COSINE AND SINE FAMILIES ON TIME SCALES AND ABSTRACT SECOND ORDER DYNAMIC EQUATIONS ON TIME SCALES

JAQUELINE G. MESQUITA‡, ALDO PEREIRA†, AND RODRIGO PONCE*

Abstract. Abstract cosine and sine functions defined on a Banach space are useful tools in the study of wide classes of abstract evolution equations. In this paper, we introduce a definition of cosine and sine functions on time scales, which unify the continuous, discrete and the cases "in between". Our definition includes several types of time scales such as real numbers set, integers numbers set, quantum scales, among others. We investigate the relationship between the cosine function on time scales and its infinitesimal generator, proving several properties concerning it. Also, we investigate the sine functions on time scales, presenting their main properties. Finally, we apply our theory to study the homogeneous and inhomogeneous abstract Cauchy problem on time scales in Banach spaces.

1. Introduction

Recently, it was introduced in the literature the concept of abstract C_0-semigroup on time scales (see [14]), which encompasses its definition in all the classical cases, that is, $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$, as well as several important time scales for applications such as the quantum scales, hybrid scales which mixes continuous and discrete behaviors, among others. To address this definition, the authors employed Laplace transform, which avoids the requirement of the group property for the time scale. Also, the theory of semigroup on time scales plays an important role for the study of the first order abstract Cauchy problem on time scales, being a first step in the direction of deeper investigations about this type of problems in the setting of time scales.

On the other hand, it is a known fact that the theory of abstract cosine and sine families plays an important role in the study of the existence of solutions of second order equations and to investigate different problems in many fields of knowledge. Due to this fact, this theory has been attracting the attention of several authors through the years, see [2, 10, 11, 12, 15, 16, 17] and the references therein. However,

2010 *Mathematics Subject Classification.* 34N05; 47D06; 34G10; 34K30.

Key words and phrases. dynamic equations on time scales; cosine function; sine function; abstract Cauchy Problem.

‡ Universidade de Brasília, Departamento de Matemática, Campus Universitario Darcy Ribeiro, Asa Norte 70910-900, Brasília-DF, Brazil, jgmesquita@unb.br.

The first author is partially supported by FAPDF grant 0193.001300/2016 and CNPq 407952/2016-0.

† Universidad de La Serena, Departamento de Matemáticas, Avenida Cisternas 1200, La Serena, Chile, aldo.pereira@userena.cl.

*Corresponding author. Universidad de Talca, Instituto de Matemáticas, Casilla 747, Talca, Chile, rponce@inst-mat.utalca.cl.

the theory of discrete abstract cosine functions has not been completely developed yet. For instance, the correspondence with the set of all discrete cosine functions defined by means of D'Alembert functional equations is still an open problem as well as the description of the generator of abstract cosine families has not been given yet. Also, to the best of our knowledge, the definition of abstract cosine functions on quantum or hybrid scales was not introduced in the literature until now. Therefore, motivated by these facts, the main goal of this paper is to fill this gap in the literature and to present a unified and extended theory for abstract cosine and sine families which encompasses many types of time scales such as continuous, discrete, hybrid and quantum scales, bringing also new developments and features for the discrete case.

In this paper, we start by introducing the concepts of cosine and sine families defined on a time scale $\mathbb{T}$, that is, defined on any closed and nonempty subset of $\mathbb{R}$, in order to investigate the homogeneous abstract second order Cauchy problem on time scales

$$u^{\Delta\Delta}(t) = Au(t), \quad t \in \mathbb{T}_0^+, \text{ and } \quad u(0) = x, \quad u^{\Delta}(0) = y, \tag{1.1}$$

the inhomogeneous abstract second order Cauchy problem on time scales

$$u^{\Delta\Delta}(t) = Au(t) + f(t), \quad t \in \mathbb{T}_0^+, \text{ and } \quad u(0) = x, \quad u^{\Delta}(0) = y, \tag{1.2}$$

and the nonlinear abstract second order Cauchy problem on time scales

$$u^{\Delta\Delta}(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{T}_0^+, \text{ and } \quad u(0) = x, \quad u^{\Delta}(0) = y, \tag{1.3}$$

where A is a closed linear operator in a Banach space X, $x, y \in X$ and $\mathbb{T}_0$ is a time scale such that $0 \in \mathbb{T}_0$ and $\sup \mathbb{T}_0 = +\infty$ and $\mathbb{T}_0^+ = \mathbb{T}_0 \cap \mathbb{R}^+$.

It is a known fact that when we are dealing with second order dynamic equations on time scales, we can formulate the problem using many different ways. Therefore, it may be a challenge to find out an appropriate formulation to investigate each problem. In our case, since we are interested to study abstract cosine and sine families as well as their properties, our equation given by (1.1) is the most appropriate to investigate this problem. See Remark 3.1 for more details.

Let us recall that the classical way to define cosine function is through the following property:

$$2C(t)C(s) = C(t+s) + C(t-s), \quad t, s \in \mathbb{R}, \text{ and } C(0) = I, \tag{1.4}$$

which is known as *D'Alembert functional equation*, which makes sense when we are dealing with $t, s \in \mathbb{R}$ or even $t, s \in \mathbb{Z}$. However, when we deal with the domain being a time scale, that is, a closed and nonempty subset of the real numbers, to satisfy (1.4) implies to require a strong condition on the time scale, since to make sense (1.4) and to ensure the well-posedness of this type of problem, it is necessary to require that the time scale has the *group property*, i.e., $0 \in \mathbb{T}$ and if $t, s \in \mathbb{T}$, then $t - s \in \mathbb{T}$. These two conditions lead to many restrictions concerning the class of time scales that could be considered. See Remark 2.24.

Taking it into account, we present the definitions of the cosine and sine functions using Laplace transform on time scales. This approach allows us to present a much more general definition in the setting of time scales, avoiding the group property. Despite of all these advantages, to prove the results using only this general definition represents a challenge, since to prove most of the results concerning abstract cosine and sine functions, we need to employ completely different and new arguments to the ones found in the literature, since these last ones usually employ the functional equation (1.4) to get them.

The paper is organized as follows. The second section has two subsections: the first one is devoted to remember the basic concepts concerning the theory of time scales, and the second subsection shows how restrictive it is to deal with a class of time scales satisfying the group property. The third section introduces the concept of abstract cosine functions C on time scales, which generalize the properties of the classical theory.

We notice that the definition of the generator A considers all the different possibilities for 0 and $\sigma(0)$, obtaining three options for the domain of A depending on the continuous or discrete behavior of 0 and $\sigma(0)$.

These descriptions of A are very surprising and bring a unified definition for all time scales such that $0 \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$, and are consistent with the known formulations in the discrete and continuous cases.

The fourth section is devoted to introduce the concept of abstract sine function on time scales and prove its properties. Also, we prove a result which ensures that if $\mu(0) > 0$ and the homogeneous Cauchy problem (1.1) has a solution, then A is a bounded linear map (see Theorem 4.3).

In the fifth section, we apply our results to study the inhomogeneous abstract second order Cauchy problems (1.2) and (1.3), obtaining a version of the variation of constants formula for the solution of each problem.

2. Time scales theory

2.1. **Basic definitions and results.** Now, we present some basic concepts and properties about time scales. For more details, applications and recent results, we refer to [7, Chapters 1 and 2], the survey [1] and [8, 6, 13, 14].

A *time scale* $\mathbb{T}$ is a closed and nonempty subset of $\mathbb{R}$.

Definition 2.1. We define the *forward* and *backward jump operators* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, respectively, by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, for $t \in \mathbb{T}$. In this definition, we consider $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

Definition 2.2. If $\sigma(t) > t$, we say that $t \in \mathbb{T}$ is *right-scattered*. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*. If $\rho(t) < t$, we say that $t \in \mathbb{T}$ is *left-scattered*, whereas if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is *left-dense*. The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$.

Throughout this paper, we will denote a Banach space by $X \equiv (X, \|\cdot\|)$. Also, we will denote a closed interval in $\mathbb{T}$ by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}$, where $a, b \in \mathbb{T}$. Similarly, we can define open intervals and half-open intervals, among others.

Definition 2.3. A function $f : \mathbb{T} \to X$ is called *regulated* if its right-sided limits exist at right-dense points in $\mathbb{T}$, and its left-sided limits exist at left-dense points in $\mathbb{T}$.

Definition 2.4. A function $f : \mathbb{T} \to X$ is called *rd-continuous* if it is continuous at right-dense points in $\mathbb{T}$, and its left-sided limits exist at left-dense points in $\mathbb{T}$. We denote the class of all rd-continuous functions $f : \mathbb{T} \to X$ by $\mathcal{C}_{rd} = \mathcal{C}_{rd}(\mathbb{T}, X)$.

We define the set $\mathbb{T}^\kappa$, which is derived from $\mathbb{T}$, as follows:

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus \{m\}, & \text{if } \mathbb{T} \text{ has a left-scattered maximum } m, \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

Definition 2.5. For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, we define the *delta derivative* of y to be the number (if it exists) $y^\Delta(t)$ with the following property: given $\epsilon > 0$, there exists a neighborhood U of t such that $|y(\sigma(t)) - y(s) - y^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|$ for all $s \in U$.

Below, we list some important results concerning the delta–integrals. For further details on these integrals, we refer to [7, Chapter 1, Section 4].

Theorem 2.6. *If $f, g \in \mathcal{C}_{rd}$, then $\int_t^{\sigma(t)} f(s)\,\Delta s = \mu(t)f(t)$.*

Theorem 2.7. *If $f, g \in \mathcal{C}_{rd}$, then*

$$\int_a^b f(\sigma(t))g^\Delta(t)\,\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\,\Delta t$$

and

$$\int_a^b f(t)g^\Delta(t)\,\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(\sigma(t))\,\Delta t.$$

We denote the class of all functions $f : \mathbb{T} \to X$ that are twice delta-differentiable with the second delta-derivative being rd-continuous by $\mathcal{C}^2_{rd} = \mathcal{C}^2_{rd}(\mathbb{T}, X)$.

Next, we recall the notions and the results concerning the Hilger complex plane and generalized exponential function. All the results can be found in [7, Chapter 2].

Definition 2.8. For a given $h > 0$, the *Hilger complex numbers* are defined by $\mathbb{C}_h := \left\{z \in \mathbb{C} : z \neq -\frac{1}{h}\right\}$. For $h = 0$, let $\mathbb{C}_0 := \mathbb{C}$.

Definition 2.9. For $h > 0$ and $z \in \mathbb{C}_h$, we define the *Hilger real part* of z by $Re_h(z) := \frac{|zh+1|-1}{h}$ and $Re_0(z) := \mathrm{Re}(z)$ in the usual sense.

We use the following notation in the rest of the paper: $Re_\mu(\lambda)(t) := Re_{\mu(t)}(\lambda)$.

Theorem 2.10. *If we define the circle plus addition $\oplus$ on $\mathbb{C}_h$ by $z \oplus w := z + w + zwh$, then $(\mathbb{C}_h, \oplus)$ is an Abelian group.*

Definition 2.11. If $z \in \mathbb{C}_h$, the *additive inverse* of z under the operation $\oplus$ is $\ominus z := \frac{-z}{1+zh}$, and we define the *circle minus substraction* $\ominus$ on $\mathbb{C}_h$ by $z \ominus w = z \oplus (\ominus w)$.

Definition 2.12. We say that a function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Definition 2.13. For $p \in \mathcal{R}$, we define the *generalized exponential function* by $e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(r)}(p(r))\,\Delta r\right)$ for $s, t \in \mathbb{T}$. Moreover, the cylinder transformation ξ_h is given by $\xi_h(z) = \frac{1}{h}\log(1 + zh)$, where log is the principal logarithm function. For $h = 0$, we define $\xi_0(z) = z$ for all $z \in \mathbb{C}$.

For the properties of the generalized exponential function see [7, Theorem 2.36].

Next, let us denote by $\mathbb{T}_0$ a time scale such that $0 \in \mathbb{T}_0$ and $\sup \mathbb{T}_0 = +\infty$, and $\mathbb{T}_0^+ = \mathbb{T}_0 \cap [0, +\infty)$. Also, note that if $\lambda \in \mathcal{R}$ is constant, then $\ominus\lambda \in \mathcal{R}$ and $e_{\ominus\lambda}(t, 0)$ is well defined on $\mathbb{T}_0$ (see [7, Chapter 2]). Now, we recall the notion of Laplace transform on time scales and its main properties. For more details, we refer to [7, Chapter 3] and [4, 5, 9, 14].

Definition 2.14. Given a regulated function $x : \mathbb{T}_0 \to \mathbb{R}$, its *Laplace transform* is defined by $\widehat{x}(\lambda) = \mathcal{L}\{x\}(\lambda) := \int_0^\infty x(t)e^\sigma_{\ominus\lambda}(t,0)\,\Delta t$, for $\lambda \in D\{x\}$, where $D\{x\} := \left\{\lambda \in \mathbb{C} : \int_0^\infty x(t)e^\sigma_{\ominus\lambda}(t,0)\,\Delta t \text{ exists}\right\}$.

Theorem 2.15. *Assume f and g are regulated functions on $\mathbb{T}_0$, and α and β are constants. Then $\mathcal{L}\{\alpha x + \beta y\}(\lambda) = \alpha\mathcal{L}\{x\}(\lambda) + \beta\mathcal{L}\{y\}(\lambda)$ for $\lambda \in D\{x\} \cap D\{y\}$.*

Theorem 2.16 ([14, Theorem 3.12])**.** *Let $f : \mathbb{T}_0^+ \to X$ be an rd-continuous and delta-differentiable function. If $\lambda \in \mathbb{C}$ is such that $\lim_{t\to\infty} f(t)e_{\ominus\lambda}(t,0) = 0$ whenever $Re_\mu(\lambda)(t) > 0$ for all $t \in \mathbb{T}_0^+$ and $\widehat{f^\Delta}(\lambda)$ exists, then $\widehat{f}(\lambda)$ exists and $\widehat{f^\Delta}(\lambda) = \lambda\widehat{f}(\lambda) - f(0)$.*

Corollary 2.17 ([14, Corollary 3.13])**.** Let $f \in \mathcal{C}_{rd}(\mathbb{T}_0^+, X)$ and $F(t) = \int_0^t f(s)\,\Delta s$. Let $\lambda \in \mathbb{C}\setminus\{0\}$ for all $t \in \mathbb{T}_0^+$ such that $\widehat{f}(\lambda)$ exists and $\lim\limits_{t\to\infty} f(t)e_{\ominus\lambda}(t,0) = 0$, then $\widehat{F}(\lambda) = \widehat{f}(\lambda)/\lambda$.

Definition 2.18. A function $T : \mathbb{T}_0^+ \to \mathcal{L}(X)$ is called *strongly rd-continuous* if T satisfies $\|T(t)x - x\| \to 0$ as $t \to 0^+$ for each $x \in X$, whenever 0 is right-dense.

The following result will be essential to our purposes.

Theorem 2.19 ([14, Theorem 3.15])**.** *Let A be a closed linear operator in X, and $f, g \in L^1_{loc}(\mathbb{T}_0^+, X)$ such that $\omega \in D\{f\} \cap D\{g\}$. Then, the following are equivalent*

i) *$f(t) \in D(A)$ and $Af(t) = g(t)$ a.e. on $\mathbb{T}_0^+$.*
ii) *$\widehat{f}(\lambda) \in D(A)$ and $A\widehat{f}(\lambda) = \widehat{g}(\lambda)$ whenever $Re_\mu(\lambda)(t) > Re_\mu(\omega)(t)$, for all $t \in \mathbb{T}_0^+$.*

2.2. **Remarks on time scales with group property.** In this subsection, we will present some important remarks concerning time scales with group property.

The classical definition of cosine function using the *D'Alembert functional equation* is very restrictive in terms of time scales, since it requires that a time scale satisfies the group property in order to be well–defined, that is, $\mathbb{T}$ must satisfy:

(i) $0 \in \mathbb{T}$
(ii) if $a, b \in \mathbb{T}$, then $a - b \in \mathbb{T}$.

In the sequel, let us show some consequences of these assumptions.

Theorem 2.20. *If $\mathbb{T}$ has the group property, then for every $a, b \in \mathbb{T}$, we have $a + b \in \mathbb{T}$.*

Proof. Since $0 \in \mathbb{T}$ and $a, b \in \mathbb{T}$, it follows that $-a, -b \in \mathbb{T}$ and therefore, $a-(-b) = a + b \in \mathbb{T}$. □

Theorem 2.21. *If $\mathbb{T}$ has the group property, then every point in $\mathbb{T}$ is right-dense or every point in $\mathbb{T}$ is right-scattered.*

Proof. Suppose that there exist $a, b \in \mathbb{T}$ such that a is right-dense and b is right-scattered. Since a is right-dense, there exists a sequence $(t_n) \subset \mathbb{T}$ such that $t_n \to a^+$ as $n \to \infty$. It implies that the sequence $s_n := t_n - a \in \mathbb{T}$ for each $n \in \mathbb{N}$ and converges to zero. Therefore, $s_n + b \in \mathbb{T}$ for each $n \in \mathbb{N}$ and converges to b^+ as $n \to \infty$. Therefore, b is right-dense, which is a contradiction, proving the result. □

Remark 2.22. Theorem 2.21 shows that time scales which fulfill the group property cannot be hybrid, that is, the time scale cannot contain simultaneously right-dense and right-scattered points. It excludes the time scale $\mathbb{T} = \bigcup_{k\in\mathbb{Z}}[a_k, b_k]$ for $a_k < b_k < a_{k+1}$, which is crucial to describe population models (see [7], for more details).

Lemma 2.23. *If $\mathbb{T}$ has the group property, and there exists $0 \neq a \in \mathbb{T}$, then $\sup\mathbb{T} = +\infty$ and $\inf\mathbb{T} = -\infty$.*

Proof. It follows directly from the definition. □

Remark 2.24. If $\mathbb{T}$ has the group property and $a \neq 0 \in \mathbb{T}$ is right-dense, then by Theorems 2.21 and 2.23, we get $\mathbb{T} = \mathbb{R}$.

Theorem 2.25. *Suppose $\mathbb{T}$ has the group property, then*

$$\sigma(a+b) = \sigma(a) + b \quad \text{and} \quad \sigma(a+b) = \sigma(b) + a \quad \textit{for every} \quad a, b \in \mathbb{T}. \tag{2.1}$$

Proof. If a is right-dense, then by Theorem 2.21, $a+b$ is also right-dense. Therefore, the equality (2.1) follows. If a is right-scattered, then $a + b$ is also right-scattered by Theorem 2.21. Also, notice that $a + b < \sigma(a) + b$ and $\sigma(a) + b \in \mathbb{T}$. This, and the definition of the forward jump operator imply that $\sigma(a + b) \leq \sigma(a) + b$. Reciprocally, we notice that $a = a+b-b < \sigma(a+b)-b$ and since $\sigma(a+b)-b \in \mathbb{T}$, we conclude that $\sigma(a) \leq \sigma(a + b) - b$. Therefore, $\sigma(a) + b \leq \sigma(a + b)$. Combining these two inequalities, we get $\sigma(a+b) = \sigma(a)+b$, concluding the result. The other equality is proved analogously. □

Finally, by the results above, we conclude that the graininess function is constant.

Corollary 2.26. Suppose $\mathbb{T}$ has the group property, then for every $a, b \in \mathbb{T}$, we have $\mu(a) = \mu(b)$.

3. Abstract Cosine function on time scales

In this section, we investigate the abstract second order Cauchy problem

$$\begin{cases} u^{\Delta\Delta}(t) &= Au(t), \quad t \in \mathbb{T}_0^+, \\ u(0) &= x, \\ u^{\Delta}(0) &= y, \end{cases} \tag{3.1}$$

where $x, y \in X$, and A is a closed linear operator in a Banach space X.

Remark 3.1. When one is studying a second-order dynamic equation on time scales it is necessary to choose an appropriate formulation of the problem, since it is a known fact that there are several ways to formulate a second order dynamic equation on the setting of time scales. For instance, the formulation

$$u^{\Delta\Delta}(t) = Au^{\sigma}(t), \quad t \in \mathbb{T}_0^+, \text{ and } u(0) = x, u^{\Delta}(0) = y, \tag{3.2}$$

does not lead to the natural definition of abstract cosine function on time scales (see Remark 3.7). Therefore, it seems to be more natural to consider the formulation (3.1) instead of (3.2), since here we are interested in the study of cosine and sine functions on time scales. Further, another natural question which appears when we are dealing with second order dynamic equations on time scales concerns about the possibility to consider the problem using both nabla and delta derivatives, instead of only using delta-derivatives or only nabla-derivatives. In this case, our problem would have the following formulation:

$$u^{\Delta\nabla}(t) = Au(t), \quad t \in \mathbb{T}_0^+, \text{ and } u(0) = x, u^{\nabla}(0) = y. \tag{3.3}$$

The formulation given by (3.3) is not appropriate in our case again, because the well–known property described in Corollary 2.17 does not remain true if we define $F(t) = \int_0^t f(s)\nabla s$ instead of $F(t) = \int_0^t f(s)\Delta s$. It perhaps comes from the formulation of Laplace transform that we are considering. In this case, it is defined by means delta–integrals instead of nabla integrals. This implies that the usual and classical properties of abstract cosine function on time scales do not coincide in the case $\mathbb{T} = \mathbb{R}$, which is not interesting for our purpose of unification and extension of the theories. Other motivation for not considering (3.3) follows from the fact that neither hyperbolic cosine, nor hyperbolic sine, nor cosine, nor sine functions on time scales are considered through equations of this type (see [7]). Finally, see Remark 3.7 for other considerations about the formulation of the problem.

Definition 3.2. A *classical solution* of (3.1) is a function $u \in \mathcal{C}^2_{rd}(\mathbb{T}_0^+, X)$ such that $u(t) \in D(A)$ for every $t \in \mathbb{T}_0^+$ and satisfies the problem (3.1).

In the sequel, we introduce a general definition of the solution for the problem (3.1).

Definition 3.3. A *mild solution* of (3.1) is a function $u \in \mathcal{C}_{rd}(\mathbb{T}_0^+, X)$ which satisfies $\int_0^t \int_0^s u(r)\,\Delta r\,\Delta s = \int_0^t (t-\sigma(s))u(s)\,\Delta s \in D(A)$ and for all $t \in \mathbb{T}_0^+$

$$u(t) = x + ty + A\int_0^t (t-\sigma(s))u(s)\,\Delta s. \tag{3.4}$$

In the sequel, we provide an example which gives a explicit formula for the mild solution of (3.1) in the case of hybrid time scale.

Example 3.4. Consider the following problem

$$\left\{\begin{array}{rll} u^{\Delta\Delta}(t) & = Au(t), & t \in \mathbb{T}_0^+, \\ u(0) & = x, & \\ u^{\Delta}(0) & = y, & \end{array}\right. \tag{3.5}$$

where $\mathbb{T}_0^+ = \displaystyle\bigcup_{k=0}^{\infty}[a_k, b_k]$ is a hybrid time scale. Assuming that $a_0 = 0$, let $t \in \displaystyle\bigcup_{k=0}^{\infty}[a_k, b_k]$. Therefore, there exists $k_0 \in \mathbb{N}$ such that $t \in [a_{k_0}, b_{k_0}]$. Suppose the first case, that is, $t \in [a_{k_0}, b_{k_0})$, then a mild solution of (3.5) by the Definition 3.3 is given by

$$\begin{aligned} u(t) &= x + ty + A\int_0^t (t-\sigma(s))u(s)\,\Delta s \\ &= x + ty + A\sum_{k=0}^{k_0-1}\int_{a_k}^{b_k}(b_k - s)u(s)ds + A\sum_{k=0}^{k_0-1}\int_{b_k}^{\sigma(b_k))}(b_k-\sigma(s))u(s)ds + \\ &\qquad + \int_{a_{k_0}}^t (t-\sigma(s))u(s)ds \\ &= x + ty + A\sum_{k=0}^{k_0-1}\int_{a_k}^{b_k}(b_k - s)u(s)ds - A\sum_{k=0}^{k_0-1} u(b_k)(\mu(b_k))^2 + \\ &\qquad + \int_{a_{k_0}}^t (t-\sigma(s))u(s)ds. \end{aligned}$$

On the other hand, if $t = b_{k_0}$, then

$$\begin{aligned} u(t) &= x + ty + A\int_0^t (t-\sigma(s))u(s)\,\Delta s \\ &= x + ty + A\sum_{k=0}^{k_0}\int_{a_k}^{b_k}(b_k - s)u(s)ds + A\sum_{k=0}^{k_0-1}\int_{b_k}^{\sigma(b_k))}(b_k-\sigma(s))u(s)ds + \\ &= x + ty + A\sum_{k=0}^{k_0}\int_{a_k}^{b_k}(b_k - s)u(s)ds - A\sum_{k=0}^{k_0-1}u(b_k)(\mu(b_k))^2 \end{aligned}$$

The next result relates strongly a mild and a classical solution of problem (3.1).

Theorem 3.5. *A mild solution u of the problem* (3.1) *is a classical solution if, and only if, $u \in \mathcal{C}^2_{rd}(\mathbb{T}_0^+, X)$.*

Proof. If u is a classical solution of (3.1), by the definition it follows directly that $u \in \mathcal{C}^2_{rd}(\mathbb{T}_0^+, X)$, obtaining the desired result. Conversely, assume $u \in \mathcal{C}^2_{rd}(\mathbb{T}_0^+, X)$ is a mild solution of (3.1) and let $t \in \mathbb{T}_0^+$. Integrating and applying the Fundamental Theorem of Calculus for Δ-integrals [8, Theorem 5.34] in (3.1), we have $u^{\Delta}(t) - y = A\int_0^t u(s)\,\Delta s$, and then, integrating and applying [8, Theorem 5.34], we obtain $u(t) - x - ty = A\int_0^t\int_0^s u(r)\Delta r\Delta s = A\int_0^t\int_{\sigma(r)}^t u(r)\Delta s\Delta r = A\int_0^t (t-\sigma(r))u(r)\Delta r$, where in the second equality, we use the change of order of integration (see [3]). $\square$

The next result brings an important property of a mild solution of (3.1).

Theorem 3.6. *Let $u \in \mathcal{C}_{rd}(\mathbb{T}_0^+, X)$, $\omega \in D\{u\}$. Assume $\lambda \in \mathbb{C}$ is such that*

$$\lim_{t\to\infty} u(t)e_{\ominus\lambda}(t,0) = 0, \quad \lim_{t\to\infty} e_{\ominus\lambda}(t,0)\int_0^t u(s)\,\Delta s = 0, \quad \lim_{t\to\infty} e_{\ominus\lambda}(t,0) = 0, \tag{3.6}$$

and $Re_\mu(\lambda)(t) > Re_\mu(\omega)(t)$ for all $t \in \mathbb{T}_0^+$. Then u is a mild solution of (3.1) *if, and only if, $\widehat{u}(\lambda) \in D(A)$ and $\lambda x + y = (\lambda^2 - A)\widehat{u}(\lambda)$, for all λ which satisfies* (3.6).

Proof. Define the following functions $v(t) := \int_0^t u(s)\,\Delta s$ and $z(t) = \int_0^t v(s)\,\Delta s$ for every $t \in \mathbb{T}_0^+$. Therefore, by hypotheses and by Corollary 2.17, we get $\widehat{v}(\lambda) = \frac{\widehat{u}(\lambda)}{\lambda}$ and $\widehat{z}(\lambda) = \frac{\widehat{v}(\lambda)}{\lambda}$, which implies that $\widehat{z}(\lambda) = \frac{\widehat{u}(\lambda)}{\lambda^2}$. On the other hand, we have $\frac{\widehat{u}(\lambda)}{\lambda^2} = \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\int_0^t v(s)\,\Delta s\,\Delta t$. Then, by hypotheses and by Theorem 2.19, we get $\frac{\widehat{u}(\lambda)}{\lambda^2} = \widehat{z}(\lambda) = \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\int_0^t v(s)\,\Delta s\,\Delta t \in D(A)$. If u is a mild solution of (3.1) and by applying Theorem 2.19 again, we have

$$\widehat{u}(\lambda) = \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)x\Delta t + \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)ty\Delta t + A\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\int_0^t (t-\sigma(s))u(s)\Delta s\Delta t. \tag{3.7}$$

By the properties of the exponential function [7, Theorem 2.36], we obtain

$$\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)x\,\Delta t = \int_0^\infty -\frac{\ominus\lambda}{\lambda}e_{\ominus\lambda}(t,0)x\,\Delta t = -\frac{1}{\lambda}\big(e_{\ominus\lambda}(t,0)x\big)_{t=0}^{t\to+\infty} = \frac{x}{\lambda}, \tag{3.8}$$

and using integration by parts (Theorem 2.7), we have

$$\begin{aligned} \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)ty\,\Delta t &= \int_0^\infty -\frac{\ominus\lambda}{\lambda}e_{\ominus\lambda}(t,0)ty\,\Delta t = \int_0^\infty -\frac{1}{\lambda}e^{\Delta}_{\ominus\lambda}(t,0)ty\,\Delta t \\ &= -\frac{1}{\lambda}\big(e_{\ominus\lambda}(t,0)ty\big)_{t=0}^{t\to+\infty} + \frac{1}{\lambda}\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)y\,\Delta t \end{aligned}$$

$$= \frac{1}{\lambda}\int_0^\infty -\frac{\ominus\lambda}{\lambda}e_{\ominus\lambda}(t,0)y\,\Delta t = -\frac{1}{\lambda^2}\big(e_{\ominus\lambda}(t,0)y\big)_{t=0}^{t\to+\infty} = \frac{y}{\lambda^2}. \tag{3.9}$$

Finally

$$\begin{aligned} A\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\int_0^t(t-\sigma(s))u(s)\Delta\Delta t &= A\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\int_0^t\int_0^s u(r)\Delta r\Delta s\Delta t \\ &= A\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\int_0^t v(s)\Delta s\Delta t = A\frac{\widehat{u}(\lambda)}{\lambda^2}. \end{aligned} \tag{3.10}$$

Combining (3.7) with (3.8), (3.9) and (3.10), we get

$$\widehat{u}(\lambda) = \frac{x}{\lambda} + \frac{y}{\lambda^2} + A\frac{\widehat{u}(\lambda)}{\lambda^2}, \tag{3.11}$$

obtaining the result. To prove the other part, it is only to follow the same steps. $\square$

Remark 3.7. Considering equation (3.2) instead of (3.1), the equation (3.11) is not satisfied. In that case, we obtain $\widehat{u}(\lambda) = \frac{x}{\lambda} + \frac{y}{\lambda^2} + A\frac{\widehat{u^\sigma}(\lambda)}{\lambda^2}$. Choosing $\mathbb{T}=\mathbb{Z}$ and using the property $\widehat{u^\sigma}(\lambda) = (\lambda+1)\widehat{u}(\lambda) - u(0)$, we have $\widehat{u}(\lambda) = \frac{x}{\lambda} + \frac{y}{\lambda^2} + A\frac{(\lambda+1)\widehat{u}(\lambda)-u(0)}{\lambda^2}$. Therefore, if $y=0$, then we have $\widehat{u}(\lambda) = (\lambda - A)(\lambda^2 - (\lambda+1)A)^{-1}x$ instead of $\widehat{u}(\lambda) = (\lambda^2 - A)^{-1}x$.

As an immediate consequence, we have the following result.

Corollary 3.8. *Let $u \in \mathcal{C}_{rd}(\mathbb{T}_0^+, X)$, $\omega \in D\{u\}$ and suppose the conditions (3.6) hold for every λ with $Re_\mu(\lambda)(t) > Re_\mu(\omega)(t)$ and $Re_\mu(\lambda^2)(t) > Re_\mu(\omega)(t)$ for all $t \in \mathbb{T}_0^+$. Then u is a mild solution of the problem (3.1), if and only if,*

$$\widehat{u}(\lambda) \in D(A) \;\; and \;\; \widehat{u}(\lambda) = \lambda(\lambda^2 - A)^{-1}x + (\lambda^2 - A)^{-1}y. \tag{3.12}$$

From now on, we assume $u \in \mathcal{C}_{rd}(\mathbb{T}_0^+, X)$ and $\omega \in D\{u\}$. Now, let us present the definition of abstract cosine function on time scales.

Definition 3.9. A strongly rd-continuous function $C : \mathbb{T}_0^+ \to \mathcal{L}(X)$ is called a *cosine function with generator* A if the following conditions are satisfied: $C(0) = I$ and there exists $\omega \in \mathbb{R}$ such that $(\omega,\infty)_\mathbb{T} \subset \rho(A)$, $\lambda^2 \in D\{C\}$, and for all $x \in X$,

$$\widehat{C}(\lambda)x = \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)C(t)x\Delta t = \lambda(\lambda^2 - A)^{-1}x, \;\; \text{for all } Re_\mu(\lambda^2)(t) > \omega, t \in \mathbb{T}_0^+. \tag{3.13}$$

Next, we present some important properties of a cosine function on time scales and their relations with the generator A.

Proposition 3.10. *Let $C : \mathbb{T}_0^+ \to \mathcal{L}(X)$ be a cosine function on X and let A be its generator. Let $\omega \in D\{C\}$, and suppose that $\lambda \in \mathbb{C}$ is such that (3.6) holds, $Re_\mu(\lambda^2)(t) > \omega$ and $Re_\mu(\lambda)(t) > \omega$ for all $t \in \mathbb{T}_0^+$. Then*

a) $\int_0^t (t-\sigma(s))C(s)x\,\Delta s \in D(A)$ and $A\int_0^t (t-\sigma(s))C(s)x\,\Delta s = C(t)x - x$, for all $x \in X, t \in \mathbb{T}_0^+$.
b) Let $x \in D(A)$, $\omega \in D\{C\}$, suppose $A\widehat{C}(\lambda) = \widehat{C}(\lambda)A$, then $C(t)x \in D(A)$ and $AC(t)x = C(t)Ax$ for all $t \in \mathbb{T}_0^+$.
c) Let $x, y \in X$, then $x \in D(A)$ and $Ax = y$ if and only if $C(t)x - x = \int_0^t (t-\sigma(s))C(s)y\,\Delta s$ for all $t \in \mathbb{T}_0^+$.

d) If 0 is right-dense, then $D(A) = \left\{x \in X : \lim\limits_{h\to 0^+} \frac{2(C(h)x-x)}{h^2} \text{ exists}\right\}$, and $Ax = \lim_{h\to 0^+} \frac{2(C(h)x-x)}{h^2}$.

e) If 0 and $\sigma(0)$ are right-scattered, then

$$D(A) = \left\{x \in X : \frac{(C(\sigma(\sigma(0))) - C(\sigma(0)))x}{\mu(\sigma(0))\mu(0)} + \frac{(C(0) - C(\sigma(0)))x}{\mu(0)^2} \text{ is well-defined}\right\},$$

and $Ax = \dfrac{(C(\sigma(\sigma(0))) - C(\sigma(0)))x}{\mu(\sigma(0))\mu(0)} + \dfrac{(C(0) - C(\sigma(0)))x}{\mu(0)^2}$.

f) If 0 is right-scattered and $\sigma(0)$ is right-dense, then

$$D(A) = \left\{x \in X : \lim_{h\to 0^+} \frac{(C(\sigma(0)+h) - C(\sigma(0)))x}{\mu(0)h} + \frac{(C(0) - C(\sigma(0))x}{\mu(0)} \text{ exists}\right\},$$

and $Ax = \lim\limits_{h\to 0^+} \dfrac{(C(\sigma(0)+h) - C(\sigma(0)))x}{\mu(0)h} + \dfrac{(C(0) - C(\sigma(0))x}{\mu(0)}$.

Proof. a) Considering $y = 0$ in the equation (3.12), it follows by the definition of the abstract cosine function that C satisfies the equation (3.12). Then by Corollary 3.8, we have that C is a mild solution of (3.1) for $y = 0$. Therefore, the result follows by applying the definition of a mild solution.

b) For $x \in X$ and $Re_\mu(\lambda)(t) > \omega$ for $t \in \mathbb{T}_0^+$, we have $\int_0^\infty e^\sigma_{\ominus\lambda}(s,0)C(s)Ax\,\Delta s = \widehat{C}(\lambda)Ax = A\widehat{C}(\lambda)x = \int_0^\infty e^\sigma_{\ominus\lambda}(s,0)AC(s)x\,\Delta s$, which implies by the uniqueness of the Laplace transform [14, Theorem 3.14], that $AC(t)x = C(t)Ax$ for all $t \in \mathbb{T}_0^+$.

c) We define $C_1(t) := \int_0^t C(s)\,\Delta s$ and $C_2(t) := \int_0^t C_1(s)\,\Delta s$ for all $t \in \mathbb{T}_0^+$. By hypotheses and by Corollary 2.17, we obtain

$$\widehat{C_1}(\lambda) = \frac{\widehat{C}(\lambda)}{\lambda}, \quad \widehat{C_2}(\lambda) = \frac{\widehat{C_1}(\lambda)}{\lambda}, \quad \text{and} \quad \widehat{C_2}(\lambda) = \frac{\widehat{C}(\lambda)}{\lambda^2}. \tag{3.14}$$

Assume that $\int_0^t (t-\sigma(s))C(s)y\,\Delta s = C(t)x - x$ for all $t \in \mathbb{T}_0^+$. Using Laplace transform, we get by (3.14) and (3.13) that

$$\begin{aligned}
\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\int_0^t (t-\sigma(s))C(s)y\,\Delta s\,\Delta t &= \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\int_0^t\int_0^s C(r)y\,\Delta r\,\Delta s\,\Delta t\\
&= \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\int_0^t C_1(s)y\,\Delta s\,\Delta t\\
&= \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)C_2(t)y\,\Delta t\\
&= \widehat{C_2}(\lambda)y = \frac{\widehat{C}(\lambda)y}{\lambda^2} = \frac{1}{\lambda}(\lambda^2 - A)^{-1}y. \qquad (3.15)
\end{aligned}$$

On the other hand, applying (3.13) and using (3.8), we have $\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\big(C(t)x - x\big)\,\Delta t = \lambda(\lambda^2 - A)^{-1}x - \frac{x}{\lambda}$, which implies by hypothesis, by (3.15) and from the uniqueness of the Laplace transform [14, Theorem 3.14] that $\frac{1}{\lambda}(\lambda^2 - A)^{-1}y = \lambda(\lambda^2 - A)^{-1}x - \frac{x}{\lambda}$. Therefore, $y = (\lambda^2 - A)(\lambda^2 - A)^{-1}y = (\lambda^2 - A)\lambda^2(\lambda^2 - A)^{-1}x - (\lambda^2 - A)x = \lambda^2 x - (\lambda^2 - A)x = Ax$, for $Re_\mu(\lambda^2)(t) > \omega$, proving the first part. Conversely, let $x, y \in X$ be such that $x \in D(A)$ and $Ax = y$. Since $\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)C(t)Ax\,\Delta t = \widehat{C}(\lambda)Ax$, by (3.13) we have $A\widehat{C}(\lambda)x = A\lambda(\lambda^2 - A)^{-1}x =$

$\lambda(\lambda^2 - A)^{-1}y = \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)C(t)y\,\Delta t = \widehat{C}(\lambda)Ax$, and by part a) and b), we conclude that

$$C(t)x - x = A\int_0^t (t-\sigma(s))C(s)x\Delta s = \int_0^t (t-\sigma(s))AC(s)x\Delta s = \int_0^t (t-\sigma(s))C(s)y\Delta s.$$

d) Let $x \in D(A)$ and $Ax = y$. From part c), it follows that

$$\frac{2}{t^2}(C(t)x - x) - y = \frac{2}{t^2}\int_0^t (t-\sigma(s))C(s)y\Delta s - y = \frac{2}{t^2}\int_0^t (t-\sigma(s))\big(C(s)y - y\big)\Delta s.$$

Since 0 is right-dense, then it is possible to find a sequence $\{\delta_n\} \subset \mathbb{T}_0^+$ such that $\delta_n \to 0^+$ as $n \to \infty$. Since σ is rd-continuous and 0 is right-dense, it follows that $\lim_{n\to\infty}\sigma(\delta_n) = \sigma(0) = 0$. Now, we notice that

$$\begin{aligned}
\left\|\frac{2(C(\sigma(\delta_n))x - x)}{(\sigma(\delta_n))^2} - y\right\| = &\left\|\frac{2}{(\sigma(\delta_n))^2}\int_0^{\sigma(\delta_n)}(\sigma(\delta_n) - \sigma(s))(C(s)y - y)\,\Delta s\right\| \\
&\leq \frac{2}{(\sigma(\delta_n))^2}\int_0^{\delta_n}(\sigma(\delta_n) - \sigma(s))\,\|(C(s)y - y)\|\,\Delta s \\
&+ \frac{2}{(\sigma(\delta_n))^2}\int_{\delta_n}^{\sigma(\delta_n)}(\sigma(\delta_n) - \sigma(s))\,\|(C(s)y - y)\|\,\Delta s.
\end{aligned} \tag{3.16}$$

The first integral in the last inequality goes to 0 as $n \to \infty$ by applying L'Hôpital theorem, and in the second one we have

$$\begin{aligned}
\int_{\delta_n}^{\sigma(\delta_n)}(\sigma(\delta_n) - \sigma(s))\,\|(C(s)y - y)\|\,\Delta s &= (\sigma(\delta_n) - \sigma(\delta_n))\|(C(\delta_n)y - y)\|\mu(\delta_n) \\
&= 0,
\end{aligned}$$

by Theorem 2.6. Therefore, we have

$$\lim_{t\to 0^+}\frac{2}{t^2}(C(t)x - x) = y. \tag{3.17}$$

Conversely, let $x, y \in X$ be such that (3.17) holds. From the properties in part a), we obtain that $A\frac{2}{t^2}\int_0^t (t-\sigma(s))C(s)x\Delta s = \frac{2}{t^2}\big(C(t)x - x\big) \to y$ as $t \to 0^+$. Therefore, by (3.16) and since A is closed, we have $x \in D(A)$ and $Ax = y$.

e) Let $x \in D(A)$, and $Ax = y$. From part c), it follows that

$$\begin{aligned}
&\frac{\big(C(\sigma(\sigma(0))) - C(\sigma(0))\big)x}{\mu(\sigma(0))\mu(0)} + \frac{\big(C(0) - C(\sigma(0))\big)x}{\mu(0)^2} \\
&= \frac{\big(C(\sigma(\sigma(0))) - C(0) + C(0) - C(\sigma(0))\big)x}{\mu(\sigma(0))\mu(0)} + \frac{\big(C(0) - C(\sigma(0))\big)x}{\mu(0)^2} \\
&= \frac{1}{\mu(\sigma(0))\mu(0)}\left[\int_0^{\sigma(\sigma(0))}(\sigma(\sigma(0)) - \sigma(s))C(s)y\Delta s - \int_0^{\sigma(0)}(\sigma(0) - \sigma(s))C(s)y\Delta s\right] \\
&+ \frac{1}{\mu(0)^2}\int_0^{\sigma(0)}(\sigma(0) - \sigma(s))C(s)y\Delta s \\
&= \frac{1}{\mu(\sigma(0))\mu(0)}\left[\int_0^{\sigma(0)}(\sigma(\sigma(0)) - \sigma(s))C(s)y\Delta s - \int_0^{\sigma(0)}(\sigma(0) - \sigma(s))C(s)y\,\Delta s\right]
\end{aligned}$$

$$+\frac{1}{\mu(\sigma(0))\mu(0)}\int_{\sigma(0)}^{\sigma(\sigma(0))}(\sigma(\sigma(0))-\sigma(s))C(s)y\Delta s+\frac{1}{\mu(0)^2}(\sigma(0)-\sigma(0))C(0)\mu(0)y$$
$$=\frac{1}{\mu(\sigma(0))\mu(0)}\left[(\sigma(\sigma(0))-\sigma(0))\mu(0)C(0)y+(\sigma(\sigma(0))-\sigma(\sigma(0)))\mu(\sigma(0))C(\sigma(0))y\right]$$
$$-\frac{1}{\mu(\sigma(0))\mu(0)}\left[(\sigma(0)-\sigma(0))\mu(0)C(0)y\right]$$
$$=\frac{1}{\mu(\sigma(0))\mu(0)}\mu(\sigma(0))\mu(0)C(0)y=y,$$

proving the claim. Conversely, let $x, y \in X$ be such that

$$\frac{(C(\sigma(\sigma(0)))-C(\sigma(0)))x}{\mu(\sigma(0))\mu(0)}+\frac{(C(0)-C(\sigma(0)))x}{\mu(0)^2}=y.$$

From part a), we have

$$A\frac{1}{\mu(\sigma(0))\mu(0)}\left[\int_0^{\sigma(\sigma(0))}(\sigma(\sigma(0))-\sigma(s))C(s)y\,\Delta s-\int_0^{\sigma(0)}(\sigma(0)-\sigma(s))C(s)y\,\Delta s\right]$$
$$-A\frac{1}{\mu(0)^2}\int_0^{\sigma(0)}(\sigma(0)-\sigma(s))C(s)y\,\Delta s$$
$$=\frac{\big(C(\sigma(\sigma(0)))-C(\sigma(0))\big)x}{\mu(\sigma(0))\mu(0)}+\frac{\big(C(0)-C(\sigma(0))\big)x}{\mu(0)^2}=y,$$

getting the desired result. Therefore, $x \in D(A)$ and $Ax = y$.

f) Let $x \in D(A)$, and $Ax = y$. From c) and using Theorem 2.6, it follows that

$$\begin{aligned}
&\frac{C(\sigma(0)+h)x-C(\sigma(0))x}{\mu(0)h}\\
=\;&\frac{1}{\mu(0)h}\Big(C(\sigma(0)+h)x-x-C(\sigma(0))x+x\Big)\\
=\;&\frac{1}{\mu(0)h}\left(\int_0^{\sigma(0)+h}(\sigma(0)+h-\sigma(s))C(s)y\,\Delta s-\int_0^{\sigma(0)}(\sigma(0)-\sigma(s))C(s)y\,\Delta s\right)\\
=\;&\frac{1}{\mu(0)h}\left(\int_0^{\sigma(0)}hC(s)y\,\Delta s+\int_{\sigma(0)}^{\sigma(0)+h}(\sigma(0)+h-\sigma(s))C(s)y\,\Delta s\right)\\
=\;&\frac{1}{\mu(0)h}\left(h\mu(0)C(0)y+\int_{\sigma(0)}^{\sigma(0)+h}(\sigma(0)+h-\sigma(s))C(s)y\,\Delta s\right)\\
=\;&y+\frac{1}{\mu(0)h}\left(\int_{\sigma(0)}^{\sigma(0)+h}(\sigma(0)+h-\sigma(s))C(s)y\,\Delta s\right),
\end{aligned}$$

from which, applying limit when $h \to 0^+$ and using L'Hopital theorem, we get

$$\begin{aligned}
\lim_{h\to0^+}\frac{C(\sigma(0)+h)x-C(\sigma(0))x}{\mu(0)h} &= \lim_{h\to0^+}\left[y+\int_{\sigma(0)}^{\sigma(0)+h}(\sigma(0)+h-\sigma(s))C(s)y\,\Delta s\right]\\
&= y+\lim_{h\to0^+}\frac{1}{\mu(0)}(\sigma(0)+h-\sigma(\sigma(0)))C(\sigma(0))y\\
&= y+\lim_{h\to0^+}\frac{1}{\mu(0)}(\sigma(0)+h-\sigma(0))C(\sigma(0))y
\end{aligned}$$

$$= y + \lim_{h\to 0^+} \frac{1}{\mu(0)} hC(\sigma(0))y = y, \tag{3.18}$$

since $\sigma(0)$ is right-dense. On the other hand, we get

$$\begin{aligned} -\frac{(C(\sigma(0)) - C(0))x}{\mu(0)^2} &= -\frac{1}{\mu(0)^2}\int_0^{\sigma(0)} (\sigma(0)-\sigma(s))C(s)y\Delta s \\ &= -\frac{1}{\mu(0)^2}(\sigma(0)-\sigma(0))\mu(0)C(0)y = 0. \end{aligned}$$

Therefore, we have

$$\lim_{h\to 0^+} \frac{(C(\sigma(0)+h) - C(\sigma(0)))x}{\mu(0)h} + \frac{(C(0)-C(\sigma(0))x}{\mu(0)} = y, \tag{3.19}$$

obtaining the result. Conversely, suppose that x, y satisfy (3.19). From a), we have

$$\begin{aligned} & A\left[\frac{1}{\mu(0)h}\left(\int_0^{\sigma(0)} hC(s)y\,\Delta s + \int_{\sigma(0)}^{\sigma(0)+h} (\sigma(0)+h-\sigma(s))C(s)y\,\Delta s\right)\right] \\ & -A\left[\frac{1}{\mu(0)^2}\int_0^{\sigma(0)} (\sigma(0)-\sigma(s))C(s)y\Delta s\right] \\ = {} & \frac{(C(\sigma(0)+h)-C(\sigma(0)))x}{\mu(0)h} + \frac{(C(0)-C(\sigma(0))x}{\mu(0)} \to y \quad \text{as } t \to 0^+. \end{aligned}$$

Hence, by (3.18) and since A is closed, it follows that $x \in D(A)$ and $Ax = y$. □

Remark 3.11. Notice that Theorem 3.6 describes all the possibilities for the definition of the generator A. Indeed, the case where 0 is right-dense and $\sigma(0)$ is right-scattered at the same time is not possible, since if 0 is right-dense, then $\sigma(0) = 0$ which implies that $\sigma(0)$ has to be right-dense.

4. Abstract Sine family on time scales

In this section, we introduce the notion of abstract sine family on time scales.

Definition 4.1. We say that a strongly rd-continuous function $S : \mathbb{T}_0^+ \to \mathcal{L}(X)$ is a *sine function with generator* A if the following conditions are satisfied: $S(0) = 0$, and there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty)_{\mathbb{T}} \subset \rho(A)$, $\lambda^2 \in D\{S\}$, and for all $x \in X$,

$$\widehat{S}(\lambda)x = \int_0^\infty e_{\ominus\lambda}^\sigma(t,0)S(t)x\Delta t = (\lambda^2 - A)^{-1}x, \text{ for all } Re_\mu(\lambda^2)(t) > \omega, t \in \mathbb{T}_0^+.$$

If A generates a cosine function C, then $(\lambda^2 - A)^{-1} = \frac{1}{\lambda}\int_0^\infty e_{\ominus\lambda}^\sigma(t,0)C(t)\,\Delta t = \int_0^\infty e_{\ominus\lambda}^\sigma(t,0)\int_0^t C(s)\,\Delta s\,\Delta t$, for $Re_\mu(\lambda^2)(t) > \omega$. Thus, A generates the abstract sine function S given by $S(t)x := \int_0^t C(s)x\,\Delta s$. It implies that this definition is consistent with Definition 3.9.

The next result establishes some relations between a sine function and its generator. Its proof follows as in the proof of Proposition 3.10. We omit the details.

Proposition 4.2. Let $S : \mathbb{T}_0^+ \to \mathcal{L}(X)$ be a sine function on X and let A be its generator. Let $\omega \in D\{S\}$, and suppose that $\lambda \in \mathbb{C}$ is such that (3.6) holds, $Re_\mu(\lambda^2)(t) > \omega$ and $Re_\mu(\lambda)(t) > \omega$ for all $t \in \mathbb{T}_0^+$. Then

a) $\int_0^t (t-\sigma(s))S(s)x\,\Delta s \in D(A)$ and $A\int_0^t (t-\sigma(s))S(s)x\,\Delta s = S(t)x - tx$, for all $x \in X, t \in \mathbb{T}_0^+$.

b) Let $x \in D(A)$ and $A\widehat{S}(\lambda) = \widehat{S}(\lambda)A$, then $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$ for all $t \in \mathbb{T}_0^+$.
c) Let $x, y \in X$, then $x \in D(A)$ and $Ax = y$ if and only if $S(t)x - tx = \int_0^t (t - \sigma(s))S(s)y\Delta s$ for all $t \in \mathbb{T}_0^+$.

Our definitions of abstract sine and cosine functions on time scales are directly related with the existence of mild solutions to problem (3.1). Now, we discuss about the existence of mild solutions to (3.1).

Theorem 4.3. *Let $A \in \mathcal{L}(X)$. Then the problem* (3.1) *has a unique mild solution.*

Proof. We fix $a \in \mathbb{T}_0^+$, and define the operator $\Gamma : \mathcal{C}_{rd}([0, a], X) \to \mathcal{C}_{rd}([0, a], X)$ by $(\Gamma u)(t) = x + ty + \int_0^t (t - \sigma(s))u(s)\,\Delta s$, where $t \in \mathbb{T}_0^+$. For $t \leq a \in \mathbb{T}_0^+$, it is not difficult to prove that Γ has a unique fixed point $u(\cdot)$, which is the solution to (3.1). Moreover, since $u(t) = x + ty + \int_0^t (t - \sigma(s))u(s)\,\Delta s$, for all $t \in \mathbb{T}_0^+$, for such fixed point $u(\cdot)$, we have that $u(\cdot)$ is a mild solution of (3.1), concluding the proof. $\square$

Now, we give an example. Assume that A generates a cosine family $C : \mathbb{T}_0^+ \to \mathcal{L}(X)$ with associated sine $S : \mathbb{T}_0^+ \to \mathcal{L}(X)$, and let $\lambda \in \mathbb{R}$ be an eigenvalue of A.

Example 4.4. If $\lambda \in \mathbb{R}$, then $C(t)x = \cosh_{\sqrt{\lambda}}(t, 0)x, \quad S(t)y = \frac{1}{\sqrt{\lambda}} \sinh_{\sqrt{\lambda}}(t, 0)y$, for all $t \in \mathbb{T}_0^+$. Indeed, let $u(t) = \cosh_{\sqrt{\lambda}}(t, 0)x + \frac{1}{\sqrt{\lambda}} \sinh_{\sqrt{\lambda}}(t, 0)y$. From the definition of hyperbolic functions on time scales [7, Definition 3.17], it is immediate that $u(0) = x$. Applying delta derivative, we get $u^\Delta(t) = \sqrt{\lambda} \sinh_{\sqrt{\lambda}}(t, 0)x + \cosh_{\sqrt{\lambda}}(t, 0)y$, and from this formula we obtain $u^\Delta(0) = y$. Applying delta derivative again, we have $u^{\Delta\Delta}(t) = \lambda \cosh_{\sqrt{\lambda}}(t, 0)x + \sqrt{\lambda} \sinh_{\sqrt{\lambda}}(t, 0)y = \lambda u(t)$. From the uniqueness of solutions of the problem (3.1) for the case $A := \lambda$, we conclude that $u(t) = C(t)x + S(t)y$, and the result is proved.

5. Inhomogeneous second order abstract Cauchy problem

In this section, we investigate the existence of solutions of the following inhomogeneous abstract Cauchy problem on time scales

$$u^{\Delta\Delta}(t) = Au(t) + f(t), \quad t \in \mathbb{T}_0^+, \text{ and } u(0) = x, u^\Delta(0) = y, \tag{5.1}$$

where $x, y \in X$. Assume that $u(t) \in X$ and $f : \mathbb{T}_0^+ \to X$ is rd-continuous. Also, we assume that A generates a cosine function $C : \mathbb{T}_0^+ \to X$ and a sine function $S : \mathbb{T}_0^+ \to X$, and that there exists an rd-continuous function g such that $(\lambda^2 I - A)\widehat{g}(\lambda) = \widehat{f}(\lambda)$, $\lambda > \omega$. Now, we introduce the definition of a mild solution of (5.1).

Definition 5.1. An rd-continuous function $u : \mathbb{T}_0^+ \to X$ is a *mild solution of* (5.1) if $u(t) = x + ty + A\int_0^t (t - \sigma(s))u(s)\,\Delta s + \int_0^t (t - \sigma(s))f(s)\,\Delta s$ for all $t \in \mathbb{T}_0^+$.

We restrict us to consider the operator $A \in \mathcal{L}(X)$. In this case, for each $s \in \mathbb{T}_0^+$, we consider the abstract Cauchy problem given by

$$u^{\Delta\Delta}(t) = Au(t), \quad t \in \mathbb{T}_0^+, \text{ and } u(s) = x,\ u^\Delta(s) = y. \tag{5.2}$$

Definition 5.2. We say that an rd-continuous function $u : [s, \infty)_\mathbb{T} \to X$ is a *mild solution of* (5.2) if $u(t) = x + ty + A\int_s^t (t - \sigma(r))u(r)\,\Delta r$, for all $t \geq s$ and $t, s \in \mathbb{T}_0^+$.

It is easy to show that problem (5.2) has a unique solution $u(t, s)$ for all $x, y \in X$. Next, we define $C(t, s)x + S(t, s)y = u(t, s)$. Also, $C : \{(t, s) : t \geq s, t, s \in \mathbb{T}_0^+\} \to \mathcal{L}(X)$ and $S : \{(t, s) : t \geq s, t, s \in \mathbb{T}_0^+\} \to \mathcal{L}(X)$ are strongly rd-continuous maps.

Theorem 5.3. *Let $A \in \mathcal{L}(X)$ and assume that $f : \mathbb{T}_0^+ \to X$ is an rd-continuous function. Then the mild solution $u(t)$ of the problem* (5.1) *is given by* $u(t) = C(t,t_0)x + S(t,t_0)y + \int_{t_0}^t S(t,\sigma(r))f(r)\,\Delta r$.

Remark 5.4. In the following proof, we only seek the solution for the case $t \geq t_0$. Hence, $C(t,t_0)$ and $S(t,t_0)$ are only defined for $t \geq t_0$. On the other hand, by the definition of abstract sine function, let us consider $S(t) := S(t,t_0) = \int_{t_0}^t C(s)\,\Delta s$. Then, $S(t,s) = \int_s^t C(r)\,\Delta r$. This implies that $S(t,s) = 0$ whenever $t = s$. On the other hand, we also consider $C(t) := C(t,t_0)$ for all $t \geq t_0$.

Proof of Theorem 5.3. For $t \in \mathbb{T}_0^+$ we define the following function $u(t) := C(t)x + S(t)y + \int_0^t S(t,\sigma(r))f(r)\,\Delta r$. By Proposition 4.2 a), we have

$$\begin{aligned}
A\int_0^t (t-\sigma(s))\int_0^s S(s,\sigma(r))f(r)\Delta r\Delta s &= A\int_0^t\int_0^s (t-\sigma(s))S(s,\sigma(r))f(r)\Delta r\Delta s \\
&= \int_0^t A\int_{\sigma(r)}^t (t-\sigma(s))S(s,\sigma(r))f(r)\Delta s\Delta r \\
&= \int_0^t \left(S(t,\sigma(r)) - t\right) f(r)\Delta r \\
&- \int_0^t \left(S(\sigma(r),\sigma(r)) - \sigma(r)\right) f(r)\Delta r \\
&= \int_0^t \left(S(t,\sigma(r)) - (t-\sigma(r))\right) f(r)\Delta r.
\end{aligned}$$

where in the second equality, we used the change of order of the integration (see [3]). Replacing u given in Definition 5.1, we obtain by Propositions 3.10 and 4.2

$$\begin{aligned}
u(t) &= x + ty + A\int_0^t (t-\sigma(s))u(s)\,\Delta s + \int_0^t (t-\sigma(s))f(s)\Delta s \\
&= x + ty + A\int_0^t (t-\sigma(s))C(s)x\,\Delta s + A\int_0^t (t-\sigma(s))S(s)y\,\Delta s \\
&+ \quad A\int_0^t (t-\sigma(s))\int_0^s S(s,\sigma(r))f(r)\,\Delta r\,\Delta s + \int_0^t (t-\sigma(s))f(s)\,\Delta s \\
&= C(t)x + S(t)y + \int_0^t S(t,\sigma(r))f(r)\Delta r.
\end{aligned}$$

□

Now, we also investigate the existence of solutions of the following nonlinear abstract Cauchy problem on time scales

$$u^{\Delta\Delta}(t) = Au(t) + f(t,u(t)), \quad t \in \mathbb{T}_0^+, \text{ and } u(0) = x,\ u^{\Delta}(0) = y, \tag{5.3}$$

where $x, y \in X$. We assume that the values $u(t) \in X$ and $f : \mathbb{T}_0^+ \times X \to X$ is an rd-continuous function with respect to the first variable. Also, we assume that A generates a cosine function $C : \mathbb{T}_0^+ \to X$ and a sine function $S : \mathbb{T}_0^+ \to X$.

We start by introducing the definition of a mild solution of problem (5.3).

Definition 5.5. An rd-continuous function $u : \mathbb{T}_0^+ \to X$ is a *mild solution of* (5.3) if $u(t) = x + ty + A\int_0^t (t-\sigma(s))u(s)\,\Delta s + \int_0^t (t-\sigma(s))f(s,u(s))\,\Delta s$ for all $t \in \mathbb{T}_0^+$.

The next theorem plays an important role in the study of nonlinear second order abstract Cauchy problems on time scales. The proof of the next theorem follows similarly to the proof of Theorem 5.3. We will omit it here.

Theorem 5.6. *Let $A \in \mathcal{L}(X)$ and assume that $f : \mathbb{T}_0^+ \times X \to X$ is an rd-continuous function. Then, the mild solution $u(t)$ of the problem* (5.1) *is given by* $u(t) = C(t,t_0)x + S(t,t_0)y + \int_{t_0}^t S(t,\sigma(r))f(r,u(r))\,\Delta r$.

Acknowledgements. The authors thank to the referees for their carefully reading of the manuscript and for making suggestions which have improved the previous version of this paper.

Conflict of interest. This work does not have any conflicts of interest.

References

[1] R. Agarwal, M. Bohner, D. O'Regan, A. Peterson *Dynamic equations on time scales: a survey,* J. of Computational and Applied Mathematics **141** (2002) 1–26.

[2] W. Arendt, C. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems,* Monographs in Math. 96, Birkhäuser, Basel, Switzerland, 2001.

[3] M. Bohner, G. Guseinov, *Multiple integration on time scales,* Dynam. Systems Appl. 14 (2005), no. 3-4, 579-606.

[4] M. Bohner, G. Guseinov, *The Laplace transform on isolated time scales*, Comput. Math. Appl. 60 (2010), no. 6, 1536–1547.

[5] M. Bohner, G. Guseinov, *The h-Laplace and q-Laplace transforms*, J. Math. Anal. Appl. 365 (2010), no. 1, 75–92.

[6] M. Bohner, J. G. Mesquita, *Massera's theorem in quantum calculus,* Proc. Amer. Math. Soc. 146 (2018), no. 11, 4755-4766.

[7] M. Bohner, A. Peterson, *Dynamic equations on time scales: An introduction with applications,* Birkhäuser, Boston, 2001.

[8] M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scales,* Birkhäuser, 2003.

[9] M. Bohner, A. Peterson, *Laplace transform and Z-transform: unification and extensions*, Methods Appl. Anal. 9 (2002), no. 1, 151–157.

[10] W. Chojnacki, *On operator-valued cosine sequences on UMD spaces,* Studia Math. 199 (2010), no. 3, 267-278.

[11] H. Fattorini, *A note on fractional derivatives of semigroups and cosine functions,* Pacific J. of Math., **109**, Vol. 2, 335-347, 1983.

[12] C. Goodrich, A. Peterson, *Discrete fractional calculus,* Springer, Cham, 2015. xiii+556 pp.

[13] H. Henríquez, J. G. Mesquita, *Self-accessible states for linear systems on time scales,* Proc. Amer. Math. Soc. 146 (2018), no. 3, 1257–1269.

[14] H. Henríquez, C. Lizama, J. G. Mesquita, *Semigroups on time scales and applications to abstract Cauchy problems*, Top. Methods in Nonlinear Analysis, **56** (1) (2020), 83-115.

[15] R. Hoppe, *Discrete approximations of cosine operator functions. I,* SIAM J. Numer. Anal. 19 (1982), no. 6, 1110-1128.

[16] C. C. Travis, G. F. Webb, *An abstract second–order semilinar Volterra integro–differential equation,* SIAM J. Math. Anal. 10 (1979), no. 2, 412–424.

[17] C. C. Travis, G. F. Webb, *Cosine families and abstract nonlinear second order differential equations,* Acta Math. Acad. Sci. Hungar. 32 (1978), no. 1–2, 75–96.