MAXIMAL REGULARITY FOR DEGENERATE DIFFERENTIAL EQUATIONS WITH INFINITE DELAY IN PERIODIC VECTOR-VALUED FUNCTION SPACES.

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Abstract. Let A and M be closed linear operators defined on a complex Banach space X and $a \in L^1(\mathbb{R}_+)$ be an scalar kernel. We use operator-valued Fourier multipliers techniques to obtain necessary and sufficient conditions to guarantee the existence and uniqueness of periodic solutions to the equation

$$\frac{d}{dt}(Mu(t)) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t), \quad t > 0,$$

with initial condition $Mu(0) = Mu(2\pi)$, solely in terms of spectral properties of the data. Our results are obtained in the scales of periodic Besov, Triebel-Lizorkin and Lebesgue vector-valued function spaces.

1. Introduction

In this paper, we study maximal regularity in Lebesgue, Besov and Triebel-Lizorkin vector-valued function spaces for the following class of differential equation with infinite delay

$$\frac{d}{dt}(Mu(t)) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t), \qquad 0 \le t \le 2\pi, \tag{1.1}$$

where $(A, D(A))$ and $(M, D(M))$ are (unbounded) closed linear operators defined on a Banach space X, with $D(A) \subseteq D(M)$, $a \in L^1(\mathbb{R}_+)$ an scalar-valued kernel, and f an X-valued function defined on $[0, 2\pi]$.

The model (1.1) corresponds to problems related with viscoelastic materials; that is, materials whose stresses at any instant depend on the complete history of strains that the material has undergone (see [21]) or heat conduction with memory. For more details, see, for instance, [14], [15] and [23].

The recent linear theory of maximal regularity is not only important on its own but it is also the indispensable basis for the theory of nonlinear evolution equations, see e.g. [1], [13], [23] and references therein. In case $M = I$ (the identity in X) and $a \equiv 0$, equation (1.1) with periodic initial conditions have been studied by Arendt-Bu, Bu-Kim

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and characterizations of the maximal regularity in Lebesgue, Besov and Triebel-Lizorkin vector-valued function spaces were obtained using the resolvent set of A. See [3], [4] and [7].

On the other hand, characterizations of maximal regularity for equation (1.1) in case $M = I$ and $a \in L^1(\mathbb{R})$ have been obtained by Keyantuo-Lizama [18] in Lebesgue and Besov vector-valued function spaces and by Bu-Fang [8] in Triebel-Lizorkin vector valued spaces. We note that periodic solutions have been also studied by other authors, for example in [11] using topological methods.

Characterizations of maximal regularity in these vector-valued function spaces, for the degenerate abstract equation (1.1) with periodic initial conditions

$$Mu(0) = Mu(2\pi), \tag{1.2}$$

and $a \equiv 0$ has been studied recently by the authors [22]. The method used in [22] is based on the results given by Arendt-Bu ([3, 4]) and Bu-Kim ([7]), for operator-valued Fourier multipliers in Lebesgue, Besov and Triebel-Lizorkin vector-valued function spaces. It is worthwhile to mention that this method enables to improve and extend results on degenerate abstract equations obtained previously in the literature. Compare e.g. Barbu-Favini [5] and [22].

In this article, we apply the same method to obtain characterizations of maximal regularity for (1.1) in the above mentioned vector-valued function spaces. The advantage of our approach is clear. We recover, as special cases, the results in [3], [4], [7], [8], [18] and [22]. In addition, we are also able to improve the results in [22] where is assumed the closedness of the operator $ikM - A$ for all $k \in \mathbb{Z}$ to prove boundedness of $(ikM - A)^{-1}$. Indeed, we give a simple argument to show that this condition is not necessary under the presence of maximal regularity (cf. the proof of Theorem 3.4).

It is remarkable that in the characterizations that we will obtain, no conditions on the commutativity of operators A and M, or in the existence of bounded inverse of A or M, are needed. Also, in the case of periodic Besov and Triebel-Lizorkin function spaces, no geometrical assumption on the underlying Banach space X is needed.

The plan of the paper is the following: After some preliminaries in the second section, assuming that X is an UMD space, we characterize in Section 3 the uniqueness and existence of a strong L^p-solution for the problem (1.1)-(1.2) solely in terms of a property of R-boundedness for the sequence of operators $ikM(ikM - (1 + \tilde{a}(ik))A)^{-1}$. Here the tilde denotes Laplace transform of $a(t)$. In Section 4, we obtain a characterization in the context of Besov spaces. We notice that, as particular case of this characterization, a simply condition to guarantee the existence and uniqueness of solution in Hölder spaces $C^s((0, 2\pi); X)$, $0 < s < 1$, in general Banach spaces X, is obtained. Namely

Theorem 1.1. *Let $s > 0$ and $A : D(A) \subseteq X \to X$, $M : D(M) \subseteq X \to X$ be closed linear operators on a Banach space X. Suppose that $D(A) \subseteq D(M)$, the sequence $\{\tilde{a}(ik)\}_{k\in\mathbb{Z}}$ is 2-regular and $(ikM - (1 + \tilde{a}(ik))A)$ are closed operators for all $k \in \mathbb{Z}$. Then, the following assertions are equivalent:*

(i) For every $f \in C^s((0,2\pi);X)$ *there is a unique strong* C^s*-solution of (1.1) such that* $Mu(0) = Mu(2\pi)$*;*

(ii) $(ikM - (1+\tilde{a}(ik))A)^{-1}$ *exists for all* $k \in \mathbb{Z}$ *and*

$$\sup_{k\in\mathbb{Z}} ||ikM(ikM - (1+\tilde{a}(ik))A)^{-1}|| < \infty.$$

In Section 5 we give the corresponding characterization in case of the scale of Triebel-Lizorkin vector valued spaces. The difference with the scale of Besov vector valued spaces is only that we need more regularity of the sequence $\tilde{a}(ik)$. In Section 6, we apply our results in two concrete examples.

2. Preliminaries

Given $1 \le p < \infty$, we denote by $L^p_{2\pi}(\mathbb{R}, X)$ the space of all 2π-periodic Bochner measurable X-valued functions f, such that the restriction of f to $[0, 2\pi]$ is p-integrable. For a function $f \in L^1_{2\pi}(\mathbb{R}, X)$ we denote by $\hat{f}(k)$, the k-th Fourier coefficient of f:

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt} f(t)dt,$$

for all $k \in \mathbb{Z}$. We remark that the Fourier coefficients determine the function f, that is, $\hat{f}(k) = 0$ for all $k \in \mathbb{Z}$ if and only if $f(t) = 0$ a.e. Let X, Y be Banach spaces. We denote by $\mathcal{B}(X,Y)$ be the space of all bounded linear operators from X to Y. When $X = Y$, we write simply $\mathcal{B}(X)$. For a linear operator A on X, we denote domain by $D(A)$ and its resolvent set by $\rho(A)$. By $[D(A)]$ we denote the domain of A equipped with the graph norm.

We begin with some preliminaries about operator-valued Fourier multipliers. More information can be found in [4] in the periodic case and for the non-periodic case, see for instance, [25].

Definition 2.1. [3] *For* $1 \le p < \infty$*, we say that a sequence* $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ *is an* L^p*-multiplier if, for each* $f \in L^p_{2\pi}(\mathbb{R}, X)$*, there exists* $u \in L^p_{2\pi}(\mathbb{R}, Y)$ *such that*

$$\hat{u}(k) = M_k \hat{f}(k) \text{ for all } k \in \mathbb{Z}.$$

It follows from the uniqueness theorem of Fourier series that u is uniquely determined by f.

For $j \in \mathbb{N}$, denote by r_j the j-th Rademacher function on $[0,1]$ i.e. $r_j(t) = sgn(\sin(2^j\pi t))$ and for $x \in X$, $r_j \otimes x$, denote the vector valued function $t \to r_j(t)x$.

Definition 2.2. *A family of operators* $\mathcal{T} \subset \mathcal{B}(X,Y)$ *is called R-bounded, if there is a constant* $C_p > 0$ *and* $p \in [1,\infty)$ *such that for each* $N \in \mathbb{N}, T_j \in \mathcal{T}, x_j \in X, j = 1, ..., N$ *the inequality*

$$||\sum_{j=1}^N r_j \otimes T_j x_j||_{L^p((0,1);Y)} \le C_p||\sum_{j=1}^N r_j \otimes x_j||_{L^p((0,1);X)} \tag{2.1}$$

is valid.

If (2.1) holds for some $p \in [1,\infty)$ then if holds for all $p \in [1,\infty)$. The smallest C_p in (2.1) is called R-bound of $\mathcal{T}$, we denote it by $R_p(\mathcal{T})$.

We remark that large classes of classical operators are R-bounded (cf. [16] and references therein). Hence, this assumption is not too restrictive for the applications that we consider in this article.

Remark 2.3.

Several properties of R-bounded families can be founded in the monograph of Denk-Hieber-Prüss [13]. For the reader's convenience, we summarize here from [13, Section 3] some results.

(a) If $\mathcal{T} \subset \mathcal{B}(X,Y)$ is R-bounded then it is uniformly bounded, with

$$\sup\{||T|| : T \in \mathcal{T}\} \leq R_p(\mathcal{T}).$$

(b) The definition of R-boundedness is independent of $p \in [1,\infty)$.

(c) When X and Y are Hilbert spaces, $\mathcal{T} \subset \mathcal{B}(X,Y)$ is R-bounded if and only if $\mathcal{T}$ is uniformly bounded.

(d) Let X, Y be Banach spaces and $\mathcal{T}, \mathcal{S} \subset \mathcal{B}(X,Y)$ be R-bounded. Then

$$\mathcal{T} + \mathcal{S} = \{T + S : T \in \mathcal{T}, S \in \mathcal{S}\}$$

is R-bounded as well, and $R_p(\mathcal{T} + \mathcal{S}) \leq R_p(\mathcal{T}) + R_p(\mathcal{S})$.

(e) Let X, Y, Z be Banach spaces, and $\mathcal{T} \subset \mathcal{B}(X,Y)$ and $\mathcal{S} \subset \mathcal{B}(Y,Z)$ be R-bounded. Then

$$\mathcal{S}\mathcal{T} = \{ST : T \in \mathcal{T}, S \in \mathcal{S}\}$$

is R-bounded, and $R_p(\mathcal{S}\mathcal{T}) \leq R_p(\mathcal{S})R_p(\mathcal{T})$.

(g) Let X, Y be Banach spaces and $\mathcal{T} \subset \mathcal{B}(X,Y)$ be R-bounded. If $\{\alpha_k\}_{k\in\mathbb{Z}}$ is a bounded sequence, then $\{\alpha_k T : T \in \mathcal{T}\}$ is R-bounded.

Proposition 2.4. [3] *Let X be a Banach space and $\{M_k\}_{k\in\mathbb{Z}}$ be an L^p-multiplier, where $1 \leq p < \infty$. Then, the set $\{M_k : k \in \mathbb{Z}\}$ is R-bounded.*

A Banach space X is said to be UMD, if the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for some (and then all) $p \in (1,\infty)$. Here the Hilbert transform H of a function $f \in \mathcal{S}(\mathbb{R}, X)$, the Schwartz space of rapidly decreasing X-valued functions, is defined by

$$Hf := \frac{1}{\pi} PV(\frac{1}{t}) * f.$$

These spaces are also called $\mathcal{HT}$ spaces. It is a well known that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces. This has been shown by Bourgain [6] and Burkholder [9]. Some examples of UMD-spaces include the Hilbert spaces, Sobolev spaces $W_p^s(\Omega)$, $1 < p < \infty$, Lebesgue spaces $L^p(\Omega, \mu)$, $1 < p < \infty$, $L^p(\Omega, \mu; X)$, $1 < p < \infty$, when X is a UMD-space. Moreover, a UMD-space is reflexive and therefore, $L^1(\Omega, \mu)$, $L^\infty(\Omega, \mu)$ (in the case infinite dimensional) and $C^s([0, 2\pi]; X)$ are not UMD. More information on UMD spaces can be found in [6, 9] and [10].

We recall the following theorem, due to Arendt-Bu [3, Theorem 1.3], in the context of UMD-spaces.

Theorem 2.5. [3] *Let X, Y be UMD spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$. If the sets $\{M_k\}_{k\in\mathbb{Z}}$ and $\{k(M_{k+1}-M_k)\}_{k\in\mathbb{Z}}$ are R-bounded, then $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1<p<\infty$.*

We will need the following Lemmas.

Lemma 2.6. [3] *Let $f, g \in L^p_{2\pi}(\mathbb{R}; X)$, where $1 \le p < \infty$ and A is a closed operator on a Banach space X. Then, the following assertions are equivalent.*

(i) $f(t) \in D(A)$ and $Af(t) = g(t)$, a.e.

(ii) $\hat{f}(k) \in D(A)$ and $A\hat{f}(k) = \hat{g}(k)$, for all $k \in \mathbb{Z}$.

Lemma 2.7. [22] *Let M be a closed operator, $u \in L^p_{2\pi}(\mathbb{R}; [D(M)])$ and $u' \in L^p_{2\pi}(\mathbb{R}; X)$ for $1 \le p < \infty$. Then, the following assertions are equivalent,*

(i) $\int_0^{2\pi} (Mu)'(t)dt = 0$ and there exist $x \in X$ such that $Mu(t) = x + \int_0^t (Mu)'(s)ds$ a.e. on $[0, 2\pi]$;

(ii) $\widehat{(Mu)'}(k) = ikM\hat{u}(k)$ for all $k \in \mathbb{Z}$.

Let a be a complex valued function. We define the set

$$\begin{aligned}\rho_{M,a}(A) = \{\lambda \in \mathbb{C} : (\lambda M - (1+a(\lambda))A) : D(A) \cap D(B) \to X \\ \text{is invertible and } (\lambda M - (1+a(\lambda))A)^{-1} \in \mathcal{B}(X)\},\end{aligned}$$

and denote by $\sigma_{M,a}(A)$ the complementary set $\mathbb{C} \setminus \rho_{M,a}(A)$. If $M = I$, is the identity operator on X and $a \equiv 0$, we denote simply the set $\rho_{M,a}(A)$ by $\rho(A)$ and as usual we call this set, the resolvent set of A. Denote by $\tilde{a}(\lambda)$ the Laplace transform of a. In what follows, we always assume that $\tilde{a}(ik)$ exists for all $k \in \mathbb{Z}$.

Henceforth, we use the following notation:

$$a_k := \tilde{a}(ik)$$

and we assume that $a_k \neq -1$ for all $k \in \mathbb{Z}$.

Remark 2.8. *Note that by the Riemann-Lebesgue lemma, we have that the sequences $\{a_k\}_{k\in\mathbb{Z}}$ and $\{\frac{1}{\alpha + a_k}\}_{k\in\mathbb{Z}}$ ($\alpha \neq 0$) are bounded.*

Finally, from [20] we recall the concept of n-regularity for $n = 1, 2, 3$. The general notion of n-regularity is the discrete analogue for the notion of n-regularity related to Volterra integral equations (see [23, Chapter I, Section 3.2]).

Definition 2.9. *A sequence $\{c_k\}_{k\in\mathbb{Z}} \subset \mathbb{C} \setminus \{0\}$ is said to be:*

(i) 1-regular, if the sequence $\left\{k\frac{(c_{k+1}-c_k)}{c_k}\right\}_{k\in\mathbb{Z}}$ is bounded.

(ii) 2-regular, if it is 1-regular and the sequence $\left\{k^2\dfrac{(c_{k+1}-2c_k+c_{k-1})}{c_k}\right\}_{k\in\mathbb{Z}}$ *is bounded.*

(iii) 3-regular, if it is 2-regular and the sequence $\left\{k^3\dfrac{(c_{k+2}-3c_{k+1}+3c_k-c_{k-1})}{c_k}\right\}_{k\in\mathbb{Z}}$ *is bounded.*

Note that if $\{c_k\}_{k\in\mathbb{Z}}$ is 1-regular, then $\lim_{|k|\to\infty} c_{k+1}/c_k = 1$. For more details of n-regularity sequences, see [20].

3. Maximal regularity on vector-valued Lebesgue spaces

To characterize the maximal regularity, we begin with the study of the relation between multipliers and R-boundedness of a sequences of operators. Consider the problem

$$\begin{cases} \dfrac{d}{dt}(Mu(t)) = Au(t) + \displaystyle\int_{-\infty}^{t} a(t-s)Au(s)ds + f(t),\ 0\le t\le 2\pi, \\ \\ Mu(0) = Mu(2\pi), \end{cases} \tag{3.1}$$

where $(A, D(A))$ and $(M, D(M))$ are closed linear operators on X, $D(A)\subseteq D(M)$, $a\in L^1(\mathbb{R}_+)$ is a scalar-valued kernel, and $f\in L^p_{2\pi}(\mathbb{R},X), p\ge 1$.

From [22], we recall that for a given closed operator M, and $1\le p<\infty$, the set $H^{1,p}_{per,M}(\mathbb{R};[D(M)])$ is defined by

$$\begin{aligned} H^{1,p}_{per,M}(\mathbb{R};[D(M)]) \quad = \quad & \{u\in L^p_{2\pi}(\mathbb{R};[D(M)]) : \exists v\in L^p_{2\pi}(\mathbb{R};X), \\ & \text{such that } \ \hat{v}(k) = ikM\hat{u}(k) \text{ for all } k\in\mathbb{Z}\}. \end{aligned}$$

When $M = I$, we denote $H^{1,p}_{per}(\mathbb{R};X)$; see [3].

Lemma 3.1. *Let X be a UMD-space. Suppose that the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular. Then, $\{\frac{1}{1+a_k}I\}_{k\in\mathbb{Z}}$ is an L^p-multiplier.*

Proof. By Remarks 2.8 and 2.3 (g), $\{\frac{1}{1+a_k}I\}_{k\in\mathbb{Z}}$ is R-bounded. Moreover,

$$k\left(\frac{1}{1+a_{k+1}}-\frac{1}{1+a_k}\right) \quad = \quad -k\left(\frac{a_{k+1}-a_k}{a_k}\right)\cdot a_k\cdot\frac{1}{1+a_{k+1}}\cdot\frac{1}{1+a_k}.$$

Since $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular, we conclude the proof of Lemma by Remark 2.8 and Theorem 2.5. ■

The following Proposition is an extension of [22, Proposition 3.2].

Proposition 3.2. *Suppose that the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular. Let $A : D(A)\subseteq X\to X$ and $M : D(M)\subseteq X\to X$ be linear closed operators defined on a UMD space X. Suppose that $D(A)\subseteq D(M)$. Then, the following assertions are equivalent*

(i) $\{ik\}_{k\in\mathbb{Z}}\subset\rho_{M,\tilde{a}}(A)$ and $\{ikM(ikM-(1+a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1<p<\infty$;

(ii) $\{ik\}_{k\in\mathbb{Z}}\subset\rho_{M,\tilde{a}}(A)$ and $\{ikM(ikM-(1+a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.

Proof. Define $N_k := (ikM - (1+a_k)A)^{-1}$ and $M_k := ikM(ikM - (1+a_k)A)^{-1}$, $k \in \mathbb{Z}$. By the Closed Graph Theorem we can show that if $ik \in \rho_{M,\tilde{a}}(A)$, then M_k are bounded operators for each $k \in \mathbb{Z}$. By Proposition 2.4 it follows that (i) implies (ii). Conversely, by Theorem 2.5 is sufficient to prove that the set $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ is R-bounded. In fact, we note the following

$$\begin{aligned} k[M_{k+1} - M_k] &= k\big[i(k+1)MN_{k+1} - ikMN_k\big] \\ &= ikMN_{k+1}\big[(k+1)[ikM - (1+a_k)A] - \\ &\quad k[i(k+1)M - (1+a_{k+1})A]\big]N_k \\ &= ikMN_{k+1}\left[k\tfrac{(a_{k+1}-a_k)}{1+a_k}\right](1+a_k)AN_k - M_k(1+a_k)AN_k. \end{aligned} \tag{3.2}$$

Since $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular, the sequence $\{k(\frac{a_{k+1}-a_k}{1+a_k})\}_{k\in\mathbb{Z}}$ is bounded. The identity $(1+a_k)AN_k = M_k - I$, imply that $\{(1+a_k)AN_k\}_{k\in\mathbb{Z}}$ is R-bounded. We conclude the proof using the Remark 2.3. ∎

Next, we introduce the following definition of solution of (3.1).

Definition 3.3. *We say that a function $u \in H^{1,p}_{per,M}(\mathbb{R};[D(M)]) \cap L^p_{2\pi}(\mathbb{R};[D(A)])$ is a strong L^p-solution of (3.1) if $u(t) \in D(A)$ and equation (3.1) holds for a.e. $t \in [0,2\pi]$.*

Let $a \in L^1(\mathbb{R}_+)$ and A be a closed operator defined on X. For $u \in L^p_{2\pi}(\mathbb{R};[D(A)])$ denote by $F(t) := (a{*}Au)(t) = \int_{-\infty}^t a(t-s)Au(s)ds$. An easy computation show that $\hat{F}(k) = A\tilde{a}(ik)\hat{u}(k), k \in \mathbb{Z}$ (where the hat $\hat{}$ denotes Fourier transform). It is remarkable that in case $a \equiv 0$ the next theorem improves the main result in [22].

Theorem 3.4. *Let X be a UMD space. Let $A : D(A) \subseteq X \to X$, $M : D(M) \subseteq X \to X$ be linear closed operators. Suppose that $D(A) \subseteq D(M)$ and the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular. Then, the following assertions are equivalent*

(i) For every $f \in L^p_{2\pi}(\mathbb{R}, X)$, there exist a unique strong L^p-solution of (3.1);

(ii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\{ikM(ikM - (1+a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$;

(iii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\{ikM(ikM - (1+a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.

Proof. $(i) \Rightarrow (ii)$. Let $k \in \mathbb{Z}$ and $y \in X$. Define $f(t) = e^{ikt}y$. By hypothesis, there exists $u \in H^{1,p}_{per,M}(\mathbb{R};[D(M)]) \cap L^p_{2\pi}(\mathbb{R};[D(A)])$ such that $u(t) \in D(A)$ and $(Mu)'(t) = Au(t) + (a{*}Au)(t) + f(t)$. Taking Fourier transform on both sides, we have $\hat{u}(k) \in D(A)$ and,

$$\begin{aligned} ikM\hat{u}(k) &= (1+a_k)A\hat{u}(k) + \hat{f}(k) \\ &= (1+a_k)A\hat{u}(k) + y. \end{aligned}$$

Thus, $(ikM - (1+a_k)A)\hat{u}(k) = y$ for all $k \in \mathbb{Z}$ and we conclude that $(ikM - (1+a_k)A)$ is surjective. Let $x \in D(A)$. If $(ikM - (1+a_k)A)x = 0$, then $u(t) = e^{ikt}x$ defines a periodic solution of (3.1). Hence $u \equiv 0$ by the assumption of uniqueness, and thus $x = 0$. Therefore, $(ikM - (1+a_k)A)$ is bijective.

Now, we must prove that $(ikM-(1+a_k)A)^{-1}$ is a bounded operator for all $k\in\mathbb{Z}$. Suppose that $(ikM-(1+a_k)A)^{-1}$ has no bounded inverse. Then, for each $k\in\mathbb{Z}$ there exists a sequence $(y_{n,k})_{n\in\mathbb{Z}}\subset X$ such that $||y_{n,k}||\leq 1$ and

$$||(ikM-(1+a_k)A)^{-1}y_{n,k}||\geq n^2, \text{ for all } n\in\mathbb{Z}.$$

Thus, we obtain that the sequence $x_k:=y_{k,k}$, satisfies

$$||(ikM-(1+a_k)A)^{-1}x_k||\geq k^2, \text{ for all } k\in\mathbb{Z}.$$

Let $f(t):=\sum_{k\in\mathbb{Z}\setminus\{0\}}e^{ikt}\frac{x_k}{k^2}$. Observe that $f\in L^p_{2\pi}(\mathbb{R},X)$ and so, by hypothesis, there exists a unique strong solution $u\in L^p_{2\pi}(\mathbb{R},X)$ of (3.1). One can check that $u(t)=\sum_{k\in\mathbb{Z}\setminus\{0\}}(ikM-(1+a_k)A)^{-1}e^{ikt}\frac{x_k}{k^2}$. Since

$$\left\|(ikM-(1+a_k)A)^{-1}e^{ikt}\frac{x_k}{k^2}\right\|\geq 1, \text{ for all } k\in\mathbb{Z}\setminus\{0\},$$

we obtain $u\notin L^p_{2\pi}(\mathbb{R},X)$. A contradiction. Thus, we conclude that $(ikM-(1+a_k)A)^{-1}$ is a bounded operator for all $k\in\mathbb{Z}$, and therefore $ik\in\rho_{M,\tilde{a}}(A)$ for all $k\in\mathbb{Z}$.

Using the Closed Graph Theorem, we have that there exist a constant $C>0$ independent of $f\in L^p_{2\pi}(\mathbb{R};X)$ such that

$$||(Mu)'||_{L^p}+||a\dot{*}Au||_{L^p}+||Au||_{L^p}\leq C||f||_{L^p}.$$

Note that for $f(t)=e^{itk}y, y\in X$, the solution u of (3.1) is given by $u(t)=(ikM-(1+a_k)A)^{-1}e^{ikt}y$. Hence,

$$||ikM(ikM-(1+a_k)A)^{-1}y||\leq C||y||.$$

We obtain that for $k\in\mathbb{Z}$, $ikM(ikM-(1+a_k)A)^{-1}$ is a bounded operator. Let $f\in L^p_{2\pi}(\mathbb{R},X)$, by hypothesis, there exist $u\in H^{1,p}_{per,M}(\mathbb{R};[D(M)])\cap L^p_{2\pi}(\mathbb{R};[D(A)])$ such that $u(t)\in D(A)$ and $(Mu)'(t)=Au(t)+(a\dot{*}Au)(t)+f(t)$. Taking Fourier transform on both sides, and using that $(ikM-(1+a_k)A)$ is invertible, we have $\hat{u}(k)\in D(A)$ and $\hat{u}(k)=(ikM-(1+a_k)A)^{-1}\hat{f}(k)$. Now, since $u\in H^{1,p}_{per,M}(\mathbb{R};[D(M)])$ and by definition of $H^{1,p}_{per,M}(\mathbb{R};[D(M)])$, there exist $v\in L^p_{2\pi}(\mathbb{R},X)$ such that $\hat{v}(k)=ikM\hat{u}(k)$ for all $k\in\mathbb{Z}$. Therefore, we have $\hat{v}(k)=ikM\hat{u}(k)=ikM(ikM-(1+a_k)A)^{-1}\hat{f}(k)$.

(ii) $\Rightarrow$ *(i)*. Define $M_k=ikM(ikM-(1+a_k)A)^{-1}$ and $N_k=(ikM-(1+a_k)A)^{-1}$. Suppose that $\{ik\}_{k\in\mathbb{Z}}\subset\rho_{M,\tilde{a}}(A)$ and $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. For $f\in L^p_{2\pi}(\mathbb{R},X)$ there exist $u\in L^p_{2\pi}(\mathbb{R},X)$ such that $\hat{u}(k)=ikM(ikM-(1+a_k)A)^{-1}\hat{f}(k)$, for all $k\in\mathbb{Z}$. The identity $I=M_k-(1+a_k)AN_k$ imply that

$$\begin{aligned}\hat{u}(k)&=ikM(ikM-(1+a_k)A)^{-1}\hat{f}(k)\\&=(I+(1+a_k)AN_k)\hat{f}(k).\end{aligned}$$

So, we obtain $\widehat{(u-f)}(k)=(1+a_k)AN_k\hat{f}(k)$. By Lemma 3.1, the sequence $\{\frac{1}{1+a_k}I\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. Thus, for $u-f\in L^p_{2\pi}(\mathbb{R},X)$ there exists $v\in L^p_{2\pi}(\mathbb{R},X)$ such that $\hat{v}(k)=\frac{1}{1+a_k}\widehat{(u-f)}(k)=AN_k\hat{f}(k)$. Since that $0\in\rho_{M,\tilde{a}}(A)$ we obtain that $A^{-1}\in\mathcal{B}(X)$, and therefore $w := A^{-1}v \in L^p_{2\pi}(\mathbb{R}, X)$ and $\hat{w}(k) = N_k\hat{f}(k)$. Hence $ikM\hat{w}(k) - (1 + a_kA)\hat{w}(k) = \hat{f}(k)$. Now, observe that for all $k \in \mathbb{Z}$, we have

$$\hat{u}(k) = ikM(ikM - (1 + a_k)A)^{-1}\hat{f}(k) = ikM\hat{w}(k).$$

Thus $w \in H^{1,p}_{per,M}(\mathbb{R}, [D(M)]) \cap L^p_{2\pi}(\mathbb{R}; [D(A)])$. Moreover $Mw(0) = Mw(2\pi)$, since $w(0) = w(2\pi)$ and $w(t) \in D(A)$. Since A and M are closed operators and $\widehat{(Mw)'}(k) = ikM\hat{w}(k) = (1 + a_k)A\hat{w}(k) + \hat{f}(k)$, for all $k \in \mathbb{Z}$, one has $(Mw)'(t) = Aw(t) + (a \dot{*} Au)(t) + f(t)$ *a.e.* by Lemmas 2.6 and 2.7. Thus, we conclude that w is a strong L^p-solution of (3.1).

Finally, to see the uniqueness, let $u \in H^{1,p}_{per,M}(\mathbb{R}, [D(M)]) \cap L^p_{2\pi}(\mathbb{R}, [D(A)])$ such that $(Mu)'(t) = Au(t) + (a \dot{*} Au)(t)$. Taking, Fourier transform on both sides we have $\hat{u}(k) \in D(A)$, and $(ikM - (1 + a_k)A)\hat{u}(k) = 0$, for all $k \in \mathbb{Z}$. Since $(ikM - (1 + a_k)A)$ is invertible for all $k \in \mathbb{Z}$, we obtain $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$, and thus $u \equiv 0$.

$(ii) \Leftrightarrow (iii)$. Proposition 3.2. ■

In the Hilbert case, we obtain a simple condition to existence and uniqueness of solutions of (3.1). Since in Hilbert spaces the concepts of R-boundedness and boundedness are equivalent [13], the proof of the next Corollary follows from Theorem 3.4.

Corollary 3.5. *Let H be a Hilbert space, $A : D(A) \subset H \to H$, and $M : D(M) \subset H \to H$ closed linear operators satisfying $D(A) \subseteq D(M)$. Suppose that the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular. Then, for $1 < p < \infty$, the following assertions are equivalent*

(i) For every $f \in L^p_{2\pi}(\mathbb{R}, H)$, there exists a unique strong L^p-solution of (3.1);

(ii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\sup_{k\in\mathbb{Z}} ||ikM(ikM - (1 + a_k)A)^{-1}|| < \infty$.

Note that the solution $u(\cdot)$ given in Theorem 3.4 satisfies the following maximal regularity property.

Corollary 3.6. *In the context of Theorem 3.4, if condition (iii) is fulfilled, we have $(Mu)', Au, a \dot{*} Au \in L^p_{2\pi}(\mathbb{R}, X)$. Moreover, there exists a constant $C > 0$ independent of $f \in L^p_{2\pi}(\mathbb{R}; X)$ such that*

$$||(Mu)'||_{L^p} + ||Au||_{L^p} + ||a \dot{*} Au||_{L^p} \le C||f||_{L^p}. \tag{3.3}$$

Remark 3.7.

The Fejer's Theorem (see [3, Proposition 1.1]) can be used to construct the solution u given in the Theorem 3.4. More precisely, if $M_k = ikM(ikM - (1 + a_k)A)^{-1}$ satisfies the condition (ii) or (iii) in the Theorem 3.4, then for $f \in L^p_{2\pi}(\mathbb{R}, X)$, the solution $u \in L^p_{2\pi}(\mathbb{R}, X)$ of (3.1) is given by

$$u(\cdot) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \otimes M_k\hat{f}(k),$$

where $e_k(t) := e^{ikt}$, $t \in \mathbb{R}$ and the convergence holds in $L^p_{2\pi}(\mathbb{R}, X)$.

4. Maximal Regularity on Hölder and Besov Spaces

In this section, we formulate analogous theorems to the above section, in the context of Hölder and Besov Spaces.

Besov spaces form one class of function spaces that are of special interest. The relatively complicated definition is recompensed by useful applications to differential equations (see [2] for a concrete model).

We consider solutions to equation (3.1) in $B^s_{p,q}((0,2\pi);X)$, the vector-valued periodic Besov spaces for $1 \le p \le \infty$, $s > 0$. Is remarkable that in this case, there are no geometrical conditions on the Banach space X. For the definition and main properties of these spaces we refer to [4] or [19]. For the scalar case, see [12], [24]. Contrary to the L^p case, the multiplier theorems established for vector-valued Besov spaces are valid for arbitrary Banach spaces X; see [1], [4] and [17]. Note also that, for $0 < s < 1$, $B^s_{\infty,\infty}$ is the the usual Hölder space C^s. We summarize some useful properties of $B^s_{p,q}((0,2\pi);X)$. See [4, Section 2] for a proof.

(i) $B^s_{p,q}((0,2\pi);X)$ is a Banach space;

(ii) If $s > 0$, then $B^s_{p,q}((0,2\pi);X) \hookrightarrow L^p((0,2\pi);X)$, and the natural injection from $B^s_{p,q}((0,2\pi);X)$ into $L^p((0,2\pi);X)$ is a continuos linear operator;

(iii) Let $s > 0$. Then $f \in B^{s+1}_{p,q}((0,2\pi);X)$ if and only if f is differentiable a.e. and $f' \in B^s_{p,q}((0,2\pi);X)$.

We begin with the definition of operator valued Fourier multipliers in the context of periodic Besov spaces.

Definition 4.8. *Let $1 \le p, q \le \infty, s > 0$. A sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ is a $B^s_{p,q}$-multiplier if for each $f \in B^s_{p,q}((0,2\pi);X)$ there exists a function $g \in B^s_{p,q}((0,2\pi);Y)$ such that*

$$\hat{g}(k) = M_k \hat{f}(k), \quad k \in \mathbb{Z}.$$

Definition 4.9. [20] *We say that $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ satisfies the Marcinkiewicz condition of order 2 if*

$$\sup_{k\in\mathbb{Z}} ||M_k|| < \infty, \ \sup_{k\in\mathbb{Z}} ||k(M_{k+1} - M_k)|| < \infty, \tag{4.4}$$

$$\sup_{k\in\mathbb{Z}} ||k^2(M_{k+1} - 2M_k + M_{k-1})|| < \infty. \tag{4.5}$$

We recall the following operator-valued Fourier multiplier theorem on Besov spaces.

Theorem 4.10. [4] *Let X, Y be Banach spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$ be a M-bounded sequence. Then for $1 \le p, q \le \infty$, $s \in \mathbb{R}$, $\{M_k\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}$-multiplier.*

The following Proposition is the analogous of the Proposition 3.2.

Proposition 4.11. *Let $A : D(A) \subseteq X \to X$, $M : D(M) \subseteq X \to X$ be linear closed operators on a Banach space X. Suppose that $D(A) \subseteq D(M)$ and the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular. Then, the following assertions are equivalent* *(i)* $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ *and* $\{ikM(ikM-(1+a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ *is an* $B^s_{p,q}$*-multiplier for* $1 \le p,q \le \infty$;

(ii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ *and* $\sup_{k\in\mathbb{Z}} ||ikM(ikM-(1+a_k)A)^{-1}|| < \infty$.

Proof. $(i) \Rightarrow (ii)$. Follows the same lines as the proof in [18, Proposition 3.4]. $(ii) \Rightarrow (i)$ For $k \in \mathbb{Z}$, define $M_k = ikM(ikM-(1+a_k)A)^{-1}$ and $N_k = (ikM-(1+a_k)A)^{-1}$. From the identity (3.2) we obtain:

$$\sup_{k\in\mathbb{Z}} ||k(M_{k+1}-M_k)|| < \infty, \tag{4.6}$$

proving (4.4). To verify (4.5), note that:

$k^2[M_{k+1}-2M_k+M_{k-1}] =$

$$\begin{aligned}
&= ik^2M\Big[(k+1)N_{k+1} - 2kN_k + (k-1)N_{k-1}\Big]\\
&= ik^2MN_{k+1}\Big[(k+1)N_k^{-1} - 2kN_{k+1}^{-1} + (k-1)N_{k+1}^{-1}N_{k-1}N_k^{-1}\Big]N_k\\
&= ik^2MN_{k+1}\Big[(k+1)N_k^{-1} - 2k\big[(ikM-(1+a_k)A) - (a_{k+1}-a_k)A + iM\big] +\\
&\quad (k-1)\big[(i(k-1)M-(1+a_{k-1})A) + 2iM - (a_{k+1}-a_{k-1})A\big]N_{k-1}N_k^{-1}\Big]N_k\\
&= ik^2MN_{k+1}\Big[(k+1)N_k^{-1} - 2kN_k^{-1} + 2k(a_{k+1}-a_k)A - 2ikM +\\
&\quad \big[(k-1)N_{k-1}^{-1} + 2i(k-1)M - (k-1)(a_{k+1}-a_{k-1})A\big]N_{k-1}N_k^{-1}\Big]N_k\\
&= ik^2MN_{k+1}\Big[(k+1)I - 2kI + 2k(a_{k+1}-a_k)AN_k - 2ikMN_k +\\
&\qquad \big[(k-1)I + 2i(k-1)MN_{k-1} - (k-1)(a_{k+1}-a_{k-1})AN_{k-1}\big]\Big]\\
&= ik^2MN_{k+1}\Big[2k(a_{k+1}-a_k)AN_k - 2(M_k-M_{k-1}) - (k-1)(a_{k+1}-a_{k-1})AN_{k-1}\Big]\\
&= ikMN_{k+1}\Big[2k^2(a_{k+1}-a_k)AN_k - 2k(M_k-M_{k-1}) -\\
&\qquad k(k-1)(a_{k+1}-a_{k-1})AN_{k-1}\Big]\\
&= ikMN_{k+1}\Big[2k^2\frac{(a_{k+1}-a_k)}{1+a_k}(M_k-I) - 2k(M_k-M_{k-1}) -\\
&\qquad k(k-1)\frac{(a_{k+1}-a_{k-1})}{1+a_{k-1}}(M_{k-1}-I)\Big].
\end{aligned}$$

Using the identities,

$$2k^2(a_{k+1}-a_k) = k^2(a_{k+1}-2a_k+a_{k-1}) + k^2(a_{k+1}-a_{k-1}),$$

and

$$k(k-1)(a_{k+1}-a_{k-1}) = k^2(a_{k+1}-a_{k-1}) - k(a_{k+1}-a_{k-1}),$$

we obtain

$$
\begin{aligned}
&2k^2\frac{(a_{k+1}-a_k)}{1+a_k}[M_k-I]-k(k-1)\frac{(a_{k+1}-a_{k-1})}{1+a_{k-1}}[M_{k-1}-I]=\\
=\ & k^2\frac{(a_{k+1}-2a_k+a_{k-1})}{1+a_k}[M_k-I]+k^2\frac{(a_{k+1}-a_{k-1})}{1+a_k}\cdot\\
&\cdot\Big[\frac{(M_k-M_{k-1})+(a_k-a_{k-1})I+a_{k-1}M_k-a_kM_{k-1}}{1+a_{k-1}}\Big]+k\frac{(a_{k+1}-a_{k-1})}{1+a_{k-1}}[M_{k-1}-I]\\
=\ & k^2\frac{(a_{k+1}-2a_k+a_{k-1})}{1+a_k}[M_k-I]+k\frac{(a_{k+1}-a_{k-1})}{1+a_k}\Big[\frac{1}{1+a_{k-1}}k(M_k-M_{k-1})\\
&+k\frac{(a_k-a_{k-1})}{1+a_{k-1}}I+\frac{k}{1+a_{k-1}}[a_{k-1}M_k-a_kM_{k-1}]\Big]+k\frac{(a_{k+1}-a_{k-1})}{1+a_{k-1}}[M_{k-1}-I].
\end{aligned}
$$

Since the identities

$$\frac{k}{1+a_{k-1}}\Big[a_{k-1}M_k-a_kM_{k-1}\Big]=\frac{a_{k-1}}{1+a_{k-1}}k[M_k-M_{k-1}]+k\frac{(a_{k-1}-a_k)}{1+a_{k-1}}M_{k-1},$$

$$k\frac{(a_{k+1}-a_{k-1})}{1+a_k}=k\frac{(a_{k+1}-a_k)}{1+a_k}+\frac{k}{k-1}(k-1)\frac{(a_k-a_{k-1})}{a_{k-1}}a_{k-1}\frac{1}{1+a_k},$$

$$k\frac{(a_{k+1}-a_{k-1})}{1+a_{k-1}}=k\frac{(a_{k+1}-a_k)}{1+a_{k-1}}+k\frac{(a_k-a_{k-1})}{1+a_{k-1}},$$

and

$$k[M_k-M_{k-1}]=\frac{k}{k-1}(k-1)[M_k-M_{k-1}],$$

are valid, and from the fact that $\{k(M_{k+1}-M_k)\}_{k\in\mathbb{Z}}$ is bounded and $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular, we conclude from the above identities and Remark 2.8 that,

$$\sup_{k\in\mathbb{Z}}||k^2(M_{k+1}-2M_k+M_{k-1})||<\infty. \tag{4.7}$$

Thus, $\{M_k\}_{k\in\mathbb{Z}}$, satisfies the Marcinkiewicz condition of order 2 and therefore, by Theorem 4.10, $\{M_k\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}$-multiplier.

■

Lemma 4.12. *Let X be a Banach space. Suppose that the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular. Then, $\{\frac{1}{1+a_k}I\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}$-multiplier for $1\le p,q\le\infty$.*

Proof. Define $m_k:=\frac{1}{1+a_k}$, $k\in\mathbb{Z}$. By Remark 2.8, the sequence $\{m_k\}_{k\in\mathbb{Z}}$ is bounded. Moreover, $\{m_k\}_{k\in\mathbb{Z}}$ satisfies the identities,

$$k[m_{k+1}-m_k]=-k\frac{a_{k+1}-a_k}{1+a_k}\frac{1}{1+a_{k+1}},$$

and

$$\begin{aligned}
k^2[m_{k+1} - 2m_k + m_{k-1}] &= -\frac{1}{(1+a_{k+1})(1+a_k)(1+a_{k-1})} k^2(a_{k+1} - 2a_k + a_{k-1}) + \\
&\quad \frac{2}{(1+a_{k+1})(1+a_k)(1+a_{k-1})} k(a_k - a_{k-1}) k(a_{k+1} - a_k) - \\
&\quad \frac{a_k}{(1+a_{k+1})(1+a_k)(1+a_{k-1})} k^2(a_{k+1} - 2a_k + a_{k-1}).
\end{aligned}$$

Since $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular, we conclude that $\{m_k\}_{k\in\mathbb{Z}}$ satisfies the Marcinkiewicz condition of order 2 and therefore $\{m_k\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}$-multiplier. ∎

Definition 4.13. *Let $1 \le p, q \le \infty$ and $s > 0$. A function $u \in B^s_{p,q}((0, 2\pi); [D(A)])$ is said to be a strong $B^s_{p,q}$-solution of (3.1) if $Mu \in B^{s+1}_{p,q}((0, 2\pi); X)$ and equation (3.1) holds for a.e. $t \in (0, 2\pi)$.*

The next Theorem, is the main result of this section and is the analogous version of Theorem 3.4 in the context of Besov spaces. We remark that there are no special conditions in the space X.

Theorem 4.14. *Let $1 \le p, q \le \infty$ and $s > 0$. Let $A : D(A) \subseteq X \to X$, $M : D(M) \subseteq X \to X$ be linear closed operators on a Banach space X. Suppose that $D(A) \subseteq D(M)$ and the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular. Then, the following assertions are equivalent*

(i) For every $f \in B^s_{p,q}((0, 2\pi); X)$ there exist a unique strong $B^s_{p,q}$-solution of (3.1);

(ii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\{ikM(ikM - (1 + a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}$-multiplier;

(iii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\sup_{k\in\mathbb{Z}} ||ikM(ikM - (1 + a_k)A)^{-1}|| < \infty$.

Proof. $(i) \Rightarrow (iii)$. Suppose that for every $f \in B^s_{p,q}((0, 2\pi); X)$ there exist a unique strong $B^s_{p,q}$-solution of (3.1). Fix $x \in X$ and $k \in \mathbb{Z}$. Define $f(t) = e^{itk}x$. Then $f \in B^s_{p,q}((0, 2\pi); X)$. By hypothesis there exist $u \in B^s_{p,q}((0, 2\pi); [D(A)])$ with $Mu \in B^{s+1}_{p,q}((0, 2\pi); X)$ such that $u(t) \in D(A)$ and $(Mu)'(t) = Au(t) + (a\dot{*}Au)(t) + f(t)$ a.e. $t \in (0, 2\pi)$. By Lemma 2.7 we have $ikM\hat{u}(k) = A\hat{u}(k) + a_kA\hat{u}(k) + x$. Following the same reasoning that in the proof of Theorem 3.4, we obtain that $ik \in \rho_{M,\tilde{a}}(A)$ for all $k \in \mathbb{Z}$. Let $M_k := ikM(ikM - (1 + a_k)A)^{-1}$. We will see that $\{M_k\}_{k\in\mathbb{Z}}$ is bounded. Using the Closed Graph Theorem, we have that there exist a constant C independent of f such that

$$||Mu||_{B^{s+1}_{p,q}((0,2\pi);X)} + ||Au||_{B^s_{p,q}((0,2\pi);[D(A)])} + ||a\dot{*}Au||_{B^s_{p,q}((0,2\pi);[D(A)])} \le C||f||_{B^s_{p,q}((0,2\pi);X)}.$$

Note that for $f(t) = e^{itk}x$, the solution u of (3.1) is given by $u(t) = (ikM - (1 + a_k)A)^{-1}e^{ikt}x$. Hence,

$$\sup_{k\in\mathbb{Z}} ||ikM(ikM - (1 + a_k)A)^{-1}x|| \le C||x||.$$

$(ii) \Rightarrow (i)$. Define $M_k = ikM(ikM - (1+a_k)A)^{-1}$ and $N_k = (ikM - (1+a_k)A)^{-1}$. Suppose that $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\{M_k\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}$-multiplier. For $f \in B^s_{p,q}((0, 2\pi); X)$

there exist $u \in B^s_{p,q}((0,2\pi);X)$ such that $\hat{u}(k) = ikM(ikM - (1+a_k)A)^{-1}\hat{f}(k)$, for all $k \in \mathbb{Z}$. The identity $I = M_k - (1+a_k)AN_k$ imply that

$$\begin{aligned}\hat{u}(k) &= ikM(ikM - (1+a_k)A)^{-1}\hat{f}(k)\\ &= (I + (1+a_k)AN_k)\hat{f}(k).\end{aligned}$$

So, we obtain $\widehat{(u-f)}(k) = (1+a_k)AN_k\hat{f}(k)$. By Lemma 4.12, the sequence $\{\frac{1}{1+a_k}I\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}$-multiplier. Thus, for $u - f \in B^s_{p,q}((0,2\pi);X)$ there exists $v \in B^s_{p,q}((0,2\pi);X)$ such that $\hat{v}(k) = \frac{1}{1+a_k}\widehat{(u-f)}(k) = AN_k\hat{f}(k)$. Since that $0 \in \rho_{M,\tilde{a}}(A)$ we obtain that $A^{-1} \in \mathcal{B}(X)$, and therefore $w := A^{-1}v \in B^s_{p,q}((0,2\pi);X)$ and $\hat{w}(k) = N_k\hat{f}(k)$. Hence $ikM\hat{w}(k) - (1+a_kA)\hat{w}(k) = \hat{f}(k)$. Observe that for all $k \in \mathbb{Z}$, we have

$$\hat{u}(k) = ikM(ikM - (1+a_k)A)^{-1}\hat{f}(k) = ikM\hat{w}(k).$$

Thus, by uniqueness of Fourier coefficients, $u(t) = (Mw)'(t)$. Since $u \in B^s_{p,q}((0,2\pi);X)$, then $(Mw)' \in B^s_{p,q}((0,2\pi);X)$ and therefore, $Mw \in B^{s+1}_{p,q}((0,2\pi);X)$. Moreover $Mw(0) = Mw(2\pi)$, since $w(0) = w(2\pi)$ and $w(t) \in D(A)$.

Since A and M are closed operators and $\widehat{(Mw)'}(k) = ikM\hat{w}(k) = (1+a_k)A\hat{w}(k) + \hat{f}(k)$, for all $k \in \mathbb{Z}$, one has $(Mw)'(t) = Aw(t) + (a*Au)(t) + f(t)$ *a.e.* by Lemmas 2.6 and 2.7. We conclude that $w \in B^s_{p,q}((0,2\pi);X)$ is a strong $B^s_{p,q}$-solution to (3.1). Finally, the uniqueness follows the same way as in the proof of Theorem 3.4.

$(iii) \Leftrightarrow (ii)$. Follows from Proposition 4.11. ∎

5. Maximal Regularity on Triebel-Lizorkin Spaces

In this section, we study the existence and uniqueness of solutions to (3.1) in the context of Triebel-Lizorkin spaces; $F^s_{p,q}((0,2\pi);X)$, where X is a Banach space, $1 \le p,q \le \infty$ and $s \in \mathbb{R}$. More details of theses spaces can be found it in [7] and the references therein.

The next definition and theorem are the analogous versions mentioned in the above sections.

Definition 5.15. *Let $1 \le p,q \le \infty, s \in \mathbb{R}$. A sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ is a $F^s_{p,q}$-multiplier if for each $f \in F^s_{p,q}((0,2\pi);X)$ there exists a function $g \in F^s_{p,q}((0,2\pi);Y)$ such that*

$$\hat{g}(k) = M_k\hat{f}(k), \quad k \in \mathbb{Z}.$$

We recall the following result due to [7].

Theorem 5.16. [7] *Let X, Y be Banach spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$. Assume that*

$$\sup_{k\in\mathbb{Z}} ||M_k|| < \infty, \ \sup_{k\in\mathbb{Z}} ||k(M_{k+1} - M_k)|| < \infty, \tag{5.8}$$

$$\sup_{k\in\mathbb{Z}} ||k^2(M_{k+1} - 2M_k + M_{k-1})|| < \infty, \tag{5.9}$$

$$\sup_{k\in\mathbb{Z}} ||k^3(M_{k+2} - 3M_{k+1} + 3M_k - M_{k-1})|| < \infty, \tag{5.10}$$

where $1 \leq p < \infty$ and $1 \leq q \leq \infty$, $s \in \mathbb{R}$, $\{M_k\}_{k\in\mathbb{Z}}$. Then $\{M_k\}_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier.

Remark 5.17.

We remark that, if X, Y are UMD spaces in the above theorem, then the conditions (5.8) and (5.9) are sufficient for $\{M_k\}_{k\in\mathbb{Z}}$ to be an $F^s_{p,q}$-multiplier.

The definition of solution of the equation (3.1) in the Triebel-Lizorkin spaces is the same that in the Besov case. The proof of following theorem is similar to Theorem 4.14. We omit the details.

Theorem 5.18. *Let $1 \leq p, q \leq \infty$ and $s > 0$. Let $A : D(A) \subseteq X \to X$, $M : D(M) \subseteq X \to X$ be linear closed operators on a Banach space X. Suppose that $D(A) \subseteq D(M)$ and the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 3-regular. Then, the following assertions are equivalent*

(i) For every $f \in F^s_{p,q}((0, 2\pi); X)$ there exist a unique strong $F^s_{p,q}$-solution of (3.1);

(ii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\{ikM(ikM - (1 + a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier;

(iii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\sup_{k\in\mathbb{Z}} ||ikM(ikM - (1 + a_k)A)^{-1}|| < \infty$.

6. Applications

We conclude the paper, with some applications of the above results.

Example 6.19.

Let us consider the boundary value problem

$$\frac{\partial(m(x)u(t,x))}{\partial t} - \Delta u = \int_{-\infty}^{t} a(t-s)\Delta u(s,x)ds + f(t,x), \ \text{ in } [0, 2\pi] \times \Omega \tag{6.11}$$

$$u = 0, \ \text{ in } [0, 2\pi] \times \partial\Omega, \tag{6.12}$$

$$m(x)u(0,x) = m(x)u(2\pi, x) \ \text{ in } \Omega, \tag{6.13}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $m(x) \geq 0$ is a given measurable bounded function on Ω and f is a function on $[0, 2\pi] \times \Omega$.

Let M be the multiplication operator by m. If we take $X = H^{-1}(\Omega)$ then by [5, p.38] (see also references therein), we have that there exists a constant $c > 0$ such that

$$||M(zM - \Delta)^{-1}|| \leq \frac{c}{1 + |z|},$$

whenever $Re(z) \geq -c(1+|Im(z)|)$. Thus, the inequality

$$||ikM(ikM-((1+a_k)\Delta)^{-1}|| = \frac{|k|}{|1+a_k|}\left\|M\left(\frac{ik}{1+a_k}M-\Delta\right)^{-1}\right\| \leq c,$$

holds, if $Re\big(\frac{ik}{1+a_k}\big) \geq -c(1+|Im\big(\frac{ik}{1+a_k}\big)|)$, for all $k \in \mathbb{Z}$, that is, if

$$k\beta_k \geq -c((1+\alpha_k)^2+\beta_k^2+|k(1+\alpha_k)|), \tag{6.14}$$

is valid for all $k \in \mathbb{Z}$, where α_k and β_k denotes the real and imaginary part of a_k, respectively. In particular, if $a(t) := \frac{t^{b-1}}{\Gamma(b)}$, with b an even integer, then one can check that $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular and $\beta_k = 0$ for all $k \in \mathbb{Z}$, thus the inequality (6.14) holds. Therefore, by Theorem 4.14 (or Corollary 3.5), we conclude that that for all $f \in L^p_{2\pi}(\mathbb{R}, H^{-1}(\Omega))$ there exists a unique solution for (6.11)-(6.12).

Example 6.20.

Consider, for $t \in [0,2\pi]$ and $x \in [0,\pi]$, the problem

$$\frac{\partial}{\partial t}\Big(\frac{\partial^2}{\partial x^2}+1\Big)u(t,x) = -b\frac{\partial^2}{\partial x^2}u(t,x) - cu(t,x) + \tag{6.15}$$

$$\int_{-\infty}^{t} a(t-s)\big(b\frac{\partial^2}{\partial x^2}+c\big)u(s,x)ds + f(t,x) \tag{6.16}$$

$$u(t,0) = u(t,\pi) = \frac{\partial^2}{\partial x^2}u(t,0) = \frac{\partial^2}{\partial x^2}u(t,\pi) = 0 \tag{6.17}$$

$$\Big(\frac{\partial^2}{\partial x^2}+1\Big)u(0,x) = \Big(\frac{\partial^2}{\partial x^2}+1\Big)u(2\pi,x), \tag{6.18}$$

where b is positive constant and $-2b < c < 4b$. If we take $X = C_0([0,\pi]) = \{u \in C([0,\pi]) : u(0) = u(\pi)\}$ and K the realization of $\frac{\partial^2}{\partial x^2}$ with domain

$$D(K) = \{u \in C^2([0,\pi]) : u(0) = u(\pi) = \frac{\partial^2}{\partial x^2}u(0) = \frac{\partial^2}{\partial x^2}u(\pi) = 0\},$$

then we take $M = K + I,\ A = bM + (c-b)I$. By [5, p.39, Ex.1.2] we have that:

$$||M(zM-A)^{-1}|| \leq \frac{d}{1+|z|}$$

for all $Re(z) \geq -d(1+|Im(z)|)$, and d being a suitable positive constant. Therefore, as in the Example 6.20, if for all $k \in \mathbb{Z}$, the inequality

$$k\beta_k \geq -d((\alpha_k-1)^2+\beta_k^2+|k(\alpha_k-1)|), \tag{6.19}$$

is valid, then for all $f \in B^s_{p,q}((0,2\pi), C_0([0,\pi]))$, $s > 0$, $1 \leq p,q \leq \infty$, by Theorem 4.14, we conclude that the problem (6.15)-(6.18) has a unique strong solution u with regularity $\frac{\partial^2 u}{\partial x^2} \in B^s_{p,q}((0,2\pi), C_0([0,\pi]))$. In particular, if $a(t) := e^{\gamma t}$, where $\gamma \in \mathbb{R}$, we can check that $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular and the inequality (6.19) holds with $d = 1$.

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