MAXIMAL L^p-REGULARITY FOR FRACTIONAL DIFFERENTIAL EQUATIONS ON THE LINE

VERÓNICA POBLETE AND RODRIGO PONCE

Abstract. We characterize the L^p-maximal regularity of an abstract fractional differential equation with delay on the Lebesgue spaces. The method is based on the theory of operator-valued Fourier multipliers and weighted Sobolev spaces on the line.

1. Introduction

In this paper, we consider the following fractional differential equation with delay

$$D^\alpha u(t) = Au(t) + Fu_t + f(t), \qquad t \in \mathbb{R}, \tag{1.1}$$

where A is a closed linear operator defined on a Banach space X, $f \in L^p(\mathbb{R}; X)$, $1 < p < \infty$ and the fractional derivative for $\alpha > 0$ is taken in the sense of Caputo. Here the delay $F : L^p([-r, 0]; X) \to X$ is supposed to be a bounded linear operator, and $u_t(\cdot) = u(t + \cdot)$. Recent investigations into physics, engineering, biological sciences and other fields have demonstrated that the dynamics of many systems are described more accurately using fractional differential equations and that fractional differential equations with delay are often more realistic to describe natural phenomena than those without delay [27, 28, 43, 44].

We notice that in the case $F \equiv 0$, the fractional differential equation (1.1) is equivalent to the integral equation

$$u(t) = \int_{-\infty}^{t} a(t - s)[Au(s) + f(s)]ds, \quad t \in \mathbb{R}, \tag{1.2}$$

where $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} =: g_\alpha(t)$. The study of maximal regularity on $L^p(\mathbb{R}; X)$ for the integral equation (1.2) (in the sense that for all $f \in L^p(\mathbb{R}; X)$ there exists a unique strong solution u of equation (1.2)), for general kernels $a \in L^1(\mathbb{R}_+)$, goes back to Ph. Clément and G. Da Prato, [16]. However, since $g_\alpha \notin L^1(\mathbb{R}_+)$, the results in [16] are not applicable to equation (1.1). When $F \not\equiv 0$, the equation (1.1) can not be equivalent to an integral equation. A complete study of equations in the form of (1.2) can be found in the monograph [40]. We remark that the behavior of fractional differential equations with and without delay are completely different, even in the case when F is a bounded operator, see [31]. To the knowledge of the authors, time fractional differential equations in $L^p(\mathbb{R}; X)$ and with delay have not been studied until now. Moreover, we notice that one of the difficulties is to determine the right definition of fractional derivative

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to be used in this case. Here, we consider the so-called Caputo (or Weyl) fractional derivative [25].

Time fractional differential equations with delay on periodic Lebesgue spaces $L^p_{2\pi}(X)$, (X being a UMD space and $1 < p < \infty$) have been treated in [7, 8, 9, 10, 24, 31]. See also [11, 12, 13, 15, 32, 33, **?**, 38, 39].

In [3], Arendt and Duelli study the L^p-maximal regularity to equation (1.1) when $\alpha = 1$, that is,

$$u'(t) = Au(t) + f(t), \qquad t \in \mathbb{R}. \tag{1.3}$$

The authors showed, using Weis's operator valued Fourier multiplier theorem [42], that if X is a UMD space, then the problem (1.3) is maximal L^p-regular ($1 < p < \infty$) if and only if the operator A is $\mathcal{R}$-bisectorial and invertible. This result extends Mielke's characterisation of maximal L^p-regularity of (1.3) when the underlying space is a Hilbert space, see [35]. Applying this results, the authors study the second-order problem

$$u''(t) = Au(t) + f(t), \qquad t \in \mathbb{R}, \tag{1.4}$$

by transforming this equation into a first-order system. Other approaches of maximal regularity to equation (1.3), using interpolation spaces, can be found in [45].

Maximal regularity on $L^p(J; X)$ where $J = [0, T], T > 0$, or $J = [0, \infty)$ to fractional differential equation (1.1) and have been studied, see for example [6, 19, 23].

In this paper, we study the problem of characterize the L^p-maximal regularity of the fractional differential equation with delay (1.1) in $L^p(\mathbb{R}; X)$, the vector-valued L^p-spaces for $1 < p < \infty$.

The paper is organized as follows. In Section 2, we review some results about vector-valued Fourier multipliers and we recall the definition and some basic properties on fractional calculus. Section 3 is devoted to our main result (Theorem 3.9), where a characterization of maximal regularity of problem (1.1) is obtained under some suitable assumptions. Finally, some applications are examined in Section 4 and 5.

2. Preliminaries

Let X and Y be complex Banach spaces. We denote the space of all linear and bounded operators from X to Y by $\mathcal{B}(X, Y)$. In the case $X = Y$, we will write briefly $\mathcal{B}(X)$. Let A be an operator defined on X. We will denote its domain by $D(A)$, its domain endowed with the graph norm by $[D(A)]$, its resolvent set by $\rho(A)$, and its spectrum set by $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

For $n \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, $C^n(\mathbb{R}; X)$ denotes the set of X-valued functions which are n-times differentiable on $\mathbb{R}$.

Given $\alpha > 0$, the *Liouville fractional integrals of order* α, $D_-^{-\alpha} f$ and $D_+^{-\alpha} f$ are defined, respectively, by

$$D_-^{-\alpha} f(t) := \int_{-\infty}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds, \quad t \in \mathbb{R}, \tag{2.1}$$

and

$$D_+^{-\alpha} f(t) := \int_{t}^{\infty} \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds, \quad t \in \mathbb{R}. \tag{2.2}$$

A sufficient condition for that the fractional integrals (2.1) and (2.2) exist is that $f(t) = O(|t|^{-\alpha-\epsilon})$ for $\epsilon > 0$ and $t \to \infty$. Integrable functions satisfying this property are sometimes referred to as functions of Liouville class, see [36].

The *Caputo left and right-sided fractional derivatives,* corresponding to those in (2.1) and (2.2) are defined, respectively, by

$$D^{\alpha}_{-}f(t) := D^{-(n-\alpha)}_{-}\frac{d^n}{dt^n}f(t) = \int_{-\infty}^{t} \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(s)ds$$

and

$$D^{\alpha}_{+}f(t) := (-1)^n D^{-(n-\alpha)}_{+}\frac{d^n}{dt^n}f(t) = (-1)^n\int_{t}^{\infty} \frac{(s-t)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(s)ds,$$

where $t \in \mathbb{R}$, $f \in C^n(\mathbb{R}; X)$ and $n = \lceil \alpha \rceil$. Here $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal to α. More details of Caputo fractional calculus can be found in [25, Section 2.4] and [14].

We notice that the Caputo fractional calculus can also be applied to functions not belonging to the Liouville class (see [36, p. 237]). For example, let g and h be measurable functions on $\mathbb{R}$ such that $D^{-\alpha}_{\pm}g$ exists and $h = D^{-\alpha}_{\pm}g$ a.e. Then we set $D^{\alpha}_{\pm}h = g$.

It is known that $D^{\alpha+\beta}_{\pm} = D^{\alpha}_{\pm}(D^{\beta}_{\pm})$ for any $\alpha, \beta \in \mathbb{R}$, where $D^0_{\pm} = \mathrm{Id}$ denotes the identity operator and $(-1)^n D^n_{+} = D^n_{-} = \frac{d^n}{dt^n}$ holds with $n \in \mathbb{N}$. See [36].

In what follows, we refer to the Caputo left-sided fractional derivative, $D^{\alpha}_{-}f$, as the *Caputo fractional derivative of order* $\alpha > 0$ *of* f and we write $D^{\alpha}f := D^{\alpha}_{-}f$. For example, for the function $e^{\lambda t}$ we have

$$D^{-\alpha}_{-}e^{\lambda t} = \lambda^{-\alpha}e^{\lambda t} \quad \text{and} \quad D^{\alpha}e^{\lambda t} = \lambda^{\alpha}e^{\lambda t}, \quad \mathrm{Re}\lambda > 0.$$

For $\alpha > 0$, we define $W^{\alpha,p}(\mathbb{R}; X)$ as the Banach space consisting of all $u \in L^p(\mathbb{R}; X)$, for which there exists $u', u'', ..., u^n \in L^p(\mathbb{R}; X)$, $n = \lceil \alpha \rceil$, such that

$$\int_{\mathbb{R}} u(t)D^{\alpha}\phi(t)dt = \int_{\mathbb{R}} D^{\alpha}u(t)\phi(t)dt$$

for all $\phi \in \mathcal{D}(\mathbb{R})$.

Thus, if $u \in L^p(\mathbb{R}; [D(A)])$ is a *weak solution* of equation (1.1), i.e.

$$\int_{\mathbb{R}} u(t)D^{\alpha}\phi(t)dt = \int_{\mathbb{R}} (Au(t)+Fu_t + f(t))dt$$

for all $\phi \in \mathcal{D}(\mathbb{R})$, then $u \in W^{\alpha,p}(\mathbb{R}; X)$ and $D^{\alpha}u = Au + Fu_{\cdot} + f$.

We denote by $\hat{f}$ the *Fourier transform* of f, that is

$$\hat{f}(s) := \int_{\mathbb{R}} e^{-ist}f(t)dt,$$

for $s \in \mathbb{R}$ and $f \in L^1(\mathbb{R}; X)$.

We denote by $\mathcal{D}(\mathbb{R}; X)$ the space of X-valued $C^{\infty}-$functions with compact support on $\mathbb{R}$. $\mathcal{S}'(\mathbb{R}; X) = \mathcal{B}(\mathcal{S}(\mathbb{R}); X)$ is the space of all tempered distributions. Then the Fourier transform $\mathcal{F}$ on $\mathcal{S}'(\mathbb{R}; X)$ is defined by

$$\langle \mathcal{F}u, \phi \rangle = \langle u, \hat{\phi} \rangle,$$

where $u \in \mathcal{S}'(\mathbb{R}, X)$ and $\phi \in \mathcal{S}(\mathbb{R})$. If we identify $\mathcal{S}(\mathbb{R}; X)$ with a subspace of $\mathcal{S}'(\mathbb{R}; X)$ by letting

$$\langle u, \phi \rangle = \int_{\mathbb{R}} u(t)\phi(t)dt, \quad \phi \in \mathcal{S}(\mathbb{R}),$$

for all $u \in \mathcal{S}(\mathbb{R}; X)$, then $\hat{u} = \mathcal{F}u$, i.e.,

$$\int_{\mathbb{R}} u(t)\hat{\phi}(t)dt = \int_{\mathbb{R}} \hat{u}(s)\phi(s)ds,$$

for all $u \in \mathcal{S}(\mathbb{R}; X), \phi \in \mathcal{S}(\mathbb{R})$. Thus $\mathcal{F} : \mathcal{S}'(\mathbb{R}; X) \to \mathcal{S}'(\mathbb{R}; X)$ is an isomorphism extending the isomorphism $u \mapsto \hat{u}$ on $\mathcal{S}(\mathbb{R}; X)$. We refer to [1] for all these properties.

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}; X)$ of subexponential growth, that is

$$\int_{-\infty}^{\infty} e^{-\epsilon|t|}\|f(t)\|dt < \infty, \quad \text{for each } \epsilon > 0,$$

we denote by $\tilde{f}(\lambda)$ for the *Carleman transform* of f :

$$\tilde{f}(\lambda) = \begin{cases} \int_0^{\infty} e^{-\lambda t} f(t)dt, & \mathrm{Re}\lambda > 0, \\ -\int_{-\infty}^{0} e^{-\lambda t} f(t)dt, & \mathrm{Re}\lambda < 0. \end{cases}$$

Definition 2.1. *Let X, Y be Banach spaces, $1 < p < \infty$. A function $M \in C^{\infty}(\mathbb{R}; \mathcal{B}(X, Y))$ is an $L^p_{X,Y}-$multiplier if there exists a bounded operator $T : L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; Y)$ such that for all $f \in \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}; X)$*

$$Tf \in \mathcal{S}(\mathbb{R}; Y), \quad \textit{and} \quad (Tf)^{\wedge}(s) = M(s)\hat{f}(s), \quad s \in \mathbb{R}.$$

Definition 2.2. *A family of operators $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called $\mathcal{R}$-bounded if there is a constant $C > 0$ such that for all $T_1, ..., T_n \in \mathcal{T}$, $x_1, ..., x_n \in X$, $n \in \mathbb{N}$,*

$$\int_0^1 \left\| \sum_{j=1}^{n} r_j(t)T_j x_j \right\|_Y dt \leq C \int_0^1 \left\| \sum_{j=1}^{n} r_j(t)x_j \right\|_X dt, \tag{2.3}$$

where (r_j) is a sequence of independent symmetric $\{-1, 1\}$-valued random variables on $[0, 1]$, e.g. the Rademacher functions $r_j(t) = \mathrm{sgn}(\sin(2^j\pi t))$. The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$ and we denote it by $R_p(\mathcal{T})$.

We note that in a Hilbert space every normbounded set $\mathcal{T}$ is $\mathcal{R}$-bounded. Several properties of $\mathcal{R}$-bounded families can be found in the monograph of Denk, Hieber and Prüss [20]. See moreover [26].

The following operator-valued multiplier theorem is due to Weis [42, Theorem 3.4].

Theorem 2.3. *Let X, Y be UMD-spaces and $1 < p < \infty$. Suppose that $M \in C^1(\mathbb{R}; \mathcal{B}(X, Y))$, and that the sets*

$$\{M(s) : s \in \mathbb{R}\} \quad \textit{and} \quad \{sM'(s) : s \in \mathbb{R}\},$$

are $\mathcal{R}$-bounded. Then M is an $L^p_{X,Y}-$multiplier.

3. The Maximal L^p-regularity. The UMD case.

Let $A : D(A) \subseteq X \to X$ be a linear closed operator on the Banach spaces X. We consider the following fractional differential equation

$$D^\alpha u(t) = Au(t) + Fu_t + f(t), \quad t \in \mathbb{R}, \tag{3.1}$$

where $\alpha \geq 1$, and $f \in L^p(\mathbb{R}; X)$ and for $r > 0$, $F : L^p([-r, 0]; X) \to X$ is a linear and bounded operator. Moreover u_t is an element of $L^p([-r, 0]; X)$ which is defined as $u_t(\theta) = u(t + \theta)$ for $-r \leq \theta \leq 0$.

Definition 3.1. *Let $1 < p < \infty$. For $f \in L^p(\mathbb{R}; X)$, we call $u \in L^p(\mathbb{R}; X)$ a solution of equation (3.1) if $u \in W^{\alpha,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; [D(A)])$ and u satisfies equation (3.1) for a.e. $t \in \mathbb{R}$.*

Definition 3.2. *We say that the equation (3.1) has maximal L^p-regularity if for each $f \in L^p(\mathbb{R}; X)$ there exists a unique solution u of equation (3.1).*

Remark 3.3.

Observe that if equation (3.1) has maximal L^p-regularity, it follows from the closed graph theorem that the map $M : L^p(\mathbb{R}; X) \to W^{\alpha,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; [D(A)])$, which associates to f the unique solution u of equation (3.1) is linear and continuous.

Indeed, since A is a closed operator, we have that the space $H := W^{\alpha,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; [D(A)])$ endowed with the norm

$$||u||_H := \|D^\alpha u\|_{L^p} + \|Au\|_{L^p} + \|u\|_{L^p}$$

is a Banach space.

For appropriate functions, the Caputo left and right-sided fractional derivatives are adjoint in the sense of the following lemma.

Lemma 3.4. *If $D^\alpha f$ and $D_+^{-\alpha} g$ exist, then*

$$\int_{\mathbb{R}} f(t)g(t)dt = \int_{\mathbb{R}} D^\alpha f(t) D_+^{-\alpha} g(t)dt.$$

Proof. The proof is similar to [25, p. 89]. ∎

Denote by $e_\lambda(t) := e^{i\lambda t}$ for all $\lambda \in \mathbb{R}$, and define the operators $\{F_\lambda\}_{\lambda\in\mathbb{R}} \subseteq \mathcal{B}(X)$ by

$$F_\lambda x := F(e_\lambda x), \quad \text{for all } \lambda \in \mathbb{R} \quad \text{and} \quad x \in X.$$

An easy computation shows that $\widehat{Fu_t}(s) = F_s \hat{u}(s)$ for all $s \in \mathbb{R}$ and $u \in L^1(\mathbb{R}; X)$.

For $s \in \mathbb{R}$, we recall that $(is)^\alpha$ is defined by $(is)^\alpha = |s|^\alpha e^{\frac{\pi\alpha i}{2}\operatorname{sgn}(s)}$, where $\operatorname{sgn}(s)$ denotes the sign of s. We define the *real resolvent set* $\rho(A, F)$ by

$$\rho(A, F) := \{s \in \mathbb{R} : (is)^\alpha I - F_s - A \text{ has a bounded inverse }\}.$$

The *real spectrum set* $\sigma(A, F)$ is defined by

$$\sigma(A, F) := \mathbb{R} \setminus \rho(A, F).$$

Proposition 3.5. *Let $1 < p < \infty$, and $f \in \mathcal{F}^{-1}(\mathbb{R}; X)$, and $u \in L^p(\mathbb{R}; [D(A)])$. Assume that $\sigma(A, F) = \emptyset$. The following assertions are equivalent.*

(*i*) *$u \in W^{\alpha,p}(\mathbb{R}; X)$ and u is a solution of equation (3.1);*
(*ii*) *$u \in \mathcal{S}(\mathbb{R}; [D(A)])$ and $\hat{u}(s) = ((is)^\alpha - F_s - A)^{-1} \hat{f}(s)$ for $s \in \mathbb{R}$.*

Proof. $(ii) \Rightarrow (i)$. Observe that $\widehat{D^\alpha u}(s) = (is)^\alpha \hat{u}(s)$, for all $s \in \mathbb{R}$. In fact, since $D_+^\alpha(e^{-ist}) = (is)^\alpha e^{-ist}$ for all $s \in \mathbb{R}$, we obtain by Lemma 3.4 that

$$(is)^\alpha \hat{u}(s) = \int_{\mathbb{R}} (is)^\alpha e^{-ist} u(t)dt = \int_{\mathbb{R}} e^{-ist} D^\alpha u(t)dt = \widehat{D^\alpha u}(s).$$

Using that A is closed operator we have $\widehat{Au}(s) = A\hat{u}(s)$. Hence $(D^\alpha u - Fu. - Au)^\wedge(s) = ((is)^\alpha - F_s - A)\hat{u}(s) = \hat{f}(s)$, for all $s \in \mathbb{R}$. Consequently $D^\alpha u - Fu. - Au = f$.

$(i) \Rightarrow (ii)$. Let $u \in L^p(\mathbb{R}; [D(A)]) \cap W^{\alpha,p}(\mathbb{R}; X)$ be a solution of equation (3.1). Let $\phi \in \mathcal{S}(\mathbb{R})$ and $R(A, F, s) := ((is)^\alpha - F_s - A)^{-1}$, $s \in \mathbb{R}$. From Lemma 3.4 and Fubini's theorem, we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \phi(s)\, R(A,F,s) \hat{f}(s)ds &= \int_{\mathbb{R}} \phi(s)\, R(A,F,s) \int_{\mathbb{R}} e^{-ist} f(t)dtds \\
&= \int_{\mathbb{R}} \phi(s)\, R(A,F,s) \int_{\mathbb{R}} e^{-ist}[D^\alpha u(t) - Fu_t - Au(t)]dtds \\
&= \int_{\mathbb{R}} \phi(s)\, R(A,F,s) \int_{\mathbb{R}} u(t) D_+^\alpha(e^{-ist}) - e^{-ist} Fu_t - e^{-ist} Au(t)dtds \\
&= \int_{\mathbb{R}} \phi(s)\, R(A,F,s) \int_{\mathbb{R}} ((is)^\alpha - F_s - A)\, u(t) e^{-ist} dtds \\
&= \int_{\mathbb{R}} \phi(s) \int_{\mathbb{R}} u(t) e^{-ist} dtds \\
&= \int_{\mathbb{R}} u(t) \int_{\mathbb{R}} \phi(s) e^{-ist} dsdt \\
&= \int_{\mathbb{R}} u(t) \hat{\phi}(t)dt.
\end{aligned}$$

Therefore, identifying $L^p(\mathbb{R}; [D(A)])$ with a subspace of $\mathcal{S}'(\mathbb{R}; [D(A)])$ by letting

$$\langle v, \phi \rangle = \int_{\mathbb{R}} v(t)\phi(t)dt,$$

for $v \in L^p(\mathbb{R}; [D(A)])$ and $\phi \in \mathcal{S}(\mathbb{R})$. From the identity above, we have that $\hat{u}(s) = ((is)^\alpha - F_s - A)^{-1}\hat{f}(s)$, for all $s \in \mathbb{R}$, and $\mathcal{F}u = ((i\cdot)^\alpha - F. - A)^{-1}\hat{f}(\cdot) \in \mathcal{D}(\mathbb{R}; [D(A)])$. Hence $u \in \mathcal{S}(\mathbb{R}; [D(A)])$. □

Lemma 3.6. *Let $\alpha, \beta > 0$. If f belongs to the Liouville class, then*

$$D^\alpha(e^{-\beta|t|} f(t)) = e^{-\beta|t|} \sum_{k=0}^{\infty} \binom{\alpha}{k} (-\mathrm{sgn}(t)\beta)^k D^{\alpha-k} f(t), \quad t \in \mathbb{R},$$

where $\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-k+1)}{k!}$.

Proof. Similar to [21, Lemma 5.3]. □

For $\beta > 0$ we define the following weighted L^p and Sobolev spaces on $\mathbb{R}$ with values in the Banach spaces X

$$L^p_\beta(\mathbb{R}; X) := \{f : \mathbb{R} \to X \text{ measurable } : \|f\|_{\beta,p} < \infty\},$$

$$W^{\alpha,p}_\beta(\mathbb{R}; X) := \{f : \mathbb{R} \to X \text{ measurable } : f, f', \ldots, f^n \in L^p_\beta(\mathbb{R}; X)\} \ \text{ with } \ n = \lceil \alpha \rceil,$$

where $\|f\|_{\beta,p} := \left(\int_{\mathbb{R}} \|e^{-\beta|t|}f(t)\|^p dt\right)^{1/p}$ is the norm in $L^p_\beta(\mathbb{R};X)$ and $\|f\|_{\beta,p} + \|f'\|_{\beta,p} + \cdots + \|f^{(n)}\|_{\beta,p}$ is the norm in $W^{\alpha,p}_\beta(\mathbb{R};X)$.

As in Definition 3.1, for $f \in L^p_\beta(\mathbb{R};X)$ we call $u \in L^p_\beta(\mathbb{R};X)$ a *solution* of equation (3.1) if $u \in W^{\alpha,p}_\beta(\mathbb{R};X) \cap L^p_\beta(\mathbb{R};[D(A)])$ and u satisfies equation (3.1) for a.e. $t \in \mathbb{R}$.

Further, we define the following mapping

$$\begin{array}{rcl} ^{-}: L^p_\beta(\mathbb{R};X) & \to & L^p(\mathbb{R};X) \\ u & \mapsto & \bar{u}, \text{ where } \bar{u}(t) := e^{-\beta|t|}u(t). \end{array}$$

The function $^{-}$ is an isomorphism between $L^p_\beta(\mathbb{R};X)$ and $L^p(\mathbb{R};X)$.

The following lemma establishes the connection between solutions in $L^p(\mathbb{R};X)$ and solutions in $L^p_\beta(\mathbb{R};X)$.

Lemma 3.7. *Let $1 < p < \infty$, $\beta > 0$ and $f \in L^p_\beta(\mathbb{R};X)$. Then $u \in W^{\alpha,p}_\beta(\mathbb{R};X)\cap L^p_\beta(\mathbb{R};[D(A)])$ is solution of equation (3.1) if and only if $\bar{u} \in W^{\alpha,p}(\mathbb{R};X)\cap L^p(\mathbb{R};[D(A)])$ is solution of*

$$D^\alpha \bar{u}(t) = A\bar{u}(t) + F\overline{u_t} + \bar{f}(t) + e^{-\beta|t|}\sum_{k=1}^{\infty}\binom{\alpha}{k}(-\operatorname{sgn}(t)\beta)^k D^{\alpha-k}(e^{\beta|t|}\bar{u}(t)). \tag{3.2}$$

Proof. Follows directly from Lemma 3.6. □

This result includes the cases of first and second order (for $F = 0$) treated in [41]. Let $f \in L^p_\beta(\mathbb{R};X)$. In the particular case $\alpha = 1$, we have that $u \in L^p_\beta(\mathbb{R};X)$ is solution of $u'(t) = Au(t) + f(t)$ if and only if $\bar{u} \in L^p(\mathbb{R};X)$ is solution of

$$\bar{u}'(t) = A\bar{u}(t) + \bar{f}(t) - \beta \operatorname{sgn}(t)\, \bar{u}(t).$$

For $\alpha = 2$, it follows that $u \in L^p_\beta(\mathbb{R};X)$ is solution of $u''(t) = Au(t) + f(t)$ if and only if $\bar{u} \in L^p(\mathbb{R};X)$ is solution of

$$\bar{u}''(t) = A\bar{u}(t) + \bar{f}(t) - \beta^2\, \bar{u}(t) - 2\operatorname{sgn}(t)\, \bar{u}'(t).$$

Lemma 3.8. *If the equation (3.1) has maximal L^p-regularity, then there exists $\beta > 0$ such that for all $f \in L^p_\beta(\mathbb{R};X)$ there exists an unique solution $u \in W^{\alpha,p}_\beta(\mathbb{R};X)\cap L^p_\beta(\mathbb{R};[D(A)])$ of equation (3.1) and the solution operator $M_\beta : L^p_\beta(\mathbb{R};X) \to W^{\alpha,p}_\beta(\mathbb{R};X)\cap L^p_\beta(\mathbb{R};[D(A)])$ is bounded.*

Proof. Let $f \in L^p_\beta(\mathbb{R};X)$. From Lemma 3.7 we obtain that $u \in W^{\alpha,p}_\beta(\mathbb{R};X)\cap L^p_\beta(\mathbb{R};[D(A)])$ is solution of equation (3.1) if and only if $\bar{u} \in W^{\alpha,p}(\mathbb{R};X)\cap L^p(\mathbb{R};[D(A)])$ is solution of equation (3.2). Define the mapping $T_\beta : W^{\alpha,p}(\mathbb{R};X) \to W^{\alpha,p}(\mathbb{R};X)\cap L^p(\mathbb{R};[D(A)])$ by

$$T_\beta g := M\left(-h_g\right),$$

where M is the solution operator of equation (3.1) and

$$h_g(t) = e^{-\beta|t|}\sum_{k=1}^{\infty}\binom{\alpha}{k}(-\operatorname{sgn}(t)\beta)^k \beta^{-1} D^{\alpha-k}(e^{\beta|t|}g(t)).$$

From Lemma 3.6 in follows that $h_g \in L^p(\mathbb{R};X)$, hence T_β is well-defined and is a bounded operator. On the other hand, by (3.2) we have

$$\begin{aligned} D^\alpha((1+\beta T_\beta)\bar{u})(t) &= D^\alpha \bar{u}(t) + \beta D^\alpha(T_\beta \bar{u})(t) \\ &= A\bar{u}(t) + F\overline{u_t} + \bar{f}(t) + \beta h_{\bar{u}}(t) + \beta D^\alpha(T_\beta \bar{u})(t) \\ &= A\bar{u}(t) + F\overline{u_t} + \bar{f}(t) + \beta h_{\bar{u}}(t) + \beta D^\alpha(M(-h_{\bar{u}}))(t) \\ &= A\bar{u}(t) + F\overline{u_t} + \bar{f}(t) + \beta h_{\bar{u}}(t) + \beta[AM(-h_{\bar{u}})(t) + F(M(-h_{\bar{u}}))_t - h_{\bar{u}}(t)] \\ &= A\bar{u}(t) + F\overline{u_t} + \bar{f}(t) + \beta[AM(-h_{\bar{u}})(t) + F(M(-h_{\bar{u}}))_t] \\ &= A[\bar{u}(t) + \beta M(-h_{\bar{u}})(t)] + F\overline{u_t} + \beta F(M(-h_{\bar{u}}))_t + \bar{f}(t) \\ &= A(1+\beta T_\beta)\bar{u}(t) + F\overline{u_t} + \beta F(T_\beta \bar{u})_t + \bar{f}(t) \\ &= A(1+\beta T_\beta)\bar{u}(t) + F((1+\beta T_\beta)\bar{u})_t + \bar{f}(t) + F\overline{u_t} - F(\bar{u})_t, \end{aligned}$$

since $F((1+\beta T_\beta)\bar{u})_t = F(\bar{u})_t + \beta F(T_\beta \bar{u})_t$. Therefore, $M(\bar{f}+g) = (1+\beta T_\beta)\bar{u}$, where $g(\cdot) = F\overline{u_\cdot} - F(\bar{u})_\cdot \in L^p(\mathbb{R};X)$. If β is small enough, then $(1+\beta T_\beta)$ is invertible. For this β, we have that

$$M_\beta f = (\bar{\ })^{-1}(1+\beta T_\beta)^{-1} M(\bar{f}+g),$$

and by the closed graph theorem the operator M_β which takes $f \in L^p_\beta(\mathbb{R};X)$ into the unique solution $u \in W^{\alpha,p}_\beta(\mathbb{R};X) \cap L^p_\beta(\mathbb{R};[D(A)])$ of equation (3.1) is a bounded operator. □

The main result in this section is the following theorem.

Theorem 3.9. *Assume that X is a UMD-space and $1 < p < \infty$. The following assertions are equivalent.*

(*i*) *Equation (3.1) has maximal L^p-regularity;*

(*ii*) *$\sigma(A,F) = \emptyset$ and $\{(is)^\alpha((is)^\alpha - F_s - A)^{-1}\}_{s\in\mathbb{R}}$ is $\mathcal{R}$-bounded.*

Proof. (*i*) ⇒ (*ii*). Assume that equation (3.1) has maximal L^p-regularity. Let $s \in \mathbb{R}$ and suppose

$$((is)^\alpha - F_s - A)x = 0, \tag{3.3}$$

for $x \in D(A)$. Let $u(t) := e^{ist}x$. Then $u \in W^{\alpha,p}_\beta(\mathbb{R};X) \cap L^p_\beta(\mathbb{R};[D(A)])$ for all $\beta > 0$. Observe that u is a solution to equation (3.1) with $f \equiv 0$. In fact, $D^\alpha u(t) = (is)^\alpha e^{ist}x$ (see [36, p. 248]). Moreover, an easy computation shows that $Fu_t = e^{ist}F_s x$ and therefore by (3.3) we have

$$Au(t) = e^{ist}Ax = e^{ist}[(is)^\alpha - F_s]x = D^\alpha u(t) - Fu_t.$$

Hence, choosing the number $\beta > 0$ given in Lemma 3.8, we obtain by uniqueness that $u \equiv 0$, that is, $x = 0$. Hence $((is)^\alpha - F_s - A)$ is injective.

In order to show the surjectivity, let $y \in X$ be arbitrary. Let $s \in \mathbb{R}$ and β be small enough as in Lemma 3.8. Let f_s defined by $f_s(t) := e^{ist}y$. Clearly $f_s \in L^p_\beta(\mathbb{R};X)$. Let $M_\beta : L^p_\beta(\mathbb{R};X) \to W^{\alpha,p}_\beta(\mathbb{R};X) \cap L^p_\beta(\mathbb{R};[D(A)])$ be the bounded operator which takes each $f \in L^p_\beta(\mathbb{R};X)$ to the unique solution u of equation (3.1).

Let $u = M_\beta f_s$. For fixed $r \in \mathbb{R}$ we have that $v_1(t) := u(t+r)$ and $v_2(t) := e^{isr}u(t)$ are both solutions of (3.1) with $g(t) = e^{isr}f_s(t)$. Hence, $v_1 = v_2$, that is, $u(t+r) = e^{isr}u(t)$ for all $r,t \in \mathbb{R}$. Let $x = u(0) \in D(A)$. If $r = -t$, then $u(t) = e^{ist}x$ for all $t \in \mathbb{R}$. Since $D^\alpha u(t) = (is)^\alpha e^{ist}x$ we have $D^\alpha u(0) = (is)^\alpha x$ and therefore,

$$((is)^\alpha - F_s - A)x = D^\alpha u(0) - Fu_0 - Au(0).$$

Since $u(t)$ satisfies the equation (3.1) for all $t \in \mathbb{R}$, we obtain,

$$((is)^\alpha - F_s - A)x = D^\alpha u(0) - Fu_0 - Au(0) = f_s(0) = y. \tag{3.4}$$

Therefore $((is)^\alpha - F_s - A)$ is surjective for all $s \in \mathbb{R}$. Since A is a closed operator, we have that $\sigma(A, F) = \emptyset$.

Now, we show that $\{(is)^\alpha((is)^\alpha - F_s - A)^{-1} : s \in \mathbb{R}\}$ is $\mathcal{R}$-bounded. In fact, since the operator solution M of equation (3.1) is bounded, we have that if $f \in \mathcal{F}^{-1}(\mathbb{R}; X)$ then it follows from Proposition 3.5 that $u = Mf \in \mathcal{S}(\mathbb{R}; [D(A)])$ and $\hat{u}(s) = \widehat{Mf}(s) = ((is)^\alpha - F_s - A)^{-1}\hat{f}(s)$, for all $s \in \mathbb{R}$. Therefore, the function $N : \mathbb{R} \to \mathcal{B}(X; [D(A)])$ given by $N(s) = ((is)^\alpha - F_s - A)^{-1}$ is an $L^p_{X;[D(A)]}-$multiplier and thus $\{N(s) : s \in \mathbb{R}\}$ is $\mathcal{R}$-bounded (see [17, Proposition 1]). Since $A : D(A) \to X$ is an isomorphism and F_s are bounded operators for all $s \in \mathbb{R}$, we obtain that $\{AN(s) + F_sN(s) : s \in \mathbb{R}\}$ $\mathcal{R}$-bounded. The identity $(is)^\alpha N(s) = I + AN(s) + F_sN(s)$ shows that $\{(is)^\alpha((is)^\alpha - F_s - A)^{-1} : s \in \mathbb{R}\}$ is $\mathcal{R}$-bounded.

$(ii) \Rightarrow (i)$. For $s \in \mathbb{R}$, define the operator $N(s) := ((is)^\alpha - F_s - A)^{-1}$. Observe that by hypothesis $N \in C^1(\mathbb{R}; \mathcal{B}(X, [D(A)]))$. We claim that $\{N(s) : s \in \mathbb{R}\}$ is a $L^p_{X,[D(A)]}-$multiplier.

In fact, the hypothesis shows that $\{N(s) : s \in \mathbb{R}\}$ is $\mathcal{R}$-bounded. Moreover,

$$isN'(s) = \alpha(is)^\alpha N(s)N(s) - isN(s)F'_sN(s).$$

An easy computation shows that $\{F'_s : s \in \mathbb{R}\}$ is $\mathcal{R}$-bounded (see [29, Proposition 3.2]) and as consequence, $\{isN'(s) : s \in \mathbb{R}\}$ is $\mathcal{R}$-bounded. By Theorem 2.3, $\{N(s) : s \in \mathbb{R}\}$ is an $L^p_{X,[D(A)]}-$multiplier. Therefore, there exits a bounded operator

$$T : L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; [D(A)])$$

such that for $f \in \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}; X)$, $u := Tf \in \mathcal{S}(\mathbb{R}; [D(A)])$ and $\hat{u}(s) = ((is)^\alpha - F_s - A)^{-1}\hat{f}(s)$ and thus, by Proposition 3.5 it follows that u is a solution of equation (3.1). Observe that,

$$\|u\|_{L^p(\mathbb{R};[D(A)])} \le \|T\| \, \|f\|_{L^p(\mathbb{R};X)}.$$

Now, let $f \in L^p(\mathbb{R}; X)$ be an arbitrary function. Then there exist $f_n \in \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}; X)$ such that $f_n \to f$ in $L^p(\mathbb{R}; X)$. Let $u_n = Tf_n$. Then u_n is a solution of equation (3.1) for f_n. Moreover $u_n \to u := Tf$ in $L^p(\mathbb{R}; [D(A)])$. For $\phi \in \mathcal{D}(\mathbb{R})$, one has by Lemma 3.4,

$$\int_{\mathbb{R}} (Au_n(t) + F(u_n)_t + f_n(t))\phi(t)dt = \int_{\mathbb{R}} D^\alpha u_n(t)\phi(t)dt = \int_{\mathbb{R}} u_n(t)D^\alpha_+\phi(t)dt.$$

Letting $n \to \infty$ we have by Lemma 3.4 that u is a weak solution of equation (3.1) and therefore $D^\alpha u = Au + Fu_\cdot + f$, that is, the equation (3.1) has maximal L^p-regularity.

To see the uniqueness, suppose that

$$D^\alpha u(t) = Au(t) + Fu_t, \quad t \in \mathbb{R}, \tag{3.5}$$

with $u \in W^{\alpha,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; [D(A)])$.

A simply computation shows that the Carleman transform of fractional derivative of u satisfies

$$\widetilde{D^\alpha u}(\lambda) = \lambda^\alpha \tilde{u}(\lambda) - \sum_{k=0}^{n-1} u^{(k)}(0)\lambda^{\alpha-1-k}, \text{ for } \mathrm{Re}\lambda \ne 0,\ n = \lceil \alpha \rceil.$$

Moreover, it is easy to show that $\widetilde{u_\cdot} \in L^p([-r, 0]; X)$ (see [37]) and $\widetilde{Fu_\cdot}(\lambda) = Fg\tilde{u}(\lambda) + Fgh$, for all $\mathrm{Re}\lambda \ne 0$, where $g(\theta) = e^{\lambda\theta}$ and $h(\theta) = \int_\theta^0 e^{-\lambda t}u(t)dt$, see [30].

Taking Carleman transform in (3.5), we have

$$(\lambda^\alpha - Fg - A)\tilde{u}(\lambda) = \sum_{k=0}^{n-1} u^{(k)}(0)\lambda^{\alpha-1-k} + Fgh, \text{ for } \mathrm{Re}\lambda \neq 0,\ n = \lceil \alpha \rceil.$$

Since $\sigma(A, F) = \emptyset$, it follows that the Carleman spectrum $\tilde{u}$ of u is empty and therefore $u = 0$ (see [4, Theorem 4.8.2]).

$\square$

Corollary 3.10. *Let H be Hilbert space and let $A : D(A) \subseteq H \to H$ be a closed linear operator. Then, the following assertions are equivalent for $1 < p < \infty$.*

(*i*) *Equation (3.1) has maximal L^p-regular;*

(*ii*) $\sigma(A, F) = \emptyset$ *and* $\sup_{s\in\mathbb{R}} \|(is)^\alpha((is)^\alpha - F_s - A)^{-1}\| < \infty.$

Corollary 3.11. *In the context of Theorem 3.9, if condition (ii) is fulfilled, we have that $D^\alpha u$, $Au \in L^p(\mathbb{R}; X)$. Moreover, there exists a constant $C > 0$ independent of $f \in L^p(\mathbb{R}; X)$ such that*

$$\|D^\alpha u\|_{L^p(\mathbb{R};X)} + \|Au\|_{L^p(\mathbb{R};X)} \leq C\|f\|_{L^p(\mathbb{R};X)}. \tag{3.6}$$

The inequality (3.6) is a consequence of the closed graph theorem and known as the *maximal regularity property* for equation (3.1). We deduce that the operator L defined by:

$$(Lu)(t) := D^\beta u(t) - Au(t) - Fu_t, \quad t \in \mathbb{R},$$

with domain

$$D(L) = W^{\alpha,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; [D(A)]),$$

is an isomorphism onto. In fact, by Remark 3.3 we have that the space $H := W^{\alpha,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; [D(A)])$ becomes a Banach space under the norm

$$\|u\|_H := \|D^\alpha u\|_{L^p(\mathbb{R};X)} + \|Au\|_{L^p(\mathbb{R};X)} + \|u\|_{L^p(\mathbb{R};X)}.$$

We remark that such isomorphisms are crucial for the handling of nonlinear evolution equations (see [1]). Indeed, assume that X is a Banach space and A, F satisfy the condition (*ii*) in Theorem 3.9. Consider the semilinear problem

$$D^\alpha u(t) = Au(t) + Fu_t + f(t, u(t)), \quad t \in \mathbb{R}. \tag{3.7}$$

Define the Nemytskii's operator $N : H \to L^p(\mathbb{R}; X)$ given by $N(v)(t) = f(t, v(t))$ and the bounded linear operator

$$T := L^{-1} : L^p(\mathbb{R}; X) \to H$$

by $T(g) = u$ where u is the unique solution to linear problem

$$D^\alpha u(t) = Au(t) + Fu_t + g(t).$$

To solve (3.7) we need to show that the operator $R : H \to H$ defined by $R = TN$ has a fixed point. For more details, we refer to Amann [1, 2].

For $y \in X$ and $r \in \mathbb{R}$, we define $f_r(t) := e^{irt}y$. Is clear that $f_r \in L^p_\beta(\mathbb{R}; X)$ for all $r \in \mathbb{R}$ since

$$||f_r||_{\beta,p} = \left(\int_{\mathbb{R}} e^{-\beta|t|p} dt\right)^{1/p} ||y|| =: C_{\beta,p}\, ||y||\,.$$

Theorem 3.12. *Let A be a linear operator on a Banach space X. Assume that equation (3.1) has maximal L^p-regularity for equation (3.1) for some $p \in (1,\infty)$. Then $\sigma(A,F)=\emptyset$ and there exists a constant $C>0$ such that*

$$\|((is)^\alpha - F_s - A)^{-1}\| \leq \frac{C}{1+|s|^\alpha}, \quad s \in \mathbb{R}.$$

Proof. Let $s \in \mathbb{R}$ and $y \in X$. As in the proof of Theorem 3.9 we have that $((is)^\alpha - F_s - A)$ is biyective and hence there is $z \in D(A)$ such that

$$((is)^\alpha - F_s - A)z = y.$$

Let β be small enought as in Lemma 3.8. From proof of Theorem 3.9 we have that for $f_s(t) := e^{ist}y$, the unique solution of equation (3.1) is $u_s(t) := e^{ist}z$. Moreover,

$$\|u_s\|_{\beta,p} = C_{\beta,p}\,\|z\|.$$

Let $n = \lceil \alpha \rceil$. Observe that

$$\begin{aligned}
\|u_s\|_{\beta,p} &= C_{\beta,p}\,\|z\| \\
\|u_s'\|_{\beta,p} &= |s| C_{\beta,p}\,\|z\| \\
\|u_s''\|_{\beta,p} &= |s|^2 C_{\beta,p}\,\|z\| \\
&\vdots \\
\|u_s^{(n)}\|_{\beta,p} &= |s|^n C_{\beta,p}\,\|z\|
\end{aligned}$$

By Lemma 3.8 we have

$$\begin{aligned}
(1+|s|+|s|^2+\ldots+|s|^n)C_{\beta,p}\|z\| &= \|u_s\|_{\beta,p} + \|u_s'\|_{\beta,p} + \ldots + \|u_s^{(n)}\|_{\beta,p} \\
&= \|u_s\|_{W_\beta^{\alpha,p}(\mathbb{R};X)\cap L_\beta^p(\mathbb{R};[D(A)])} \\
&= \|M_\beta f_s\|_{W_\beta^{\alpha,p}(\mathbb{R};X)\cap L_\beta^p(\mathbb{R};[D(A)])} \\
&\leq \|M_\beta\|\|f_s\|_{L_\beta^{\alpha,p}(\mathbb{R};X)} \\
&= CC_{\beta,p}\|y\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|((is)^\alpha - F_s - A)^{-1}y\| &\leq \frac{C}{1+|s|+|s|^2+\ldots+|s|^n}\|y\| \\
&\leq \frac{C}{1+|s|^\alpha}\|y\|.
\end{aligned}$$

□

4. Maximal regularity of a particular abstract equation

In this section, we consider the following equation

$$D^\alpha u(t) + A^\varepsilon u(t) = f(t), \quad t \in \mathbb{R}, \tag{4.8}$$

where A is a sectorial operator, $1<\alpha<2$, and $\varepsilon>0$. Maximal regularity to this class of equations have been studied in [18, 24].

We begin with some preliminaries on sectorial operators. We recall that a closed, densely defined operator A is *sectorial of angle* $\beta \in (0,\pi)$ if $\sigma(A) \subset \overline{\Sigma}_\beta$, and for every $\beta' \in (\beta,\pi)$

$$\sup_{z\in\mathbb{C}\setminus\Sigma_{\beta'}} \|z(z-A)^{-1}\| < \infty,$$

where $\Sigma_\beta := \{z \in \mathbb{C} : |\arg z| < \beta\}$. For a sectorial operator, define the *sectorial angle* $\omega(A)$ by

$$\omega(A) := \inf\{\beta \in (0,\pi) : A \text{ is sectorial of angle } \beta\}.$$

For every $\beta \in (0,\pi)$ we write

$$H^\infty(\Sigma_\beta) := \{f : \Sigma_\beta \to \mathbb{C} \text{ holomorphic } : \|f\|_\infty < \infty\},$$

$$H_0^\infty(\Sigma_\beta) := \left\{ f \in H^\infty(\Sigma_\beta) : \exists\varepsilon > 0 \text{ such that } \sup_{z\in\Sigma_\beta} |f(z)| \left|\frac{1+z^2}{z}\right|^\varepsilon < \infty \right\}.$$

If A is a sectorial operator of angle $\beta \in (0,\pi)$, then

$$\Phi_A(f) := f(A) := \frac{1}{2\pi i}\int_{\partial\Sigma_{\beta'}} f(z)(z-A)^{-1}dz$$

defines a functional calculus from $H_0^\infty(\Sigma_{\beta'})$ into $\mathcal{B}(X)$ for every $\beta' > \beta$. This functional calculus may be extended in a natural way in order to define the fractional powers A^ε for every $\varepsilon > 0$, see [22, 34].

A sectorial operator A admits a *bounded* H^∞ *functional calculus of angle* $\beta \in [\omega(A),\pi)$ if the functional calculus on $H_0^\infty(\Sigma_{\beta'})$ extends to a bounded linear operator on $H^\infty(\Sigma_{\beta'})$ for every $\beta' \in (\beta,\pi)$. The infimum of all such β is denoted by $\omega_H(A)$.

The well-known examples for general classes of closed linear operator with a bounded H^∞ calculus are

(1) normal sectorial operators in a Hilbert space;
(2) m-accreative operators in a Hilbert space;
(3) generators of bounded C_0-groups on L_p-spaces;
(4) negative generators of positive contraction semigroups on L_p-spaces.

The class of sectorial operators A which admit a bounded H^∞ calculus is denoted by $H^\infty(X)$. The operator A is said to admit a *bounded* RH^∞*-functional calculus* of angle $\beta \in [\omega_H(A),\pi)$ if, in addition, for every $\beta \in (\beta',\pi)$, the set

$$\{f(A) : \|f\|_{H^\infty(\Sigma'_\beta)} \le 1\}$$

is $\mathcal{R}$-bounded.

The main result of this section is the following.

Theorem 4.13. *Let A be a sectorial operator which admits a bounded RH^∞ functional calculus of angle $\omega \in \left(0, \frac{\pi}{\varepsilon}(1-\frac{\alpha}{2})\right)$ on a UMD Banach space X, where $1 < \alpha < 2$ and $\varepsilon > 1$. If $0 \in \rho(A)$ and $1 < p < \infty$, then (4.8) has maximal L^p-regularity.*

Proof. Follow the same lines of [24, Theorem 4.6]. Since $\omega \in \left(0, \frac{\pi}{\varepsilon}(1-\frac{\alpha}{2})\right)$, there exists $\beta > 0$ such that $\beta < \frac{\pi}{\varepsilon}(1-\frac{\alpha}{2})$. For each $z \in \Sigma_\beta$ and $s \in \mathbb{R}$, define $N(is,z) := (is)^\alpha((is)^\alpha + z^\varepsilon)^{-1}$. Note that $\frac{z^\varepsilon}{(is)^\alpha}$ belongs to the sector $\Sigma_{\frac{\pi\alpha}{2}+\beta\varepsilon}$, where $\frac{\pi\alpha}{2} + \beta\varepsilon < \pi$. Hence the distance from the sector $\Sigma_{\frac{\pi\alpha}{2}+\beta\varepsilon}$, to -1 is always positive. Therefore, there exists a constant $M > 0$ independent of $s \in \mathbb{R}$ and $z \in \Sigma_\beta$, such that

$$|N(is,z)| = \left|\frac{1}{1+\frac{z^\varepsilon}{(is)^\alpha}}\right| \leq M.$$

Since A admits a RH^∞ functional calculus of angle ω, we conclude by [20, Proposition 4.10] that the set $\{N(is,A) : s \in \mathbb{R}\setminus\{0\}\}$ is $\mathcal{R}$-bounded. Since A invertible, the operators $((is)^\alpha + A^\varepsilon)^{-1}$ exist for all $s \in \mathbb{R}$. Therefore, $\{N(is,A) : s \in \mathbb{R}\}$ is $\mathcal{R}$-bounded. We conclude by Theorem 3.9, that the equation (4.8) has maximal L^p-regularity. □

We recall that a linear operator A defined on X is called *non-negative* if $(-\infty, 0) \in \rho(A)$ and there exists $M > 0$ such that

$$\|\lambda(\lambda - A)^{-1}\| \leq M, \qquad \text{for all} \qquad \lambda < 0,$$

and A is said to be *positive* if it is non-negative and if, in addition, $0 \in \rho(A)$. See [34] for more details.

Since each self-adjoint, positive operator admits a bounded RH^∞ calculus of angle 0, we obtain the following Corollary.

Corollary 4.14. *Let A be a selfadjoint, positive operator defined on a Hilbert space H, $1 < \alpha < 2$ and $\varepsilon > 1$. Then for every $f \in L^p(\mathbb{R}; H)$ there exists a unique $u \in L^p(\mathbb{R}; [D(A)]) \cap W^{\alpha,p}(\mathbb{R}; H)$ such that (4.8) holds for all $t \in \mathbb{R}$.*

Since $A := -\Delta$ with domain $D(A) := \{f \in L^2(\mathbb{R};\mathbb{C}) : \frac{d^2 f}{dx^2} \in L^2(\mathbb{R};\mathbb{C})\}$ is a self-adjoint and positive operator, we have the following result.

Corollary 4.15. *Let $1 < \alpha < 2$ and $\varepsilon > 1$. Then for every $f \in L^p(\mathbb{R}; L^2(\mathbb{R};\mathbb{C}))$ there exists a unique $u \in L^p(\mathbb{R}; [D(A)]) \cap W^{\alpha,p}(\mathbb{R}; L^2(\mathbb{R};\mathbb{C}))$ such*

$$D^\alpha u(t) + (-\Delta)^\varepsilon u(t) = f(t),$$

holds for all $t \in \mathbb{R}$.

5. Examples

We conclude the paper, with an application of the previous results. Let $\eta : [-h, 0] \to \mathcal{B}(X)$ be a strongly continuous function. Let $F : L^p([-h, 0]; X) \to X$ be the bounded linear operator given by

$$F(\phi) = \int_{-h}^{0} \eta(\theta)\phi(\theta)d\theta, \phi \in L^p([-r, 0], X).$$

We notice that an important special case consists of those operators F defined by

$$F(\phi) = \sum_{k=0}^{n} C_k \phi(\tau_k), \quad \phi \in L^p([-r, 0]; X),$$

where $C_k \in \mathcal{B}(X)$ and $\tau_k \in [-r, 0]$ for $k = 0, ..., n$. For concrete equations dealing with the above classes of delay operators see the monograph of Bátkai and Piazzera [5, Chapter 3].

Consider the following problem with delay

$$\text{(5.9)} \qquad \begin{cases} D^{\alpha}u(t,x) = \dfrac{\partial^2}{\partial x^2}u(t,x) + \displaystyle\int_{-1}^{0} g(\theta)u(t+\theta,x)d\theta + f(t,x), \quad (t,x) \in \mathbb{R}\times[0,\pi], \\ u(t,0) = u(t,\pi) = 0, \quad t \in \mathbb{R}, \end{cases}$$

where $1 < \alpha < 2$, $g : [-1,0] \to \mathbb{R}$ is a continuous function and $r = 1$. We assume that the map defined by $\mathbb{R} \ni t \to f(\cdot,t) \in X := L^2([0,\pi])$ belongs to $L^p(\mathbb{R},X)$. On X define the operator A by

$$Av := \frac{d^2}{dx^2}v(x), \quad \text{with domain} \quad D(A) := \{v \in X : v \in H^2([0,\pi]), v(0) = v(\pi)\}.$$

The delay operator F is defined by

$$F(\phi) = \int_{-1}^{0} g(\theta)\phi(\theta)d\theta, \quad \phi \in L^p([-r,0];X).$$

With these notations, the problem (5.9) adopts the abstract form of equation (3.1).

It is well known that A is the infinitesimal generator of an analytic semigroup, that A has discrete spectrum with eigenvalues of the form $-n^2, n \in \mathbb{N}$, and the corresponding normalized eigenfunctions are given by $z_n(\xi) := (\frac{2}{\pi})^{\frac{1}{2}}\sin(n\xi)$. In addition, $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis for X, and thus

$$Ax = \sum_{n=1}^{\infty} -n^2\langle x, z_n\rangle z_n,$$

for all $x \in D(A)$. Therefore

$$((is)^{\alpha} - A)^{-1}x = \sum_{n=1}^{\infty} \frac{1}{(is)^{\alpha}+n^2}\langle x, z_n\rangle z_n, \quad \text{for all } x \in X.$$

We notice that if $1 < \alpha < 2$, then $\mathrm{Re}((is)^{\alpha}) < 0$, and therefore,

$$|(is)^{\alpha} + n^2| \geq |\mathrm{Im}((is)^{\alpha})| = |s|^{\alpha}\sin\left(\frac{\pi}{2}\alpha\right), \quad (s \neq 0),$$

and thus,

$$\|(is)^{\alpha} - A)^{-1}\| \leq \frac{1}{|s|^{\alpha}\sin\left(\frac{\pi}{2}\alpha\right)}, \quad (s \neq 0).$$

Observe that,

$$\|F\| \leq \|g\|_{\infty} := C < \infty.$$

Since the identity

$$(is)^{\alpha}((is)^{\alpha} - F_s - A)^{-1} = (I - ((is)^{\alpha} - A)^{-1}F_s)^{-1}(is)^{\alpha}((is)^{\alpha} - A)^{-1},$$

is valid for all $s \in \mathbb{R}$ we have

$$\|(is)^{\alpha}((is)^{\alpha} - F_s - A)^{-1}\| < \infty, \text{ when } \frac{C}{\sin\left(\frac{\pi}{2}\alpha\right)} < 1 \text{ and } 1 < \alpha < 2.$$

By Theorem 3.9, we conclude that the problem (5.9) has maximal L^p-regularity $(1 < p < \infty)$. Moreover, the solution u of (5.9) satisfies $D^{\alpha}u, \frac{\partial^2 u}{\partial x^2} \in L^p(\mathbb{R}; L^2([0,\pi]))$.

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Universidad de Chile, Facultad de Ciencias, Departamento de Matemática, Las Palmeras 3425, Santiago-Chile.

E-mail address: vpoblete@uchile.cl

Universidad de Talca, Instituto de Matemática y Física, Casilla 747, Talca-Chile.

E-mail address: rponce@inst-mat.utalca.cl