SOLUTIONS OF ABSTRACT INTEGRO-DIFFERENTIAL EQUATIONS VIA POISSON TRANSFORMATION

CARLOS LIZAMA AND RODRIGO PONCE

Abstract. We study the initial value problem

$$(*)\begin{cases} \mathbf{u}(n+1)-\mathbf{u}(n) &= A\mathbf{u}(n+1)+\displaystyle\sum_{k=0}^{n+1}\mathbf{a}(n+1-k)A\mathbf{u}(k), \quad n\in\mathbb{N}_0 \\ \mathbf{u}(0) &= x, \end{cases}$$

where A is closed linear operator defined on a Banach space X, x belongs to the domain of A and the kernel $\mathbf{a}$ is a particular discretization of an integrable kernel $a\in L^1(\mathbb{R}_+)$. Assuming that A generates a resolvent family, we find an explicit representation of the solution to the initial value problem $(*)$ as well as for its inhomogeneous version, and then we study the stability of such solutions. We also prove that for a special class of kernels $\mathbf{a}$, it suffices to assume that A generates an immediately norm continuous C_0-semigroup. We employ a new computational method based on the Poisson transformation.

1. Introduction

Consider the homogeneous initial value problem

$$\begin{cases} u'(t) &= Au(t)+\displaystyle\int_0^t a(t-s)Au(s)ds, \; t\geq 0 \\ u(0) &= x, \end{cases} \tag{1.1}$$

where A is a closed linear operator defined in a Banach space X, x belongs to X and the function $a:[0,\infty)\to\mathbb{R}$, known as *the relaxation function*, is an integrable kernel. This kind of initial value problems arise, for instance, in the study of the heat conduction in material with fading memory or in some population models, see [9, 10, 22, 23, 27] for more details. In such applications, the operator A is typically the Laplacian operator or the elasticity operator and a typical choice of the relaxation function is $a(t)=\alpha\frac{t^{\mu-1}}{\Gamma(\mu)}e^{-\beta t}$ for $\alpha\in\mathbb{R}$ and $\mu,\beta>0$.

In order to solve abstract integro-differential equations in the form of (1.1), Da Prato and Ianelli introduced in [6] the concept of *resolvent family* $\{S(t)\}_{t\geq 0}$ generated by the operator A (see Section 2 below for its definition). See also the monograph of J. Prüss [23]. By using this concept, the *mild solution* to (1.1) can be written as

$$u(t):=S(t)x. \tag{1.2}$$

We notice here that the mild solution (1.2) is a classical solution to (1.1) if $x\in D(A)$. Now, if we consider the inhomogeneous initial problem

$$\begin{cases} u'(t) &= Au(t)+\displaystyle\int_0^t a(t-s)Au(s)ds+f(t), \; t\geq 0 \\ u(0) &= x, \end{cases} \tag{1.3}$$

2010 *Mathematics Subject Classification.* Primary 65M22; 45N05; Secondary 45D05; 39A14; 47D06.

Key words and phrases. Volterra integral equations; C_0-Semigroups, Poisson transformation; resolvent families of operators.

The first author is partially supported by CONICYT under FONDECYT Grant number 1180041.

where A, a and x are as before and the forcing term f is a continuous function, then the mild solution to problem (1.3) is given by

$$u(t) = S(t)x + \int_0^t S(t-r)f(r)dr. \tag{1.4}$$

Time discretizations of inhomogeneous integro-differential equations with memory terms of convolution type have been considered by several authors in the past decades. We mention here some few works. In [24] the authors considered the equation (1.3) with A being an unbounded positive-definite self-adjoint operator with dense domain in a Hilbert space and the operator A in the convolution term in (1.3) is taken as a different operator B with domain $D(B) \supset D(A)$ and a and f are smooth functions.

More recently, in [20] was considered the equation (1.3) where A is a closed linear operator in a complex Banach space satisfying the resolvent estimate $\|(z-A)^{-1}\| \le M_\delta/(1+|z|)$, for $z \in \Sigma_\delta := \{z \neq 0, |\arg(z)| < \delta\} \cup \{0\}$ for some $\delta \in (\frac{1}{2}\pi, \pi)$, where M_δ is a positive constant and the kernel a satisfies some suitable assumptions. A typical example of such kernels is $a(t) = \kappa e^{-\nu t}$ with $\kappa \in \mathbb{R}$ and $\nu \ge 0$, see [20, Section 2].

In the paper [17] the authors study discretizations in time for the integro-differential equation

$$\begin{cases} u'(t) &= \displaystyle\int_0^t a(t-s)Au(s)ds + f(t), \ t \ge 0 \\ u(0) &= x, \end{cases} \tag{1.5}$$

in the context of real Hilbert spaces and self-adjoint positive-definite linear operators A. In Banach spaces, the authors in [18] study time discretizations for the abstract equation (1.5) where a is the weakly singular function $a(t) = t^{\alpha-1}/\Gamma(\alpha)$, $0 < \alpha < 1$ and the operator A satisfies the same conditions as in [19, 20] and [21]. In contrast, the case $1 < \alpha < 2$ is studied in [5] by assuming that A is an operator of sectorial type.

Unfortunately, it is well known that the numerical discretization result in cumulate errors even in shortterm domains. It becomes difficult in the realworld applications for longterm issues from the practical point of view. This paper aims to address this problem and propose a new kind of discretetime Volterra equation by use of the Poisson transformation on an isolated time scale.

We study abstract Volterra difference equations having the form

$$\mathbf{u}(n+1) - \mathbf{u}(n) = A\mathbf{u}(n+1) + \sum_{k=0}^{n+1} \mathbf{a}(n+1-k)A\mathbf{u}(k), \quad n \in \mathbb{N}_0, \tag{1.6}$$

where the scalar sequence $\mathbf{a}(n)$ is given in terms of the so-called Poisson transformation [13], and is defined as follows

$$\mathbf{a}(n) := \mathcal{P}(a)(n) := \int_0^\infty p_n(t)a(t)dt, \quad p_n(t) := e^{-t}\frac{t^n}{n!}, \quad n \in \mathbb{N}_0,$$

where $a(t)$ is such that the integral in the right hand side exists. Equation (1.6) arises by application of the Poisson transformation to the equation (1.1). The Poisson transformation constitutes a new method of sampling. It was introduced in [13] in the context of fractional abstract difference equations and their main properties studied in [1, Section 4].

One of the main novelties in this paper is that we consider A as the generator of a resolvent family $\{S(t)\}_{t\ge 0}$ defined on a Banach space X. In this way, we allow unbounded operators A and, consequently, our results are applicable to mixed Volterra equations, i.e. where the time variable is discrete but the space variable is continuous.

It is remarkable that Volterra difference equations in the form (1.6) appears very recently in discrete chaos [4, 25] and in the analysis of discretetime neural networks [26, Formula (10)]. In such cases, $\mathbf{a}(n) = k^\mu(n) := \frac{\Gamma(n+\mu)}{\Gamma(\mu)n!}$ are the Cesáro numbers [13]. We notice that this sequence has fundamental importance in the recent theory of fractional sum and fractional difference operators of order μ. An account of the main properties of k^μ can be found in the recent paper [8]. We recall that in the finite dimensional case, i.e. $dim(X) < \infty$, the study of Volterra difference equations in the form (1.6), with A being a matrix, dates back to many years ago. See e.g. [7] and references therein.

The first and main contributions in this paper shows that if the initial value $\mathbf{u}(0) = x$ belongs to $D(A)$, then an explicit classical solution of the problem (1.6) is given by:

$$\mathbf{u}(n) = \int_0^\infty p_n(t)S(t)(I - (1+\mathbf{a}(0))A)xdt, \quad n \in \mathbb{N}.$$

See Theorem 3.4. Concerning the important issue of stability of the solutions, we consider the special case:

$$\mathbf{a}(n) = \frac{\alpha}{(\beta+1)^{\mu+n}}k^\mu(n), \quad n \in \mathbb{N}_0,$$

where $\alpha \in \mathbb{R}, \mu, \beta > 0$. A second contribution is that assuming the following conditions in the parameters of the kernel $\mathbf{a}$ and on the operator A :

(S1) $\alpha \neq 0$, $\beta > 0$ and $\mu \geq 1$ such that $\alpha + \beta^\mu > 0$;
(S2) A generates an immediately norm continuous C_0-semigroup on a Banach space X;
(S3) $\sup\{\text{Re}\lambda, \lambda \in \mathbb{C} : \lambda(\lambda+\beta)^\mu((\lambda+\beta)^\mu + \alpha)^{-1} \in \sigma(A)\} < 0$;

we prove that the solution $\mathbf{u}(n)$ of (1.6) exists and satisfies $\|\mathbf{u}(n)\| \to 0$ as $n \to \infty$. See Theorem 4.7 below.

The plan of the paper is the following: In Section 2 we give some preliminaries on resolvent families and their main properties. In Section 3, we study the Poisson discretization for the equations (1.1) and (1.3). In Section 4 we show our main findings on the stability of the solutions, giving explicit representations in the special case $A = \rho I$ where $\rho \in \mathbb{R}$ and I is the identity operator.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space. We denote by $\mathcal{B}(X)$ the space of all bounded and linear operators from X into X. If A is a closed linear operator on X we denote by $\rho(A)$ the resolvent set of A and $R(\lambda, A) = (\lambda - A)^{-1}$ the resolvent operator of A defined for all $\lambda \in \rho(A)$. The spectrum of A is defined by $\sigma(A) := \mathbb{C} \setminus \rho(A)$. A family of operators $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be *exponentially bounded* if there exist real numbers $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\| \leq Me^{\omega t}, \quad t \geq 0. \tag{2.7}$$

We notice that if $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is exponentially bounded, then the Laplace transform of $S(t)$, namely

$$\hat{S}(\lambda)x := \int_0^\infty e^{-\lambda t}S(t)xdt$$

exists for all $\text{Re}\lambda > \omega$. Observe that if $\omega = 0$, then $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is bounded. On the other hand, if the constant ω in (2.7) is strictly negative, then $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called *uniformly exponentially stable.*

Let $a \in L^1_{loc}(\mathbb{R}_+)$ be given. In this paper, we will assume that the Laplace transform of a exists and satisfies $1 + \hat{a}(\lambda) \neq 0$ for all $\text{Re}\lambda \geq 0$.

Definition 2.1. *Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X, $a \in L^1_{loc}(\mathbb{R}_+)$ be Laplace transformable and $b(t) := 1 + \int_0^t a(s)ds$. We say that the pair (b, A) is the generator of a* resolvent family *if there exist $\omega \geq 0$ and a strongly continuous function $S : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\frac{1}{\hat{b}(\lambda)}\}_{\text{Re}\lambda>\omega} \subseteq \rho(A)$ and*

$$\frac{1}{\lambda\hat{b}(\lambda)}\left(\frac{1}{\hat{b}(\lambda)} - A\right)^{-1} x = \int_0^\infty e^{-\lambda t}S(t)xdt, \quad \text{Re}\lambda > \omega, \quad x \in X.$$

In this case the family $\{S(t)\}_{t\geq 0}$ is called a resolvent family generated by A.

From the general theory [23] it is easy to see that if a resolvent family exists then it is unique. We notice that by the uniqueness of the Laplace transform, if $a(t) = 0$ for all $t \geq 0$, then the resolvent family $\{S(t)\}_{t\geq 0}$ is the same as a C_0-semigroup. On the other hand, if $k(t) = 1$ and $b(t) := 1 + \int_0^t a(s)ds$, then the resolvent family $\{S(t)\}_{t\geq 0}$ is a particular case of the theory of (b, k)-regularized families defined in [12].

It is easy to show (see [23] or [12, Lemma 2.2 and Proposition 3.1])) that if A generates a resolvent family $\{S(t)\}_{t\geq 0}$ and $b(t) := 1 + \int_0^t a(s)ds$, then it satisfies the following properties:

(1) $S(0) = I$.

(2) If $x \in D(A)$, then
$$S(t)x = x + \int_0^t b(t-r)S(r)Axdr.$$

(3) For all $x \in X$, $\int_0^t S(r)xdr \in D(A)$, and
$$S(t)x = x + A\int_0^t b(t-r)S(r)xdr. \tag{2.8}$$

(4) For each $t \geq 0$, $S(t)x \in D(A)$ and $S(t)Ax = AS(t)x$, for all $x \in D(A)$ and $t \geq 0$.

We denote by $\mathbb{N}_0 := \{0, 1, 2, ...\}$, the set of non-negative integer numbers. For a given Banach space X, we denote by $s(\mathbb{N}_0, X)$ the vectorial space consisting of all vector-valued sequences $s : \mathbb{N}_0 \to X$. Let $f : \mathbb{R}_+ \to X$ be a bounded and locally integrable function. We recall from [13] (see also [1]) that the *Poisson transformation* of f is the vector-valued sequence defined by
$$\mathbf{f}(n) := \mathcal{P}(f)(n) := \int_0^\infty p_n(t)f(t)dt, \quad n \in \mathbb{N}_0,$$
where $p_n(t) := \frac{t^n e^{-t}}{n!}$ is the Poisson distribution. For a resolvent family $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$, we denote
$$\mathcal{S}(n)x := \mathcal{P}(S)(n)x := \int_0^\infty p_n(t)S(t)xdt, \quad n \in \mathbb{N}_0, x \in X.$$
If $c : \mathbb{R}_+ \to \mathbb{C}$ is a continuous and bounded function, we write
$$\mathbf{c}(n) := \mathcal{P}(c)(n) := \int_0^\infty p_n(t)c(t)dt, \quad n \in \mathbb{N}_0,$$
and the finite convolution of a scalar-valued sequence s and an operator-valued sequence Q is then defined as
$$(s \star Q)(n)x := \sum_{k=0}^{n} s(n-k)Q(k)x, \quad n \in \mathbb{N}_0, \quad x \in X.$$

For further use, we recall one of the following main results in [13].

Theorem 2.2. [13, Theorem 3.4] *Let $c : \mathbb{R}_+ \to \mathbb{C}$ be Laplace transformable such that $\hat{c}(1)$ exists, and let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be strongly continuous and Laplace transformable such that $\hat{S}(1)$ exists. Then for all $x \in X$,*
$$\mathcal{P}(c * S)(n)x = (\mathcal{P}(c) \star \mathcal{P}(S))(n)x, \quad n \in \mathbb{N}_0.$$

We recall the sequence
$$k^\alpha(n) := \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)\Gamma(n+1)} = \int_0^\infty p_n(t)g_\alpha(t)dt, \quad n \in \mathbb{N}_0, \quad \alpha > 0, \tag{2.9}$$
where $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$. This sequence has a strong connection with fractional difference equations, see for instance [1, 8, 13] and references therein for more details.

3. Solutions of Volterra difference equations in Banach spaces

Let A be a closed linear operator defined on a Banach space X. Consider the following initial value problem

$$\left\{\begin{array}{rcl}\mathbf{u}(n+1)-\mathbf{u}(n) &=& A\mathbf{u}(n+1)+\sum_{k=0}^{n+1}\mathbf{a}(n+1-k)A\mathbf{u}(k), \quad n\in\mathbb{N}_0\\ \mathbf{u}(0) &=& x,\end{array}\right. \tag{3.10}$$

We will assume the following hypothesis:
(H1) There exists $a(t)$ such that $\mathbf{a}=\mathcal{P}(a)$;
(H2) the pair (b,A), where $b(t):=1+\int_0^t a(s)ds$, is the generator of a resolvent family $\{S(t)\}_{t\geq 0}$ in X;
(H3) the initial value x belongs to $D(A)$.

Definition 3.3. *Given $x\in X$, we say that a vector valued sequence $\mathbf{u}\in s(\mathbb{N}_0,X)$ is a* classical solution *to* (3.10) *if $\mathbf{u}(0)=x$ and $\mathbf{u}$ satisfies the difference equation in* (3.10) *for all $n\in\mathbb{N}_0$.*

We begin with the following representation of the solutions for the linear problem (3.10).

Theorem 3.4. *Suppose that (H1), (H2) and (H3) hold. Then, the Volterra difference equation* (3.10) *admits the unique classical solution*

$$\mathbf{u}(n)=\int_0^\infty p_n(t)S(t)(I-(1+\mathbf{a}(0))A)xdt,\quad n\in\mathbb{N}.$$

Proof. Following [13, Theorem 4.4] we first prove that if $x\in X$, then $\mathcal{S}(n)x:=\mathcal{P}(S)(n)x\in D(A)$ for all $n\in\mathbb{N}_0$. In fact, if $x\in X$ and $n\in\mathbb{N}_0$, then by [2, Theorem 1.5.1]

$$\mathcal{S}(n)x=\int_0^\infty p_n(t)S(t)xdt=\frac{(-1)^n}{n!}[\hat{S}(\lambda)]^{(n)}x\mid_{\lambda=1},$$

where

$$\hat{S}(\lambda)x=\frac{1}{1+\hat{a}(\lambda)}\left(\frac{\lambda}{1+\hat{a}(\lambda)}-A\right)^{-1}x:=c(\lambda)(d(\lambda)-A)^{-1}x,$$

and the functions c and d are defined by $c(\lambda):=1/(1+\hat{a}(\lambda))$ and $d(\lambda):=\lambda/(1+\hat{a}(\lambda))$. Now, if $R(\lambda):=(\lambda-A)^{-1}$ then $\hat{S}(\lambda)=c(\lambda)R(d(\lambda))$ and by the Leibniz's rule for the n^{th}-derivative of a product we have

$$[\hat{S}(\lambda)]^{(n)}x=\sum_{k=0}^{n}\binom{n}{k}c(\lambda)^{(n-k)}[(R\circ d)(\lambda)]^{(k)}.$$

Moreover, by the rule for the n^{th}-derivative for the composition of functions, we obtain

$$[(R\circ d)(\lambda)]^{(k)}=\sum_{j=1}^{k}\frac{U_j(\lambda)}{j!}[R(\lambda)]^{(j)},$$

where

$$U_j(\lambda):=[d(\lambda)^j]^{(n)}-\frac{j}{1!}d(\lambda)[d(\lambda)^{j-1}]^{(n)}+\frac{j(j-1)}{2!}d(\lambda)^2[d(\lambda)^{j-2}]^{(n)}-...+(-1)^{j-1}jd(\lambda)^{j-1}[d(\lambda)]^{(n)}.$$

On the other hand, since $[R(\lambda)]^{(m)}x=(\lambda-A)^{-(m+1)}x$ for all $x\in X$ and $m\in\mathbb{N}_0$, we obtain $[R(\lambda)]^{(j)}x\mid_{\lambda=1}=(I-A)^{-(j+1)}x\in D(A)$ and therefore $[(R\circ d)(\lambda)]^{(k)}x\mid_{\lambda=1}\in D(A)$. We conclude that $[\hat{S}(\lambda)]^{(n)}x\mid_{\lambda=1}\in D(A)$ and thus $\mathcal{S}(n)x\in D(A)$ proving the claim.

Now, we take $x\in X$. Since, by (H2), $b(t)=1+\int_0^t a(r)dr$ we obtain from (H1) that $\mathbf{b}(n):=\mathcal{P}(b)(n)=1+\sum_{k=0}^{n}\mathbf{a}(k)=:1+(1\star\mathbf{a})(n)$, for each $n\in\mathbb{N}_0$. From the resolvent identity

$$S(t)x=x+A\int_0^t b(t-r)S(r)xdr,\quad t\geq 0,$$

and taking Poisson transformation, it follows by Theorem 2.2, that

$$\mathcal{S}(n)x = x + A(\mathbf{b}\star\mathcal{S})(n)x, \quad n\in\mathbb{N}_0. \tag{3.11}$$

Therefore, for $x\in X$ we obtain

$$\begin{aligned}\mathcal{S}(n)x &= x + A\sum_{k=0}^{n}\mathbf{b}(n-k)\mathcal{S}(k)x = x + A\sum_{k=0}^{n}(1+(1\star\mathbf{a}))(n-k)\mathcal{S}(k)x\\ &= x + A\sum_{k=0}^{n}\mathcal{S}(k)x + A\sum_{k=0}^{n}(1\star\mathbf{a})(n-k)\mathcal{S}(k)x = x + A\sum_{k=0}^{n}\mathcal{S}(k)x + A\sum_{k=0}^{n}(1\star\mathbf{a})(k)\mathcal{S}(n-k)x.\end{aligned}$$

Since $(1\star\mathbf{a})(n)=\sum_{k=0}^{n}\mathbf{a}(k)$, we obtain

$$\mathcal{S}(n)x = x + A\sum_{k=0}^{n}\mathcal{S}(k)x + A\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\mathbf{a}(j)\right)\mathcal{S}(n-k)x. \tag{3.12}$$

Now, by (3.12) we have

$$\mathcal{S}(n+1)x-\mathcal{S}(n)x = A\mathcal{S}(n+1)x + A\left[\sum_{k=0}^{n+1}\left(\sum_{j=0}^{k}\mathbf{a}(j)\right)\mathcal{S}(n+1-k)x - \sum_{k=0}^{n}\left(\sum_{j=0}^{k}\mathbf{a}(j)\right)\mathcal{S}(n-k)x\right].$$

And a simply computation shows that

$$\sum_{k=0}^{n+1}\left(\sum_{j=0}^{k}\mathbf{a}(j)\right)\mathcal{S}(n+1-k)x - \sum_{k=0}^{n}\left(\sum_{j=0}^{k}\mathbf{a}(j)\right)\mathcal{S}(n-k)x = \sum_{k=0}^{n+1}\mathbf{a}(k)\mathcal{S}(n+1-k) = (\mathbf{a}\star\mathcal{S})(n+1)x,$$

which implies that for $x\in X$ we have $\mathcal{S}(n+1)x-\mathcal{S}(n)x = A\mathcal{S}(n+1)x+(\mathbf{a}\star A\mathcal{S})(n+1)x$, for all $n\in\mathbb{N}_0$. We conclude that $\mathcal{S}(n)x$ verifies the difference equation in (3.10).

Now, if $x\in D(A)$ we define $\mathbf{u}(n):=\mathcal{S}(n)(I-\mathbf{b}(0)A)x$, where $\mathbf{b}(0)=1+\mathbf{a}(0)$. It then follows that $\mathbf{u}(n)\in D(A)$ for all $n\in\mathbb{N}$ and $\mathbf{u}(n)$ solves the difference equation in (3.10). On the other hand, from the identity $\mathcal{S}(0)x = x + A(\mathbf{b}\star\mathcal{S})(0)x = x+\mathbf{b}(0)A\mathcal{S}(0)x$, it follows that $\mathbf{u}(0)=\mathcal{S}(0)(I-\mathbf{b}(0)A)x = x$, which means that $\mathbf{u}(n)$ is solution to the Volterra difference equation (3.10). Finally, the uniqueness of $\mathbf{u}$ follows from the uniqueness of the resolvent family $\{S(t)\}_{t\geq 0}$, see [23, Chapter I, Section 1]. □

As a consequence, we prove the following result for the non homogeneous problem.

Theorem 3.5. *Suppose that (H1), (H2) and (H3) hold. Then, the following Volterra difference equation*

$$\begin{cases}\mathbf{u}(n+1)-\mathbf{u}(n) &= A\mathbf{u}(n+1)+\displaystyle\sum_{k=0}^{n+1}\mathbf{a}(n+1-k)A\mathbf{u}(k)+\mathbf{f}(n+1), \quad n\in\mathbb{N}_0\\ \mathbf{u}(0) &= x,\end{cases} \tag{3.13}$$

where $\mathbf{f}:\mathbb{N}_0\to X$, *admits the unique classical solution*

$$\mathbf{u}(n)=\mathcal{S}(n)(I-(1+\mathbf{a}(0))A)x+(\mathcal{S}\star\mathbf{f})(n), \quad n\in\mathbb{N}.$$

Proof. We write $y:=(I-\mathbf{b}(0)A)x$, where $\mathbf{b}(0)=1+\mathbf{a}(0)$, and define $\mathbf{u}(n):=\mathcal{S}(n)y+(\mathcal{S}\star\mathbf{f})(n)$ for $n\geq 1$ and $\mathbf{u}(0):=\mathcal{S}(0)(I-\mathbf{b}(0)A)x$. By (3.11) we have $\mathbf{u}(0)=x$. Moreover, as in the proof of Theorem 3.4 we obtain

$$\begin{aligned}\mathbf{u}(n+1)-\mathbf{u}(n) &= (\mathcal{S}(n+1)-\mathcal{S}(n))y+(\mathcal{S}\star\mathbf{f})(n+1)-(\mathcal{S}\star\mathbf{f})(n)\\ &= A\mathcal{S}(n+1)y+(\mathbf{a}\star A\mathcal{S})(n+1)y+(\mathcal{S}\star\mathbf{f})(n+1)-(\mathcal{S}\star\mathbf{f})(n).\end{aligned}$$

We claim that

$$(\mathcal{S}\star\mathbf{f})(n+1)-(\mathcal{S}\star\mathbf{f})(n)=A(\mathcal{S}\star\mathbf{f})(n+1)+A(\mathbf{a}\star\mathcal{S}\star\mathbf{f})(n+1)+\mathbf{f}(n+1).$$

In fact, by (3.11) we obtain

$$(\mathcal{S}\star\mathbf{f})(m)=\sum_{k=0}^{m}\mathcal{S}(m-k)\mathbf{f}(k)=\sum_{k=0}^{m}\mathbf{f}(k)+A\sum_{k=0}^{m}(\mathbf{b}\star\mathcal{S})(m-k)\mathbf{f}(k)=\sum_{k=0}^{m}\mathbf{f}(k)+A(\mathbf{b}\star\mathcal{S}\star\mathbf{f})(m),$$

for all $m\in\mathbb{N}_0$. Hence $(\mathcal{S}\star\mathbf{f})(n+1)-(\mathcal{S}\star\mathbf{f})(n)=\mathbf{f}(n+1)+A\left[(\mathbf{b}\star\mathcal{S}\star\mathbf{f})(n+1)-(\mathbf{b}\star\mathcal{S}\star\mathbf{f})(n)\right]$. On the other hand, since $\mathbf{b}(m)=1+(1\star\mathbf{a})(m)$ with $(1\star\mathbf{a})(m)=\sum_{j=0}^{m}\mathbf{a}(j)$ for all $m\in\mathbb{N}_0$, we obtain

$$\begin{aligned}(\mathbf{b}\star\mathcal{S}\star\mathbf{f})(n+1) &- (\mathbf{b}\star\mathcal{S}\star\mathbf{f})(n)=\sum_{k=0}^{n+1}(\mathbf{b}\star\mathcal{S})(n+1-k)\mathbf{f}(k)-\sum_{k=0}^{n}(\mathbf{b}\star\mathcal{S})(n-k)\mathbf{f}(k)\\ &= (\mathcal{S}\star\mathbf{f})(n+1)+\sum_{k=0}^{n+1}(1\star\mathbf{a})(n+1-k)(\mathcal{S}\star\mathbf{f})(k)-\sum_{k=0}^{n}(1\star\mathbf{a})(n-k)(\mathcal{S}\star\mathbf{f})(k)\\ &= (\mathcal{S}\star\mathbf{f})(n+1)+\sum_{k=0}^{n+1}\left(\sum_{j=0}^{n+1-k}\mathbf{a}(j)\right)(\mathcal{S}\star\mathbf{f})(k)-\sum_{k=0}^{n}\left(\sum_{j=0}^{n-k}\mathbf{a}(j)\right)(\mathcal{S}\star\mathbf{f})(k)\\ &= (\mathcal{S}\star\mathbf{f})(n+1)+\sum_{k=0}^{n+1}\mathbf{a}(n+1-k)(\mathcal{S}\star\mathbf{f})(k)=(\mathcal{S}\star\mathbf{f})(n+1)+(\mathbf{a}\star\mathcal{S}\star\mathbf{f})(n+1),\end{aligned}$$

which proves the claim. Therefore,

$$\begin{aligned}\mathbf{u}(n+1)-\mathbf{u}(n)&=A\mathcal{S}(n+1)y+(\mathbf{a}\star A\mathcal{S})(n+1)y+A(\mathcal{S}\star\mathbf{f})(n+1)+A(\mathbf{a}\star\mathcal{S}\star\mathbf{f})(n+1)+\mathbf{f}(n+1)\\ &=A[\mathcal{S}(n+1)y+(\mathcal{S}\star\mathbf{f})(n+1)]+(\mathbf{a}\star A\mathcal{S})(n+1)y+(\mathbf{a}\star A\mathcal{S}\star\mathbf{f})(n+1)+\mathbf{f}(n+1)\\ &=A\mathbf{u}(n+1)+(\mathbf{a}\star A\mathbf{u})(n+1)+\mathbf{f}(n+1),\end{aligned}$$

for all $n\in\mathbb{N}$. The uniqueness follows as in Theorem 3.4, proving the Theorem. □

4. Stability Properties

In this section, we study the stability of the solution to the integro-difference equation (3.10). We recall that a vector-valued sequence $\mathbf{u}\in s(\mathbb{N}_0,X)$ is said to be *stable* if $\|\mathbf{u}(n)\|\to 0$ as $n\to\infty$. From [16, Section 2.3] we recall the following definition.

Definition 4.6. *Let X be a Banach space. A strongly continuous function $T:\mathbb{R}_+\to\mathcal{B}(X)$ is said to be* immediately norm continuous *if $T:(0,\infty)\to\mathcal{B}(X)$ is continuous.*

Now, we consider the equation (3.10) where the kernel $\mathbf{a}(t)$ is given by

$$\mathbf{a}(n)=\frac{\alpha}{(\beta+1)^{\mu+n}}k^{\mu}(n),\quad n\in\mathbb{N}_0.$$

with $\alpha\in\mathbb{R}$ and $\mu,\beta>0$. We assume the following hypothesis in the parameters of the kernel $\mathbf{a}$ and on the operator A :

- (S1) $\alpha\neq 0$, $\beta>0$ and $\mu\geq 1$ such that $\alpha+\beta^{\mu}>0$.
- (S2) A generates an immediately norm continuous C_0-semigroup.
- (S3) $\sup\{\mathrm{Re}\lambda,\lambda\in\mathbb{C}:\lambda(\lambda+\beta)^{\mu}((\lambda+\beta)^{\mu}+\alpha)^{-1}\in\sigma(A)\}<0$.

Theorem 4.7. *Let $x\in D(A)$. Suppose that the hypothesis (S1), (S2) and (S3) hold. Then, there exists a unique stable solution to the problem* (3.10).

Proof. We verify (H1) because $\mathbf{a}=\mathcal{P}(a)$ where $a(t)=\alpha\frac{t^{\mu-1}}{\Gamma(\mu)}e^{-\beta t}$. Since (S1)-(S3) hold, we have by [3, Proposition 3.1] (see also [14]), that the pair (b,A) generates a resolvent family $\{S^{\mu}_{\alpha,\beta}(t)\}_{t\geq 0}\subset\mathcal{B}(X)$ where $b(t)=1+\alpha\int_0^t\frac{r^{\mu-1}}{\Gamma(\mu)}e^{-\beta r}dr$, and therefore (H3) holds. Moreover, Proposition 3.1 in [3] asserts that

$S^{\mu}_{\alpha,\beta}(t)$ is uniformly exponentially stable, that is, there exist $M,\omega>0$ such that $\|S^{\mu}_{\alpha,\beta}(t)\|\leq Me^{-\omega t}$ for all $t\geq 0$. Consequently, by Theorem 3.4 we have

$$\begin{aligned}\|\mathbf{u}(n)\| &\leq \int_0^\infty p_n(t)\|S^{\mu}_{\alpha,\beta}(t)(I-(1-\mathbf{a}(0)A)x\|dt\\ &\leq M\int_0^\infty p_n(t)e^{-\omega t}dt\|(I-(1-\mathbf{a}(0)A)x\| = \frac{M}{(\omega+1)^{n+1}}\|(I-(1-\mathbf{a}(0)A)x\|\to 0\end{aligned}$$

as $n\to\infty$. This proves the theorem. □

Now, we consider the scalar case, that is, we assume that $A=\rho I$, where ρ is a real number. In such case, we will show that we can find an explicit representation of the resolvent families $\{S^{\mu}_{\alpha,\beta}(t)\}_{t\geq 0}$ and $\mathcal{S}^{\mu}_{\alpha,\beta}(n)$ for all $n\in\mathbb{N}_0$. In fact, from (2.8) it is not difficult to see that the Laplace transform of the family $\{S^{\mu}_{\alpha,\beta}(t)\}_{t\geq 0}$ is given by

$$\hat{S}^{\mu}_{\alpha,\beta}(\lambda)=\frac{(\lambda+\beta)^{\mu}}{(\lambda+\beta)^{\mu+1}-(\rho+\beta)(\lambda+\beta)^{\mu}-\alpha\rho}.$$

Using the properties of the Laplace transform, we obtain $S^{\mu}_{\alpha,\beta}(t)=e^{-\beta t}R^{\mu}_{\alpha,\beta}(t)$, where $R^{\mu}_{\alpha,\beta}(t)$ is a function whose Laplace transform is given by

$$\widehat{R}^{\mu}_{\alpha,\beta}(\lambda)=\frac{\lambda^{\mu}}{\lambda^{\mu+1}-(\rho+\beta)\lambda^{\mu}-\alpha\rho}.$$

Since

$$\left|\frac{(\rho+\beta)\lambda^{\mu}}{\lambda^{\mu+1}-\alpha\rho}\right|<1,$$

for λ large enough, we obtain by [11, Formula 17.6],

$$R^{\mu}_{\alpha,\beta}(t)=\sum_{j=0}^{\infty}(\rho+\beta)^j t^j E^{j+1}_{\mu+1,j+1}(\alpha\rho t^{\mu+1}),\quad t\geq 0,$$

where $E^{\gamma}_{p,q}(z):=\sum_{j=0}^{\infty}\frac{k^{\gamma}(j)z^j}{\Gamma(q+pj)}$ is the generalized Mittag-Leffer type function defined for $p,q,\gamma>0$, see for instance [11, Section 11]. Therefore, we conclude that

$$S^{\mu}_{\alpha,\beta}(t)=e^{-\beta t}\sum_{j=0}^{\infty}(\rho+\beta)^j t^j E^{j+1}_{\mu+1,j+1}(\alpha\rho t^{\mu+1}),\quad t\geq 0. \tag{4.14}$$

Now, we compute $\mathcal{S}^{\mu}_{\alpha,\beta}(n)$ when $A=\rho I$. By (4.14) we have

$$\begin{aligned}\mathcal{S}^{\mu}_{\alpha,\beta}(n) &= \int_0^\infty p_n(t)S^{\mu}_{\alpha,\beta}(t)dt=\int_0^\infty e^{-t}\frac{t^n}{n!}e^{-\beta t}\sum_{j=0}^{\infty}(\rho+\beta)^j t^j E^{j+1}_{\mu+1,j+1}(\alpha\rho t^{\mu+1})dt\\ &= \frac{1}{\Gamma(n+1)}\sum_{j=0}^{\infty}(\rho+\beta)^j\int_0^\infty e^{-(\beta+1)t}t^{(n+j+1)-1}E^{j+1}_{\mu+1,j+1}(\alpha\rho t^{\mu+1})dt.\end{aligned}$$

Observe that the last integral corresponds to the Laplace transform of the function

$$h(t):=t^{(n+j+1)-1}E^{j+1}_{\mu+1,j+1}(\alpha\rho t^{\mu+1})$$

evaluated at $\lambda=(\beta+1)$. Using [11, Formula 11.15] we obtain

$$\mathcal{S}^{\mu}_{\alpha,\beta}(n)=\frac{1}{\Gamma(n+1)}\sum_{j=0}^{\infty}(\rho+\beta)^j\frac{(\beta+1)^{-(n+j+1)}}{\Gamma(j+1)}{}_2\psi_1\left[\begin{array}{c|c}(j+1,1);(n+j+1,\mu+1) & \multirow{2}{*}{$\dfrac{\alpha\rho}{(\beta+1)^{\mu+1}}$}\\ (j+1,\mu+1) & \end{array}\right],$$

where ${}_2\psi_1[\,\cdot\,|z]$ is the generalized Wright function. From [11, Formula 8.5] it follows that

$$
{}_2\psi_1\left[\begin{array}{l|}(j+1,1);(n+j+1,\mu+1)\\(j+1,\mu+1)\end{array}\ \frac{\alpha\rho}{(\beta+1)^{\mu+1}}\right] = \sum_{r=0}^{\infty}\frac{\Gamma(j+1+r)\Gamma(j+1+(\mu+1)r+n)}{\Gamma(j+1+(\mu+1)r)}\left(\frac{\alpha\rho}{(\beta+1)^{\mu+1}}\right)^r\frac{1}{r!}
$$

Using the sequence k^α defined in (2.9), we can write

$$
\begin{aligned}
\mathcal{S}^{\mu}_{\alpha,\beta}(n) &= \sum_{j=0}^{\infty}(\rho+\beta)^j(\beta+1)^{-(n+j+1)}\times\\
&\quad\sum_{r=0}^{\infty}\frac{\Gamma(j+1+r)}{\Gamma(j+1)\Gamma(r+1)}\frac{\Gamma(j+1+(\mu+1)r+n)}{\Gamma(j+1+(\mu+1)r)\Gamma(n+1)}\left(\frac{\alpha\rho}{(\beta+1)^{\mu+1}}\right)^r\\
&= \sum_{j=0}^{\infty}\frac{(\rho+\beta)^j}{(\beta+1)^{(n+j+1)}}\sum_{r=0}^{\infty}k^{j+1}(r)k^{(j+1)+(\mu+1)r}(n)\left(\frac{\alpha\rho}{(\beta+1)^{\mu+1}}\right)^r.
\end{aligned}
$$

This gives the claimed representation.

Now, we consider the difference equation

$$
\left\{\begin{array}{rcl}\mathbf{u}(n+1)-\mathbf{u}(n) &=& \rho\mathbf{u}(n+1)+\dfrac{\alpha\rho}{(\beta+1)^{\mu}}\displaystyle\sum_{k=0}^{n+1}\frac{k^{\mu}(n+1-k)}{(\beta+1)^{n+1-k}}\mathbf{u}(k),\ n\in\mathbb{N}_0\\ \mathbf{u}(0) &=& x,\end{array}\right. \tag{4.15}
$$

Since $A=\rho I$ trivially generates an immediately norm continuous C_0-semigroup, we obtain the following result.

Proposition 4.8. *Assume the hypothesis (S1) and $x\in X$. Then, the difference equation* (4.15) *has the classical solution*

$$
\mathbf{u}(n)=\frac{(\beta+1)^{\mu}(1-\rho)-\alpha\rho}{(\beta+1)^{\mu}}\mathcal{S}^{\mu}_{\alpha,\beta}(n)x.
$$

Moreover, $\|\mathbf{u}(n)\|\to 0$ *as* $n\to\infty$.

Proof. An easy computation shows that the pair $(a,\rho I)$ generates the resolvent family $\{S^{\mu}_{\alpha,\beta}(t)\}_{t\geq 0}$, where $a(t)=\alpha\frac{t^{\mu-1}}{\Gamma(\mu)}e^{-\beta t}$. Since $\mathbf{a}=\mathcal{P}(a)$ and $\mathbf{b}(0)=1+\mathbf{a}(0)=1+\frac{\alpha}{(\beta+1)^{\mu}}k^{\mu}(0)=1+\frac{\alpha}{(\beta+1)^{\mu}}$, we have $(1-\mathbf{b}(0)\rho)=\frac{(\beta+1)^{\mu}(1-\rho)-\alpha\rho}{(\beta+1)^{\mu}}$ and the conclusion follows from Theorem 3.4. □

For the next result, we need to recall the following definition.

Definition 4.9. [23, Section 3.2] *Let $k\in\mathbb{N}$ and $a\in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. The kernel a is called* k-regular *if there is a constant $c>0$ such that $|\lambda^n\hat{a}^{(n)}(\lambda)|\leq c|\hat{a}(\lambda)|$, for all* $\mathrm{Re}\lambda>0$ *and* $1\leq n\leq k$.

We obtain the following criteria for the stability of the solutions for (3.10) in Hilbert spaces.

Proposition 4.10. *Let H be a Hilbert space. We assume the following hypothesis:*

(P1) $\mathbf{a}=\mathcal{P}(a)$ *where $a\in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is 1-regular.*
(P2) *the pair (b,A) is the generator of an exponentially bounded resolvent family $\{S(t)\}_{t\geq 0}$ in H.*
(P3) $0\in\rho(A)$
(P4) $\frac{1}{\hat{b}(\lambda)}\in\rho(A)$ *for all* $\mathrm{Re}(\lambda)\geq 0,\lambda\neq 0$.
(P5) $H(\lambda):=(\lambda-(1+\hat{a}(\lambda))A)^{-1}$ *is uniformly bounded in* $\mathbb{C}_+:=\{\lambda\in\mathbb{C}:\mathrm{Re}(\lambda)>0\}$.

Then, the classical solution to the difference initial value problem (3.10) *exists and is stable.*

Proof. The 1-regularity of the kernel a implies the existence of a constant $d > 0$ such that $|\lambda\widehat{a}'(\lambda)| \leq d|\hat{a}(\lambda)|$. Since $b(t) = 1 + (1 * a)(t)$, we obtain

$$|\lambda \hat{b}'(\lambda)| = \left| -\frac{1}{\lambda} - \frac{1}{\lambda}\hat{a}(\lambda) + \hat{a}'(\lambda) \right| \leq |\hat{b}(\lambda)| + d\frac{|\hat{a}(\lambda)|}{|\lambda|} \leq |\hat{b}(\lambda)| + d\frac{|1 + \hat{a}(\lambda)|}{|\lambda|} = (d+1)|\hat{b}(\lambda)|,$$

for all $\mathrm{Re}\lambda > 0$, which means that b is a 1-regular kernel. On the other hand, $\lim_{\lambda\to 0} \lambda\hat{b}(\lambda) = \lim_{\lambda\to 0}(1 + \hat{a}(\lambda)) = 1 + \hat{a}(0) \neq 0$. By [15, Theorem 1] we conclude that the resolvent family $\{S(t)\}_{t\geq 0}$ is uniformly stable. Then, the dominated convergence theorem shows that $\lim_{n\to\infty} \|\mathcal{S}(n)\| = 0$. Finally, an application of Theorem 3.4 finishes the proof. ☐

References

[1] L. Abadías, C. Lizama, P. J. Miana, M. P. Velasco, *On well-posedness of vector-valued fractional differential-difference equations,* Discrete Contin. Dyn. Systems, Series A, 39 (2019), 2679–2708.

[2] W. Arendt, C. Batty, M. Hieber, F. Neubrander, *Vector-Valued Laplace transforms and Cauchy problems.* Monogr. Math., vol. **96**, Birkhäuser, Basel, 2011.

[3] Y. K. Chang, R. Ponce, *Uniform exponential stability and applications to bounded solutions of integro-differential equations in Banach spaces,* J. Integral Equations Appl. **30** (3) (2018), 347–369.

[4] A. Conejero, A. Mira-Iglesias, C. Lizama, C. Rodero. *Visibility graphs of fractional Wu-Baleanu time series.* J. Difference Equations Appl., to appear. DOI: 10.1080/10236198.2019.1619714.

[5] E. Cuesta, C. Palencia, *A numerical method for an integro-differential equation with memory in Banach spaces: Qualitative properties,* SIAM J. Numer. Anal., 41 (2003), 1232-1241.

[6] G. Da Prato, M. Iannelli, *Linear integrodifferential equations in Banach space,* Rend. Sem. Mat. Univ. Padova 62 (1980), 207–219.

[7] S. Elaydi, *Stability and asymptoticity of Volterra difference equations: A progress report,* J. Comput. Appl. Math. 228 (2009), 504–513.

[8] C. Goodrich, C. Lizama. *A transference principle for nonlocal operators using a convolutional approach: Fractional monotonicity and convexity.* Israel Journal of Mathematics, In Press.

[9] G. Gripenberg, S-O Londen, O. Staffans, *Volterra Integral and Functional Equations.* Encyclopedia of Mathematics and Applications, **34**, Cambridge University Press, Cambridge-New York, 1990.

[10] M. Gurtin, A Pipkin, *A general theory of heat conduction with finite wave speeds,* Arch. Ration. Mech. Anal., **31** (1968), 113–126.

[11] H. Haubold, A. Mathai, R. Saxena, *Mittag-Leffler Functions and Their Applications,* Journal of Applied Mathematics, Volume 2011, Article ID 298628, 51 pages.

[12] C. Lizama, *Regularized solutions for abstract Volterra equations,* J. Math. Anal. Appl. **243** (2000), 278-292.

[13] C. Lizama, *The Poisson distribution, abstract fractional difference equations, and stability,* Proc. Amer. Math. Soc. **145** (9) (2017), 3809–3827.

[14] C. Lizama, R. Ponce, *Bounded solutions to a class of semilinear integro-differential equations in Banach spaces,* Nonlinear Anal. **74** (2011), 3397-3406.

[15] C. Lizama, V. Vergara, *Uniform stability of resolvent families*, Proc. Amer. Math. Soc. **132** (1) (2013), 175–181.

[16] C. Lizama. *Abstract Linear Fractional Evolution Equations,* in: Handbook of Fractional Calculus with Applications. Volume 2: Fractional Differential Equations. Ed. by A. Kochubei and Y. Luchko, De Gruyter, Boston, 2019, pp. 465-498.

[17] W. McLean, V. Thomee, L. Wahlbin, *Discretization with variable time steps of an evolution equation with a positive-type memory term,* J. Comput. Appl. Math. **69** (1996) 49-69.

[18] W. McLean, V. Thomée, *Time discretization of an evolution equation via Laplace transforms,* IMA J. of Numerical Analysis **24** (2004), 439-463.

[19] W. McLean, V. Thomée, *Numerical solution via Laplace transforms of a fractional order evolution equation.* J. Integral Equations Appl. **22** (2010), 57-94.

[20] W. McLean, I. Sloan, V. Thomée, *Time discretization via Laplace transformation of an integro-differential equation of parabolic type.* Numer. Math. **102** (3) (2006), 497–522.

[21] K. Mustapha, W. McLean, *Uniform convergence for a discontinuous Galerkin, time-stepping method applied to a fractional diffusion equation,* IMA J. Numer. Anal. **32** (2012), 906-925.

[22] J. W. Nunziato, *On heat conduction in materials with memory,* Quart. Appl. Math. **29** (1971), 187-304.

[23] J. Prüss, *Evolutionary Integral Equations and Applications,* Monographs Math., **87**, Birkhäuser, Boston, 1993.

[24] I. Sloan, V. Thomée, *Time discretization of an integro-differential equation of parabolic type,* SIAM J. Numer. Anal. **23** (5) (1986), 1052-1061.

[25] G.C. Wu, D. Baleanu- *Discrete chaos in fractional delayed logistic maps.* Nonlinear Dyn (80) (2015), 16971703

[26] G. C. Wu, T. Abdeljawad, J. Liu, D. Baleanu, K. T. Wu, *MittagLeffler stability analysis of fractional discretetime neural networks via fixed point technique,* Nonlinear Analysis: Modelling and Control, (2019). In Press.
[27] D. Xu, *Uniform* l^1*-convergence in the Crank-Nicolson method of a linear integro-differential equation for viscoelastic rods and plates,* Math. Comp., **83** (286) (2014), 735–769.

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