# COSINE SEQUENCES GENERATED BY CLOSED LINEAR OPERATORS.

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ABSTRACT. In this paper, we introduce the notion of a cosine sequence $\{C_\tau^n\}_{n\in\mathbb{N}_0}$ generated by a closed linear operator $A$ in a Banach space $X$. We provide a systematic study of the properties of $\{C_\tau^n\}_{n\in\mathbb{N}_0}$, showing that it satisfies a discrete d'Alembert's functional equation. We also explore its connections with its generator $A$, the resolvent operator $R_\tau := \tau^{-2}(\tau^{-2}-A)^{-1}$, and its corresponding sine family $\{S_\tau^n\}_{n\in\mathbb{N}_0}$.

Moreover, we show that the solution to the abstract discrete system of second order

$$\nabla_\tau^2 u^n = Au^n + f^n, \quad n \geq 2,$$

subject to the initial conditions $u^0 = x_0$, $u^1 = x_1$, where $f : \mathbb{N}_0 \to X$ is a given sequence, $\tau > 0$ is a specified step size, and $\nabla_\tau^2 u^n$ is the backward operator of order two, can be expressed as a discrete variation parameter formula in terms of $C_\tau^n$ and its corresponding sine sequence.

## 1. INTRODUCTION

A one-parameter family $\{C(t)\}_{t\in\mathbb{R}}$ of bounded linear operators defined in a Banach space $X$ is called a strongly continuous *cosine* family if $C(0) = I$ (the identity operator), the mapping $t \mapsto C(t)x$ is a continuous function for all $x \in X$, and for every $t, s \in \mathbb{R}$, the family $\{C(t)\}_{t\in\mathbb{R}}$ satisfies the d'Alembert functional equation:

$$C(t+s) + C(t-s) = 2C(t)C(s). \tag{1.1}$$

The notion of a strongly continuous cosine family was introduced by Sova in [39], who defined the infinitesimal generator as $A := C''(0)$ and proved a generation theorem. Subsequently, many authors have studied cosine families and their connections with the existence of solutions to abstract second-order differential equations. For example, Da Prato and Giusti [12] provided a characterization of the generator of a cosine family, while Fattorini in [13, 14] studied cosine families in certain locally convex spaces and their connections with differential equations of order $n$. Travis and Webb [40] unified and simplified various ideas from the theory of cosine families and investigated their relation to the abstract second-order problem:

$$\begin{cases} u''(t) &= Au(t) + f(t), \quad t > 0, \\ u(0) &= x_0, \\ u'(0) &= x_1. \end{cases} \tag{1.2}$$

Here, $A : D(A) \subset X \to X$ is a closed linear operator defined on $X$, and $f$ is a given vector-valued function. In fact, it can be shown (see, for instance, [4, 40]) that if $\{C(t)\}_{t\in\mathbb{R}}$ is a cosine family with generator $A = C''(0)$, then the solution to the Cauchy problem (1.2) is given by:

$$u(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)f(s)\,ds, \tag{1.3}$$

where $S(t) := \int_0^t C(s)\,ds$ defines the sine family associated with $C(t)$.

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Since then, cosine families have been studied extensively by many authors, not only from an abstract perspective but also in their connections with the study of solutions to second-order Cauchy problems. See, for instance, [1, 7, 8, 23, 24, 27, 33, 28, 29, 30, 36, 37].

Equivalently, a cosine family can be defined in terms of its generator. Given a closed linear operator $A$, a strongly continuous family $\{C(t)\}_{t\geq 0}$ is called a *cosine family generated by* $A$ if it satisfies the following conditions:

(1) $C(0)=I$,
(2) $C(t)x \in D(A)$ for all $x\in X$, and $C(t)Ax = AC(t)x$ for all $x\in D(A)$, and
(3) $C(t)x = x + A(g_2 * C)(t)x$, for $t\geq 0$ and $x\in X$.

Here, $g_2$ is defined by the function $g_2(t) := t$, and $(g_2 * C)(t)x := \int_0^t (t-s)C(s)x\,ds$. Moreover, Lizama and Poblete [32] showed that if $\{C(t)\}_{t\geq 0}$ is a cosine family according to this definition, then it satisfies the functional equation:

$$C(t)(g_2 * C)(s)x - (g_2 * C)(s)C(t)x = (g_2 * C)(t)x - (g_2 * C)(s)x, \quad t,s\geq 0,\, x\in X. \tag{1.4}$$

In the same paper, the authors demonstrated that (1.4) is equivalent to the d'Alembert functional equation (1.1).

On the other hand, we can use a discrete version of the d'Alembert functional equation (1.1) to define the notion of a cosine sequence of operators. A sequence of bounded linear operators in $X$, $\{C^n\}_{n\in\mathbb{Z}}$, is called a *cosine sequence* if $C^0 = I$ and it satisfies the discrete d'Alembert functional equation:

$$C^{n+m} + C^{n-m} = 2C^nC^m, \quad n,m\in\mathbb{Z}, \tag{1.5}$$

see for instance [15, Chapter V]. The sequence $\{C^n\}_{n\in\mathbb{Z}}$ is an even sequence, the generator of $\{C^n\}_{n\in\mathbb{Z}}$ is the operator $C^1$, and the cosine sequence $\{C^n\}_{n\in\mathbb{Z}}$ can be completely determined from $C^1$ using Chebyshev polynomials, as shown in [9].

Now, if we consider the discrete problem of second order:

$$\left\{\begin{array}{rcl} \Delta^2 u^n &=& (I-T)u^n, \quad n\geq 2,\\ u^0 &=& x_0,\\ u^1 &=& 0, \end{array}\right.$$

where $\Delta^2 u^n := u^{n+2} - 2u^{n+1} + u^n$ is the forward operator of order two, $T: X\to X$ is a given bounded operator, and $x_0\in X$, then the unique solution $\mathcal{C}^n x_0$ $(n\in\mathbb{N}_0)$ to this problem is sometimes referred to as the *discrete-time cosine* operator generated by $I-T$, see for instance [3, Chapter 1] and [11]. The representation of the solution to the inhomogeneous problem

$$\left\{\begin{array}{rcl} \Delta^2 u^n &=& (I-T)u^n + f^n, \quad n\geq 2,\\ u^0 &=& x_0,\\ u^1 &=& x_1, \end{array}\right. \tag{1.6}$$

where $f:\mathbb{N}\to X$ is a given sequence and $x_0, x_1\in X$, can be expressed in terms of $\mathcal{C}^n$ and its corresponding sine sequence $\mathcal{S}^n$ by:

$$u^n = \mathcal{C}^n x_0 + \mathcal{S}^n x_1 + \sum_{j=0}^{n-1} \mathcal{S}^{n-1-j} f^j, \tag{1.7}$$

as demonstrated in [3, Proposition 1.3.1].

Since the forward operator $\Delta^2 u^n$ can be considered a time discretization of the second-order derivative $u''(t)$, cosine sequences $\{C^n\}_{n\in\mathbb{Z}}$ (or $\{\mathcal{C}^n\}_{n\in\mathbb{Z}}$) can be seen as a natural discretization of cosine families $\{C(t)\}_{t\in\mathbb{R}}$. Discretizations of cosine families of operators have been known for over four decades. Hoppe [22] introduced a method based on a discrete cosine function $C_h^n$ $(h>0)$ defined on a uniform partition $\Delta_h = \{nh : n\in\mathbb{Z}\}$ that satisfies the d'Alembert equation (1.1). The author then approximates the strongly continuous cosine functions $C(t)$ to study the existence of solutions to (1.2) by using the discrete cosine functions $C_h^n$ and a generalized version of the approximation theorem for semigroups of linear operators proposed by Trotter in [41]. However, there is no known relation between $C_h^n$ and the generator $A$. See also [5, 6, 16, 38] for related references.

On the other hand, we note that for a given vector-valued function $v : \mathbb{R}_+ \to X$ and for sufficiently small $\tau > 0$, the sequence $(v^n)_{n\in\mathbb{N}_0}$ defined by $v^n := \int_0^\infty \rho_n^\tau(t)v(t)dt$ with $\rho_n^\tau(t) := e^{-\frac{t}{\tau}}\left(\frac{t}{\tau}\right)^n \frac{1}{\tau n!}$ corresponds intuitively to an approximation of $v$ at $t_n := \tau n$. Therefore, it seems natural to approximate the cosine function $t \mapsto C(t)x$ by $C^n x := \int_0^\infty \rho_n^\tau(t)C(t)xdt$, for $x \in X$ and $n \in \mathbb{N}_0$. However, although the cosine family $\{C(t)\}_{t\in\mathbb{R}}$ satisfies the equation (1.1), the corresponding approximation $C^n$ does not satisfy the discrete d'Alembert equation (1.5). In fact, considering $C(t) = \cos(\sqrt{a}t)$, we find that $\{C(t)\}_{t\in\mathbb{R}}$ forms a cosine family in $X = \mathbb{R}$, but (taking, for instance, $a = 1$, $m = 2$, and $n = 4$) we have:

$$C^{n+m}x + C^{n-m}x = \int_0^\infty (\rho_{n+m}^\tau(t) + \rho_{n-m}^\tau(t))C(t)xdt \neq 2\int_0^\infty \rho_n^\tau(t)C(t)dt\int_0^\infty \rho_m^\tau(t)C(t)xdt = 2C^nC^m x.$$

A natural question arises here: *(Q1) Is there any relation between cosine families and cosine sequences?* Additionally, as $\Delta^2 u$ approximates $u''$, and the solution to (1.2) is given by the variation of parameters formula (1.3), we have the question: *(Q2) Can we express the solution to the discrete system* (1.6) *as a discrete variation of parameters formula if we consider $A$ as a closed linear operator instead of* $(I - T)$ *(with $T$ being a bounded operator)?* Furthermore, if for a given closed linear operator $A$, the cosine family $\{C(t)\}_{t\in\mathbb{R}}$ can be defined using the conditions (1)–(3) above, then another question arises: *(Q3) How can we define a discrete cosine sequence generated by $A$?*

Finally, as the authors mention in [3, p. 17], *not much is known about discrete cosine functions* and *the correspondence with the set of all discrete cosine functions defined by means of d'Alembert functional equations remains an open problem.*

Thus, we observe that the study of discrete cosines of operators is not fully understood, indicating a need for tools to connect cosine families generated by closed linear operators $A$ with cosine sequences (whose generator is $A$) and how they relate to the solution of the discrete second-order equation (1.6).

This paper aims, on one hand, to answer questions *(Q1)-(Q3)*, and on the other hand, to provide a new framework for studying discrete cosine families by introducing a new notion of a cosine sequence of operators generated by a closed linear operator $A$. More specifically, given a closed linear operator $A$ and $\tau > 0$, we use a discrete version of conditions (1)–(3) above to define a cosine sequence $\{C_\tau^n\}_{n\in\mathbb{N}_0}$ generated by $A$. We study the properties of $\{C_\tau^n\}_{n\in\mathbb{N}_0}$, and in particular, we show that if $A$ is a closed linear operator that generates the cosine sequence $\{C_\tau^n\}_{n\in\mathbb{N}_0}$, then:

a) $\tau^{-2} \in \rho(A)$ (the resolvent set of $A$) and $\{C_\tau^n\}_{n\in\mathbb{N}_0}$ satisfies the discrete d'Alembert-type functional equation
$$(*) \quad C_\tau^{n+m} + R_\tau^m C_\tau^{n-m} = 2C_\tau^n C_\tau^m, \quad n, m \in \mathbb{N}_0,$$
where $R_\tau$ is the resolvent operator defined by $R_\tau := \tau^{-2}(\tau^{-2} - A)^{-1}$;
b) $\{C_\tau^n\}_{n\in\mathbb{N}_0}$ satisfies a discrete version of the functional equation (1.4);
c) $\{C_\tau^n\}_{n\in\mathbb{N}_0}$ can be expressed as
$$C_\tau^n x = \sum_{j=1}^n a_{n,j} R_\tau^j x, \quad n \in \mathbb{N}, x \in X,$$
where $(a_{n,l})$ is a known sequence of real numbers (determined using the coefficients in the Chebyshev polynomials of the first kind) and $R_\tau^j$ is the $j$-th power of $R_\tau$; and
d) The (unique) solution to the discrete second-order problem
$$(**) \quad \begin{cases} \nabla_\tau^2 u^n &= Au^n + f^n, \quad n \geq 2, \\ u^0 &= x_0, \\ u^1 &= x_1, \end{cases}$$
where $\nabla_\tau^2$ is an approximation of $u''(t)$ for $\tau > 0$ small enough, can be expressed as a discrete variation of parameters formula involving $\{C_\tau^n\}_{n\in\mathbb{N}_0}$ and its corresponding sine family $S_\tau^n := \sum_{j=1}^n C_\tau^j$.

Additionally, we show that if $A$ is any closed linear operator such that $\tau^{-2} \in \rho(A)$ and $\{C^n\}_{n\in\mathbb{N}_0}$ is a sequence of linear operators satisfying $(*)$, then $\{C^n\}_{n\in\mathbb{N}_0}$ is a cosine sequence generated by $A$.

Moreover, we demonstrate that if $A$ is the generator of a cosine family $\{C(t)\}_{t\in\mathbb{R}}$, then the sequence $\{C^n_\tau\}_{n\in\mathbb{N}_0}$ defined by

$$C^0_\tau := I, \quad \text{and} \quad C^n_\tau x := \int_0^\infty \rho^\tau_{n-1}(t)C(t)x dt, \quad x \in X, \, n \geq 1,$$

is a cosine sequence generated by $A$ and satisfies the functional equation

$$C^{m+n}_\tau + \sum_{l=1}^{n} \frac{1}{2^{m+n-l}} \binom{m+n-l-1}{n-l} C^l_\tau + \sum_{l=1}^{m} \frac{1}{2^{m+n-l}} \binom{m+n-l-1}{m-l} C^l_\tau = 2C^n_\tau C^m_\tau,$$

for all $n, m \in \mathbb{N}$.

The rest of the paper is organized as follows. In Section 2, we provide some preliminaries and define the notion of a cosine sequence generated by a closed operator $A$ in a Banach space $X$. We study its main properties, the corresponding sine sequence, recursive relations, finite representations in terms of the resolvent operator, compactness, among others. Additionally, we present a subordination principle, which shows that if $A$ is the generator of a cosine family, then $A$ is also the generator of a cosine sequence.

Finally, in Section 3, we investigate the existence and uniqueness of solutions to Problem $(**)$ and solve it for the second-order operator $A = \Delta$.

## 2. Discrete cosine sequences

The set of non-negative integer numbers will be denoted by $\mathbb{N}_0$. For any $n \in \mathbb{N}_0$ and $\tau > 0$, we define the sequence of functions $\rho^\tau_n : [0, \infty) \to \mathbb{R}$, given by

$$\rho^\tau_n(t) := e^{-\frac{t}{\tau}} \left(\frac{t}{\tau}\right)^n \frac{1}{\tau n!},$$

We notice that $\rho^\tau_n$ are non-negative functions and $\int_0^\infty \rho^\tau_n(t)dt = 1$, for all $n \in \mathbb{N}_0$.

For a given Banach space $X$, $s(\mathbb{N}_0, X)$ denotes the space of all vector-valued sequences $v : \mathbb{N}_0 \to X$. $\nabla_\tau : s(\mathbb{N}_0, X) \to s(\mathbb{N}_0, X)$ denotes *backward Euler operator of order one,* defined by

$$\nabla_\tau v^n := \frac{v^n - v^{n-1}}{\tau}, \quad n \in \mathbb{N}. \tag{2.8}$$

The *backward difference operator of order* $m$, $\nabla^m_\tau : s(\mathbb{N}_0, X) \to s(\mathbb{N}_0, X)$, is defined by

$$(\nabla^m_\tau v)^n := \nabla^{m-1}_\tau (\nabla_\tau v)^n, \quad n \geq m.$$

for any $m \geq 2$. Here, $\nabla^1_\tau$ is understood as $\nabla^1_\tau := \nabla_\tau$ and $\nabla^0_\tau$ as the identity operator. We notice that if $v \in s(\mathbb{N}_0, X)$ then

$$(\nabla^m_\tau v)^n = \frac{1}{\tau^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j v^{n-j}, \quad n \in \mathbb{N}.$$

For any $\alpha > 0$, we define the sequence

$$k^\alpha_\tau(n) := \frac{\tau^{\alpha-1}\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)} = \int_0^\infty \rho^\tau_n(t) g_\alpha(t)dt, n \in \mathbb{N}_0, \tag{2.9}$$

where $g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $\Gamma(\cdot)$ denotes the Gamma function. We note that, for non-integer values of $\alpha$, the sequence (2.9) has many interesting properties on discrete fractional calculus, see for instance [2, 18, 19, 20] and [31].

Given a sequence of linear operators $\{T^n\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ and a scalar sequence $c = (c^n)_{n\in\mathbb{N}_0}$, we define its discrete convolution $c \star T$ as

$$(c\star T)^n := \sum_{j=0}^{n} c^{n-j}T^j, \quad n \in \mathbb{N}_0.$$

Similarly, for given scalar valued sequences $b = (b^n)_{n\in\mathbb{N}_0}$ and $c = (c^n)_{n\in\mathbb{N}_0}$, we define $(b \star c \star T)^n := (b \star (c \star T))^n$ for all $n \in \mathbb{N}_0$.

By [34, Corollary 2.9], we have that if $\alpha, \beta > 0$, then the following semigroup property holds

$$k_\tau^{\alpha+\beta}(n) = \tau \sum_{j=0}^{n} k_\tau^{\alpha}(n-j)k_\tau^{\beta}(j) = \tau(k_\tau^\alpha \star k_\tau^\beta)^n, \quad \text{for any } n \in \mathbb{N}_0. \tag{2.10}$$

The $Z$-transform of a sequence $s \in s(\mathbb{N}_0, X)$, $\tilde{s}$, is defined by $\tilde{s}(z) := \sum_{j=0}^\infty z^{-j}s^j$, where $s^j := s(j)$ and $z \in \mathbb{C}$. For any $\alpha > 0$, the $Z$-transform of the sequence $\{k_\tau^\alpha(n)\}_{n\in\mathbb{N}_0}$ is given by (see [34])

$$\widetilde{k_\tau^\alpha}(z) = \tau^{\alpha-1}\frac{z^\alpha}{(z-1)^\alpha}, \quad \text{for all } |z| > 1. \tag{2.11}$$

In addition, the $Z$-transform is a linear operator on $s(\mathbb{N}_0, X)$ and satisfies the finite discrete convolution property (see for instance [3]):

$$\widetilde{s_1 \star s_2}(z) = \tilde{s}_1(z)\tilde{s}_2(z), \quad s_1, s_2 \in s(\mathbb{N}_0, X). \tag{2.12}$$

Now, we introduce the notion of a cosine sequence generated by a closed linear operator $A$ in a Banach space $X$.

**Definition 2.1.** *Let $\tau > 0$ be given. The closed linear operator $A$ is called the generator of a* cosine sequence $\{C_\tau^n\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ *if it satisfies the following conditions*

1. *$C_\tau^0 = I$.*
2. *$C_\tau^n x \in D(A)$ for all $x \in X$ and $AC_\tau^n x = C_\tau^n Ax$ for all $x \in D(A)$, and $n \in \mathbb{N}$.*
3. *For each $x \in X$ and $n \geq 1$,*

$$C_\tau^n x = x + \tau A \sum_{j=1}^{n} k_\tau^2(n-j)C_\tau^j x. \tag{2.13}$$

Since $k_\tau^2(n) = \tau(n+1)$ for all $n \in \mathbb{N}_0$, the condition (3) in Definition 2.1 can be written as

$$\begin{aligned} C_\tau^n x &= x + \tau^2 A \sum_{j=1}^{n}(n+1-j)C_\tau^j x \\ &= x + \tau^2 A\left[nC_\tau^1 x + (n-1)C_\tau^2 x + \ldots + C_\tau^n x\right] \\ &= x + \tau^2 A\left[C_\tau^1 x + (C_\tau^1 x + C_\tau^2 x) + \ldots + (C_\tau^1 x + \ldots + C_\tau^n x)\right] \\ &= x + \tau^2 A \sum_{j=1}^{n}\sum_{l=1}^{j} C_\tau^l x, \end{aligned} \tag{2.14}$$

for all $n \in \mathbb{N}$ and $x \in X$.

**Lemma 2.2.** *Let $\tau > 0$ be given. Assume that $A$ is the generator of a cosine sequence $\{C_\tau^n\}_{n\in\mathbb{N}_0}$. Then,*

$$\left(\sum_{j=1}^{m} k_\tau^2(m-j)C_\tau^j\right) C_\tau^n x - C_\tau^m\left(\sum_{j=1}^{n} k_\tau^2(n-j)C_\tau^j x\right) = \sum_{j=1}^{m} k_\tau^2(m-j)C_\tau^j x - \sum_{j=1}^{n} k_\tau^2(n-j)C_\tau^j x, \tag{2.15}$$

*for all $m, n \in \mathbb{N}$ and $x \in X$.*

*Proof.* By Definition 2.1, we have that for any $m, n \in \mathbb{N}$, and $x \in X$,

$$\begin{aligned}\left(\sum_{j=1}^{m} k_\tau^2(m-j)C_\tau^j\right) C_\tau^n x &= \sum_{j=1}^{m} k_\tau^2(m-j)C_\tau^j x + \left(\tau A \sum_{j=1}^{m} k_\tau^2(m-j)C_\tau^j\right)\left(\sum_{j=1}^{n} k_\tau^2(n-j)C_\tau^j x\right)\\ &= \sum_{j=1}^{m} k_\tau^2(m-j)C_\tau^j x + (C_\tau^m - I)\left(\sum_{j=1}^{n} k_\tau^2(n-j)C_\tau^j x\right)\\ &= \sum_{j=1}^{m} k_\tau^2(m-j)C_\tau^j x + C_\tau^m\left(\sum_{j=1}^{n} k_\tau^2(n-j)C_\tau^j x\right) - \sum_{j=1}^{n} k_\tau^2(n-j)C_\tau^j x.\end{aligned}$$

This means that the cosine sequence verifies (2.15). □

For any $\tau > 0$, we define the *resolvent operator* $R_\tau : X \to D(A)$ by

$$R_\tau := \tau^{-2}\left(\tau^{-2} - A\right)^{-1}, \tag{2.16}$$

and for $m \in \mathbb{N}$, $R_\tau^m := \tau^{-2m}\left(\tau^{-2} - A\right)^{-m}$.

The next Proposition shows that a cosine sequence generated by $A$ verifies a nice recurrence relation, which will be an important result to study the properties of $\{C_\tau^n\}_{n\in\mathbb{N}_0}$.

**Proposition 2.3** (Recurrence relation)**.** *Let $\tau > 0$ be given. Assume that $A$ is the generator of a cosine sequence $\{C_\tau^n\}_{n\in\mathbb{N}_0}$. Then,*

(1) *$\tau^{-2} \in \rho(A)$ and $C_\tau^1 = R_\tau$.*
(2) *For all $n \geq 2$, $\{C_\tau^n\}_{n\in\mathbb{N}_0}$ verifies the recurrence relation*

$$C_\tau^n = 2R_\tau C_\tau^{n-1} - R_\tau C_\tau^{n-2}. \tag{2.17}$$

*Proof.* By (2.13), we have $C_\tau^1 x = x + \tau A k_\tau^2(0) C_\tau^1 x = x + \tau^2 A C_\tau^1 x$, which implies that $(\tau^{-2} - A)C_\tau^1 x = \tau^{-2}x$ for all $x \in X$. Since $A$ and $C_\tau^n$ commute on $D(A)$ we have that for each $x \in D(A)$,

$$\tau^{-2}x = (\tau^{-2} - A)C_\tau^1 x = C_\tau^1(\tau^{-2} - A)x.$$

Since $A$ is a closed operator, $\tau^{-2} \in \rho(A)$ and

$$C_\tau^1 x = \tau^{-2}(\tau^{-2} - A)^{-1}x, \tag{2.18}$$

for all $x \in X$. To prove the second assertion we proceed by induction. For $n = 2$ and $x \in X$, by (2.13) and (1) we have

$$C_\tau^2 x = x + \tau A k_\tau^2(1) C_\tau^1 x + \tau A k_\tau^1(0) C_\tau^2 x = x + 2\tau^2 A C_\tau^1 x + \tau^2 A C_\tau^2 x = x + 2\tau^2 A R_\tau x + \tau^2 A C_\tau^2 x.$$

As $\tau^{-2} \in \rho(A)$ we get

$$C_\tau^2 x = R_\tau x + 2\tau^2 A R_\tau^2 x.$$

The identity $(\tau^{-2} - A)(\tau^{-2} - A)^{-1}x = x$ for all $x \in X$, implies that $\tau^2 A R_\tau x = (R_\tau - I)x$ and therefore

$$C_\tau^2 x = 2R_\tau^2 x - R_\tau x.$$

Now, assume that (2.17) holds for $l \leq n$. As $k_\tau^2(n+1-j) = k_\tau^2(n-j) + k_\tau^2(0)$ for all $n \in \mathbb{N}_0$ and $j \leq n$, by (2.13) we can write

$$C_\tau^{n+1}x = x + \tau A \sum_{j=1}^{n+1} k_\tau^2(n+1-j)C_\tau^j x = x + \tau A \sum_{j=1}^{n} k_\tau^2(n-j)C_\tau^j x + \tau^2 A \sum_{j=1}^{n} C_\tau^j x + \tau^2 A C_\tau^{n+1}x.$$

As $\tau^{-2} \in \rho(A)$, we get

$$C_\tau^{n+1}x = R_\tau C_\tau^n x + \tau^2 R_\tau A \sum_{j=1}^{n} C_\tau^j x.$$

Since $C_\tau^n x \in D(A)$ for all $n \in \mathbb{N}$, the identity $(\tau^{-2} - A)^{-1}(\tau^{-2} - A)x = x$ for all $x \in D(A)$, and the induction hypothesis imply that

$$
\begin{aligned}
C_\tau^{n+1}x =& R_\tau C_\tau^n x + (R_\tau - I)\sum_{j=1}^{n} C_\tau^j x \\
=& 2R_\tau C_\tau^n x + R_\tau C_\tau^{n-1}x + R_\tau C_\tau^{n-2}x + \dots + R_\tau C_\tau^2 x - C_\tau^n x - C_\tau^{n-1}x - \dots - C_\tau^2 x + (R_\tau - I)C_\tau^1 x \\
=& 2R_\tau C_\tau^n x + R_\tau C_\tau^{n-1}x + R_\tau C_\tau^{n-2}x + \dots + R_\tau C_\tau^2 x \\
&-(2R_\tau C_\tau^{n-1}x - R_\tau C_\tau^{n-2}x) - (2R_\tau C_\tau^{n-2}x - R_\tau C_\tau^{n-3}x) - \dots - (2R_\tau C_\tau^1 x - R_\tau C_\tau^0 x) + (R_\tau - I)C_\tau^1 x \\
=& 2R_\tau C_\tau^n x - R_\tau C_\tau^{n-1}x.
\end{aligned}
$$

This completes the proof. □

**Proposition 2.4.** *Assume that $\{C_\tau^n\}_{n\in\mathbb{N}_0}$ and $\{\mathcal{C}_\tau^n\}_{n\in\mathbb{N}_0}$ are cosine sequences generated by $A$. Then $C_\tau^n = \mathcal{C}_\tau^n$ for all $n \in \mathbb{N}_0$.*

*Proof.* Let $x \in X$. By definition and Proposition 2.3, $C_\tau^0 x = \mathcal{C}_\tau^0 x = x$ and $C_\tau^1 x = \mathcal{C}_\tau^1 x = R_\tau x$. Assume, by induction hypothesis, that $C_\tau^l x = \mathcal{C}_\tau^l x$, for all $l \le n$. By Proposition 2.3 $C_\tau^{n+1}x = 2R_\tau C_\tau^n x - R_\tau C_\tau^{n-1}x$ and $\mathcal{C}_\tau^{n+1}x = 2R_\tau \mathcal{C}_\tau^n x - R_\tau \mathcal{C}_\tau^{n-1}x$. By hypothesis, we get $C_\tau^{n+1}x = \mathcal{C}_\tau^{n+1}x$. □

The next theorem shows that a cosine sequence generated by $A$ verifies a discrete functional equation of D'Alembert type.

**Theorem 2.5** (Cosine relation)**.** *Let $\tau > 0$ be given. Assume that $A$ is the generator of a cosine sequence $\{C_\tau^n\}_{n\in\mathbb{N}_0}$. Then,*

$$C_\tau^{n+m} + R_\tau^m C_\tau^{n-m} = 2C_\tau^n C_\tau^m. \tag{2.19}$$

*for all $n, m \in \mathbb{N}_0$ such that $n \ge m$.*

*Proof.* Let $n \in \mathbb{N}$. We proceed by induction on $m$. By Proposition 2.3, $C_\tau^1 = R_\tau$, and replacing $n$ by $n+1$ in the relation (2.17) we get

$$C_\tau^{n+1} + R_\tau C_\tau^{n-1} = 2R_\tau C_\tau^n = 2C_\tau^1 C_\tau^n, \tag{2.20}$$

which means that (2.19) holds for $m = 1$. Assume that (2.19) is valid for all $l \le m$. Now, replacing $n$ by $n+m$ and $n$ by $n-m$ in (2.20), we obtain, respectively

$$\begin{cases} C_\tau^{n+m+1} + R_\tau C_\tau^{n+m-1} = 2R_\tau C_\tau^{n+m} \\ C_\tau^{n-m+1} + R_\tau C_\tau^{n-m-1} = 2R_\tau C_\tau^{n-m}. \end{cases}$$

This, implies that

$$\begin{cases} C_\tau^{n+m+1} + R_\tau C_\tau^{n+m-1} = 2R_\tau C_\tau^{n+m} \\ R_\tau^m C_\tau^{n-m+1} + R_\tau^{m+1} C_\tau^{n-m-1} = 2R_\tau^{m+1} C_\tau^{n-m}. \end{cases}$$

Adding these two equations we obtain

$$C_\tau^{n+(m+1)} + R_\tau^{m+1} C_\tau^{n-(m+1)} + R_\tau(C_\tau^{n+m-1} + R_\tau^{m-1} C_\tau^{n-(m-1)}) = 2R_\tau(C_\tau^{n+m} + R_\tau^m C_\tau^{n-m}).$$

By the induction hypothesis and Proposition 2.3 we have

$$C_\tau^{n+(m+1)} + R_\tau^{m+1} C_\tau^{n-(m+1)} = 4R_\tau C_\tau^n C_\tau^m - 2R_\tau C_\tau^n C_\tau^{m-1} = 2C_\tau^n(2R_\tau C_\tau^m - R_\tau C_\tau^{m-1}) = 2C_\tau^n C_\tau^{m+1}.$$

□

If we take $n = 0$ in (2.19) we get $C_\tau^m + R_\tau^m C_\tau^{-m} = 2C_\tau^m$, which implies that

$$R_\tau^m C_\tau^{-m} = C_\tau^m, \quad m \in \mathbb{N}_0. \tag{2.21}$$

Thus, for $m \le 0$, $C_\tau^n$ can be defined through the relation (2.21). This means that a cosine sequence generated by $A$ is an even sequence up to a power of $R_\tau$.

If we extend the backward operator (2.8) for all $n\in\mathbb{Z}$, using (2.21) we have that the generator $A$ of $C^n_\tau$ verifies

$$(\nabla^2_\tau C_\tau)^0 x = Ax, \tag{2.22}$$

for all $x\in D(A)$. If fact, for $x\in D(A)$, by (2.21) we obtain $C^{-1}_\tau x = C^1_\tau R^{-1}_\tau x$, and by Proposition 2.3, $C^{-1}_\tau x = R_\tau R^{-1}_\tau x = x$. Similarly, $C^{-2}_\tau x = C^2_\tau R^{-2}_\tau x$, and $C^{-2}_\tau x = (2R^2_\tau - R_\tau)R^{-2}_\tau x = I + \tau^2 Ax$. Thus

$$(\nabla^2_\tau C_\tau)^0 x = \frac{1}{\tau^2}\left(C^0_\tau - 2C^{-1}_\tau + C^{-2}_\tau\right)x = Ax.$$

Let $\tau>0$ be given. We denote by $\{S^n_\tau\}_{n\in\mathbb{N}_0}$ the *sine sequence* associated with the cosine sequence $\{C^n_\tau\}_{n\in\mathbb{N}_0}$, which is defined by

(1) $S^0_\tau = 0$,
(2) For any $n\in\mathbb{N}$, and $x\in X$,

$$S^n_\tau x = \tau\sum_{j=1}^{n} C^j_\tau x. \tag{2.23}$$

As $S^n_\tau$ is given in terms of finite sum, the sine sequence associated with the cosine sequence $\{C^n_\tau\}_{n\in\mathbb{N}_0}$ is well defined. The next result gives the main properties of the cosine and sine sequences, $\{C^n_\tau\}_{n\in\mathbb{Z}}$ and $\{S^n_\tau\}_{n\in\mathbb{N}_0}$, generated by a closed linear operator $A$.

**Proposition 2.6.** *Let $\tau>0$ be given. Assume that $A$ is the generator of a cosine sequence $\{C^n_\tau\}_{n\in\mathbb{N}_0}$. Then, for each $x\in X$,*

(1) $(\nabla_\tau S_\tau)^n x = C^n_\tau x$, *for all* $n\in\mathbb{N}$.
(2) $(\nabla_\tau C_\tau)^n x = AS^n_\tau x$, *for all* $n\in\mathbb{N}$.
(3) $C^n_\tau$, $C^m_\tau$, $S^n_\tau$ *and* $S^m_\tau$ *commute, for all* $n,m\in\mathbb{N}_0$.
(4) $S^n_\tau x = \tau R_\tau C^{n-1}_\tau x + \tau R^2_\tau C^{n-2}_\tau x + ... + \tau R^n_\tau C^0_\tau x$, *for all* $n\in\mathbb{N}$.
(5) *For all* $n,m\in\mathbb{N}_0$ *with* $n\ge m$,

$$S^{n+m}_\tau + R^m_\tau S^{n-m}_\tau = 2S^n_\tau C^m_\tau. \tag{2.24}$$

*Proof.* To prove (1) we notice that $(\nabla_\tau S_\tau)^n x = \frac{1}{\tau}(S^n_\tau - S^{n-1}_\tau)x = \sum_{j=1}^n C^j_\tau x - \sum_{j=1}^{n-1} C^j_\tau x = C^n_\tau x$. By (2.14) it follows that $C^n_\tau x = x + \tau A\sum_{j=1}^n S^j_\tau x$ and thus $(\nabla_\tau C_\tau)^n x = \frac{1}{\tau}(C^n_\tau - C^{n-1}_\tau)x = A\sum_{j=1}^n S^j_\tau x - \sum_{j=1}^{n-1} S^j_\tau x = AS^n_\tau x$, which proves (2). To prove (3), by Theorem 2.5 we have

$$\begin{cases} 2C^n_\tau C^m_\tau = C^{n+m}_\tau + R^m_\tau C^{n-m}_\tau \\ 2C^m_\tau C^n_\tau = C^{m+n}_\tau + R^n_\tau C^{m-n}_\tau. \end{cases}$$

Subtracting both equations, we get by (2.21) that

$$2(C^n_\tau C^m_\tau - C^m_\tau C^n_\tau) = R^m_\tau C^{n-m}_\tau - R^m_\tau R^{n-m}_\tau C^{-(n-m)}_\tau = 0.$$

That is, $C^n_\tau$ and $C^m_\tau$ commute. Now, as $C^n_\tau S^m_\tau = \tau C^n_\tau\sum_{j=0}^m C^j_\tau = \tau\sum_{j=0}^m C^j_\tau C^n_\tau = S^m_\tau C^n_\tau$, we deduce that $C^n_\tau$ and $S^m_\tau$ commute. Similarly, we have $S^n_\tau S^m_\tau = \tau\sum_{j=1}^m S^n_\tau C^j_\tau = S^m_\tau S^n_\tau$, that is, $S^n_\tau$ and $S^m_\tau$ commute. To prove (4) we proceed by induction. In fact, $\tau R_\tau C^0_\tau x = \tau R_\tau x = S^1_\tau x$. If (4) holds for $n$, then by (2.17) we get

$$\begin{aligned}
-S_\tau^{n+1}x + \tau R_\tau C_\tau^n x + \tau R_\tau^2 C_\tau^{n-1}x + ... + \tau R_\tau^{n+1}C_\tau^0 x & \\
= & -S_\tau^n x - \tau C_\tau^{n+1} + \tau R_\tau C_\tau^n x + R_\tau[\tau R_\tau C_\tau^{n-1}x + \tau R_\tau^2 C_\tau^{n-2}x + ... + \tau R_\tau^n C_\tau^0 x] \\
= & -S_\tau^n x - \tau C_\tau^{n+1} + \tau R_\tau C_\tau^n x + R_\tau S_\tau^n \\
= & -S_\tau^n x - \tau C_\tau^{n+1} + \tau R_\tau C_\tau^1 x + ... + \tau R_\tau C_\tau^{n-1}x + 2\tau R_\tau C_\tau^n \\
= & -S_\tau^n x - \tau C_\tau^{n+1} + \tau R_\tau C_\tau^1 x + ... + \tau R_\tau C_\tau^{n-1}x + \tau(2R_\tau C_\tau^n - R_\tau C_\tau^{n-1}x) + \tau R_\tau C_\tau^{n-1}x \\
= & -(\tau C_\tau^1 x + ... + \tau C_\tau^{n-1}x) - \tau C_\tau^n x + \tau R_\tau C_\tau^1 x + ... + \tau R_\tau C_\tau^{n-2}x + 2\tau R_\tau C_\tau^{n-1}x \\
\vdots & \\
= & -(\tau C_\tau^1 x + \tau C_\tau^2 x) - \tau C_\tau^3 x + \tau R_\tau C_\tau^1 x + \tau(2R_\tau C_\tau^2 x - R_\tau C_\tau^1 x) + \tau R_\tau C_\tau^1 x \\
= & -\tau C_\tau^1 x - \tau C_\tau^2 x + \tau(2R_\tau C_\tau^1 x - R_\tau C_\tau^0 x) + \tau R_\tau C_\tau^0 x \\
= & -\tau R_\tau x + \tau R_\tau C_\tau^0 x \\
= & \ 0.
\end{aligned}$$

This proves (4). Finally, to prove (5) we notice that for (2.19) we have for any $j \in \mathbb{N}$,

$$C_\tau^{j+m} + R_\tau^m C_\tau^{j-m} = 2C_\tau^j C_\tau^m.$$

Summing up over $j = 1, ..., n$ we obtain

$$\tau \sum_{l=m+1}^{n+m} C_\tau^l + \tau R_\tau^m \sum_{l=1-m}^{n-m} C_\tau^l = 2S_\tau^n C_\tau^m,$$

which is equivalent to

$$S_\tau^{n+m} - S_\tau^m + R_\tau^m S_\tau^{n-m} + \tau R_\tau^m[C_\tau^{-m+1} + \ldots + C_\tau^{-1} + C_\tau^0] = 2S_\tau^n C_\tau^m.$$

By (2.21), we can write

$$\begin{aligned}
R_\tau^m[C_\tau^{-m+1} + \ldots + C_\tau^{-1} + C_\tau^0] &= R_\tau R_\tau^{m-1}C_\tau^{-(m-1)} + R_\tau^2 R_\tau^{m-2}C_\tau^{-(m-2)} + \ldots + R_\tau^{m-1}R_\tau C_\tau^{-1} + R_\tau^m C_\tau^0 \\
&= R_\tau C_\tau^{m-1} + R_\tau^2 C_\tau^{m-2} + \ldots R_\tau^m C_\tau^0.
\end{aligned}$$

By (4) it follows that

$$S_\tau^{n+m} + R_\tau^m S_\tau^{n-m} = 2S_\tau^n C_\tau^m.$$

□

Taking $n = 0$ in (2.24) we get $S_\tau^m + R_\tau^m S_\tau^{-m} = 2S_\tau^0 C_\tau^m$, that is,

$$R_\tau^m S_\tau^{-m} = -S_\tau^m, \quad m \in \mathbb{N}_0. \tag{2.25}$$

Thus, we can use the relation (2.25) to define $S_\tau^n$ for $n \leq 0$.

The next Proposition corresponds to a discrete counterpart of [33, Theorem 2].

**Proposition 2.7.** *Let $\tau > 0$ be given. Assume that $A$ is the generator of a cosine sequence $\{C_\tau^n\}_{n\in\mathbb{N}_0}$. Then,*

$$S_\tau^{n+m} = S_\tau^n C_\tau^m + S_\tau^m C_\tau^n. \tag{2.26}$$

*Proof.* By (2.24) we have

$$\left\{ \begin{array}{l} S_\tau^{n+m} = 2S_\tau^n C_\tau^m - R_\tau^m S_\tau^{n-m} \\ S_\tau^{m+n} = 2S_\tau^m C_\tau^n - R_\tau^n S_\tau^{m-n}. \end{array} \right.$$

Adding both equations and using (2.25) we get

$$2S_\tau^{n+m} = 2S_\tau^n C_\tau^m + 2S_\tau^m C_\tau^n - [R_\tau^m S_\tau^{n-m} + R_\tau^m R_\tau^{n-m} S_\tau^{-(n-m)}],$$

which implies (2.26). □

The next Proposition gives the $Z$-transform of a cosine sequence generated by $A$.

**Proposition 2.8.** *Let $\tau > 0$ be given. Assume that $A$ is the generator of a cosine sequence $\{C_\tau^n\}_{n\in\mathbb{N}_0}$. Then, its $Z$-transform satisfies*

$$\widetilde{C}(z)x = \left(\frac{z-1}{\tau^2 z} - A\right)\left(\left(\frac{z-1}{\tau z}\right)^2 - A\right)^{-1} x,$$

*for all $x \in X$ and $z \in \mathbb{C}$ such that $\left(\frac{z-1}{\tau z}\right)^2 \in \rho(A)$.*

*Proof.* Let $x \in X$. We first compute $\widetilde{C}(z)R_\tau x$. By (2.11) and (2.12) we have

$$\begin{aligned}
\widetilde{C}(z)R_\tau x &= \sum_{j=0}^{\infty} C_\tau^j R_\tau x \\
&= R_\tau x + \sum_{j=1}^{\infty} z^{-j}\left[R_\tau x + \tau A \sum_{l=1}^{j} k_\tau^2(j-l) C_\tau^l R_\tau x\right] \\
&= R_\tau x + \sum_{j=1}^{\infty} z^{-j} R_\tau x + \tau A \sum_{j=0}^{\infty} z^{-j} \sum_{l=0}^{j} k_\tau^2(j-l) C_\tau^l R_\tau x - \tau A \sum_{j=0}^{\infty} z^{-j} k_\tau^2(j) R_\tau x \\
&= \frac{z}{z-1} R_\tau x + \left(\frac{\tau z}{z-1}\right)^2 A\widetilde{C}(z)R_\tau x - \left(\frac{\tau z}{z-1}\right)^2 AR_\tau x,
\end{aligned}$$

which implies that

$$\left(\left(\frac{z-1}{\tau z}\right)^2 - A\right)\widetilde{C}(z)R_\tau x = \frac{z-1}{\tau^2 z} R_\tau x - AR_\tau x.$$

Hence,

$$\widetilde{C}(z)R_\tau x = \left(\left(\frac{z-1}{\tau z}\right)^2 - A\right)^{-1}\left(\frac{z-1}{\tau^2 z} - A\right) R_\tau x.$$

By (2) in Definition 2.1, we deduce that $R_\tau^{-1}\widetilde{C}(z)R_\tau x = \widetilde{C}(z)x$ for all $x \in X$. Thus,

$$\begin{aligned}
\widetilde{C}(z)x &= R_\tau^{-1}\widetilde{C}(z)R_\tau x \\
&= \left(\left(\frac{z-1}{\tau z}\right)^2 - A\right)^{-1}\left(\frac{z-1}{\tau^2 z} - A\right) R_\tau x - \tau^2 A\left(\left(\frac{z-1}{\tau z}\right)^2 - A\right)^{-1}\left(\frac{z-1}{\tau^2 z} - A\right) R_\tau x.
\end{aligned}$$

Now, we notice that $(\lambda - A)(\lambda - A)^{-1}w = (\lambda - A)^{-1}(\lambda - A)w$ for all $w \in D(A)$ and $\lambda \in \rho(A)$, which implies $A(\lambda - A)^{-1}w = (\lambda - A)^{-1}Aw$. As $R_\tau x \in D(A)$, we have

$$\widetilde{C}(z)x = \left(\frac{z-1}{\tau^2 z} - A\right)\left(\left(\frac{z-1}{\tau z}\right)^2 - A\right)^{-1} R_\tau x - \tau^2 A\left(\frac{z-1}{\tau^2 z} - A\right)\left(\left(\frac{z-1}{\tau z}\right)^2 - A\right)^{-1} R_\tau x.$$

Since for any $w \in D(A)$, $(\lambda - A)^{-1}w \in D(A^2)$, we get

$$\begin{aligned}\widetilde{C}(z)x &= \left(\frac{z-1}{\tau^2 z} - A\right)\left(\left(\frac{z-1}{\tau z}\right)^2 - A\right)^{-1} R_\tau x - \left(\frac{z-1}{\tau^2 z} - A\right)\left(\left(\frac{z-1}{\tau z}\right)^2 - A\right)^{-1} \tau^2 A R_\tau x \\ &= \left(\frac{z-1}{\tau^2 z} - A\right)\left(\left(\frac{z-1}{\tau z}\right)^2 - A\right)^{-1} [I - \tau^2 A] R_\tau x \\ &= \left(\frac{z-1}{\tau^2 z} - A\right)\left(\left(\frac{z-1}{\tau z}\right)^2 - A\right)^{-1} x.\end{aligned}$$

□

The next result gives a representation of $C_\tau^n$ in terms of a finite sum of powers of $R_\tau$.

**Theorem 2.9.** *Let $\tau > 0$ be given. Assume that $A$ is the generator of a cosine sequence $\{C_\tau^n\}_{n\in\mathbb{N}_0}$. Then,*

$$C_\tau^n x = \sum_{j=1}^{n} a_{n,j} R_\tau^j x, \quad n \in \mathbb{N}, \tag{2.27}$$

*where, $(a_{n,l})$ is the sequence of real numbers defined by*

$$a_{n,1} := \begin{cases} 1, & n = 1, \\ \left(k_\tau^2(0) - \displaystyle\sum_{j=1}^{n-1} k_\tau^2(n-j) a_{j,1}\right) k_\tau^2(0)^{-1}, & n \geq 2, \end{cases}$$

*and*

$$a_{n,l} := \begin{cases} \left(\displaystyle\sum_{j=l-1}^{n-1} k_\tau^2(n-j) a_{j,l-1} - \sum_{j=l}^{n-1} k_\tau^2(n-j) a_{j,l}\right) k_\tau^2(0)^{-1}, & 2 \leq l \leq n-1, \\ k_\tau^2(1) a_{n-1,n-1} k_\tau^2(0)^{-1}, & l = n. \end{cases}$$

*Proof.* It follows similarly to the proof of [17, Theorem 3.7], but we give it here for the sake of completeness. We proceed by induction on $n$. By Proposition 2.3, the result is valid for $n = 1$. Assume that (2.27) holds for all $l \leq n$. By (2.15) we have

$$k_\tau^2(0) C_\tau^1 C_\tau^{n+1} x - C_\tau^1 \left(\sum_{j=1}^{n+1} k_\tau^2(n+1-j) C_\tau^j x\right) = k_\tau^2(0) C_\tau^1 x - \sum_{j=1}^{n+1} k_\tau^2(n+1-j) C_\tau^j x,$$

which implies

$$\begin{aligned} k_\tau^2(0) C_\tau^{n+1} x &= k_\tau^2(n) C_\tau^1 C_\tau^1 x + k_\tau^2(n-1) C_\tau^1 C_\tau^2 x + \cdots + k_\tau^2(1) C_\tau^1 C_\tau^n x + k_\tau^2(0) C_\tau^1 \\ &\quad - k_\tau^2(n) C_\tau^1 x - k_\tau^2(n-1) C_\tau^2 x - \cdots - k_\tau^2(1) C_\tau^n x. \end{aligned}$$

By Proposition 2.3 and hypothesis induction we can write

$$
\begin{aligned}
k_\tau^2(0)C_\tau^{n+1}x &= \big[k_\tau^2(n)R_\tau + k_\tau^2(n-1)\left(a_{2,1}R_\tau + a_{2,2}R_\tau^2\right) + k_\tau^2(n-2)\left(a_{3,1}R_\tau + a_{3,2}R_\tau^2 + a_{3,3}R_\tau^3\right) \\
&+ \cdots + k_\tau^2(1)\left(a_{n,1}R_\tau + a_{n,2}R_\tau^2 + \cdots + a_{n,n}R_\tau^n\right)\big] R_\tau x + k_\tau^2(0)R_\tau x \\
&- k_\tau^2(n)R_\tau x - k_\tau^2(n-1)\left(a_{2,1}R_\tau + a_{2,2}R_\tau^2\right)x - \cdots - k_\tau^2(1)\left(a_{n,1}R_\tau + \cdots + a_{n,n}R_\tau^n\right)x \\
&= \left[k_\tau^2(0) - k_\tau^2(n)a_{1,1} - k^2(n-1)a_{2,1} - \cdots - k_\tau^2(1)a_{n,1}\right] R_\tau x \\
&+ \left[k_\tau^2(n)a_{1,1} + \cdots + k_\tau^2(1)a_{n,1} - k_\tau^2(n-1)a_{2,2} - \cdots - k_\tau^2(1)a_{n,2}\right] R_\tau^2 x \\
&+ \left[k_\tau^2(n-1)a_{2,2} + \cdots + k_\tau^2(1)a_{n,2} - k_\tau^2(n-2)a_{3,3} - \cdots - k_\tau^2(1)a_{n,3}\right] R_\tau^3 x \\
&\vdots \\
&+ \left[k_\tau^2(0)a_{n-1,n-1} + k_\tau^2(1)a_{n,n-1} - k_\tau^2(1)a_{n,n}\right] R_\tau^n x \\
&+ \left[k_\tau^2(1)a_{n,n}\right] R_\tau^{n+1} x \\
&= \left[k_\tau^2(0) - \sum_{j=1}^{n} k_\tau^2(n+1-j)a_{j,1}\right] R_\tau x \\
&+ \left[\sum_{j=1}^{n} k_\tau^2(n+1-j)a_{j,1} - \sum_{j=2}^{n} k_\tau^2(n+1-j)a_{j,2}\right] R_\tau^2 x \\
&+ \left[\sum_{j=2}^{n} k_\tau^2(n+1-j)a_{j,2} - \sum_{j=3}^{n} k_\tau^2(n+1-j)a_{j,3}\right] R_\tau^3 x \\
&\vdots \\
&+ \left[\sum_{j=n-1}^{n} k_\tau^2(n+1-j)a_{j,n-1} - \sum_{j=n}^{n} k_\tau^2(n+1-j)a_{j,n}\right] R_\tau^n x \\
&+ \left[k_\tau^2(1)a_{n,n}\right] R_\tau^{n+1} x.
\end{aligned}
$$

Therefore,

$$C_\tau^{n+1}x = \sum_{j=1}^{n+1} a_{n,j}R_\tau^j x.$$

This completes the proof. $\square$

**Corollary 2.10.** *Let $\tau > 0$ be given. Assume that $A$ is the generator of a cosine sequence $\{C_\tau^n\}_{n\in\mathbb{N}_0}$. If $(\tau^{-2} - A)^{-1}$ is a compact operator, then $C_\tau^n$ is a compact operator for all $n \in \mathbb{N}$.*

*Proof.* The representation (2.27) as a finite sum of $(\tau^{-2} - A)^{-1}$ implies that $C_\tau^n$ is a compact operator for all $n \in \mathbb{N}$. $\square$

We notice that the terms of the sequence $(a_{n,l})$ given in Theorem 2.9 can be computed explicitly and the cosine sequence $C_\tau^n$ can be written as the matrix representation

$$C_\tau^n x = \begin{pmatrix} a_{1,1} & a_{n,2} & a_{n,3} & \dots & a_{n,n} \end{pmatrix} \begin{pmatrix} R_\tau x \\ R_\tau^2 x \\ R_\tau^3 x \\ \vdots \\ R_\tau^n x \end{pmatrix}, \quad n \in \mathbb{N}.$$

For example,

$$C_\tau^1 x = \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} R_\tau x \end{pmatrix}, \; C_\tau^2 x = \begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} R_\tau x \\ R_\tau^2 x \end{pmatrix}, \; C_\tau^3 x = \begin{pmatrix} 0 & -3 & 4 \end{pmatrix} \begin{pmatrix} R_\tau x \\ R_\tau^2 x \\ R_\tau^3 x \end{pmatrix}, \; C_\tau^4 x = \begin{pmatrix} 0 & 1 & -8 & 8 \end{pmatrix} \begin{pmatrix} R_\tau x \\ R_\tau^2 x \\ R_\tau^3 x \\ R_\tau^4 x \end{pmatrix},$$

and

$$C_\tau^5 x = \begin{pmatrix} 0 & 0 & 5 & -20 & 16 \end{pmatrix} \begin{pmatrix} R_\tau x \\ R_\tau^2 x \\ R_\tau^3 x \\ R_\tau^4 x \\ R_\tau^5 x \end{pmatrix}, \quad C_\tau^6 x = \begin{pmatrix} 0 & 0 & -1 & 18 & -48 & 32 \end{pmatrix} \begin{pmatrix} R_\tau x \\ R_\tau^2 x \\ R_\tau^3 x \\ R_\tau^4 x \\ R_\tau^5 x \\ R_\tau^6 x \end{pmatrix}.$$

Now, we define the sequence $(T_n)_{n\in\mathbb{N}_0}$ by setting $T_0 := (1)$ and, for $n \geq 1$,

$$T_n := \underbrace{(a_{n,n}, 0, a_{n,n-1}, 0, a_{n,n-2}, 0, \ldots)}_{n+1\text{-terms}},$$

which means that $T_n$ is constructed by placing the scalar $a_{n,l}$ given in Proposition 2.3 followed by a 0 in between each coefficient. For example,

$$T_0 = (1), \quad T_1 = (1, 0), \quad T_2 = (2, 0, -1), \quad T_3 = (4, 0, -3, 0), \quad T_4 = (8, 0, -8, 0, 1),$$

and

$$T_5 = (16, 0, -20, 0, 5, 0), \quad T_6 = (32, 0, -48, 0, 18, 0, -1).$$

Then, the terms in $T_n$ correspond to coefficients in the Chebyshev polynomials of first kind $T_n(x)$ :

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \quad T_4(x) = 8x^4 - 8x^2 + 1,$$

and

$$T_5(x) = 16x^5 - 20x^3 + 5x, \quad T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1.$$

Now, let $A$ be any closed linear operator. Let $\{C^n\}_{n\in\mathbb{Z}} \subset \mathcal{B}(X)$ be a sequence of linear operators satisfying the cosine relation (2.19), that is,

$$C^{n+m}x + R_\tau^m C^{n-m}x = 2C^n C^m x, \quad n, m \in \mathbb{Z}, x \in X. \tag{2.28}$$

Replacing $n$ by $n-1$ and $m$ by $m = 1$, we get

$$C^n x = 2C^{n-1}C^1 x - R_\tau C^{n-2}x, \tag{2.29}$$

for all $n \in \mathbb{Z}$ and $x \in X$. This means that $C^n$, for any $n \in \mathbb{Z}$, is completely determined by $C^0$ and $C^1$.

The next result shows that the cosine relation (2.29) can be used to define a cosine sequence generated by $A$.

**Theorem 2.11.** *Let $A$ be a closed linear operator in a Banach space $X$ such that $\tau^{-2} \in \rho(A)$. Let $\{C^n\}_{n\in\mathbb{Z}} \subset \mathcal{B}(X)$ be a sequence of linear operators such that*

(1) *$C^0 = I$,*
(2) *$C^1 = R_\tau$, and*
(3) *$C^{n+m}x + R_\tau^m C^{n-m}x = 2C^n C^m x$, for all $n, m \in \mathbb{Z}$ and $x \in X$.*

*Then, $\{C^n\}_{n\in\mathbb{N}_0}$ is a cosine sequence generated by $A$.*

*Proof.* We first show that $C^n x \in D(A)$ for all $x \in X$ and $n \in \mathbb{N}$. In fact, by (1) $C^1 x = R_\tau x$, and thus $C^1 x \in D(A)$. If we assume that $C^l x \in D(A)$, for all $l \leq n$ and $x \in X$, the identity (2.29) implies that $C^{n+1}x = 2C^n C^1 x - R_\tau C^{n-1}x \in D(A)$.

Now, the identity (2.28) implies that

$$R_\tau^{-m} C^m x = C^{-m}x, \quad m \in \mathbb{Z}, x \in X,$$

and, using the same method as in Proposition 2.6, we deduce that $C^n$ and $C^m$ commute, that is,

$$C^nC^mx = C^mC^nx, \quad n, m \in \mathbb{Z}, x \in X. \tag{2.30}$$

Next, we need to prove that $AC^nx = C^nAx$ for all $n \in \mathbb{N}$ and $x \in D(A)$. In fact, as $(\tau^{-2} - A)(\tau^{-2} - A)^{-1}x = (\tau^{-2} - A)^{-1}(\tau^{-2} - A)x = x$ for all $x \in D(A)$ we get $R_\tau Ax = AR_\tau x$, that is, $C^1Ax = AC^1x$ for all $x \in D(A)$. Now, if we assume that $AC^lx = C^lAx$ for all $l \leq n$ and $x \in D(A)$, by (2.29) we have

$$AC^{n+1}x = 2AR_\tau C^nx - AR_\tau C^{n-1}x = 2R_\tau AC^nx - R_\tau C^{n-1}x = 2R_\tau C^nAx - R_\tau C^{n-1}Ax = C^{n+1}Ax.$$

Finally, we nee to show that $C^n$ verifies the identity (2.13). As $R_\tau^{-1} = (I - \tau^2A)$, using (2.29) and (2.30), we get, for $j \geq 2$, that

$$(I - \tau^2A)C^jx = 2C^{j-1}x - C^{j-2}x, \quad x \in X.$$

Multiplying this last identity by $k_\tau^2(n-j)$ and summing up over $j = 2, ..., n$ we have

$$\sum_{j=2}^n k_\tau^2(n-j)C^jx - 2\sum_{j=2}^n k_\tau^2(n-j)C^{j-1}x + \sum_{j=2}^n k_\tau^2(n-j)C^{j-2}x = \tau^2A\sum_{j=2}^n k_\tau^2(n-j)C^jx.$$

This last identity is equivalent to

$$\begin{aligned} k_\tau^2(n-2)C^2x + \cdots + k_\tau^2(1)C^{n-1}x + k_\tau^2(0)C^nx - 2\left[k_\tau^2(n-2)C^1x + \cdots + k_\tau^2(1)C^{n-2}x + k_\tau^2(0)C^{n-1}x\right] \\ + k_\tau^2(n-2)C^0x + \cdots + k_\tau^2(0)C^{n-2}x = \tau^2A\sum_{j=2}^n k_\tau^2(n-j)C^jx. \end{aligned}$$

Hence,

$$\begin{aligned} &\left[k_\tau^2(n-2) - 2k_\tau^2(n-3) + k_\tau^2(n-4)\right]C^2x + \cdots + \left[k_\tau^2(2) - 2k_\tau^2(1) + k_\tau^2(0)\right]C^{n-2}x \\ &+ k_\tau^2(1)C^{n-1}x + k_\tau^2(0)C^nx - 2k_\tau^2(n-2)C^1x - 2k_\tau^2(0)C^{n-1}x + k_\tau^2(n-2)C^0x + k_\tau^2(n-3)C^1x \\ &= \tau^2A\sum_{j=2}^n k_\tau^2(n-j)C^jx. \end{aligned}$$

Since $k_\tau^2(n-j) - 2k_\tau^2(n-(j+1)) + k_\tau^2(n-(j+2)) = 0$ for all $j \leq n-2$, and $k_\tau^2(n) = \tau(n+1)$, and $C^0 = I, C^1 = R_\tau$, we obtain

$$k_\tau^2(0)C^nx - 2k_\tau^2(n-2)R_\tau x + k_\tau^2(n-2)x + k_\tau^2(n-3)R_\tau x = \tau^2A\sum_{j=2}^n k_\tau^2(n-j)C^jx.$$

As $\tau^2AR_\tau x = (R_\tau - I)x$ and $k_\tau^2(n) = \tau(n+1)$ for all $n \in \mathbb{N}_0$, we deduce that $-2k_\tau^2(n-2)R_\tau x + k_\tau^2(n-2)x + k_\tau^2(n-3)R_\tau x = \tau n(I - R_\tau)x - \tau x = -\tau^2k_\tau(n-1)AR_\tau x - \tau x$, and thus

$$k_\tau^2(0)C^nx = \tau x + \tau^2A\sum_{j=1}^n k_\tau^2(n-j)C^jx.$$

Therefore, $\{C^n\}_{n\in\mathbb{N}_0}$ is a cosine sequence generated by $A$.

□

**Lemma 2.12.** *Let $F : \mathbb{R} \to X$ be a given vector-valued function.*

(1) *If $n, m \in \mathbb{N}_0$, then*

$$\int_0^\infty \int_0^\infty \rho_n^\tau(t)\rho_m^\tau(s)F(t+s)dtds = \int_0^\infty \rho_{m+n+1}^\tau(t)F(t)dt.$$

(2) *In addition, if $F$ is an even function, that is, $F(-t) = F(t)$ for all $t \geq 0$, then for any $n, m \in \mathbb{N}_0$,*

$$\int_0^\infty \int_0^\infty \rho_n^\tau(t)\rho_m^\tau(s)F(t-s)dtds = \sum_{j=0}^n \frac{1}{2^{m+n-j+1}}\binom{m+n-j}{n-j}F_\tau^j + \sum_{j=0}^m \frac{1}{2^{m+n-j+1}}\binom{m+n-j}{m-j}F_\tau^j,$$

*where, for any* $l \in \mathbb{N}_0$, $F_\tau^l := \int_0^\infty \rho_l^\tau(t)F(t)dt$.

*Proof.* First, for any $s \in \mathbb{R}$, we define $F_s : \mathbb{R} \to X$, by $F_s(t) := F(t+s)$.

To prove (1), we notice that

$$\int_0^\infty \int_0^\infty \rho_n^\tau(t)\rho_m^\tau(s)F(t+s)dtds = \frac{1}{n!\tau^{n+1}} \int_0^\infty \rho_m^\tau(s) \int_0^\infty e^{-\frac{1}{\tau}t} t^n F_s(t)dtds$$
$$= \frac{(-1)^n}{n!\tau^{n+1}} \int_0^\infty \rho_m^\tau(s)[\hat{F}_s(\lambda)]^{(n)} \mid_{\lambda=\frac{1}{\tau}} ds.$$

Now,

$$\hat{F}_s(\lambda) = \int_0^\infty e^{-\lambda t} F(t+s)dt = e^{\lambda s} \int_s^\infty e^{-\lambda r} F(r)dr.$$

Thus, for any $n \in \mathbb{N}$, we have that

$$[\hat{F}_s(\lambda)]^{(n)} = \frac{d^n \hat{F}_s(\lambda)}{d\lambda^n} = e^{\lambda s} \int_s^\infty (s-r)^n e^{-\lambda r} F(r)dr.$$

This implies that

$$[\hat{F}_s(\lambda)]^{(n)} \mid_{\lambda=\frac{1}{\tau}} = e^{\frac{1}{\tau}s} \int_s^\infty (s-r)^n e^{-\frac{1}{\tau}r} F(r)dr.$$

Hence,

$$\int_0^\infty \int_0^\infty \rho_n^\tau(t)\rho_m^\tau(s)F(t+s)dtds = \frac{(-1)^n}{n!m!\tau^{n+m+2}} \int_0^\infty s^m \int_s^\infty (s-r)^n e^{-\frac{1}{\tau}r} F(r)drds$$
$$= \frac{(-1)^n}{n!m!\tau^{n+m+2}} \sum_{j=0}^n \binom{n}{j} (-1)^j \int_0^\infty s^{m+n-j} \int_s^\infty e^{-\frac{1}{\tau}r} r^j F(r)drds.$$

Integrating by parts,

$$\int_0^\infty \int_0^\infty \rho_n^\tau(t)\rho_m^\tau(s)F(t+s)dtds = \frac{(-1)^n}{n!m!\tau^{n+m+2}} \sum_{j=0}^n \binom{n}{j} \frac{(-1)^j}{m+n-j+1} \int_0^\infty e^{-\frac{1}{\tau}r} s^{m+n+1} F(s)ds$$
$$= \frac{(-1)^n}{n!m!} \sum_{j=0}^n \binom{n}{j} \frac{(-1)^j}{m+n-j+1} (m+n+1)! \int_0^\infty \rho_{m+n+1}^\tau(s)F(s)ds.$$

As $\int_0^\infty \rho_l^\tau(t)dt = 1$ for all $l \in \mathbb{N}_0$ and the above equality holds for all $F : \mathbb{R} \to X$, we can take $F : \mathbb{R} \to \mathbb{R}$ defined by $F \equiv 1$ to obtain

$$1 = \int_0^\infty \int_0^\infty \rho_n^\tau(t)\rho_m^\tau(s)dtds = \frac{(-1)^n}{n!m!} \sum_{j=0}^n \binom{n}{j} \frac{(-1)^j}{m+n-j+1} (m+n+1)!,$$

for all $n, m \in \mathbb{N}_0$. Therefore,

$$\int_0^\infty \int_0^\infty \rho_n^\tau(t)\rho_m^\tau(s)F(t+s)dtds = \int_0^\infty \rho_{m+n+1}^\tau(s)F(s)ds, \quad n, m \in \mathbb{N}_0.$$

Now, to prove (2), we use the same notation as in the proof of (1) to obtain

$$\int_0^\infty \int_0^\infty \rho_n^\tau(t)\rho_m^\tau(s)F(t-s)dtds = \frac{(-1)^n}{n!\tau^{n+1}} \int_0^\infty \rho_m^\tau(s)[\hat{F}_{-s}(\lambda)]^{(n)} \mid_{\lambda=\frac{1}{\tau}} ds,$$

and

$$[\hat{F}_{-s}(\lambda)]^{(n)} \mid_{\lambda=\frac{1}{\tau}} = (-1)^n e^{-\frac{1}{\tau}s} \int_{-s}^\infty (s+r)^n e^{-\frac{1}{\tau}r} F(r)dr.$$

Thus,

$$\begin{aligned}
\int_0^\infty\int_0^\infty \rho_n^\tau(t)\rho_m^\tau(s)F(t-s)dtds &= \frac{1}{n!m!\tau^{n+m+2}}\int_0^\infty e^{-\frac{1}{\tau}s}s^m e^{-\frac{1}{\tau}s}\int_{-s}^\infty e^{-\frac{1}{\tau}r}(s+r)^nF(r)drds\\
&= \frac{1}{n!m!\tau^{n+m+2}}\int_0^\infty e^{-\frac{1}{\tau}s}s^m e^{-\frac{1}{\tau}s}\int_0^\infty e^{-\frac{1}{\tau}r}(s+r)^nF(r)drds\\
&+ \frac{1}{n!m!\tau^{n+m+2}}\int_0^\infty e^{-\frac{1}{\tau}s}s^m e^{-\frac{1}{\tau}s}\int_{-s}^0 e^{-\frac{1}{\tau}r}(s+r)^nF(r)drds\\
&=: I_1+I_2.
\end{aligned}$$

The integral $I_1$ can be written as

$$\begin{aligned}
I_1 &= \frac{1}{m!\tau^{m+1}}\int_0^\infty e^{-\frac{1}{\tau}s}s^m\int_0^\infty\sum_{j=0}^n\frac{1}{n!}\binom{n}{j}e^{-\frac{1}{\tau}(s+r)}\frac{s^{n-j}r^j}{\tau^{n+1}}e^{-\frac{1}{\tau}r}F(r)drds\\
&= \frac{1}{m!\tau^m}\int_0^\infty e^{-\frac{1}{\tau}s}s^m\int_0^\infty\sum_{j=0}^n\rho_{n-j}^\tau(s)\rho_j^\tau(r)F(r)drds\\
&= \tau\int_0^\infty\rho_m^\tau(s)\sum_{j=0}^n\rho_{n-j}^\tau(s)\int_0^\infty\rho_j^\tau(r)F(r)drds\\
&= \tau\sum_{j=0}^n\int_0^\infty\rho_m^\tau(s)\rho_{n-j}^\tau(s)dsF_\tau^j.
\end{aligned}$$

An easy computation shows that for any $l,q\in\mathbb{N}_0$,

$$\int_0^\infty\rho_l^\tau(s)\rho_q^\tau(s)ds = \frac{1}{2^{l+q+1}\tau}\frac{(l+q)!}{l!q!}, \tag{2.31}$$

and therefore

$$I_1 = \sum_{j=0}^n\frac{1}{2^{m+n-j+1}}\frac{(m+n-j)!}{m!(n-j)!}F_\tau^j = \sum_{j=0}^n\frac{1}{2^{m+n-j+1}}\binom{m+n-j}{n-j}F_\tau^j.$$

On the other hand, as $F$ is an even function,

$$\begin{aligned}
I_2 &= \frac{1}{n!m!\tau^{n+m+2}}\int_0^\infty e^{-\frac{1}{\tau}s}s^m e^{-\frac{1}{\tau}s}\int_{-s}^0 e^{-\frac{1}{\tau}r}(s+r)^nF(r)drds\\
&= \frac{1}{n!m!\tau^{n+m+2}}\int_0^\infty e^{-\frac{1}{\tau}s}s^m\int_0^s e^{-\frac{1}{\tau}w}w^nF(w-s)dwds\\
&= \int_0^\infty\rho_m^\tau(s)\int_0^s\rho_n^\tau(w)F(s-w)dwds\\
&= \int_0^\infty\rho_m^\tau(s)(\rho_n^\tau*F)(s)ds.
\end{aligned}$$

By [34, Theorem 2.8],

$$\int_0^\infty\rho_m^\tau(s)(\rho_n^\tau*F)(s)ds = \tau\sum_{j=0}^m(\rho_n^\tau)^{m-j}F_\tau^j,$$

where, for any $l\geq 0$,

$$(\rho_n^\tau)^l := \int_0^\infty\rho_l^\tau(t)\rho_n^\tau(t)dt.$$

By (2.31) we obtain

$$\begin{aligned} I_2 &= \tau \sum_{j=0}^{m} (\rho_n^\tau)^{m-j} F_\tau^j \\ &= \sum_{j=0}^{m} \frac{1}{2^{m+n-j+1}} \frac{(m+n-j)!}{n!(m-j)!} F_\tau^j = \sum_{j=0}^{m} \frac{1}{2^{m+n-j+1}} \binom{m+n-j}{m-j} F_\tau^j. \end{aligned}$$

The proof of (2) is finished. □

The next Proposition gives a subordination result, which asserts that if $A$ generates a cosine family, then $A$ is also the generator of a cosine sequence.

**Proposition 2.13.** *If $A$ is the generator of a cosine family $\{C(t)\}_{t\in\mathbb{R}}$, then the sequence $\{C_\tau^n\}_{n\in\mathbb{N}_0}$ defined by*

$$C_\tau^0 := I, \qquad \textit{and} \qquad C_\tau^n x := \int_0^\infty \rho_{n-1}^\tau(t) C(t) x dt, \quad x \in X, \, n \geq 1, \tag{2.32}$$

*is the cosine sequence generated by $A$.*

*Proof.* Since $\{C(t)\}_{t\in\mathbb{R}}$ is a cosine family generated by $A$, $C(0) = I$, $C(t)x \in D(A)$ for all $x \in X$ and $AC(t)x = C(t)Ax$, for all $x \in D(A)$ and $t > 0$. As $\int_0^\infty \rho_{n-1}^\tau(t)dt = 1$ and $A$ is a closed linear operator, by [4, Proposition 1.1.7] we obtain

$$C_\tau^n x = \int_0^\infty \rho_{n-1}^\tau(t) C(t) x dt \in D(A), \quad \text{for any } x \in X,$$

and

$$AC_\tau^n x = \int_0^\infty \rho_{n-1}^\tau(t) AC(t) x dt = \int_0^\infty \rho_{n-1}^\tau(t) C(t) A x dt = C_\tau^n A x, \quad \text{for any } x \in D(A).$$

As $A$ generates $\{C(t)\}_{t\in\mathbb{R}}$, we have $C(t)x = x + A(g_2 * C)(t)x$, for any $x \in X$ and $t \geq 0$, where $g_2(s) := s$ for any $s \geq 0$. By [34, Theorem 2.8] we have

$$C_\tau^n x = \int_0^\infty \rho_{n-1}^\tau(t) C(t) x dt = x + A \int_0^\infty \rho_{n-1}^\tau(t) (g_2 * C)(t) x dt = x + A\tau \sum_{j=0}^{n-1} g_2^{n-1-j} C^j x,$$

where, for any $l \in \mathbb{N}_0$, $g_2^l := \int_0^\infty \rho_l^\tau(t) g_2(t) dt$, and $C^l := \int_0^\infty \rho_l^\tau(t) C(t) dt$. By (2.9), $g_2^l = k_\tau^2(l)$ and, by (2.32), $C_\tau^j = C^{j-1}$. Therefore,

$$C_\tau^n x = x + \tau A \sum_{j=1}^{n} k_\tau^2(n-j) C_\tau^j x.$$

We conclude that $\{C_\tau^n\}_{n\in\mathbb{N}_0}$ is a cosine sequence generated by $A$. □

The next result gives a new functional equation that satisfies the cosine sequence $\{C_\tau^n\}_{n\in\mathbb{N}_0}$, which does not depend explicitly on the operator $A$.

**Corollary 2.14.** *If $A$ is the generator of a cosine family $\{C(t)\}_{t\in\mathbb{R}}$, then the cosine sequence $\{C_\tau^n\}_{n\in\mathbb{N}_0}$ given in Proposition 2.13, satisfies*

$$C_\tau^{m+n} + \sum_{l=1}^{n} \frac{1}{2^{m+n-l}} \binom{m+n-l-1}{n-l} C_\tau^l + \sum_{l=1}^{m} \frac{1}{2^{m+n-l}} \binom{m+n-l-1}{m-l} C_\tau^l = 2 C_\tau^n C_\tau^m,$$

*for all $n, m \in \mathbb{N}$.*

*Proof.* Since $\{C(t)\}_{t\in\mathbb{R}}$ is a cosine family, we have

$$C(t+s)+C(t-s)=2C(t)C(s), \quad t,s\in\mathbb{R}.$$

Multiplying by $\rho_n^\tau(t)\rho_n^\tau(s)$ and integrating over $\mathbb{R}_+^2$, we get by Lemma 2.12 that

$$C^{m+n+1}+\sum_{j=0}^{n}\frac{1}{2^{m+n-j+1}}\binom{m+n-j}{n-j}C^j+\sum_{j=0}^{m}\frac{1}{2^{m+n-j+1}}\binom{m+n-j}{m-j}C^j=2C^nC^m, \tag{2.33}$$

where, for any $l\geq 0$, $C^l:=\int_0^\infty \rho_l^\tau(t)C(t)dt$. Next, we replace $n$ by $n-1$ and $m$ by $m-1$ in (2.33) and we recall that, by Proposition 2.13, $C_\tau^n=C^{n-1}$, for all $n\in\mathbb{N}$, to obtain

$$C_\tau^{m+n}+\sum_{l=1}^{n}\frac{1}{2^{m+n-l}}\binom{m+n-l-1}{n-l}C_\tau^l+\sum_{l=1}^{m}\frac{1}{2^{m+n-l}}\binom{m+n-l-1}{m-l}C_\tau^l=2C_\tau^nC_\tau^m, \quad n,m\in\mathbb{N}.$$

□

**Corollary 2.15.** *Let $A$ be the generator of a cosine family $\{C(t)\}_{t\in\mathbb{R}}$ and $\{C_\tau^n\}_{n\in\mathbb{N}_0}$ be the cosine sequence given in Proposition 2.13. Then*

$$R_\tau^mC_\tau^{n-m}=\sum_{l=1}^{n}\frac{1}{2^{m+n-l}}\binom{m+n-l-1}{n-l}C_\tau^l+\sum_{l=1}^{m}\frac{1}{2^{m+n-l}}\binom{m+n-l-1}{m-l}C_\tau^l,$$

*for all $n,m\in\mathbb{N}$.*

*Proof.* It follows from Theorem 2.5 and Corollary 2.14. □

The next result shows that, using the sequence given in Theorem 2.9, it is possible to construct a discrete cosine sequence without assuming that $A$ generates a continuous cosine family.

**Proposition 2.16.** *Let $A$ be a closed linear operator in a Banach space $X$ such that $\tau^{-2}\in\rho(A)$. Let $\{C_\tau^n\}_{n\in\mathbb{N}_0}\subset\mathcal{B}(X)$ be the sequence defined by $C_\tau^0:=I$, and*

$$C_\tau^nx:=\sum_{j=1}^{n}a_{n,j}R_\tau^jx, \quad n\geq 1, \quad x\in X, \tag{2.34}$$

*where $(a_{n,j})$ is the sequence given in Theorem 2.9. Then, $\{C_\tau^n\}_{n\in\mathbb{N}_0}$ is a cosine sequence generated by $A$.*

*Proof.* We need verify that the sequence $\{C_\tau^n\}_{n\in\mathbb{N}_0}$ defined in (2.34) satisfies conditions (1)-(3) of Definition 2.1. Conditions (1) and (2) follow directly from the definition of $C_\tau^n$ and the properties of $R_\tau$. Therefore, it remains only to prove that

$$C_\tau^nx=x+\tau A\sum_{j=1}^{n}k_\tau^2(n-j)C_\tau^jx, \quad n\geq 1. \tag{2.35}$$

From the definition of $R_\tau$, we get $\tau^{-1}R_\tau-\tau^{-1}I=\tau AR_\tau$, what in turn implies $\tau AR_\tau^j=\tau^{-1}R_\tau^j-\tau^{-1}R_\tau^{j-1}$, for all $j\in\mathbb{N}$. Let $T_n:=\sum_{j=1}^n k_\tau^2(n-j)C_\tau^j$, where $C_\tau^j$ is defined in (2.34). Then,

$$\begin{aligned}\tau AT_n&=\tau A\sum_{j=1}^{n}k_\tau^2(n-j)\sum_{l=1}^{j}a_{j,l}R_\tau^l\\&=\tau A\sum_{l=1}^{n}\left(\sum_{j=l}^{n}k_\tau^2(n-j)a_{j,l}\right)R_\tau^l\\&=\sum_{l=1}^{n}\left(\sum_{j=l}^{n}k_\tau^2(n-j)a_{j,l}\right)(\tau^{-1}R_\tau^l-\tau^{-1}R_\tau^{l-1}).\end{aligned} \tag{2.36}$$

In this way, to prove (2.35) is equivalent to showing that

$$C_\tau^n x = x + \tau A T_n, \quad \text{for all} \quad x \in X, n \in \mathbb{N}. \tag{2.37}$$

To this end, we need to compare the coefficients of $R_\tau^l$ in the left and the right hand sides of (2.37), and verify that the coefficients $a_{n,l}$ satisfy the required identities.

In fact, if $n = 1$, then the right and left hand sides of (2.37) coincide by definition. For $n \geq 2$, according to the definition of $C_\tau^n$, the left hand side of (2.37) does not have identity operator, and therefore, to verify such equality we need to obtain

$$0 = 1 - \tau^{-1} \sum_{j=1}^{n} k_\tau^2(n-j) a_{j,1}, \quad n \geq 1.$$

Firstly, for $l = 1$, by separating the term $a_{n,1}$ in the last expression, we obtain $a_{n,1} = (k_\tau^2(0) - \sum_{j=1}^{n-1} k_\tau^2(n-j)a_{j,1})k_\tau^2(0)^{-1}$, which is exactly the definition of $a_{n,1}$ given in Theorem 2.9.

Next, take $2 \leq l \leq n-1$. Comparing the coefficients of $R_\tau^l$ on the left and right hand sides of (2.37), the definition of $C_\tau^n$ and equality (2.36) imply that

$$a_{n,l} = \tau^{-1} \sum_{j=l}^{n} k_\tau^2(n-j) a_{j,l} - \tau^{-1} \sum_{j=l+1}^{n} k_\tau^2(n-j) a_{j,l+1}.$$

In this equality, after separating the last term in the first sum, we cancel out the term $a_{n,l}$. Also, by replacing $l+1$ with $l$ in the second sum, we obtain

$$\sum_{j=l+1}^{n} k_\tau^2(n-j) a_{j,l+1} = \sum_{j=l}^{n-1} k_\tau^2(n-j) a_{j,l}.$$

In this last equality, we separate the term $a_{n,l+1}$ in the left hand side to obtain

$$k_\tau^2(0) a_{n,l+1} = \sum_{j=l}^{n-1} k_\tau^2(n-j) a_{j,l} - \sum_{j=l+1}^{n-1} k_\tau^2(n-j) a_{j,l+1},$$

which is the same coefficient given in Theorem 2.9. Finally, comparing the coefficients of $R_\tau^n$ in (2.37), we obtain $a_{n,n} = k_\tau(1) a_{n-1,n-1} k_\tau(0)^{-1}$. This concludes the proof. ☐

It is well know the relation between the generator of a $C_0$-group and a cosine family, see for instance [4, Sections 3.1 and 3.14]. In the discrete case, we need the definition of a $\tau$-discrete semigroup generated by a closed linear operator $A$.

**Definition 2.17.** [35, Definition 3.3] *Let $A$ be a closed linear operator defined on a Banach space $X$. An operator-valued sequence $\{T_\tau^n\}_{n\in\mathbb{N}_0} \subset B(X)$ is called a $\tau$-*discrete semigroup* generated by $A$ if it satisfies the following conditions*

1. *$T_\tau^0 = I$.*
2. *$T_\tau^n \in D(A)$ for all $x \in X$ and $AT_\tau^n x = T_\tau^n Ax$ for all $x \in D(A)$, and $n \in \mathbb{N}_0$.*
3. *For each $x \in X$ and $n \in \mathbb{N}$,*

$$T_\tau^n x = x + \tau A \sum_{j=1}^{n} T_\tau^j x. \tag{2.38}$$

The next definition corresponds to the discrete counterpart of [4, Definition 3.1.19].

**Definition 2.18.** *An operator $A$ on $X$ is said to be the generator of a $\tau$-*discrete group* if $A$ and $-A$ are generators of $\tau$-discrete semigroups.*

In this case, we define the sequence of operators $\{U_\tau^n\}_{n\in\mathbb{Z}}$ by

$$U_\tau^n := \begin{cases} T_{\tau,+}^n & n\in\mathbb{N}_0,\\ T_{\tau,-}^{-n}, & -n\in\mathbb{N},\end{cases}$$

where $\{T_{\tau,+}^n\}_{n\in\mathbb{N}_0}$ and $\{T_{\tau,-}^n\}_{n\in\mathbb{N}_0}$ are the $\tau$-discrete semigroups generated by $A$ and $-A$, respectively. In this case, the sequence $\{U_\tau^n\}_{n\in\mathbb{Z}}$ is called the $\tau$-*discrete group* generated by $A$. We note that, the sequence $\{T_{\tau,-}^n\}_{n\in\mathbb{N}_0}$ verifies

(1) $T_{\tau,-}^0 = I$;
(2) $T_{\tau,-}^n x\in D(A)$ for all $x\in X$, and $AT_{\tau,-}^n x = T_{\tau,-}^n Ax$ for all $x\in D(A)$ and $n\in\mathbb{N}_0$; and
(3) For each $x\in X$ and $n\in\mathbb{N}$, $T_{\tau,-}^n x = x-\tau A\sum_{j=1}^n T_{\tau,-}^j x$.

The next results give properties of a $\tau$-discrete semigroup generated by $-A$.

**Lemma 2.19.** *Let $\{T_{\tau,-}^n\}_{n\in\mathbb{N}_0}$ be a $\tau$-discrete semigroup generated by $-A$. Then,*

(1) *$\tau^{-1}\in\rho(-A)$, and $T_\tau^{n,*} = \tau^{-n}(\tau^{-1}+A)^{-n}$ for all $n\in\mathbb{N}_0$.*
(2) *Its $Z$-transform satisfies*

$$\widetilde{T}_{\tau,-}(z) = \Big(\frac{1}{\tau}+A\Big)\Big(\frac{z-1}{\tau z}+A\Big)^{-1}.$$

*Proof.* Follows in the same way as in [35, Propositions 3.4 and 3.7] . □

**Proposition 2.20.** *Let $B$ be the generator of a $\tau$-discrete group $\{U_\tau^n\}_{n\in\mathbb{Z}}$. Then, $A=B^2$ is generator of the cosine sequence $\{C_\tau^n\}_{n\in\mathbb{Z}}$ defined by*

$$C_\tau^n = \frac{U_\tau^n+U_\tau^{-n}}{2},\quad n\in\mathbb{Z}.$$

*Proof.* By Proposition 2.8 and Lemma 2.19, we have

$$\widetilde{T}_{\tau,+}(z) = \Big(\frac{1}{\tau}-B\Big)\Big(\frac{z-1}{\tau z}-B\Big)^{-1},\qquad \widetilde{T}_{\tau,-}(z) = \Big(\frac{1}{\tau}+B\Big)\Big(\frac{z-1}{\tau z}+B\Big)^{-1},$$

respectively. Since $U_\tau^n+U_\tau^{-n} = T_{\tau,+}^n+T_{\tau,-}^{-n}$ for all $n\in\mathbb{Z}$, we obtain

$$\begin{aligned}
&\Big(\frac{1}{\tau}-B\Big)\Big(\frac{z-1}{\tau z}-B\Big)^{-1}+\Big(\frac{1}{\tau}+B\Big)\Big(\frac{z-1}{\tau z}+B\Big)^{-1}\\
=\;&\Big(\frac{1}{\tau}-B\Big)\Big(\frac{z-1}{\tau z}+B\Big)\Big(\Big(\frac{z-1}{\tau z}\Big)^2-A\Big)^{-1}+\Big(\frac{1}{\tau}+B\Big)\Big(\frac{z-1}{\tau z}-B\Big)\Big(\Big(\frac{z-1}{\tau z}\Big)^2-A\Big)^{-1}\\
=\;&\Big(\frac{1}{\tau}\frac{z-1}{\tau z}+\frac{1}{\tau}B-\frac{z-1}{\tau z}B-A+\frac{1}{\tau}\frac{z-1}{\tau z}-\frac{1}{\tau}B+\frac{z-1}{\tau z}B-A\Big)\Big(\Big(\frac{z-1}{\tau z}\Big)^2-A\Big)^{-1}\\
=\;&\Big(\frac{2(z-1)}{\tau^2 z}-2A\Big)\Big(\Big(\frac{z-1}{\tau z}\Big)^2-A\Big)^{-1},
\end{aligned}$$

that is, $\widetilde{T}_{\tau,+}(z)+\widetilde{T}_{\tau,-}(z) = 2\widetilde{C}_\tau(z)$, concluding the proof. □

Next result gives an inversion theorem for cosine families generated by a closed linear operator $A$.

**Theorem 2.21.** *Let $A$ be the generator of a cosine family exponentially bounded $\{C(t)\}_{t\in\mathbb{R}}$ and $\{C_\tau^n\}_{n\in\mathbb{N}_0}$ be the cosine sequence given in Proposition 2.13. Then,*

$$C(t)x = \lim_{\tau\to 0^+}\tau\sum_{n=1}^\infty \rho_{n-1}^\tau(t)C_\tau^n x,\quad x\in X, \tag{2.39}$$

*uniformly in $t$ in compact subsets of $\mathbb{R}_+$.*

*Proof.* By [34, Proposition 2.2], it follows that

$$\tilde{C}(z)x = \frac{1}{\tau z}\hat{C}\left(\frac{1}{\tau}\left(1-\frac{1}{z}\right)\right)x + x,$$

for all $|z| > 1$ and $x \in X$, where $\hat{C}$ denotes the Laplace transform of $t \mapsto C(t)$. To complete the proof we proceed similarly to [35, Theorem 4.13]. $\square$

The next proposition shows that if $A$ generates a cosine sequence on a Banach space, then it is possible to define an associated continuous cosine family whose generator is the same operator $A$.

**Proposition 2.22.** *Let $A$ be a densely defined closed linear operator. Suppose that $A$ generates the cosine sequence $\{C_\tau^n\}_{n\in\mathbb{N}_0}$ with $\|C_\tau^n\| \le M$ for all $n \in \mathbb{N}_0$. Assume that there exists $\tau_0 > 0$ such that for all $\tau \in (0,\tau_0] \subset \rho(A)$. If $\|(I-\tau^2 A)^{-1}\| \le K$ for all $\tau \in (0,\tau_0]$, then, $\{C(t)\}_{t\ge 0}$ defined by*

$$C(t)x := \lim_{\tau\to 0^+} \tau \sum_{n=1}^{\infty} \rho_{n-1}^{\tau}(t) C_\tau^n x, \quad x \in X, t \ge 0, \tag{2.40}$$

*defines a cosine family generated by $A$.*

*Proof.* Let $x \in X$ and $\tau \in (0,\tau_0]$. By Proposition 2.3,

$$\tau \sum_{n=1}^{\infty} \rho_{n-1}^{\tau}(0) C_\tau^n x = \tau \rho_0^{\tau}(0) C_\tau^1 x = R_\tau x = (I-\tau^2 A)^{-1} x. \tag{2.41}$$

For $x \in D(A)$, we have $\|(I-\tau^2A)^{-1}x - x\| = \tau^2 \|(I-\tau^2 A)^{-1}Ax\| \le \tau^2 K \|Ax\|$, which tends to 0 as $\tau \to 0^+$. As $\overline{D(A)} = X$, we get $\|(I-\tau^2 A)^{-1}x - x\| \to 0$ as $\tau \to 0^+$ for all $x \in X$. Thus, by (2.41), we deduce that $C(0)x = \lim_{\tau\to 0^+}(I-\tau^2A)^{-1}x = x$ for all $x \in X$, that is, $C(0) = I$.

Now, we show that $C(t)$ is strongly continuous. Since $\tau\sum_{n=1}^{\infty} \rho_{n-1}^{\tau}(t)x = x$, and $\lim_{\tau\to 0^+}\|R_\tau x - x\| = 0$, we can write

$$C(t)x - x = \lim_{\tau\to 0^+} \tau \sum_{n=1}^{\infty} \rho_{n-1}^{\tau}(t)[C_\tau^n x - x] = \lim_{\tau\to 0^+} \tau \sum_{n=1}^{\infty} \rho_{n-1}^{\tau}(t)[C_\tau^n x - R_\tau x].$$

From the representation (2.27) given in Theorem 2.9, and noticing that $a_{n,1} = 1$, and $C_\tau^1 x = R_\tau x$, we have

$$C(t)x - x = \lim_{\tau\to 0^+} \tau \sum_{n=1}^{\infty} \rho_{n-1}^{\tau}(t) \sum_{j=1}^{n} a_{n,j} R_\tau^j x.$$

For $l \ge 2$, we observe that each $a_{n,l}$ is a linear combination of $k_\tau^2(m)$, and $a_{j,\bar{l}}$ for certain values of $j \le n$ and $\bar{l} \le l$. Since $k_\tau^2(m) \to 0$ as $\tau \to 0^+$, we obtain $\lim_{\tau\to 0^+} a_{n,l} = 0$ for all $l \ge 2$ and $n \in \mathbb{N}$, we deduce that $\|C(t)x - x\| \to 0$ as $t \to 0^+$.

Since $AC_\tau^n x = C_\tau^n Ax$ for all $x \in X$, it is easy to see that this implies $AC(t)x = C(t)Ax$ for all $x \in D(A)$ and $t \ge 0$. Finally, we need to show that $C(t)x = x + A(g_2 * C)(t)x$, for all $t \ge 0$ and $x \in X$. In fact, according to Definition 2.1, we have

$$\begin{aligned}
\tau \sum_{n=1}^{\infty} \rho_{n-1}^{\tau}(t) C_\tau^n x &= \tau \sum_{n=1}^{\infty} \rho_{n-1}^{\tau}(t)x + \tau^2 A \sum_{n=1}^{\infty} \rho_{n-1}^{\tau}(t) \sum_{j=1}^{n} k_\tau^2(n-j) C_\tau^j x \\
&= x + \tau^2 A \sum_{j=1}^{\infty} \left( \sum_{n=j}^{\infty} \rho_{n-1}^{\tau}(t) k_\tau^2(n-j) \right) C_\tau^j x \\
&= x + \tau^2 A \sum_{j=1}^{\infty} \left( \sum_{m=0}^{\infty} \rho_{m+j-1}^{\tau}(t) k_\tau^2(m) \right) C_\tau^j x, \quad x \in X. \qquad (2.42)
\end{aligned}$$

Now, we notice that $\rho^\tau_{m+j-1}(t)=\int_0^t\rho^\tau_{m-1}(t-s)\rho^\tau_{j-1}(s)ds=(\rho^\tau_{m-1}*\rho^\tau_{j-1})(t)$, and therefore

$$\tau^2\sum_{j=1}^\infty\left(\sum_{m=0}^\infty\rho^\tau_{m+j-1}(t)k^2_\tau(m)\right)C^j_\tau x=\tau\sum_{j=1}^\infty\int_0^t\rho^\tau_{j-1}(s)\left(\tau\sum_{m=0}^\infty\rho^\tau_{m-1}(t-s)k^2_\tau(m)\right)C^j_\tau x\,ds, \tag{2.43}$$

for all $x\in X$. In addition, we have the estimate

$$\begin{aligned}\left\|\int_0^t(t-s)C(s)x\,ds-\tau^2\sum_{n=1}^\infty\rho^\tau_{n-1}(t)\sum_{j=1}^n k^2_\tau(n-j)C^j_\tau x\right\|\leq&\int_0^t(t-s)\left\|C(s)x-\tau\sum_{j=1}^\infty\rho^\tau_{j-1}(s)C^j_\tau x\right\|ds\\&+\|C^j_\tau x\|\left|\tau\sum_{j=1}^\infty\int_0^t\rho^\tau_{j-1}(s)\right.\\&\left.\times\left(\tau\sum_{m=0}^\infty\rho^\tau_{m-1}(t-s)k^2_\tau(m)-(t-s)\right)ds\right|.\end{aligned}$$

Here, we define $\rho^\tau_{-1}(t)=\delta(t)$, and thus $(\delta*\rho^\tau_{m-1})(t)=\rho^\tau_{m-1}(t)$. As $\lim_{\tau\to0^+}\tau\sum_{m=0}^\infty\rho^\tau_{m-1}(t-s)k^2_\tau(m)=(t-s)$ for any $t\geq s$, we conclude by (2.40) that

$$\int_0^t(t-s)C(s)x\,ds=\lim_{\tau\to0^+}\tau^2\sum_{n=1}^\infty\rho^\tau_{n-1}(t)\sum_{j=1}^n k^2_\tau(n-j)C^j_\tau x,\quad x\in X. \tag{2.44}$$

Finally, taking $\tau\to0^+$ in (2.42), we obtain by (2.43) and (2.44), that

$$C(t)x=x+A\int_0^t(t-s)C(s)x\,ds,\quad x\in X,\,t\geq0.$$

concluding the proof. $\square$

An operator $A:D(A)\subset X\to X$ is said to be *sectorial of angle* $\theta$, and we write $A\in\mathrm{Sect}(\theta,M)$, if there exist $M>0$ and $\theta\in(\pi/2,\pi)$ such that $\rho(A)\supset\Sigma_\theta:=\{z\in\mathbb{C}:|\arg(z)|<\theta\}$ and

$$\|(z-A)^{-1}\|\leq\frac{M}{|z|}\quad\text{for all}\quad z\in\Sigma_\theta.$$

Given linear and closed operator $A$ whose resolvent set contains the negative half-line $(-\infty,0]$, (for example, if $A$ is a sectorial operator) and $0\leq\varepsilon\leq1$, the set $X^\varepsilon$ denotes the domain of the fractional power $A^\varepsilon$, that is $X^\varepsilon:=D(A^\varepsilon)$ endowed with the graph norm $\|x\|_\varepsilon=\|A^\varepsilon x\|$. Thus, $X^1$ corresponds to the domain of $A$, and $X^0$ to the space $X$. It is a well known that if $0<\varepsilon<1$, and $x\in D(A)$, then there exists a constant $K\equiv K_\varepsilon>0$ such that (see [25])

$$\|A^\varepsilon x\|\leq K\|Ax\|^\varepsilon\|x\|^{1-\varepsilon}. \tag{2.45}$$

We have the following result.

**Lemma 2.23.** [10, Lemma 2] *Let $t>0$. For $\frac{\pi}{2}<\theta<\pi$, take $\phi$ such that $\frac12\phi<\frac{\pi}{2}<\phi<\theta$. Let $A\in\mathrm{Sect}(\theta,M)$, and let $\Gamma\equiv\Gamma_t$ be the complex path defined as the union $\Gamma^1_t\cup\Gamma^2_t$, where*

$$\Gamma^1_t:=\left\{\frac1t e^{i\psi}:-\phi<\psi<\phi\right\}\quad\text{and}\quad\Gamma^2_t:=\left\{re^{\pm i\phi}:\frac1t\leq r\right\}.$$

*If $\mu\geq0$, then*

$$\int_\Gamma\left|\frac{e^{zt}}{z^\mu}\right||dz|\leq Ct^{\mu-1},$$

*for all $t>0$, where $C:=\left(2\phi\int_{-\phi}^\phi e^{\cos(\psi)}d\psi+\frac{2}{-\cos(\phi)}\right)$.*

The next theorem shows that the cosine sequence $\{C^n_\tau\}_{n\in\mathbb{N}}$, defined in Proposition 2.13 converges to $C(t_n)$, where $t_n:=\tau n$, as $\tau\to0$, for any fixed $n\in\mathbb{N}$.

**Theorem 2.24.** *Let $0 < \varepsilon < \frac{1}{2}$ and suppose that $x \in D(A^\varepsilon)$. Let $A$ be a sectorial operator which generates a strongly continuous cosine family $\{C(t)\}_{t\geq 0}$. Let $\{C_\tau^n\}_{n\in\mathbb{N}_0}$ be the cosine sequence defined in (2.32). Then, for each $L > 0$ there exists a constant $D > 0$ such that, for $0 < t_n \leq L$,*

$$\|C(t_n)x - C_\tau^n x\| \leq D\tau t_n^{2\varepsilon-1}\|x\|_\varepsilon. \tag{2.46}$$

*In particular,*

$$\lim_{\tau\to 0^+} \|C(t_n)x - C_\tau^n x\| = 0.$$

*Proof.* Let $L > 0$ and $n \in \mathbb{N}$ such that $0 < t_n \leq L$, where $t_n := \tau n$. As $\int_0^\infty \rho_{n-1}^\tau(t)dt = 1$, for all $n \in \mathbb{N}$, we can write

$$C(t_n)x - C_\tau^n x = \int_0^\infty \rho_{n-1}^\tau(t)[C(t_n) - C(t)]xdt,$$

for all $x \in X$. As $A$ generates the cosine family $\{C(t)\}_{t\geq 0}$, $\hat{C}(z) = z(z^2 - A)^{-1}$ (see [4, Proposition 3.14.4]), and the inversion of Laplace transform allows to write

$$C(t_n)x - C(t)x = \frac{1}{2\pi i}\int_\Gamma \frac{e^{zt_n} - e^{zt}}{z} z^2(z^2 - A)^{-1}xdz,$$

where $\Gamma = \Gamma_{t_n}$ is the complex path in Lemma 2.23. As $z^2(z^2 - A)^{-1} = A(z^2 - A)^{-1} + I = A^{1-\varepsilon}(z^2 - A)^{-1}A^\varepsilon + I$, we get

$$C(t_n)x - C(t)x = \frac{1}{2\pi i}\int_\Gamma \frac{e^{zt} - e^{zt_n}}{z} xdz + \frac{1}{2\pi i}\int_\Gamma \frac{e^{zt} - e^{zt_n}}{z} A^{1-\varepsilon}(z^2 - A)^{-1}A^\varepsilon xdz.$$

As $h(z) := \frac{e^{zt} - e^{zt_n}}{z}$ has a unique removable singularity at $z = 0$, the first integral in the last identity is equal to zero. On the other hand, as $A$ is a sectorial operator, the inequality (2.45) implies that $\|A^{1-\varepsilon}(z^2 - A)^{-1}x\| \leq KM(M+1)^{1-\varepsilon}\frac{\|x\|}{|z|^{2\varepsilon}}$, for all $x \in X$. Thus

$$\|C(t_n)x - C(t)x\| \leq \frac{KM(M+1)^{1-\varepsilon}}{2\pi}\int_\Gamma \frac{|e^{zt} - e^{zt_n}|}{|z|}\frac{\|A^\varepsilon x\|}{|z|^{2\varepsilon}}|dz|.$$

By the mean value for complex-valued functions, we can find $\bar{t}_0, \bar{t}_1$ with $0 < t_n < \bar{t}_0 < \bar{t}_1 < t$ such that

$$\frac{|e^{zt} - e^{zt_n}|}{|z|} \leq (t - t_n)\left(|e^{\bar{t}_0 z}| + |e^{\bar{t}_1 z}|\right).$$

Since $0 < \varepsilon < \frac{1}{2}$ and $t_n < \bar{t}_0 < \bar{t}_1$ we have $\bar{t}_1^{2\varepsilon-1} < \bar{t}_0^{2\varepsilon-1} < t_n^{2\varepsilon-1}$. By Lemma 2.23 we get

$$\|C(t_n)x - C(t)x\| \leq \frac{KM(M+1)^{1-\varepsilon}C}{\pi}(t - t_n)t_n^{2\varepsilon-1}\|A^\varepsilon x\|.$$

As $\int_0^\infty \rho_{n-1}^\tau(t)(t - t_n)dt = \tau$, we obtain

$$\|C(t_n)x - C_\tau^n x\| \leq \int_0^\infty \rho_{n-1}^\tau(t)\|C(t_n)x - C(t)x\|dt \leq \frac{KM(M+1)^{1-\varepsilon}C}{\pi}\tau t_n^{2\varepsilon-1}\|A^\varepsilon x\|.$$

Thus, $D = \frac{KMC(M+1)^{1-\varepsilon}}{\pi}$. Finally, as $\tau t_n^{2\varepsilon-1} = n^{2\varepsilon-1}\tau^{2\varepsilon}$, we conclude that $C(t_n)x \to C_\tau^n x$ as $\tau \to 0$ for all $x \in D(A^\varepsilon)$.

□

## 3. Application to the existence of solutions to second order problems.

Consider the system of second order

$$\left\{ \begin{array}{rcl} \nabla^2_\tau u^n & = & Au^n + f^n, \quad n \geq 2, \\ u^0 & = & x_0, \\ u^1 & = & x_1, \end{array} \right. \tag{3.47}$$

where $A : D(A) \subset X \to X$ is a closed linear operator in $X$, $f : \mathbb{N}_0 \to$ is a given vector-valued sequence and $x_0, x_1 \in X$.

**Proposition 3.25.** *Assume that $A$ is the generator of a cosine sequence $\{C^n_\tau\}_{n\in\mathbb{N}_0}$. If $x_0, x_1 \in X$, then the second system* (3.47) *has a unique solution given by the sequence*

$$u^n := C^n_\tau x_0 + S^n_\tau x_1 + \tau \sum_{j=1}^{n} S^{n+1-j}_\tau f^j, \tag{3.48}$$

*for all $n \geq 2$ and $u^0 := x_0$, $u^1 := x_1$.*

*Proof.* Since $\{C^n_\tau\}_{n\geq 0}$ is a cosine sequence, the sequence $u^n$ defined in (3.48) belongs to $D(A)$ for all $n \in \mathbb{N}_0$, and by Definition 2.1, we have

$$\begin{aligned} (\nabla^2_\tau C_\tau x)^n &= \frac{1}{\tau^2}(C^n_\tau x - 2C^{n-1}_\tau x + C^{n-2}_\tau x) \\ &= \frac{A}{\tau}\Big[\sum_{j=1}^{n} k^2_\tau(n-j)C^j_\tau x - 2\sum_{j=1}^{n-1} k^2_\tau(n-1-j)C^j_\tau x + \sum_{j=1}^{n-2} k^2_\tau(n-2-j)C^j_\tau x\Big] \\ &= \frac{A}{\tau}\Big[(k^2_\tau(n-1) - 2k^2_\tau(n-2) + k^2_\tau(n-3))C^1_\tau + (k^2_\tau(n-2) - 2k^2_\tau(n-3) + k^2_\tau(n-4))C^2_\tau + \cdots \\ &\quad + (k^2_\tau(2) - 2k^2_\tau(1) + k^2_\tau(0))C^{n-2}_\tau + k^2_\tau(1)C^{n-1}_\tau + k^2_\tau(0)C^n_\tau - 2k^2_\tau(0)C^{n-1}_\tau\Big]x, \end{aligned}$$

for all $n \geq 2$. As $k^2_\tau(n-j) - 2k^2_\tau(n-1-j) + k^2_\tau(n-2-j) = 0$ for all $j \leq n-2$, we obtain

$$(\nabla^2_\tau C_\tau x)^n = \frac{A}{\tau}\left[k^2_\tau(1)C^{n-1}_\tau + k^2_\tau(0)C^n_\tau - 2k^2_\tau(0)C^{n-1}_\tau\right]x = \frac{A}{\tau}\left[2\tau C^{n-1}_\tau + \tau C^n_\tau - 2\tau C^{n-1}_\tau\right]x = AC^n_\tau x. \tag{3.49}$$

On the other hand, for $n \geq 2$, we have by Proposition 2.6 that

$$(\nabla^2_\tau S_\tau x)^n = \frac{1}{\tau^2}(S^n_\tau x - 2S^{n-1}_\tau x + S^{n-2}_\tau x) = \frac{1}{\tau}\Big[\sum_{j=1}^{n} C^j_\tau x - 2\sum_{j=1}^{n-1} C^j_\tau x + \sum_{j=1}^{n-2} C^j_\tau x\Big] = \nabla_\tau C^n_\tau x = AS^n_\tau x. \tag{3.50}$$

Finally, by (3.50) we have for all $n \geq 2$ that

$$\begin{aligned} \nabla^2_\tau\left(\sum_{j=1}^{n} S^{n+1-j}_\tau f^j\right)^n &= \frac{1}{\tau^2}\left[\sum_{j=1}^{n} S^{n+1-j}_\tau f^j - 2\sum_{j=1}^{n-1} S^{n-j}_\tau f^j + \sum_{j=1}^{n-2} S^{n-1-j}_\tau f^j\right] \\ &= \frac{1}{\tau^2}\big[(S^n_\tau - 2S^{n-2}_\tau + S^{n-2}_\tau)f^1 + \cdots + (S^3_\tau - 2S^2_\tau + S^1_\tau)f^{n-2} \\ &\quad + S^2_\tau f^{n-1} + S^1_\tau f^n - 2S^1_\tau f^{n-1}\big] \\ &= AS^n_\tau f^1 + \cdots + AS^3_\tau f^{n-2} + \frac{1}{\tau^2}\left((S^2_\tau - S^1_\tau)f^{n-1} + S^1_\tau f^n - S^1_\tau f^{n-1}\right). \end{aligned}$$

Now, by definition of $S_\tau^n$ and Proposition 2.6, $(S_\tau^2 - S_\tau^1)f^{n-1} = \tau(\nabla_\tau S_\tau)^2 f^{n-1} = \tau C_\tau^2 f^{n-1}$ and $S_\tau^1 = \tau C_\tau^1$, which implies, again by Proposition 2.6 that

$$\begin{aligned}\frac{1}{\tau^2}\left((S_\tau^2 - S_\tau^1)f^{n-1} + S_\tau^1 f^n - S_\tau^1 f^{n-1}\right) &= \frac{1}{\tau^2}\left(\tau(C_\tau^2 - C_\tau^1)f^{n-1} + S_\tau^1 f^n\right)\\ &= (\nabla_\tau C_\tau)^2 f^{n-1} + \frac{1}{\tau}C_\tau^1 f^n\\ &= AS_\tau^2 f^{n-1} + (\nabla_\tau C_\tau)^1 f^n + \frac{1}{\tau}f^n\\ &= AS_\tau^2 f^{n-1} + AS_\tau^1 f^n + \frac{1}{\tau}f^n.\end{aligned}$$

Hence,

$$\nabla^2\left(\sum_{j=1}^n S_\tau^{n+1-j}f^j\right)^n = A\sum_{j=1}^n S_\tau^{n+1-j}f^j + \frac{1}{\tau}f^n, \quad n \geq 2. \tag{3.51}$$

Defining the sequence $(u^n)_{n\in\mathbb{N}_0}$ by $u^n := C_\tau^n x_0 + S_\tau^n x_1 + \tau\sum_{j=1}^n S_\tau^{n+1-j}f^j$, for $n \geq 2$ and $u^0 := x_0$, $u^1 := x_1$, then by (3.49), (3.50) and (3.51) we get

$$\begin{aligned}(\nabla_\tau^2 u)^n &= \nabla_\tau^2\left(C_\tau^n x_0 + S_\tau^n x_1 + \tau\sum_{j=1}^n S_\tau^{n+1-j}f^j\right)\\ &= AC_\tau^n x_0 + AS_\tau^n x_1 + A\sum_{j=1}^n S_\tau^{n+1-j}f^j + f^n\\ &= Au^n + f^n,\end{aligned}$$

for all $n \geq 2$. This means that $(u^n)_{n\in\mathbb{N}_0}$ solves the equation $\nabla_\tau^2 u^n = Au^n + f^n$, $n \geq 2$, under the initial conditions $u^0 = x_0$, and $u^1 = x_1$. To see the uniqueness, suppose that $u^n$ solves (3.47) with $f^n = 0$ and $x_0 = x_1 = 0$. Then, $\nabla_\tau^2 u^n = Au^n$ for all $n \geq 2$ with $u^0 = u^1 = 0$. This is equivalent to

$$(I - \tau^2 A)u^n = 2u^{n-1} - u^{n-2}, \quad n \geq 2.$$

As $A$ is the generator of a cosine family, by Proposition 2.3, $\tau^{-2} \in \rho(A)$ and therefore,

$$u^n = 2R_\tau u^{n-1} - R_\tau u^{n-2}, \quad n \geq 2.$$

As $u^0 = u^1 = 0$, we conclude that $u^n = 0$ for all $n \geq 2$, that is, $u \equiv 0$.

□

*Example* 3.26.

Consider the space $X = L^2([0,\pi])$, define the second-order operator $A : D(A) \subset X \to X$ by $Aw(x) = w''(x)$ with domain $D(A) := H^2([0,\pi]) \cap H_0^1([0,\pi])$. It is a well-known fact that $A$ is a self-adjoint operator whose spectrum is $\sigma(A) = \{-m^2 : m \in \mathbb{N}\}$, with the corresponding eigenfunctions given by $\phi_m(x) = \sqrt{\frac{2}{\pi}}\sin(mx)$, for $x \in [0,\pi]$. Then, $A$ can be written as

$$Aw = \sum_{m=1}^\infty -m^2\langle w, \phi_m\rangle\phi_m, \quad w \in D(A).$$

Now, we consider the problem

$$\left\{\begin{array}{rcl}\nabla_\tau^2 u^n(s) &=& Au^n(s) + f^n(s), \quad n \geq 2,\\ u^0(s) &=& x_0(s),\\ u^1(s) &=& x_1(s),\end{array}\right. \tag{3.52}$$

where $s \in [0,\pi]$, $A$ is defined above, $(f^n)$ is a given sequence and $x_0, x_1 \in X$.

This problem can be written in the abstract form (3.47), by writing $u^n := u^n(\cdot), x_0 := x_0(\cdot), x_1 := x_1(\cdot), f^n := f^n(\cdot)$. We notice that (3.52) can be considered as a time-discretization of the problem

$$\left\{\begin{array}{rcl} \partial_t^2 u(t,s) & = & \dfrac{\partial^2 u(t,s)}{\partial s^2} + f(t,s), \quad t\in[0,T], \\ u(t,0) & = & u(t,\pi) = 0, \quad t\in[0,T], \\ u(0,s) & = & \varphi_0(s), \quad s\in[0,\pi], \\ u_t(0,s) & = & \varphi_1(s), \quad s\in[0,\pi], \end{array}\right.$$

where $T>0$ and $\varphi_0, \varphi_1 \in X$ are given functions.

On the other hand, it is a well-known that $A$ is the generator of the cosine family $\{C(t)\}_{t\in\mathbb{R}}$ given by (see [4, Section 3.14])

$$C(t)x = \sum_{m=1}^{\infty} \cos(mt)\langle x, \phi_m\rangle \phi_m, \quad x\in X,$$

and by Proposition 2.13, $A$ generates the cosine sequence $\{C_\tau^n\}_{n\in\mathbb{N}_0}$ defined by

$$C_\tau^0 = I, \quad \text{and} \quad C_\tau^n x = \int_0^\infty \rho_{n-1}^\tau(t) C(t)x dt, \quad x\in X, n\geq 1.$$

Since $\int_0^\infty \rho_n(t)dt = 1$ for all $n\in\mathbb{N}$, the Bessel inequality, the dominated convergence theorem, and [21, Formula 3.944-6, p.498], allow us to find an explicit expression for $C_\tau^n$ as follows

$$\begin{aligned} C_\tau^n x &= \sum_{m=1}^{\infty} \frac{1}{\tau^n (n-1)!}\int_0^\infty e^{-\frac{t}{\tau}} t^{n-1}\cos(mt)dt\langle x,\phi_m\rangle\phi_m \\ &= \sum_{m=1}^{\infty} \frac{1}{\tau^n}\frac{1}{(m^2+\tau^{-2})^{n/2}}\cos\left(n\arctan(\tau m)\right)\langle x,\phi_m\rangle\phi_m, \end{aligned}$$

for $n\geq 1, x\in X$. Therefore, by Proposition 3.25, we have an explicit representation of the solution to (3.52), given by (3.48), where $\{S_\tau^n\}_{n\in\mathbb{N}_0}$ is the sine sequence defined in (2.23).

## 4. Declarations

4.1. **Conflicts of interests.** The authors have no conflicts of interest to declare.

4.2. **Availability statement.** The paper has no associated data.

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