we do not find a formal proof of this in the existing literature.

Theorem 3.1. *Let $\{R(t)\}_{t\geq 0}$ be a uniformly continuous (a,k)-regularized resolvent family with generator A. Assume that a and k are exponentially bounded functions and a is positive. Then A must be a bounded operator and*

$$(3.2) \qquad R(t) = k(t) + \sum_{n=1}^{\infty} A^n (a^{*n} * k)(t), \quad t \geq 0.$$

Proof. We may assume that $a(t) \leq Me^{\lambda t}$ and $|k(t)| \leq Me^{\lambda t}$ for the same constants $M > 0$ and $\lambda > 0$. Denote $e_\lambda(t) = e^{\lambda t}, t \geq 0$ and observe that

$$e_\lambda^{*n}(t) = \frac{t^{n-1}}{(n-1)!} e_\lambda(t), \quad t \geq 0, \ n = 2, 3, ...$$

Hence $|(a^{*n} * k)(t)| \leq M^{n+1} \frac{t^n}{n!} e_\lambda(t)$ for all $t \geq 0$, $n \in \mathbb{N}$, and we have

$$\sum_{n=1}^{\infty} \|A^n (a^{*n} * k)(t)\| \leq \sum_{n=1}^{\infty} \|A\|^n |(a^{*n} * k)(t)| \leq M \sum_{n=1}^{\infty} \|A\|^n M^n \frac{t^n}{n!} e_\lambda(t) = M e^{\|A\|Mt} e_\lambda(t).$$

This proves that the series in the right hand side of (3.2) converges. Define

$$S(t) := k(t) + \sum_{n=1}^{\infty} A^n (a^{*n} * k)(t) \quad t \geq 0.$$

We will show that $S(t) = k(t) + A(a * S)(t)$. Indeed, we have

$$
\begin{aligned}
A(a * S)(t) &= A\Big(a * [k + \sum_{n=1}^{\infty} A^n (a^{*n} * k)]\Big)(t) \\
&= A\Big((a * k)(t) + \sum_{n=1}^{\infty} A^n (a^{*n+1} * k))(t)\Big] \\
&= A(a * k)(t) + \sum_{n=1}^{\infty} A^{n+1}(a^{*n+1} * k)(t) \\
&= \sum_{n=1}^{\infty} A^n (a^{*n} * k)(t) = S(t) - k(t),
\end{aligned}
$$

for all $t \geq 0$. This proves the claim. Now, by uniqueness, we conclude that $S(t) = R(t)$. It proves the representation (3.2).

Let $t > 0$ be fixed and define

$$f(t) := \frac{1}{(1*a)(t)} (a * R)(t) = \frac{1}{(1*a)(t)} \int_0^t a(t-s) R(s) \, ds.$$

Because $\{R(t)\}_{t\geq 0}$ is an uniformly continuous family, there exist $\delta > 0$ such that for $0 < s < \delta$ we have $\|R(s) - k(0)I\| < 1$. Let $\tau \in (0, \delta)$ be fixed. Then

$$
\begin{aligned}
\|f(\tau) - I\| &= \left\| \frac{1}{(1*a)(\tau)} \int_0^\tau a(\tau - s)R(s)\,ds - k(0)I \right\| \\
&= \left\| \frac{1}{(1*a)(\tau)} \int_0^\tau a(\tau - s)(R(s) - k(0))\,ds \right\| \\
&\leq \frac{1}{(1*a)(\tau)} \int_0^\tau a(\tau - s)\,\|R(s) - k(0)\|\,ds \\
&< \frac{1}{(1*a)(\tau)} \int_0^\tau a(\tau - s)\,ds = 1,
\end{aligned}
$$

where we have used that a is positive. Therefore, $f(\tau)$ is invertible on X. Let $x \in X$ be fixed. There exists $y \in X$ such that $x = f(\tau)y$. But, according to [19, Lemma 2.2] for $y \in X$ we have

$$
f(\tau)y = \frac{1}{(1*a)(\tau)} \int_0^\tau a(\tau - s)R(s)y\,ds \in D(A).
$$

Then $D(A) = X$ which implies that A is a bounded operator. $\qquad\square$

The following characterization is new in the context of (a, k)-regularized resolvent families with $k \neq 1$.

Theorem 3.2. *Let $\{R(t)\}_{t\geq 0}$ be an strongly continuous (a, k)-regularized resolvent family with generator A. Assume that a and k are exponentially bounded, a is positive and $k \in C^1(\mathbb{R})$. The following assertions are equivalent:*

(i) *$\{R(t)\}_{t\geq 0}$ is uniformly continuous;*
(ii) *A is a bounded operator.*

Proof. Suppose that A is a bounded operator. In order to see that the resolvent family $R(t)$ is uniformly continuous, we take $0 < t < s$ and observe that

$$
\begin{aligned}
\|R(t) - R(s)\| &= \left\| k(t) + \sum_{n=1}^\infty A^n(a^{*n} * k)(t) - k(s) - \sum_{n=1}^\infty A^n(a^{*n} * k)(s) \right\| \\
&\leq \|k(t) - k(s)\| + \left\| \sum_{n=1}^\infty A^n(a^{*n} * k)(t) - \sum_{n=1}^\infty A^n(a^{*n} * k)(s) \right\| \\
&\leq \|k(t) - k(s)\| + \sum_{n=1}^\infty \|A\|^n |(a^{*n} * k)(t) - (a^{*n} * k)(s)|.
\end{aligned}
$$

Since $k \in C^1(\mathbb{R})$, there exists $p \in C(\mathbb{R})$ such that $k(t) = \int_0^t p(r)\, dr = (p * 1)(t)$. Hence,

$$
\begin{aligned}
\|R(t) - R(s)\| &\leq \|k(t) - k(s)\| + \\
&\qquad \sum_{n=1}^{\infty} \|A\|^n \left|\big(a^{*n} * (p * 1)\big)(s) - \big(a^{*n} * (p * 1)\big)(t)\right| \\
&\leq \|k(t) - k(s)\| + \sum_{n=1}^{\infty} \|A\|^n \left|\int_t^s (a^{*n} * p)(v)\, dv\right| \\
&\leq \|k(t) - k(s)\| + |t - s| \sum_{n=1}^{\infty} \|A\|^n \sup_{t \leq v \leq s} |(a^{*n} * p)(v)| \\
&\leq \|k(t) - k(s)\| + |t - s| \sum_{n=1}^{\infty} \|A\|^n \sup_{t \leq v \leq s} \left|\int_0^v a^{*n}(\tau) p(v - \tau)\, d\tau\right|.
\end{aligned}
$$

Note that $|a^{*n}(\tau)| \leq \dfrac{M^n \tau^{n-1}}{(n-1)!}$ for $0 \leq \tau \leq s$, where $M := \sup\limits_{0 \leq \tau \leq s} |a(\tau)|$. Indeed,

$$
|a^{*n}(\tau)| \leq \int_0^\tau |a^{*n-1}(u) a(\tau - u)|\, du \leq \int_0^\tau \frac{M^{n-1} u^{n-2}}{(n-2)!} \cdot M\, du \leq \frac{M^n \tau^{n-1}}{(n-1)!}.
$$

Hence, we obtain

$$
\begin{aligned}
\|R(t) - R(s)\| &\leq \|k(t) - k(s)\| + |t - s| \sum_{n=1}^{\infty} \frac{\|A\|^n M^n s^{n-1}}{(n-1)!} \sup_{t \leq v \leq s} \left|\int_0^v p(v - \tau)\, d\tau\right| \\
&\leq \|k(t) - k(s)\| + |t - s| \sum_{n=0}^{\infty} \frac{(\|A\| M)^{n+1} s^n}{n!} \sup_{t \leq v \leq s} |k(v)| \\
&\leq \|k(t) - k(s)\| + |t - s| \|A\| M \sum_{n=0}^{\infty} \frac{(\|A\| M s)^n}{n!} \sup_{t \leq v \leq s} |k(v)| \\
&\leq \|k(t) - k(s)\| + |t - s| \cdot \sup_{t \leq v \leq s} |k(v)| \cdot \|A\| M \cdot e^{\|A\| M s},
\end{aligned}
$$

so we can conclude that $\lim\limits_{t \to s} \|R(t) - R(s)\| = 0$. This proves the theorem. $\qquad \square$

4. (a, k)-REGULARIZED RESOLVENT FAMILIES ON L^∞ AND SIMILAR SPACES

In this section we will assume that a and k are exponentially bounded and hence Laplace transformable. Moreover, we suppose that $\hat{k}(\lambda) \neq 0$ and $\hat{a}(\lambda) \neq 0$ for all λ sufficiently large. We prove the following Theorem, which shows that in certain classes of Banach spaces the uniform continuity for (a, k)-regularized resolvent families is automatic. This generalizes previous results of Lotz [24], the first author [20] and also provides new results for other classes of families. See the corollaries below.

Theorem 4.1. *Let X be a Grothendieck space with the Dunford-Pettis property. Suppose that A generates an exponentially bounded (a, k)-regularized resolvent family $\{R(t)\}_{t \geq 0}$ on X. Then $\{R(t)\}_{t \geq 0}$ is uniformly continuous on X.*

Proof. By hypothesis, we have that $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$, the resolvent set of A, for all λ sufficiently large, say $\lambda > \omega_0$. Define $T(\lambda) := (I - \hat{a}(\lambda)A)^{-1}$ for all $\lambda > \omega_0$. Observe that $\lim_{\lambda\to\infty} |\hat{a}(\lambda)| = 0$ implies $\lim_{\lambda\to\infty}(I - \hat{a}(\lambda)A)^{-1}x = x$ for all $x \in X$. Therefore, according to the uniform boundedness principle, we obtain that there exists $M_0 > 0$ such that for all $\lambda > \omega_0$ we have $\|T(\lambda)\| \le M_0$. Hence from the identity

$$\hat{a}(\mu)(T(\lambda) - T(\mu)) = T(\lambda)(T(\mu) - I)(\hat{a}(\lambda) - \hat{a}(\mu)),$$

we obtain

$$
\begin{aligned}
\|\hat{a}(\lambda)T(\lambda)(T(\mu) - I)\| &\le \|\hat{a}(\mu)T(\lambda)(T(\mu) - I)\| + \|\hat{a}(\mu)(T(\lambda) - T(\mu))\| \\
&\le |\hat{a}(\mu)| \cdot \big(\|T(\lambda)(T(\mu) - I)\| + \|(T(\lambda) - T(\mu))\|\big) \\
&\le |\hat{a}(\mu)| \cdot \big(\|T(\lambda)\|\|T(\mu)\| + 2\|T(\lambda)\| + \|T(\mu)\|\big) \\
&\le |\hat{a}(\mu)|(M_0^2 + 3M_0).
\end{aligned}
$$

Therefore

$$\|T(\lambda)(T(\mu) - I)\| \le \frac{|\hat{a}(\mu)|}{|\hat{a}(\lambda)|}(M_0^2 + 3M_0),$$

and we obtain

$$\lim_{\mu\to\infty} \|T(\lambda)(T(\mu) - I)\| = 0, \text{ for all } \lambda > \omega_0.$$

From Theorem 2.5, this implies that there exists $\lambda_2 > \omega_0$ such that $T(\lambda_2)$ is invertible on X, that is, $T(\lambda_2)^{-1} \in B(X)$. Therefore, A is a bounded operator and, by Theorem 3.2, we conclude that the family $\{R(t)\}_{t\ge0}$ is uniformly continuous. $\square$

In case $a(t) = k(t) = 1$ we recover the following result due to Lotz [24].

Corollary 4.2. *Let X be a Grothendieck space with the Dunford-Pettis property. Then every strongly continuous one-parameter semigroup of operators on X is uniformly continuous.*

In case $k(t) = 1$ and $a(t)$ a Laplace transformable kernel, we recover [20, Theorem 3.2].

Corollary 4.3. *Let X be a Grothendieck space with the Dunford-Pettis property. Then every strongly continuous resolvent family of operators on X is uniformly continuous.*

Now, we consider for $\alpha > 0$ the function $a(t) = t^{\alpha-1}/\Gamma(\alpha)$ defined for $t > 0$, where $\Gamma(\cdot)$ denotes the Gamma function. The Laplace transform is $\hat{a}(\lambda) = \lambda^{-\alpha}$. Consider the fractional abstract Cauchy problem

$$(4.1) \qquad\qquad D_t^\alpha u(t) = Au(t), \quad 0 < \alpha \le 1,$$

with initial condition $u(0) = u_0$ where A is a closed and linear operator defined on a Banach space X, and D_t^α denotes the Caputo fractional derivative. Recall that if A generates a $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, 1)$-resolvent family $\{S_\alpha(t)\}_{t\ge0}$, then the solution of (4.1) is given by $u(t) = S_\alpha(t)u_0$ whenever $u_0 \in D(A)$. An analogous result holds for $\alpha > 1$, see [4].

Corollary 4.4. *Let X be a Grothendieck space with the Dunford-Pettis property. Let $\alpha > 0$ and suppose that A generates an exponentially bounded $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, 1)$-resolvent family $\{S_\alpha(t)\}_{t\ge0}$ on X. Then $\{S_\alpha(t)\}_{t\ge0}$ is uniformly continuous on X.*

5. (a, k)-REGULARIZED RESOLVENT FAMILIES WITH COMPACT GENERATOR

In section 3, it was proved that if an operator A is bounded, then the (a, k)-resolvent family generated by A is given by

$$R(t) = k(t)I + \sum_{n=1}^{\infty} A^n (a^{*n} * k)(t), \ t > 0.$$

In this section, we develop some aspects of (a, k)-regularized resolvent families of bounded linear operators on a Banach space which have the property of being near $k(t)$ times the identity (i.e., $R(t) - k(t)I$ is compact for some positive value of t). First results on such property are due to Cuthbert [9], Henríquez [14], Lutz [25] and Lizama [20].

The following result generalizes all the above mentioned papers.

Theorem 5.1. *Let $\{R(t)\}_{t \geq 0}$ be an strongly continuous (a, k)-resolvent family of type (M, ω) with generator A. Then the following assertions are equivalent:*

(i) $R(t) - k(t)I$ *is compact for all $t > 0$.*
(ii) A *is a compact operator.*

Proof. Suppose that A is compact. Since the set of compact operators is a closed subspace of $\mathcal{B}(X)$, we have by (3.2) that

$$R(t) - k(t)I = \sum_{n=1}^{\infty} k(s)^{*n} A^n = \lim_{N \to \infty} \sum_{n=1}^{N} k(s)^{*n} A^n,$$

and hence $R(t) - k(t)I$ is a compact operator.

Conversely, suppose that $R(t) - k(t)I$ is compact for all $t > 0$. According to the hypothesis we have that for all $Re(\lambda) > \omega$ the operator $(I - \hat{a}(\lambda)A)$ is invertible and

$$\int_0^{\infty} e^{-\lambda t} R(t) \, dt = \hat{R}(\lambda) = \hat{k}(\lambda)(I - \hat{a}(\lambda)A)^{-1}$$

for $Re(\lambda) > \omega$. For $x \in X$, define $H(\lambda)x := \hat{k}(\lambda)(I - \hat{a}(\lambda)A)^{-1}x$. We have

$$\begin{aligned}
\lambda H(\lambda)x - \lambda \hat{k}(\lambda)x &= \int_0^{\infty} \lambda e^{-\lambda t} R(t)x \, dt - \lambda \hat{k}(\lambda)x \\
&= \int_0^{\infty} \lambda e^{-\lambda t} R(t)x \, dt - \int_0^{\infty} \lambda e^{-\lambda t} k(t)x \, dt \\
&= \int_0^{\infty} \lambda e^{-\lambda t} (R(t) - k(t))x \, dt.
\end{aligned}$$

Hence by [31, Corollary 2.3], we obtain that $\lambda H(\lambda)x - \lambda \hat{k}(\lambda)x$ is a compact operator. From the identity

$$\lambda H(\lambda)x - \lambda \hat{k}(\lambda) = -\lambda \hat{k}(\lambda)\left(I - \frac{H(\lambda)}{\hat{k}(\lambda)}\right),$$

we obtain that $\left(I - \frac{H(\lambda)}{\hat{k}(\lambda)}\right)$ is a compact operator, and this implies that $\text{Ran}\left(\frac{H(\lambda)}{\hat{k}(\lambda)}\right)$ is closed. On the other hand, $\text{Ran}\left(\frac{H(\lambda)}{\hat{k}(\lambda)}\right) = D(A)$ is dense on X. Therefore $D(A) = \overline{D(A)} = X$ concluding

that A is a bounded operator. Next, we observe that the following identity holds

$$A = \left(\lambda H(\lambda)x - \lambda\hat{k}(\lambda)I\right)\frac{(I - \hat{a}(\lambda)A)}{\lambda\hat{k}(\lambda)\hat{a}(\lambda)},$$

and this implies that A is a compact operator. The proof is complete. □

6. A characterization of immediate norm continuity in Hilbert spaces.

Since reflexive spaces do not have the Dunford-Pettis property, we can not apply Theorem 4.1 to characterize the uniform continuity for $t > 0$ of (a, k)-regularized resolvent families in general Banach spaces. Of course, in this case the generator A is not necessarily bounded. This is one of the reasons why a characterization only in terms of the generator is desirable but unfortunately difficult to obtain in general Banach spaces. However, we can obtain a positive result extending an important result due to O. El Mennaoui and K.-J. Engel [10] valid for the case $a \equiv k \equiv 1$ to the case of (a, k)-regularized resolvent families in Hilbert spaces (see also [21] for the case of resolvent families)

Let A be a closed operator with domain $D(A)$ densely defined, $a \in L^1_{loc}(\mathbb{R}_+)$ and $k \in C(\mathbb{R}_+)$. Moreover, suppose a, k are 2-regular kernels.

Definition 6.1. *Let $k \in L^1_{loc}(\mathbb{R}_+)$ be Laplace transformable. We say that k is an admissible kernel if there exists $lim_{\lambda\to i\rho}\hat{k}(\lambda) = \hat{k}(i\rho)$ for all $|\rho| \geq 1$ and satisfies the following condition*

(H) there exists a constant $M > 0$ such that

$$\frac{1}{|\rho\,\hat{k}(i\rho)|} \leq M$$

for all $|\rho| \geq 1$.

Example 6.2. For instance, the function $k(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ is an admissible kernel for $0 < \alpha \leq 1$, but fails to be admissible for $\alpha > 1$. Moreover, is easy to check that $k(t)$ is 2-regular (see Definition 2.4).

To prove our main result in this section, we need the following lemmata. The first Lemma, corresponds to a general result for strongly continuous families of bounded operators.

Lemma 6.3. *Let $\{R(t)\}_{t\geq 0}$ be a strongly continuous family of type (M, σ) defined in a Hilbert space H. Then for any $x \in H$ and $\omega > \sigma$, $\|\hat{R}(\omega + iu)x\|$ and $\|\hat{R}(\omega + iu)^*x\|$ are in $L^2(\mathbb{R}, H)$, viewed as functions of $u \in \mathbb{R}$.*

Proof. Without loss of generality, we can suppose that $\sigma \geq 0$. Let $\omega > \sigma$ be given and define $R_1(t) := e^{-\omega t}R(t)$. Then $\|R_1(t)\| \leq Me^{-(\omega-\sigma)t}$ for $t \geq 0$. Let $x \in H$ be fixed, and note that $\chi_{[0,\infty)}(\cdot)R_1(\cdot)x$ is in $L^2(\mathbb{R}, H)$, where $\chi_{[0,\infty)}(\cdot)$ denotes the characteristic function. In fact, we have

$$(6.1) \qquad \|\chi_{[0,\infty)}(\cdot)R_1(\cdot)x\|_2^2 \quad \leq \quad \int_0^\infty \|Me^{-(\omega-\sigma)t}x\|^2\,dt \leq \frac{M^2\|x\|^2}{2(\omega - \sigma)}.$$

On the other hand, because $\{R(t)\}_{t\geq 0}$ is an exponentially bounded family of type (M, σ), its Laplace transform $\hat{R}(\lambda)$ is well-defined for all $Re(\lambda) > \sigma$ and is holomorphic there. Hence, we

have for all $x \in H$ and $s \in \mathbb{R}$

$$
\begin{aligned}
\hat{R}(\omega + is)x &= \int_0^\infty e^{-(\omega+is)t} R(t)x \, dt = \int_0^\infty e^{-ist} R_1(t)x \, dt \\
&= \int_{-\infty}^\infty e^{-ist} \chi_{[0,\infty)}(t) R_1(t)x \, dt = \mathcal{F}\big(\chi_{[0,\infty)}(\cdot) R_1(\cdot)\big)(s).
\end{aligned}
$$

It follows from (6.1) and the Plancherel theorem that $\hat{R}(\omega + i(\cdot))x \in L^2(\mathbb{R}, H)$. Analogously, we can prove that $\hat{R}(\omega + i(\cdot))^* x \in L^2(\mathbb{R}, H)$. This proves the Lemma. $\qquad\square$

Lemma 6.4. *Let* $a, k \in L_{loc}^1(\mathbb{R}_+)$ *be Laplace transformable and* A *be a closed linear operator defined on a Banach space* X. *Assume that* $H(\lambda) := \hat{k}(\lambda)(I - \hat{a}(\lambda)A)^{-1}$ *exists for all* $Re(\lambda) > \omega$. *Then there are functions* $f_i(\lambda)$, $i = 1, 2$ *and* $h_j(\lambda)$, $j = 1, 2, 3$ *such that*

(1) $H'(\lambda) = f_1(\lambda)H(\lambda) + f_2(\lambda)H(\lambda)^2$,
(2) $H''(\lambda) = h_1(\lambda)H(\lambda) + h_2(\lambda)H(\lambda)^2 + h_3(\lambda)H(\lambda)^3$

for all $Re(\lambda) > \omega$.

Proof. A computation shows that for all $Re(\lambda) > \omega$, we have

$$
(6.2) \qquad f_1(\lambda) = \frac{\hat{k}'(\lambda)}{\hat{k}(\lambda)} - \frac{\hat{a}'(\lambda)}{\hat{a}(\lambda)}, \qquad f_2(\lambda) = \frac{\hat{a}'(\lambda)}{\hat{k}(\lambda)\hat{a}(\lambda)},
$$

and

$$
(6.3) \qquad h_1(\lambda) = \frac{\hat{k}''(\lambda)}{\hat{k}(\lambda)} - \frac{2\hat{k}'(\lambda)\hat{a}'(\lambda)}{\hat{k}(\lambda)\hat{a}(\lambda)} + \frac{2\hat{a}'(\lambda)^2}{\hat{a}(\lambda)^2} - \frac{\hat{a}''(\lambda)}{\hat{a}(\lambda)},
$$

$$
(6.4) \qquad h_2(\lambda) = \frac{\hat{a}'(\lambda)\hat{k}'(\lambda)}{\hat{k}(\lambda)^2\hat{a}(\lambda)} - \frac{4\hat{a}'(\lambda)^2}{\hat{k}(\lambda)\hat{a}(\lambda)^2} + \frac{\hat{a}''(\lambda)}{\hat{k}(\lambda)\hat{a}(\lambda)},
$$

$$
(6.5) \qquad h_3(\lambda) = \frac{2\hat{a}'(\lambda)^2}{\hat{k}(\lambda)^2\hat{a}(\lambda)^2}.
$$

$\qquad\square$

Lemma 6.5. *Let* $a, k \in L_{loc}^1(\mathbb{R}_+)$ *be Laplace transformable and 2-regular, and suppose that* k *is an admissible kernel. Then, there exists a constant* $M > 0$ *such that*

(1) $|\lambda f_1(\lambda)| < M$ *and* $|f_2(\lambda)| < M$ *for all* $Re(\lambda) > \omega$;
(2) $\sup_{|\tau| \geq N} |h_3(s + i\tau)| < M$ *for all* $s > \omega$ *and* $N \geq 1$;
(3) $\int_{|\tau| \geq N} |h_j(s + i\tau)|^j \, d\tau < M$ *for all* $s > \omega$ *and* $N \geq 1$, $j = 1, 2$.

Proof. Is a direct consequence of formulas (6.2) - (6.5). $\qquad\square$

Lemma 6.6. [10] *Let* X *be a Banach space and let* $R : [0, \infty) \to X$ *be a function which is continuous for* $t > 0$. *If there exist* $M > 0, \omega \in \mathbb{R}$ *such that* $\|R(t)\| \leq Me^{\omega t}$, *then*

$$
\lim_{|\mu| \to \infty} \|\hat{R}(\mu_0 + i\mu)\| = 0
$$

for every $\mu_0 > \omega$.

Our main result in this section is the following characterization. It extends the main result in [21, Theorem 2.2] proved in the case $k \equiv 1$. See also [32] for the same characterization in case of C_0-semigroups, i.e. $k \equiv 1$ and $a \equiv 1$.

Theorem 6.7. *Let A be a closed linear operator defined in a Hilbert space H with dense domain $D(A)$. Assume that A generates an strongly continuous (a, k)-regularized resolvent $R(t)$ of type (M, ω), with $M > 0, \omega \in \mathbb{R}$, $a \in L^1_{loc}(\mathbb{R}_+)$ and $k \in C(\mathbb{R}_+)$ Laplace transformable. Assume that a and k are 2-regular kernels, and that k is admissible. Then the following conditions are equivalent:*

(a) $\{R(t)\}_{t \geq 0}$ *is continuous in $\mathcal{B}(H)$ for $t > 0$,*

(b) $\displaystyle\lim_{|\tau| \to \infty} \|\hat{k}(s + i\tau)\left(I - \hat{a}(s + i\tau)A\right)^{-1}\| = 0$ *for some $s > \omega$.*

Proof. $(a) \Rightarrow (b)$. Follows from Lemma 6.6. $(b) \Rightarrow (a)$. Let $x \in H$ be fixed and $\mu > \omega$. Because $\|R(t)e^{-\mu t}x\| \leq Me^{-(\mu - \omega)t}\|x\|$, the function $t \mapsto \chi_{[0,\infty)}(t)R(t)e^{-\mu t}$ is in $L^2(\mathbb{R}, H)$ for all $\mu > \omega$ (compare the inequality (6.1)). Since H is a Hilbert space, by Plancherel theorem is well known that the Fourier transform is an unitary operator on $L^2(\mathbb{R}, H)$, thus we obtain

$$\mathcal{F}\left(\chi_{[0,\infty)}(\cdot)R(\cdot)e^{-\mu \cdot}x\right) = \hat{R}(\mu + i\tau)x,$$

and hence

$$(6.6) \qquad R(t)e^{-\mu t}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau t}\hat{R}(\mu + i\tau)x\, d\tau.$$

for $t > 0$ and each $x \in H$. Clearly, the resolvent $R(t)$ is continuous in $\mathcal{B}(H)$ for $t > 0$ if and only if $R(t)e^{-\mu t}$ is continuous in $\mathcal{B}(H)$ for $t > 0$. Next, note that for each $x \in H$ we have

$$\hat{R}(\mu + i\tau)x = \hat{k}(\mu + i\tau)\left(I - \hat{a}(\mu + i\tau)A\right)^{-1}x,$$

and observe that if $|\tau| \to \infty$, then we get $\hat{a}(\mu + i\tau) \to 0$ and $\hat{k}(\mu + i\tau) \to 0$, whence $\lim_{|\tau| \to \infty} \hat{R}(\mu + i\tau)x = 0$. Applying this to (6.6) and integrating by parts we have

$$R(t)e^{-\mu t}x = \frac{-1}{2\pi t}\int_{-\infty}^{\infty} e^{i\tau t}\hat{R}'(\mu + i\tau)x\, d\tau.$$

for $t > 0$ and $x \in H$. Now, by Lemma 6.4 (1) and Lemma 6.5 (1) with $H(\lambda) := \hat{R}(\lambda)$ we also have $\lim_{|\tau| \to \infty} \hat{R}'(\mu + i\tau) = 0$, whence, again integrating by parts, we get

$$(6.7) \qquad R(t)e^{-\mu t}x = \frac{1}{2\pi t^2}\int_{-\infty}^{\infty} e^{i\tau t}\hat{R}''(\mu + i\tau)x\, d\tau, \quad x \in H, \ t > 0.$$

Next we show that the operator family $\{t^2 R(t)e^{-\mu t}\}_{t>0}$ is continuous in $\mathcal{B}(H)$ for $t > 0$. Indeed, formula (6.7) shows

$$\|t^2 R(t)e^{-\mu t}x - s^2 R(s)e^{-\mu s}x\| \ = \ \frac{1}{\pi}\left\|\int_{-\infty}^{\infty}(e^{i\tau t} - e^{i\tau s})\hat{R}''(\mu + i\tau)x\, d\tau\right\|$$

$$\leq \ \frac{1}{\pi}\left\|\int_{|\tau| \geq N}(e^{i\tau t} - e^{i\tau s})\hat{R}''(\mu + i\tau)x\, d\tau\right\| + \frac{1}{\pi}\int_{|\tau| \leq N}|e^{i\tau t} - e^{i\tau s}|\|\hat{R}''(\mu + i\tau)x\|\, d\tau$$

$$=: \ I_1(N) + I_2(N).$$

We estimate $I_1(N)$: Let $\epsilon > 0$ and take $x^* \in H$. Using Lemma 6.4 (2), and the Cauchy-Schwarz and Hölder inequalities, we have

$$
\begin{aligned}
|\langle I_1(N), x^* \rangle| \quad \leq \quad & \sup_{|\tau| \geq N} \|\hat{R}(\mu + i\tau)\| \cdot \Bigg(\int_{|\tau| \geq N} |h_1(\mu + i\tau)| \, d\tau \cdot \|x\| \|x^*\| \\
& + \Big(\int_{|\tau| \geq N} \|\hat{R}(\mu + i\tau) x\|^2 \, d\tau \Big)^{1/2} \Big(\int_{|\tau| \geq N} \|h_2(\mu + i\tau) x^*\|^2 \, d\tau \Big)^{1/2} \\
& + \sup_{|\tau| \geq N} \|h_3(\mu + i\tau)\| \cdot \\
& \Big(\int_{|\tau| \geq N} \|\hat{R}(\mu + i\tau) x\|^2 \, d\tau \Big)^{1/2} \Big(\int_{|\tau| \geq N} \|\hat{R}(\mu + i\tau)^* x^*\|^2 \, d\tau \Big)^{1/2} \Bigg)
\end{aligned}
$$

Next, by the Plancherel Theorem for the Hilbert space valued Fourier transform, and Lemma 6.5 items (2) and (3), we get

$$
\begin{aligned}
|\langle I_1(N), x^* \rangle| \quad \leq \quad & \sup_{|\tau| \geq N} \|\hat{R}(\mu + i\tau)\| \left(M \cdot \|x\| \|x^*\| + \Big(2\pi \int_0^\infty \|e^{-\mu t} R(t) x\|^2 \, dt \Big)^{1/2} M \|x^*\| \right. \\
& \left. + M \cdot \Big(2\pi \int_0^\infty \|e^{-\mu t} R(t) x\|^2 \, dt \Big)^{1/2} \Big(2\pi \int_0^\infty \|e^{-\mu t} R(t)^* x^*\|^2 \, dt \Big)^{1/2} \right)
\end{aligned}
$$

Since $(R(t))_{t \geq 0}$ is of type (M, ω), we have that the families $\{e^{-\mu t} R(t)\}_{t>0}$ and $\{e^{-\mu t} R(t)^*\}_{t>0}$ are exponentially bounded of type $(M, \omega - \mu)$, and that there exists a positive constant $C > 0$ such that

$$
\int_0^\infty \|e^{-\mu t} R(t) x\|^2 \, dt \leq C^2 \|x\|, \qquad \int_0^\infty \|e^{-\mu t} R(t)^* x^*\|^2 \, dt \leq C^2 \|x\|.
$$

Combining the above with the Hahn - Banach Theorem, we obtain the existence of a constant $K > 0$ such that

$$
\begin{aligned}
I_1(N) \quad &= \quad \sup_{\|x^*\| \leq 1} |\langle \int_{|\tau| \geq N} (e^{i\tau t} - e^{i\tau s}) \hat{R}''(\mu + i\tau) x \, d\tau, x^* \rangle| \\
&\leq \quad K \cdot \sup_{|\tau| \geq N} \|\hat{R}(\mu + i\tau)\| \|x\|.
\end{aligned}
$$

Since $\lim_{|\tau| \to \infty} \|\hat{R}(\mu + i\tau)\| = 0$, there exists $N > 0$ such that

$$
K \cdot \sup_{|\tau| \geq N} \|\hat{R}(\mu + i\tau)\| < \epsilon
$$

which yields the estimate $I_1(N) < \epsilon \|x\|$ for each $x \in H$.

For estimate $I_2(N)$, we observe that $|e^{i\alpha} - 1|^2 = 4\sin^2(\alpha/2), \alpha \in \mathbb{R}$. Therefore, for the above fixed N we have

$$
\begin{aligned}
I_2(N) &= \int_{|\tau| \leq N} |e^{i\tau t} - e^{i\tau s}| \|\hat{R}''(\mu + i\tau)x\| \, d\tau \\
&\leq \Big(\int_{|\tau| \leq N} |e^{i\tau(t-s)} - 1|^2 \, d\tau \Big)^{1/2} \Big(\int_{|\tau| \leq N} \|\hat{R}''(\mu + i\tau)x\|^2 \, d\tau \Big)^{1/2} \\
&= \Big(4 \int_{|\tau| \leq N} \Big| \sin^2\Big(\frac{(s-t)\tau}{2} \Big) \Big| \Big)^{1/2} \Big(\int_{|\tau| \leq N} \|\hat{R}''(\mu + i\tau)x\|^2 \, d\tau \Big)^{1/2} \\
&\leq \Big(\int_{|\tau| \leq N} |\tau|^2 |s-t|^2 \, d\tau \Big)^{1/2} \Big(\int_{|\tau| \leq N} \|\hat{R}''(\mu + i\tau)x\|^2 \, d\tau \Big)^{1/2} \\
&\leq |s-t| \Big(\frac{2N^3}{3} \Big)^{1/2} \Big(\int_{|\tau| \leq N} \|\hat{R}''(\mu + i\tau)x\|^2 \, d\tau \Big)^{1/2}.
\end{aligned}
$$

Since $\hat{R}''(\mu + i\tau)$ is a continuous function and the integral is defined over a compact subset of $\mathbb{R}$, there exists a constant $C' > 0$ such that $\|\hat{R}''(\mu + i\tau)x\| \leq C'\|x\|$; this implies

$$
I_2(N) \leq |s-t| \Big(\frac{2N^3}{3} \Big)^{1/2} \cdot (2N)^{1/2} C' \|x\| \leq |s-t| K' N^2 \|x\|.
$$

Using these estimates for $I_1(N), I_2(N)$, yields

$$
\|t^2 R(t)e^{-\mu t} - s^2 R(s)e^{-\mu s}\| < 2\epsilon,
$$

for all $|s-t| < \delta$. This completes the proof. $\qquad\square$

We finish this paper with a direct application to results of Fan [11].

Corollary 6.8. *Let A be a closed linear operator defined in a Hilbert space H with dense domain $D(A)$. Assume that A generates an α-regularized resolvent $S_\alpha(t)$ of type (M, ω) for some $0 < \alpha < 1$ and suppose $\lim_{|\tau| \to \infty} \|(s + i\tau)^{\alpha-1}((s + i\tau)^\alpha - A)^{-1}\| = 0$ for some $s > \omega$. Then $S_\alpha(t)$ is compact for $t > 0$ if and only if $(\lambda^\alpha - A)^{-1}$ is compact for all $\lambda^\alpha \in \rho(A)$.*

Remark 6.9. *In a Hilbert space, all the results on Section 4 of Fan's paper [11] remain true when the hypothesis on the given operator A, as generator of an analytic compact α-regularized resolvent $(0 < \alpha < 1)$, is replaced by*

$$
\lim_{|\tau| \to \infty} \|(s + i\tau)^{\alpha-1}((s + i\tau)^\alpha - A)^{-1}\| = 0 \text{ for some } s > \omega.
$$

Remark 6.10. *In case $a \equiv k \equiv 1$, it is known that the characterization obtained in Theorem 6.7 cannot be extended to Banach spaces. See [7] and [26], for instance. However, we can naturally ask: There exists a class of kernels $(a, k) \neq (1, 1)$ where this characterization remains true in general Banach spaces?.*

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