BOUNDED SOLUTIONS TO A CLASS OF SEMILINEAR INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES.

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Abstract. Let A be the generator of an immediately norm continuous C_0-semigroup defined on a Banach space X. We study the existence and uniqueness of bounded solutions for the semilinear integro-differential equation with infinite delay

$$u'(t) = Au(t) + \alpha \int_{-\infty}^{t} e^{-\beta(t-s)}Au(s)ds + f(t, u(t)) \quad t \in \mathbb{R}; \quad \alpha, \beta \in \mathbb{R},$$

for each $f : \mathbb{R} \times X \to X$ satisfying diverse Lipschitz type conditions. Sufficient conditions are established for the existence and uniqueness of an almost periodic, almost automorphic and asymptotically almost periodic solution, among others types of distinguished solutions. These results have significance in viscoelasticity theory. Finally, an example is presented to illustrate the feasibility and effectiveness of the results.

1. Introduction

In this paper we consider the problem of existence, uniqueness and regularity of solutions for the following integro-differential equation

$$u'(t) = Au(t) + \alpha \int_{-\infty}^{t} e^{-\beta(t-s)}Au(s)ds + f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where $\alpha, \beta \in \mathbb{R}$, $A : \mathcal{D}(A) \subset X \to X$ is a closed linear operator defined on a Banach space X, and f belongs to a closed subspace of the space of continuous and bounded functions. Under appropriate additional assumptions on α, β, A and on the forcing function f, we want to prove that (1.1) has a unique solution u which behaves in the same way that f does. For example, we want to find conditions implying that u is almost periodic (resp. automorphic) if $f(\cdot, x)$ is almost periodic (resp. almost automorphic), that u is asymptotically periodic (resp. almost periodic) if $f(\cdot, x)$ is asymptotically periodic (resp. almost periodic), and that u is pseudo almost periodic (resp. automorphic) if $f(\cdot, x)$ is pseudo almost periodic (resp. automorphic).

This problem arises in several applied fields, like viscoelasticity or heat conduction with memory, and in such applications the operator A typically is the Laplacian in $X = L^2(\Omega)$, or the elasticity operator, the Stokes operator, or the biharmonic Δ^2, among others, and equipped with suitable boundary conditions. The exponential kernel $\alpha e^{-\beta t}$ is the typical choice when one consider Maxwell materials in viscoelasticity theory. In that context, $\alpha = \mu$ and $\beta = \mu/\nu$ where μ is the elastic modulus of the material and ν corresponds

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2000 *Mathematics Subject Classification.* Primary 43A60; Secondary 47D60, 45N05, 47J35.

Key words and phrases. periodic, almost periodic, compact almost automorphic, almost automorphic, asymptotic behavior, pseudo asymptotic behavior, semilinear integro-differential equations.

The first author is partially supported by FONDECYT grant number 1100485.

to their coefficient of viscosity. See for instance [13], [22, Section 9, Chapter II] and the references therein. Observe that the case $\alpha = 0$ leads with the semilinear problem

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

which have been studied intensively by several authors, see e.g. the monograph [4] and references therein.

The problem of existence and uniqueness of almost periodic or almost automorphic solutions, as well as the study of their behavior at infinity, is not only a very natural one for the type of non-linear evolution equations (1.1), but also there is a recent and increasing interest on this subject by many researchers, see [16, 10, 14, 18, 19, 24, 15] and references therein. In this paper, we study in an unified way the existence and uniqueness of, among others, almost periodic, almost automorphic and compact almost automorphic solutions for (1.1). Even more, as immediate consequence of our method, necessary conditions for the asymptotic and pseudo-asymptotic behavior of the equation (1.1), under the hypothesis that A generates an *immediately norm continuous* C_0-semigroup on a Banach space X, are also established.

The paper is organized as follows. In the second section, we review recent results about several intermediate Banach spaces interpolating between periodic and bounded continuous functions. In Section 3, we study the linear case of equation (1.1), necessary for our method. Assuming that A generates an immediately norm continuous C_0-semigroup we are able to give a simply spectral condition on A in order to guarantee the existence of solutions in each class of function spaces introduced in section 2 (Theorem 3.2 and Corollary 3.3). It is remarkable that in the scalar case, that is $A = \rho I$, with $\rho \in \mathbb{R} \setminus \{0\}$, an explicit form of the solution for (1.1) is given by:

$$u(t) = \int_{-\infty}^{t} S_\rho(t-s) f(s, u(s)) ds, \quad t \in \mathbb{R}, \tag{1.3}$$

where

$$S_\rho(t) = \frac{1}{2}\left(e^{t\frac{(\rho-\beta)+c}{2}} + e^{t\frac{(\rho-\beta)-c}{2}}\right) + \frac{(\beta+\rho)}{2c}\left(e^{t\frac{(\rho-\beta)+c}{2}} - e^{t\frac{(\rho-\beta)-c}{2}}\right),$$

and $c = \sqrt{(\beta+\rho)^2 + 4\alpha\rho}$. In particular, it shows that our results are a direct extension of the case $\alpha = 0$ studied in the literature but, notably, in our case the condition $\rho > 0$ even guarantee the existence of bounded solutions for the class of equations (1.1) in the linear case, in contrast with the case $\alpha = 0$ where $\rho < 0$ is necessary. Some examples and a picture of the situation completes this section. In section 4, we present our main results for the semilinear equation (1.1). There, using the previous results on the linear case and the Banach contraction principle, we present new results of existence of solutions that are based directly in the data of the problem. We finish this paper with an concrete example, to shown the feasibility of the abstract results.

2. Preliminaries: The function spaces

We denote

$$BC(X) := \{f : \mathbb{R} \to X; f \text{ is continuous }, ||f||_\infty := \sup_{t\in\mathbb{R}} ||f(t)|| < \infty\},$$

where X is a complex Banach space. In this section, we first recall the definition and properties of several function spaces of continuous and bounded functions, and then some recent results on uniform exponential stability of solutions for Volterra equations with special kernels.

To begin with, we recall that a function $f \in BC(X)$ is said to be almost periodic (in the sense of Bohr) if for any $\varepsilon > 0$, there exist $\omega = \omega(\varepsilon) > 0$ such that every subinterval $\mathbb{R}$ of length ω contains at least one point τ such that $||f(t+\tau) - f(t)||_\infty < \varepsilon$. We denote by $AP(X)$ the set of all these functions. The space of compact almost automorphic functions will be denoted by $AA_c(X)$. Recall that function $f \in BC(X)$ belongs to $AA_c(X)$ if and only if for all sequence $(s'_n)_{n\in\mathbb{N}}$ of real numbers there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that $g(t) := \lim_{n\to\infty} f(t+s_n)$ and $f(t) = \lim_{n\to\infty} g(t-s_n)$ uniformly over compact subsets of $\mathbb{R}$. Clearly the function g above is continuous on $\mathbb{R}$. Finally, a function $f \in BC(X)$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$f(t) = \lim_{n\to\infty} g(t-s_n), \quad \text{for each } t \in \mathbb{R}.$$

We denote by $AA(X)$ the set of all almost automorphic functions. We recall that $AA_c(X)$ and $AA(X)$ are Banach spaces under the norm $||\cdot||_\infty$ and

$$P_\omega(X) \subset AP(X) \subset AA(X) \subset AA_c(X) \subset BC(X).$$

Now we consider the set $C_0(X) := \{f \in BC(X) : \lim_{|t|\to\infty} ||f(t)|| = 0\}$, and define the space of asymptotically periodic functions as $AP_\omega(X) := P_\omega(X) \oplus C_0(X)$. Analogously, we define the space of asymptotically almost periodic functions,

$$AAP(X) := AP(X) \oplus C_0(X),$$

the space of asymptotically compact almost automorphic functions,

$$AAA_c(X) := AA_c(X) \oplus C_0(X),$$

and the space of asymptotically almost automorphic functions,

$$AAA(X) := AA(X) \oplus C_0(X).$$

We have the following natural proper inclusions

$$AP_\omega(X) \subset AAP(X) \subset AAA_c(X) \subset AAA(X) \subset BC(X).$$

Denote by $SAP_\omega(X) := \{f \in BC(X) : \exists\omega > 0, ||f(t+\omega) - f(t)|| \to 0 \text{ as } t \to \infty\}$. The class of functions in $SAP_\omega(X)$ is called S-asymptotically ω-periodic. Now, we next consider the following set

$$P_0(X) := \{f \in BC(X) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} ||f(s)||ds = 0\},$$

and define the following classes of spaces: The space of pseudo-periodic functions

$$PP_\omega(X) := P_\omega(X) \oplus P_0(X),$$

the space of pseudo-almost periodic functions

$$PAP(X) := AP(X) \oplus P_0(X),$$

the space of pseudo-compact almost automorphic functions

$$PAA_c(X) := AA_c(X) \oplus P_0(X),$$

and the space of pseudo-almost automorphic functions

$$PAA(X) := AA(X) \oplus P_0(X).$$

As before, we also have the following relationship between them;

$$PP_\omega(X) \subset PAP(X) \subset PAA_c(X) \subset PAA(X) \subset BC(X).$$

Denote by $\mathcal{N}(\mathbb{R}, X)$ or simply $\mathcal{N}(X)$ the following function spaces

$$\begin{aligned}\mathcal{N}(X): \; = \; & \{P_\omega(X), AP(X), AA_c(X), AA(X), AP_\omega(X), AAP(X), AAA_c(X), AAA(X),\\ & PP_\omega(X), PAP(X), PAA_c(X), PAA(X), SAP_\omega(X), BC(X)\}.\end{aligned}$$

We recall the following Theorem from [12].

Theorem 2.1 ([12])**.** *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be a uniformly integrable and strongly continuous family. If f belongs to one of the spaces of $\mathcal{N}(X)$ then*

$$\int_{-\infty}^{t} S(t-s)f(s)ds,$$

belongs to the same space as f.

We define the set $\mathcal{N}(\mathbb{R}\times X; X)$ which consists of all functions $f : \mathbb{R}\times X \to X$ such that $f(\cdot, x) \in \mathcal{N}(\mathbb{R}, X)$ uniformly for each $x \in K$, where K is any bounded subset of X.

We recall from [12] that if $\mathcal{M}(X)$ denote one of the spaces $P_\omega(X), AP_\omega(X), PP_\omega(X)$, $SAP_\omega(X), AP(X), AAP(X), PAP(X), AA(X), AAA(X), PAA(X)$ then we have the following composition theorem.

Theorem 2.2 ([12])**.** *Let $f \in \mathcal{M}(\mathbb{R}\times X, X)$ be given and assume that there exists a constant L_f such that*

$$||f(t,u) - f(t,v)|| \leq L_f||u - v||,$$

for all $t \in \mathbb{R}$ and $u, v \in X$. If $\psi \in \mathcal{M}(X)$ then $f(\cdot, \psi(\cdot)) \in \mathcal{M}(\mathbb{R}\times X, X)$.

Finally, we present some recent results of uniform exponential stability of solutions to the homogeneous abstract Volterra equation

$$\begin{cases} u'(t) = Au(t) + \alpha \displaystyle\int_0^t e^{-\beta(t-s)}Au(s)ds, \; t \geq 0 \\ u(0) = x. \end{cases} \tag{2.1}$$

We say that a solution of (2.1) is uniformly exponentially bounded if for some $\omega \in \mathbb{R}$, there exists a constant $M > 0$ such that for each $x \in D(A)$, the corresponding solution $u(t)$ satisfies

$$||u(t)|| \leq Me^{-\omega t}, \; t \geq 0. \tag{2.2}$$

In particular, we say that the solutions of (2.1) are *uniformly exponentially stable* if (2.2) holds for some $\omega > 0$ and $M > 0$.

Definition 2.3. Let X be a Banach space. A strongly continuous function $T : \mathbb{R}_+ \to \mathcal{B}(X)$ is said to be *immediately norm continuous* if $T : (0, \infty) \to \mathcal{B}(X)$ is continuous.

Finally, we recall the following remarkable result from [6]:

Theorem 2.4. *Let $\beta > 0, \alpha \neq 0$ and $\alpha + \beta > 0$ be given. Assume that*
(a) A generates an immediately norm continuous C_0-semigroup on a Banach space X;
(b) $\sup\left\{\Re\lambda, \lambda \in \mathbb{C} : \lambda(\lambda+\beta)(\lambda+\alpha+\beta)^{-1} \in \sigma(A)\right\} < 0.$
Then, the solutions of the problem (2.1) are uniformly exponentially stable.

3. The linear case

Let $\alpha, \beta \in \mathbb{R}$ be given. In this section we study bounded solutions for the linear integro-differential equation

$$u'(t) = Au(t) + \alpha \int_{-\infty}^{t} e^{-\beta(t-s)} Au(s)ds + f(t), \quad t \in \mathbb{R}, \tag{3.1}$$

where A generates an immediately norm continuous C_0-semigroup on a Banach space X. To begin our study, we note in the next proposition that under the given hypothesis on A, it is possible to construct for (3.1) an strongly continuous family of bounded and linear operators, that commutes with A and satisfy certain "resolvent equation". This class of strongly continuous families has been studied extensively in the literature of abstract Volterra equations, see e.g. Prüss [22] and references therein.

Proposition 3.1. *Let $\beta > 0, \alpha \neq 0$ and $\alpha + \beta > 0$. Assume that*
(a) A generates an immediately norm continuous C_0-semigroup on a Banach space X;
(b) $\sup\left\{\Re\lambda, \lambda \in \mathbb{C} : \lambda(\lambda+\beta)(\lambda+\alpha+\beta)^{-1} \in \sigma(A)\right\} < 0.$

Then, there exist an uniformly exponentially stable and strongly continuous family of operators $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ such that $S(t)$ commutes with A, that is, $S(t)\mathcal{D}(A) \subset \mathcal{D}(A)$, $AS(t)x = S(t)Ax$ for all $x \in \mathcal{D}(A)$, $t \geq 0$ and

$$S(t)x = x + \int_0^t b(t-s)AS(s)xds, \ \textit{for all } x \in X, \ t \geq 0, \tag{3.2}$$

where $b(t) := 1 + \frac{\alpha}{\beta}[1 - e^{-\beta t}], \ t \geq 0.$

Proof. For $t \geq 0$ and $x \in X$ define $S(t)x := u(t;x)$ where $u(t;x)$ is the unique solution of equation (2.1). See [8, Corollary 7.22, p.449] for the existence of such solution and their strong continuity. We will see that $S(\cdot)x$ satisfies the resolvent equation (3.2). Since $S(t)x$ is the solution of (2.1), we have that $S(t)x$ is differentiable and satisfies

$$S'(t)x = AS(t)x + \alpha \int_0^t e^{-\beta(t-s)} AS(s)xds. \tag{3.3}$$

Integrating (3.3), we conclude from Fubini's theorem that,

$$\begin{aligned} S(t)x - x &= \int_0^t AS(s)xds + \alpha \int_0^t \int_0^s e^{-\beta(s-\tau)} AS(\tau)xd\tau ds \\ &= \int_0^t AS(s)xds + \alpha \int_0^t \int_\tau^t e^{-\beta(s-\tau)} AS(\tau)xdsd\tau \\ &= \int_0^t AS(s)xds + \alpha \int_0^t \int_0^{t-\tau} e^{-\beta v} AS(\tau)xdvd\tau \\ &= \int_0^t AS(s)xds + \frac{\alpha}{\beta} \int_0^t (1 - e^{-\beta(t-\tau)})AS(\tau)xd\tau \\ &= \int_0^t 1 + \frac{\alpha}{\beta}[1 - e^{-\beta(t-\tau)}]AS(\tau)xd\tau \\ &= \int_0^t b(t-\tau)AS(\tau)xd\tau. \end{aligned}$$

The commutativity of $S(t)$ with A follows in the same way that [22, p. 31,32]. The uniform exponential stability follows from Corollary 2.4. ☐

We recall that a function $u \in C^1(\mathbb{R}; X)$ is called a strong solution of (3.1) on $\mathbb{R}$ if $u \in C(\mathbb{R}; D(A))$ and (3.1) holds for all $t \in \mathbb{R}$. If $u(t) \in X$ instead of $u(t) \in D(A)$, and (3.1) holds for all $t \in \mathbb{R}$ we say that u is *mild solution* of (3.1). The following is our main result in the abstract case.

Theorem 3.2. *Let $\beta > 0, \alpha \neq 0$ and $\alpha + \beta > 0$. Assume that A generates an immediately norm continuous C_0-semigroup on a Banach space X and*

$$\sup\left\{\Re\lambda : \lambda(\lambda+\beta)(\lambda+\alpha+\beta)^{-1} \in \sigma(A)\right\} < 0.$$

If f belong to some space of $\mathcal{N}(X)$ then the unique mild solution of the problem (3.1) belongs to the same space that f and is given by

$$u(t) = \int_{-\infty}^t S(t-s)f(s)ds, \quad t \in \mathbb{R},$$

where $\{S(t)\}_{t\geq 0}$ is given in Proposition 3.1.

Proof. By Proposition 3.1, the family $\{S(t)\}_{t\geq 0}$ is uniformly exponentially stable and therefore u is well defined. Since S satisfies the resolvent equation

$$S(t)x = \int_0^t b(t-s)AS(s)xds + x, \; x \in X,$$

where $b(t) = 1 + \frac{\alpha}{\beta}[1 - e^{-\beta t}]$, we have that b is differentiable and the above equation shows that for each $x \in X$, $S'(t)x$ exists and

$$S'(t)x = AS(t)x + \alpha \int_0^t e^{-\beta(t-s)} AS(s)xds.$$

It remains to prove that u is a mild solution of (3.1). Since A is a closed operator, using Fubini's theorem, we have

$$\begin{aligned}
u'(t) &= S(0)f(t) + \int_{-\infty}^{t} S'(t-s)f(s)ds \\
&= f(t) + \int_{-\infty}^{t} \Big[AS(t-s)f(s) + \alpha \int_0^{t-s} e^{-\beta(t-s-\tau)} AS(\tau)f(s)d\tau\Big]ds \\
&= f(t) + \int_{-\infty}^{t} AS(t-s)f(s)ds + \alpha \int_{-\infty}^{t}\int_0^{t-s} e^{-\beta(t-s-\tau)} AS(\tau)f(s)d\tau ds \\
&= f(t) + Au(t) + \alpha \int_{-\infty}^{t}\int_s^{t} e^{-\beta(t-v)} AS(v-s)f(s)dvds \\
&= f(t) + Au(t) + \alpha \int_{-\infty}^{t}\int_{-\infty}^{v} e^{-\beta(t-v)} AS(v-s)f(s)dsdv \\
&= f(t) + Au(t) + \alpha \int_{-\infty}^{t} e^{-\beta(t-v)} \int_{-\infty}^{v} AS(v-s)f(s)dsdv \\
&= f(t) + Au(t) + \alpha \int_{-\infty}^{t} e^{-\beta(t-v)} Au(v)dv.
\end{aligned}$$

□

In case of Hilbert spaces, we can use a result of You [23] which characterizes norm continuity of C_0-semigroups obtaining the following result.

Corollary 3.3. *Let A be the generator of a C_0-semigroup on a Hilbert space H. Let $s(A) := \sup\{\Re\lambda : \lambda \in \rho(A)\}$ denote the spectral bound of A. Let $\beta > 0, \alpha \neq 0, \alpha + \beta > 0$ be given. Assume that*

(a) $\lim_{\mu\in\mathbb{R},|\mu|\to\infty} ||(\mu_0 + i\mu - A)^{-1}|| = 0$ for some $\mu_0 > s(A)$;

(b) $\sup\{\Re\lambda : \lambda(\lambda+\beta)(\lambda+\alpha+\beta)^{-1} \in \sigma(A)\} < 0$.

If f belong to some space of $\mathcal{N}(X)$ then the unique mild solution of the problem (3.1) belongs to the same space that f.

Remark 3.4. In the case $A = \rho I$, $\rho \in \mathbb{C}$ we obtain from (3.2), using Laplace transform, that for each $x \in X$:

(3.4)
$$S_\rho(t)x = e^{\frac{(\rho-\beta)t}{2}} \left(\cosh\left(\frac{t\sqrt{(\beta+\rho)^2+4\rho\alpha}}{2}\right) + \frac{\sinh\left(\frac{t}{2}\sqrt{(\beta+\rho)^2+4\rho\alpha}\right)(\beta+\rho)}{\sqrt{(\beta+\rho)^2+4\rho\alpha}} \right) x.$$

Our following result shows the remarkable fact that in this case the conditions of the above abstract result can be considerably relaxed.

Theorem 3.5. *Let $A := \rho I$ where $\rho \in \mathbb{R}$ be given. Suppose that $\rho < \beta$ and $(\alpha+\beta)\rho < 0$. Let $f \in \mathcal{N}(X)$. Consider the equation*

$$u'(t) = \rho u(t) + \rho\alpha \int_{-\infty}^{t} e^{-\beta(t-s)}u(s)ds + f(t), \quad t \in \mathbb{R}. \tag{3.5}$$

Then the equation (3.5) has a unique solution u which belongs to the same space that f and is given by

$$u(t) = \int_{-\infty}^{t} S_\rho(t-s)f(s)ds, \quad t \in \mathbb{R}, \tag{3.6}$$

where $\{S_\rho(t)\}_{t\geq 0}$ is defined by (3.4).

Proof. Let $f \in \Omega$ where Ω is one of the spaces in $\mathcal{N}(X)$. Since $A = \rho I$ generates an immediately norm continuous C_0-semigroup and $\sigma(A) = \{\rho\}$, we have that $\lambda(\lambda+\beta)(\lambda+\alpha+\beta)^{-1} \in \sigma(A)$ if and only if $\lambda^2 + \lambda(\beta-\rho) - \rho(\alpha+\beta) = 0$. We claim that $S_\rho(t)$ is integrable. In fact, we can rewrite $S_\rho(t)$ in (3.4) as follows:

$$S_\rho(t) = \frac{1}{2}\left(e^{t\frac{(\rho-\beta)+c}{2}} + e^{t\frac{(\rho-\beta)-c}{2}}\right) + \frac{(\beta+\rho)}{2c}\left(e^{t\frac{(\rho-\beta)+c}{2}} - e^{t\frac{(\rho-\beta)-c}{2}}\right),$$

where $c = \sqrt{(\beta+\rho)^2 + 4\alpha\rho}$. Therefore,

$$|S_\rho(t)| \leq \frac{1}{2}\left(e^{t\Re\left(\frac{(\rho-\beta)+c}{2}\right)} + e^{t\Re\left(\frac{(\rho-\beta)-c}{2}\right)}\right) + \frac{|\beta+\rho|}{2c}\left(e^{t\Re\left(\frac{(\rho-\beta)+c}{2}\right)} + e^{t\Re\left(\frac{(\rho-\beta)-c}{2}\right)}\right),$$

where $\beta - \rho > c$ because $(\beta+\alpha)\rho < 0$ and $\beta > \rho$. Hence $\rho - \beta - c < \rho - \beta + c < 0$. We conclude that $S_\rho(t)$ is integrable, proving the claim. Therefore, by Proposition 3.2, there exist a unique solution of equation (3.5) which belongs to the same space that f and is explicitly given by (3.6). $\square$

Remark 3.6. Observe that the case $\alpha = 0$ implies $\rho < 0$.

Example 3.7. *Let $\rho = -1, \alpha = 1, \beta = 1$. Hence, by Theorem 3.5, for any $f \in \mathcal{N}(X)$ there exist a unique solution $u \in \mathcal{N}(X)$ of the equation*

$$u'(t) = -u(t) - \int_{-\infty}^{t} e^{s-t}u(s)ds + f(t), \quad t \in \mathbb{R}, \tag{3.7}$$

given by

$$u(t) = \int_{-\infty}^{t} e^{-(t-s)}\cos(t-s)f(s)ds, \quad t \in \mathbb{R},$$

since, $S_{-1}(t) = e^{-t}\cos(t)$.

Example 3.8. *This example is taken from* [6, Remark 3.9]. *Let $\rho = 1$, $\alpha = -3$ and $\beta = 2$. From Theorem 3.5, if $f \in \mathcal{N}(X)$, then there exist a unique solution $u \in \mathcal{N}(X)$ of the equation*

$$u'(t) = u(t) - 3\int_{-\infty}^{t} e^{-2(t-s)}u(s)ds + f(t), \quad t \in \mathbb{R}, \tag{3.8}$$

given by

$$u(t) = \int_{-\infty}^{t} S_1(t-s)f(s)ds, \quad t \in \mathbb{R},$$

where, $S_1(t) = e^{-\frac{t}{2}}\left(\cos\left(t\frac{\sqrt{3}}{2}\right) + \sqrt{3}\sin\left(t\frac{\sqrt{3}}{2}\right)\right)$. It is remarkable that even when in this case the associated C_0-semigroup $T(t)x = e^t x$ is not exponentially stable, the resolvent family $S_1(t)$ does have this property.

A complete description of the area in the complex plane where we can choose α and β in order to have exponential stability for $S_\rho(t)$ for $\rho \in \mathbb{R} \setminus \{0\}$, is shown in the following figure. Note that, depending on the sign of ρ, there are two distinguished cases.

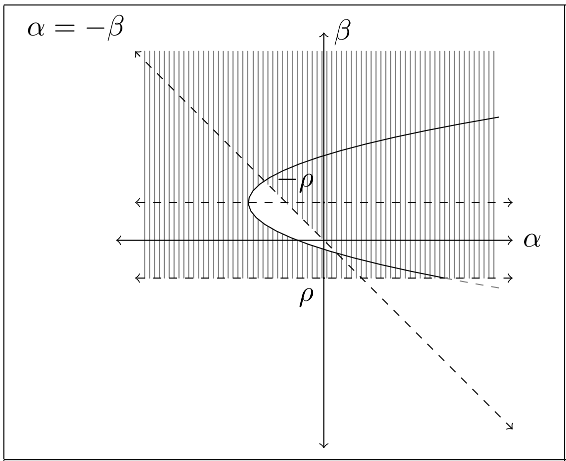

Figure 1: $\rho < 0$

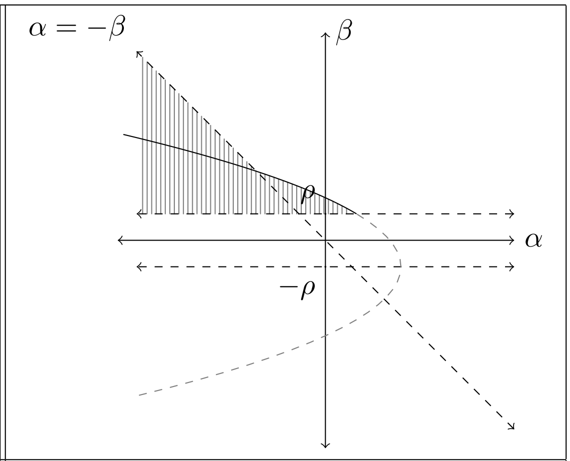

Figure 2: $\rho > 0$

Consider $A = \rho I$ for $\rho < 0$ and observe that the area shown hatched in Figure 1 includes the sector $\beta > 0, \alpha \neq 0$ and $\alpha + \beta > 0$. Hence, the area for exponential stability of $S_\rho(t)$ is considerably bigger than those guaranteed in Theorem 3.2. Note the exception of a sector located between the parabola $\beta^2 + 2\rho\beta + \rho^2 = -4\alpha\rho$ and the line $\alpha = -\beta$. Figure 2 consider the case $\rho > 0$. It shows the area where the stability of the C_0-semigroup is not necessary, in general, for the exponential stability of $S_\rho(t)$. In particular, note that the point $(-3, -2)$ belongs to the hatched area when $\rho = 1$ (cf. Example 3.8).

4. The semilinear problem

In this section we study the existence and uniqueness of solutions in $\mathcal{M}(X)$ for the semilinear integro-differential equation

$$u'(t) = Au(t) + \alpha \int_{-\infty}^{t} e^{-\beta(t-s)} Au(s)ds + f(t, u(t)), \quad t \in \mathbb{R}. \tag{4.1}$$

Definition 4.1. A function $u : \mathbb{R} \to X$ is said to be a *mild* solution to equation (4.1) if

$$u(t) = \int_{-\infty}^{t} S(t-s) f(s, u(s)) ds,$$

for all $t \in \mathbb{R}$, where $\{S(t)\}_{t \geq 0}$ is given in Proposition 3.1.

Recall that $\mathcal{M}(X)$ denote one of the spaces $P_\omega(X), AP_\omega(X), PP_\omega(X), SAP_\omega(X), AP(X), AAP(X), PAP(X), AA(X), AAA(X)$ or $PAA(X)$ defined in Section 2.

Theorem 4.2. *Let $\beta > 0, \alpha \neq 0$ and $\alpha + \beta > 0$. Assume that A generates an immediately norm continuous C_0-semigroup on a Banach space X and*

$$\sup\left\{\Re\lambda : \lambda(\lambda + \beta)(\lambda + \alpha + \beta)^{-1} \in \sigma(A)\right\} < 0. \tag{4.2}$$

If $f \in \mathcal{M}(\mathbb{R} \times X, X)$ satisfies

$$||f(t, u) - f(t, v)|| \leq L_f(t) ||u - v||, \tag{4.3}$$

for all $t \in \mathbb{R}$ and $u, v \in X$, where $L_f \in L^1(\mathbb{R})$. Then the equation (4.1) has a unique mild solution $u \in \mathcal{M}(X)$.

Proof. Define the operator $F : \mathcal{M}(X) \mapsto \mathcal{M}(X)$ by

$$(F\varphi)(t) := \int_{-\infty}^{t} S(t-s)f(s,\varphi(s))\, ds, \quad t \in \mathbb{R}, \tag{4.4}$$

where $\{S(t)\}_{t\geq 0}$ is given in Proposition 3.1. By Theorems 2.1 and 2.2 we have that F is well defined and by Proposition 3.1, there exists $\omega > 0, M > 0$ such that $||S(t)|| \leq Me^{-\omega t}$ for all $t \geq 0$. For $\varphi_1, \varphi_2 \in \mathcal{M}(X)$ and $t \in \mathbb{R}$ we have:

$$\begin{aligned}
||(F\varphi_1)(t) - (F\varphi_2)(t)|| &\leq \int_{-\infty}^{t} ||S(t-s)[f(s,\varphi_1(s)) - f(s,\varphi_2(s))]||ds \\
&\leq \int_{-\infty}^{t} L_f(s)\|S(t-s)\| \cdot \|\varphi_1(s) - \varphi_2(s)\|ds \\
&\leq M||\varphi_1 - \varphi_2||_\infty \int_{-\infty}^{t} e^{-\omega(t-s)}L_f(s)ds \\
&= M||\varphi_1 - \varphi_2||_\infty \int_{0}^{\infty} e^{-\omega\tau}L_f(t-\tau)d\tau \\
&\leq M||\varphi_1 - \varphi_2||_\infty \int_{0}^{\infty} L_f(t-\tau)d\tau \\
&= M||\varphi_1 - \varphi_2||_\infty \int_{-\infty}^{t} L_f(s)ds.
\end{aligned}$$

In general we get

$$\begin{aligned}
||(F^n\varphi_1)(t) - (F^n\varphi_2)(t)|| &\leq ||\varphi_1 - \varphi_2||_\infty \frac{M^n}{(n-1)!}\left(\int_{-\infty}^{t} L_f(s)\left(\int_{-\infty}^{s} L_f(\tau)d\tau\right)^{n-1} ds\right) \\
&\leq ||\varphi_1 - \varphi_2||_\infty \frac{M^n}{n!}\left(\int_{-\infty}^{t} L_f(s)ds\right)^n \\
&\leq ||\varphi_1 - \varphi_2||_\infty \frac{(||L_f||_1 M)^n}{n!}.
\end{aligned}$$

Hence, since $\frac{(||L_f||_1 M)^n}{n!} < 1$ for n sufficiently large, by the contraction principle F has a unique fixed point $u \in \mathcal{M}(X)$. $\square$

Some immediate consequences are the following corollaries.

Corollary 4.3. *Let $\beta > 0, \alpha \neq 0$ and $\alpha + \beta > 0$. Assume that A generates an immediately norm continuous C_0-semigroup on a Banach space X and the spectral condition (4.2). If $f \in AP(\mathbb{R} \times X, X)$ (resp. $AA(\mathbb{R} \times X, X)$) satisfies the Lipschitz condition (4.5), then the equation (4.1) has a unique mild almost periodic solution (resp. almost automorphic solution).*

Corollary 4.4. *Let $\beta > 0, \alpha \neq 0$ and $\alpha + \beta > 0$. Assume that A generates an immediately norm continuous C_0-semigroup on a Banach space X and the spectral condition (4.2). If $f \in AAP(\mathbb{R} \times X, X)$ (resp. $AAA(\mathbb{R} \times X, X)$) satisfies the Lipschitz condition (4.5),* *then the equation (4.1) has a unique mild asymptotically almost periodic solution (resp. asymptotically almost automorphic solution).*

Corollary 4.5. *Let $\beta > 0, \alpha \neq 0$ and $\alpha + \beta > 0$. Assume that A generates an immediately norm continuous C_0-semigroup on a Banach space X and the spectral condition (4.2). If $f \in PAP(\mathbb{R} \times X, X)$ (resp. $PAA(\mathbb{R} \times X, X)$) satisfies the Lipschitz condition (4.5), then the equation (4.1) has a unique mild pseudo almost periodic solution (resp. pseudo almost automorphic solution).*

In Hilbert spaces, we have the following result.

Corollary 4.6. *Let A be the generator of a C_0-semigroup on a Hilbert space H. Let $s(A) := \sup\{\Re\lambda : \lambda \in \rho(A)\}$ denote the spectral bound of A. Let $\beta > 0, \alpha \neq 0, \alpha + \beta > 0$ be given. Assume that*
(a) $\lim_{\mu\in\mathbb{R},|\mu|\to\infty} ||(\mu_0 + i\mu - A)^{-1}|| = 0$ *for some* $\mu_0 > s(A)$;
(b) $\sup\left\{\Re\lambda : \lambda(\lambda + \beta)(\lambda + \alpha + \beta)^{-1} \in \sigma(A)\right\} < 0$.
If $f \in \mathcal{M}(\mathbb{R} \times H, H)$ satisfies

$$||f(t, u) - f(t, v)|| \leq L_f(t)||u - v||,$$

for all $t \in \mathbb{R}$ and $u, v \in X$, where $L_f \in L^1(\mathbb{R})$. Then the equation (4.1) has a unique mild solution $u \in \mathcal{M}(H)$.

In the special case $A = \rho I$ we obtain the following consequence of Theorem 3.5.

Theorem 4.7. *Let $A := \rho I$ where $\rho \in \mathbb{R}$ be given. Suppose that $\rho < \beta$ and $(\alpha + \beta)\rho < 0$. Let $f \in \mathcal{M}(X)$. Consider the equation*

$$u'(t) = \rho u(t) + \rho\alpha \int_{-\infty}^{t} e^{-\beta(t-s)}u(s)ds + f(t, u(t)) \quad t \in \mathbb{R}. \tag{4.5}$$

Then the equation (4.5) has a unique solution $u \in \mathcal{M}(X)$ given by

$$u(t) = \int_{-\infty}^{t} S_\rho(t - s)f(s, u(s))ds, \quad t \in \mathbb{R}, \tag{4.6}$$

where $\{S_\rho(t)\}_{t\geq0}$ is defined in (3.4).

Remark 4.8. Observe that in case $\alpha = 0$ we must have $\rho < 0$ and hence we recover results on existence of solutions for the equation

$$u'(t) = \rho u(t) + f(t, u(t)), \quad t \in \mathbb{R}$$

in the spaces previously defined. See for instance [12, 10] and [18].

Theorem 4.9. *Let $\beta > 0, \alpha \neq 0$ and $\alpha + \beta > 0$. Assume that A generates an immediately norm continuous C_0-semigroup on a Banach space X and the spectral condition (4.2). If $f \in \mathcal{M}(\mathbb{R} \times X, X)$ satisfies*

$$||f(t, u) - f(t, v)|| \leq L_f(t)||u - v||,$$

for all $t \in \mathbb{R}$ and $u, v \in X$, where the integral $\displaystyle\int_{-\infty}^{t} L(s)ds$ exists for all $t \in \mathbb{R}$. Then the equation (4.1) has a unique mild solution $u \in \mathcal{M}(X)$.

Proof. By Proposition 3.1, there exist $M > 0, \omega > 0$ such that $||S(t)|| \leq Me^{-\omega t}$, for all $t \geq 0$. Define a new norm $|||\varphi||| := \sup_{t\in\mathbb{R}}\{v(t)||\varphi(t)||\}$, where $v(t) := e^{-k\int_{-\infty}^{t} L(s)ds}$ and k is a fixed positive constant greater than M. Define the operator F as in (4.4). Let φ_1, φ_2 be in $\mathcal{M}(X)$, then we have

$$\begin{aligned}
v(t)||(F\varphi_1)(t) - (F\varphi_2)(t)|| &= v(t)\left\|\int_{-\infty}^{t} S(t-s)[f(s,\varphi_1(s)) - f(s,\varphi_2(s))]ds\right\| \\
&\leq Mv(t)\int_{-\infty}^{t} L_f(s)e^{-\omega(t-s)}\|\varphi_1(s)-\varphi_2(s)\|ds \\
&= Mv(t)\int_{-\infty}^{t} e^{-\omega(t-s)}L_f(s)\|\varphi_1(s)-\varphi_2(s)\|ds \\
&= Mv(t)\int_{0}^{\infty} e^{-\omega\tau}L_f(t-\tau)\|\varphi_1(t-\tau)-\varphi_2(t-\tau)\|d\tau \\
&\leq Mv(t)\int_{0}^{\infty} L_f(t-\tau)\|\varphi_1(t-\tau)-\varphi_2(t-\tau)\|d\tau \\
&= M\int_{-\infty}^{t} v(s)^{-1}v(t)L_f(s)v(s)\|\varphi_1(s)-\varphi_2(s)\|ds \\
&\leq M|||\varphi_1-\varphi_2|||\int_{-\infty}^{t} v(t)v(s)^{-1}L_f(s)ds \\
&= \frac{M}{k}|||\varphi_1-\varphi_2|||\int_{-\infty}^{t} ke^{k\int_t^s L_f(\tau)d\tau}L(s)ds \\
&= \frac{M}{k}|||\varphi_1-\varphi_2|||\int_{-\infty}^{t} \frac{d}{ds}\left(e^{k\int_t^s L_f(\tau)d\tau}\right)ds \\
&= \frac{M}{k}[1-e^{-k\int_{-\infty}^{t} L_f(\tau)d\tau}]|||\varphi_1-\varphi_2||| \\
&\leq \frac{M}{k}|||\varphi_1-\varphi_2|||.
\end{aligned}$$

Hence, since $M/k < 1$, F has a unique fixed point $u \in \mathcal{M}(X)$. □

We finish this paper with the following application.

Example 4.10. *Consider the problem*

$$(4.7)\qquad \begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + \displaystyle\int_{-\infty}^{t} e^{-(t-s)}\frac{\partial^2 u}{\partial x^2}(s,x)ds + f(t,u(t)) \\ u(0,t) = u(\pi,t) = 0, \end{cases}$$

with $x \in [0,\pi], t \in \mathbb{R}$. Let $X = L^2[0,\pi]$ and define $A := \frac{\partial^2}{\partial x^2}$, with domain $D(A) = \{g \in H^2[0,\pi] : g(0) = g(\pi) = 0\}$. Then (4.7) can be converted into the abstract form (4.1) with $\alpha = \beta = 1$. It is well known that A generates an analytic (and hence immediately norm continuous) and compact C_0- semigroup $T(t)$ on X. The compactness of $T(t)$ implies that $\sigma(A) = \sigma_p(A) = \{-n^2 : n \in \mathbb{N}\}$. Since we must have $\lambda(\lambda+\beta)(\lambda+\alpha+\beta)^{-1} \in \sigma(A)$ we *need to solve the equations* $\frac{\lambda(\lambda+1)}{\lambda+2} = -n^2$, *obtaining (see* [6]*)*

$$\lambda_1 = -1 \pm i,\ \lambda_2 = \frac{-5 \pm 7i}{2},$$

and

$$\lambda_n = \frac{-(n^2+1) \pm \sqrt{(n^2-3)^2 - 8}}{2} \leq -2,$$

for all $n \geq 3$. *We conclude that*

$$\sup\left\{\Re\lambda : \lambda(\lambda+\beta)(\lambda+\alpha+\beta)^{-1} \in \sigma(A)\right\} = -1.$$

Hence, from Theorem 4.2 (resp. Theorem 4.9) we obtain that if $f \in \mathcal{M}(\mathbb{R}\times X, X)$ *satisfies*

$$||f(t,u) - f(t,v)|| \leq L_f(t)||u-v||,$$

for all $t \in \mathbb{R}$ *and* $u, v \in X$, *where* $L_f \in L^1(\mathbb{R})$ *(resp.* $\int_{-\infty}^t b(s)ds$ *exists for all* $t \in \mathbb{R}$. *), then equation (4.7) has a unique mild solution* $u \in \mathcal{M}(X)$. *In particular, if* $f(t,\varphi)(s) = b(t)\sin(\varphi(s))$, *for all* $\varphi \in X$, $t \in \mathbb{R}$ *with* $b \in \mathcal{M}(X)$, *then we observe that* $t \to f(t,\varphi)$ *belongs to* $\mathcal{M}(X)$, *for each* $\varphi \in X$, *and we have*

$$||f(t,\varphi_1) - f(t,\varphi_2)||_2^2 \leq \int_0^\pi |b(t)|^2|\sin(\varphi_1(s)) - \sin(\varphi_2(s))|ds \leq |b(t)|^2||\varphi_1 - \varphi_2||_2^2.$$

In consequence, problem (4.7) has a unique mild solution in $\mathcal{M}(X)$ *if, either* $b \in L^1(\mathbb{R})$ *(by Theorem 4.2) or* $\int_{-\infty}^t b(s)ds$ *exists for all* $t \in \mathbb{R}$ *(by Theorem 4.9).*

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