DISCRETIZATION OF C_0-SEMIGROUPS AND DISCRETE SEMIGROUPS OF OPERATORS IN BANACH SPACES.

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Abstract. In this paper we introduce the notion of a τ-discrete semigroup $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ generated by a closed linear operator A in a Banach space X. We show that $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ allows us to write the solution to an abstract discrete difference equation of first order as a discrete variation of parameters formula. Moreover, we study the main properties of $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ and its relation with the well-known notion of discrete semigroups. Finally, we characterize uniform exponential stability of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ in terms of the τ-discrete semigroup $\{T_\tau^n\}_{n\in\mathbb{N}_0}$.

1. Introduction

Let A be a closed linear operator on a Banach space X. If $x_0 \in X$ and A is the generator of the C_0-semigroup $\{T(t)\}_{t\geq 0}$, then it is well known that the mild solution to the equation

$$u'(t) = Au(t), \quad t > 0, \tag{1.1}$$

under the initial condition $u(0) = x_0$ is given by (see for instance [3, Proposition 3.1.16])

$$u(t) = T(t)x_0, \quad t \geq 0. \tag{1.2}$$

Now, let $\tau > 0$ be a time step-size. Since for each $x \in X$, the function $L : [0,\infty) \to X$, defined by $L(t) := T(t)x$ is continuous, then a natural way to approximate the function L on an interval $[0,T]$ is to define $L_\tau^n := L(t_n)$, where $t_n := \tau n$ and $1 \leq n \leq N$, and $T := \tau/N$ with $N \in \mathbb{N}$ being a fixed natural number. This means that L_τ^n approximates the C_0-semigroup and therefore it also approximates the solution $u(t)$ at t_n on the interval $[0,T]$, that is $L_\tau^n \simeq T(t_n)x_0 = u(t_n)$. As $T(t)T(s) = T(t+s)$ for all $t,s \geq 0$, then $\{L_\tau^n\}_{n\in\mathbb{N}_0}$ satisfies the semigroup law $L_\tau^{m+n} = L_\tau^n L_\tau^m$ for all $m,n \in \mathbb{N}$.

On the other hand, if we consider the same operator A and the abstract difference equation of first order

$$\nabla u^n = Au^n, \quad n \in \mathbb{N}, \tag{1.3}$$

under the initial condition $u^0 = x_0$, where ∇u^n is the backward Euler operator $\nabla u^n := \frac{u^n - u^{n-1}}{\tau}$, then (1.3) can be seen as a discretization of equation (1.1). We notice that if $\tau^{-1} \in \rho(A)$, then the solution to the difference equation (1.3) is given by

$$u^n = \tau^{-n}(\tau^{-1} - A)^{-n}x_0, \tag{1.4}$$

for all $n \in \mathbb{N}_0$. As $B_\tau := \tau^{-1}(\tau^{-1} - A)^{-1}$ is a bounded operator and $B_\tau^m B_\tau^n = B_\tau^{m+n}$ for all $m,n \in \mathbb{N}_0$, then $\{B_\tau^m\}_{m\in\mathbb{N}_0}$ is a discrete semigroup generated by B_τ, and the solution to (1.4) can be written as $u^n = B_\tau^n x_0$. The problem of the existence of solutions to equation (1.3) for bounded operators A has been studied by many authors, see for instance [5, 6, 7, 17, 19], and there is an extensive literature on the properties of discrete semigroups generated by bounded operators, see [4, 9, 10, 11, 14, 21, 23] for further details. For closed linear operators and related problems, we refer to the recent papers [2, 18, 22]. But, is there any connection between discrete semigroups and the discretization of semigroups? To the best

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of our knowledge, there are no works connecting discretization of C_0-semigroups and discrete semigroups generated by closed (possibly unbounded) linear operators A.

In this paper, we introduce the notion of τ-discrete semigroups $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ generated by a closed linear operator A in a Banach space X. We study its main properties and we prove that the inhomogeneous problem of first order

$$\nabla u^n = Au^n + f^n, \quad n \in \mathbb{N}, \tag{1.5}$$

under the initial condition $u^0 = x_0 \in X$, where $(f^n)_n$ is a given bounded sequence, can be written as a variation of parameters formula, similarly to the continuous case. Moreover, we prove that $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ is, in fact, a discrete semigroup generated by B_τ and we find a subordination relation between the C_0-semigroup and a τ-discrete semigroup generated by the same operator A. More precisely, we show that if A generates the C_0-semigroup $\{T(t)\}_{t\geq 0}$, then

$$T(t)x = \lim_{\tau\to 0^+} \tau \sum_{n=1}^{\infty} q_n^\tau(t) T_\tau^n x, \quad t \geq 0.$$

for all $x \in X$, where $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ is the τ-discrete semigroup defined by

$$T_\tau^0 x := x, \quad T_\tau^n x := \int_0^\infty q_n^\tau(t) T(t) x dt, \quad n \in \mathbb{N},$$

and $q_n^\tau(t) := e^{-t/\tau} \frac{(t/\tau)^{n-1}}{(n-1)!\tau}$, $n \in \mathbb{N}$. As a consequence, we give a characterization of uniform exponential stability of $\{T(t)\}_{t\geq 0}$ in terms of $\{T_\tau^n\}_{n\in\mathbb{N}_0}$.

The paper is organized as follows. In Section 2 we give the preliminaries. In Section 3 we introduce the τ-discrete semigroups $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ and we study its main properties. In Section 4, we study a connection between $\{T(t)\}_{t\geq 0}$ and $\{T_\tau^n\}_{n\in\mathbb{N}_0}$. Moreover, we study the existence and uniqueness of solution to the difference equation of first order (1.5), and we given necessary and sufficient conditions on $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ to obtain uniform exponential stability of $\{T(t)\}_{t\geq 0}$. Finally, in Section 5 we give some examples.

2. Preliminaries

We denote the set of non-negative integer numbers by $\mathbb{N}_0$. For a fixed $\tau > 0$ and $n \in \mathbb{N}$, we define the positive functions q_n^τ by

$$q_n^\tau(t) := e^{-\frac{t}{\tau}} \left(\frac{t}{\tau}\right)^{n-1} \frac{1}{\tau(n-1)!}.$$

Let $u : \mathbb{R}_0^+ \to X$ be a bounded and locally integrable function. Let us define the sequence $(u^n)_n$ as

$$u^n := \int_0^\infty q_n^\tau(t) u(t) dt, \quad n \in \mathbb{N}. \tag{2.6}$$

For $\tau > 0$ small enough, we notice that the function $t \mapsto q_n^\tau(t)$ behaves like a delta function at $t_n := n\tau$, and therefore, u^n approximates $u(t_n)$.

For a given Banach space X, $s(\mathbb{N}_0, X)$ denotes the vectorial space consisting of all vector-valued sequences $v : \mathbb{N}_0 \to X$. The *backward Euler operator* $\nabla_\tau : s(\mathbb{N}_0, X) \to s(\mathbb{N}_0, X)$ is defined by

$$\nabla_\tau v^n := \frac{v^n - v^{n-1}}{\tau}, \quad n \in \mathbb{N}.$$

For a given $\alpha > 0$, we define the following sequence

$$k_\tau^\alpha(n) = \frac{\tau^{\alpha-1}\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)}, \quad n \in \mathbb{N}_0. \tag{2.7}$$

In particular, we notice that $k_\tau^1(n) = 1$ for all $n \in \mathbb{N}_0$. Moreover, as in [22, Corollary 2.9] (see also [1, Proposition 3.1]) we have the following convolution property: If $\alpha, \beta > 0$, then

$$k_\tau^{\alpha+\beta}(n) = \tau \sum_{j=0}^{n} k_\tau^{\alpha}(n-j) k_\tau^{\beta}(j). \tag{2.8}$$

For a given discrete sequence of operators $\{S^n\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ and a scalar sequence $c = (c^n)_{n\in\mathbb{N}_0}$, we define the *discrete convolution* $c \star S$ as

$$(c \star S)^n := \sum_{k=0}^{n} c^{n-k} S^k, \quad n \in \mathbb{N}_0.$$

In particular, if $g_1(t) := 1$ for all $t \geq 0$, then as in [22, Corollary 2.9] we have:

$$\int_0^\infty q_n^\tau(t)(g_1 * S)(t)x dt = \tau \sum_{j=1}^{n} k_\tau^1(n-j) S^j x = \tau \sum_{j=1}^{n} S^j x, \quad x \in X, n \in \mathbb{N}, \tag{2.9}$$

where $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is a Laplace transformable family of linear operators, the $*$ denotes the usual continuous convolution, that is, $(g * S)(t) := \int_0^t g(t-r)S(r)dr$, and $S^j := \int_0^\infty q_j^\tau(t)S(t)dt$.

The Z-transform of a sequence $s \in s(\mathbb{N}_0, X)$ at $z \in \mathbb{C}$, is defined by $\tilde{s}(z) := \sum_{j=0}^\infty z^{-j} s^j$, where $s^j := s(j)$, and the convergence of this series holds for $|z| > R$, for R large enough. It is a well known fact that if $s_1, s_2 \in s(\mathbb{N}_0, X)$ and $\tilde{s_1}(z) = \tilde{s_2}(z)$ for all $|z| > R$ for some $R > 0$, then $s_1^j = s_2^j$ for all $j = 0, 1, ...$ Moreover, the Z-transform is a linear operator on $s(\mathbb{N}_0, X)$ and satisfies the finite discrete convolution property

$$\widetilde{s_1 \star s_2}(z) = \tilde{s_1}(z)\tilde{s_2}(z), \qquad s_1, s_2 \in s(\mathbb{N}_0, X). \tag{2.10}$$

We say that an operator $A : D(A) \subset X \to X$ is said to be *sectorial of angle* θ if there exist $M > 0$ and $\theta \in (\pi/2, \pi)$ such that $\rho(A) \supset \Sigma_\theta := \{z \in \mathbb{C} : |\arg(z)| < \theta\}$ and

$$\|(z - A)^{-1}\| \leq \frac{M}{|z|} \quad \text{for all} \quad z \in \Sigma_\theta.$$

In this case, we write $A \in \mathrm{Sect}(\theta, M)$. For further details on sectorial operators, see for instance [12].

For a given linear and closed operator A whose resolvent set contains the negative half-line $(-\infty, 0]$, (for example, if A is a sectorial operator) and $0 \leq \varepsilon \leq 1$, X^ε denotes the domain of the fractional power A^ε, that is $X^\varepsilon := D(A^\varepsilon)$ endowed with the graph norm $\|x\|_\varepsilon = \|A^\varepsilon x\|$. Hence, X^1 corresponds to the domain of A, and X^0 to the space X. It is a well known fact that if $0 < \varepsilon < 1$, and $x \in D(A)$, then there exists a constant $K \equiv K_\varepsilon > 0$ such that (see [20])

$$\|A^\varepsilon x\| \leq K \|Ax\|^\varepsilon \|x\|^{1-\varepsilon}. \tag{2.11}$$

Definition 2.1. *A family of linear operators $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be* exponentially bounded*, if there exist constants $M, \omega \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\omega t}$, for all $t \geq 0$.*

Proposition 2.2. *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be a family of exponentially bounded linear operators with $\|S(t)\| \leq Me^{\omega t}$, where $M > 0$ and $\omega < \frac{1}{\tau}$. Let $x \in X$. If we define the sequence $\{S^n x\}_{n\in\mathbb{N}_0}$ by*

$$S^0 x := x \quad \textit{and} \quad S^n x := \int_0^\infty q_n^\tau(t) S(t) x dt, \; n \in \mathbb{N},$$

then

$$\tilde{S}(z)x = \frac{1}{\tau z}\hat{S}\left(\frac{1}{\tau}\left(1 - \frac{1}{z}\right)\right)x + x,$$

for all $|z| > 1$ and $x \in X$.

Proof. The hypothesis implies that $\|S^n x\| \leq M \int_0^\infty q_n^\tau(t) e^{\omega t}\|x\| dt = \frac{M}{(1-\omega\tau)^n}\|x\|$ for all $n \in \mathbb{N}$. Therefore, the Z-transform of S exists for all $|z| > 1$. On the other hand, the hypothesis implies that the Laplace transform of S exists for all $\mathrm{Re}(\lambda) > 0$. Thus

$$\begin{aligned} \tilde{S}(z)x = \sum_{n=0}^{\infty} z^{-n} S^n x &= \int_0^\infty e^{-\frac{t}{\tau}} \sum_{n=1}^{\infty} z^{-n} \left(\frac{t}{\tau}\right)^{n-1} \frac{1}{\tau(n-1)!} S(t)x dt + x \\ &= \frac{1}{\tau z} \int_0^\infty e^{-\frac{t}{\tau}\left(1-\frac{1}{z}\right)} S(t)x dt + x \\ &= \frac{1}{\tau z} \hat{S}\left(\frac{1}{\tau}\left(1 - \frac{1}{z}\right)\right) x + x. \end{aligned}$$

□

3. τ-Discrete semigroups in Banach spaces

In this Section we introduce the notion of τ-discrete semigroup generated by a closed linear operator A in a Banach space and we study its main properties.

Definition 3.3. *Let A be a closed linear operator defined on a Banach space X. An operator-valued sequence $\{T_\tau^n\}_{n\in\mathbb{N}_0} \subset B(X)$ is called a τ-*discrete semigroup *generated by A if it satisfies the following conditions*

1. $T_\tau^0 = I$.
2. *$T_\tau^n \in D(A)$ for all $x \in X$ and $AT_\tau^n x = T_\tau^n Ax$ for all $x \in D(A)$, and $n \in \mathbb{N}_0$.*
3. *For each $x \in X$ and $n \in \mathbb{N}$,*

$$T_\tau^n x = x + \tau A \sum_{j=1}^{n} T_\tau^j x. \tag{3.12}$$

Moreover, we define the resolvent operator $R_\tau : X \to D(A)$ by

$$R_\tau := \tau^{-1}\left(\tau^{-1} - A\right)^{-1}.$$

Proposition 3.4. *Let $\{T_\tau^n\}_{n\in\mathbb{N}_0} \subset B(X)$ be a τ-discrete semigroup generated by A. Then,*

1. *$\tau^{-1} \in \rho(A)$, and*
2. *$T_\tau^n = R_\tau^n$ for all $n \in \mathbb{N}_0$.*

Proof. To prove (1), we notice that by (3.12), we get $T_\tau^1 x = x + \tau A T_\tau^1 x$ for all $x \in X$, which implies that $(\tau^{-1} - A)T_\tau^1 x = \tau^{-1}x$. Since A and T_τ^n commute, we obtain

$$T_\tau^1\left(\tau^{-1} - A\right)x = \tau^{-1}T_\tau^1 x - T_\tau^1 Ax = \left(\tau^{-1} - A\right)T_\tau^1 x = \tau^{-1}x,$$

for all $x \in X$. As A is a closed linear operator, we obtain that $\tau^{-1} \in \rho(A)$ and for each $x \in X$, $T_\tau^1 x = \tau^{-1}\left(\tau^{-1} - A\right)^{-1} x$. Moreover, this last equality proves (2) for $n = 1$. If we assume (2) for n, then, by definition, we get

$$T_\tau^{n+1}x = x + \tau A \sum_{j=1}^{n} T_\tau^j x + \tau A T_\tau^{n+1} x,$$

which implies that

$$\tau(\tau^{-1} - A)T_\tau^{n+1}x = x + A\sum_{j=1}^{n} T_\tau^j x = T_\tau^n x = R_\tau^n x,$$

and hence $T_\tau^{n+1}x = \tau^{-1}(\tau^{-1} - A)^{-1}R_\tau^n x = R_\tau^{n+1}x$ for all $x \in X$. □

Corollary 3.5. *If $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ is a τ-discrete semigroup generated by A, then $T_\tau^{n+m} = T_\tau^n T_\tau^m = T_\tau^m T_\tau^n$ for all $m, n \in \mathbb{N}_0$.*

Proof. In fact, by Proposition 3.4, $T_\tau^{m+n}x = R_\tau^{m+n}x = R_\tau^m R_\tau^n x = T_\tau^m T_\tau^n x$ and $T_\tau^m T_\tau^n x = T_\tau^n T_\tau^m x$ for all $m, n \in \mathbb{N}_0$ and $x \in X$. $\square$

We notice that A is the generator of a τ-discrete semigroup $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ in the sense of Definition 3.3 if and only if $T_\tau^n T_\tau^m x = T_\tau^{n+m}x$ for all $m, n \in \mathbb{N}_0$ and $x \in X$. In fact, if $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ is a τ-discrete semigroup generated by A, then

$$\left(\sum_{j=1}^m T_\tau^j\right) T_\tau^n x = \sum_{j=1}^m T_\tau^j x + \left(\tau A \sum_{j=1}^m T_\tau^j\right)\left(\sum_{j=1}^m T_\tau^j x\right) = \sum_{j=1}^m T_\tau^j x + T_\tau^m \sum_{j=1}^n T_\tau^j x - \sum_{j=1}^n T_\tau^j x,$$

that is,

$$\left(\sum_{j=1}^m T_\tau^j\right) T_\tau^n x - T_\tau^m \left(\sum_{j=1}^n T_\tau^j x\right) = \sum_{j=1}^m T_\tau^j x - \sum_{j=1}^n T_\tau^j x,$$

for all $x \in X$. As $T_\tau^0 = I$, we can write the last equality as

$$\left(\sum_{j=0}^m T_\tau^j\right) T_\tau^n x - T_\tau^n x - T_\tau^m \left(\sum_{j=0}^n T_\tau^j x\right) + T_\tau^m x = \sum_{j=0}^m T_\tau^j x - \sum_{j=0}^n T_\tau^j x.$$

By [4, Example 1.2.1], taking Z-transform (in m) we get

$$\frac{z}{z-1}\tilde{T}_\tau(z) T_\tau^n x - \frac{z}{z-1} T_\tau^n x - \tilde{T}_\tau(z)\left(\sum_{j=0}^n T_\tau^j x\right) + \tilde{T}_\tau(z)x = \frac{z}{z-1}\tilde{T}_\tau(z)x - \frac{z}{z-1}\sum_{j=0}^n T_\tau^j x,$$

and, next, taking Z-transform (in n) we obtain

$$\begin{aligned}\frac{z}{z-1}\tilde{T}_\tau(z)\tilde{T}_\tau(w)x - \frac{z}{z-1}\tilde{T}_\tau(w)x - \frac{w}{w-1}\tilde{T}_\tau(z)\tilde{T}_\tau(w)x + \frac{w}{w-1}\tilde{T}_\tau(z)x \quad &= \quad \frac{z}{z-1}\frac{w}{w-1}\tilde{T}_\tau(z)x \\ &\quad -\frac{z}{z-1}\frac{w}{w-1}\tilde{T}_\tau(w)x,\end{aligned}$$

which is equivalent to

$$(w-z)\tilde{T}_\tau(z)\tilde{T}_\tau(w)x = w\tilde{T}_\tau(z)x - z\tilde{T}_\tau(w)x,$$

for all $x \in X$. On the other hand, if we assume $T_\tau^{n+m}x = T_\tau^n T_\tau^m x$ for all $m, n \in \mathbb{N}_0$ and $x \in X$, then taking Z-transform in n we have

$$\sum_{j=0}^\infty z^{-j} T_\tau^{j+m} x = \tilde{T}_\tau(z) T_\tau^m x.$$

On the other hand, we notice that

$$\sum_{j=0}^\infty z^{-j} T_\tau^{j+m} x = z^m \tilde{T}_\tau(z)x - \sum_{r=0}^{m-1} T_\tau^r z^{m-r} x = z^m \tilde{T}_\tau(z)x - \sum_{r=0}^{m} T_\tau^r z^{m-r} x + T_\tau^m x,$$

and taking Z-transform in m, we get by [4, Example 1.2.1]

$$\frac{w}{w-z}\tilde{T}_\tau(z)x - \frac{w}{w-z}\tilde{T}_\tau(w)x + \tilde{T}_\tau(w)x = \tilde{T}_\tau(z)\tilde{T}_\tau(w)x,$$

that is,

$$w\tilde{T}_\tau(z)x - z\tilde{T}_\tau(w)x = (w-z)\tilde{T}_\tau(z)\tilde{T}_\tau(w)x.$$

Therefore, we obtain that A is the generator of a τ-discrete semigroup $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ according to Definition 3.3 if and only if $T_\tau^n T_\tau^m x = T_\tau^{n+m}x$ for all $m, n \in \mathbb{N}_0$ and $x \in X$.

The next result shows that there exists a unique τ-discrete semigroup generated by an operator A.

Proposition 3.6. *Assume that $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ and $\{Q_\tau^n\}_{n\in\mathbb{N}_0}$ are τ-discrete semigroups generated by A. Then $T_\tau^n = Q_\tau^n$ for all $n \in \mathbb{N}_0$.*

Proof. Let $x \in X$. Define $h(n) := T_\tau^n x - Q_\tau^n x$, $n \in \mathbb{N}_0$. Then, $T_\tau^0 x = Q_\tau^0 x = I$, and by Proposition 3.4 we have $T_\tau^1 x = Q_\tau^1 x = R_\tau$, and thus $h(0) = h(1) = 0$. Now, by Definition 3.3, we have

$$h(n) = \tau A \sum_{j=1}^{n} h(j), \quad n \geq 2,$$

which implies that $(I - \tau A)h(n) = \tau A \sum_{j=1}^{n-1} h(j)$. As $\tau^{-1} \in \rho(A)$ (by Proposition 3.4), we get $h(n) = 0$ for all $n \geq 2$. Therefore $T_\tau^n x = Q_\tau^n x$ for all $n \in \mathbb{N}_0$ and $x \in X$. □

Proposition 3.7. *Let $\{T_\tau^n\}_{n\in\mathbb{N}_0} \subset B(X)$ be a τ-discrete semigroup generated by A. Then, its Z-transform satisfies*

$$\widetilde{T}_\tau(z)x = \frac{1}{\tau}(I - \tau A)\left(\left(\frac{z-1}{\tau z}\right) - A\right)^{-1} x, \quad \text{for all } x \in X. \tag{3.13}$$

Proof. By Proposition 3.4, $T_\tau^n = R_\tau^n$ for all $n \in \mathbb{N}_0$. Since $R_\tau = \tau^{-1}(\tau^{-1} - A)^{-1}$ is a bounded operator, we get (by [16, Table 18.4]) for all $x \in X$ and $|z| > \|R_\tau\|$ that

$$\widetilde{T}_\tau(z)x = \sum_{n=0}^{\infty} z^{-n} T_\tau^n x = \sum_{n=0}^{\infty} z^{-n} R_\tau^n x = z(z - \tau^{-1}(\tau^{-1} - A)^{-1})^{-1} x = \frac{1}{\tau}(I - \tau A)\left(\left(\frac{z-1}{\tau z}\right) - A\right)^{-1} x.$$

□

Proposition 3.8. *Let $\tau > 0$. If A is a bounded operator with $\tau\|A\| < 1$, then A generates the τ-discrete semigroup $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ defined by*

$$T_\tau^0 := I, \quad \text{and for } n \geq 1, \quad T_\tau^n := \sum_{j=0}^{\infty} \tau^j \frac{\Gamma(j+n)}{\Gamma(j+1)\Gamma(n)} A^j = \sum_{j=0}^{\infty} \tau^j \binom{j+n-1}{j} A^j. \tag{3.14}$$

Proof. Let $x \in X$ and $n \in \mathbb{N}$. We first notice that

$$\tau^j \frac{\Gamma(j+n)}{\Gamma(j+1)\Gamma(n)} = k_\tau^{j+1}(n-1), \quad n \in \mathbb{N}.$$

Since the series in (3.14) converges for $\tau\|A\| < 1$ (see [16, Formula 8.328]) we get by (2.8) that

$$\tau A \sum_{j=1}^{n} T_\tau^j x = \sum_{l=0}^{\infty} A^{l+1} \tau \sum_{j=1}^{n} k_\tau^{l+1}(j-1)x = \sum_{l=0}^{\infty} A^{l+1} \tau \sum_{m=0}^{n-1} k_\tau^{l+1}(m)x = \sum_{l=0}^{\infty} A^{l+1} k_\tau^{l+2}(n-1)x.$$

As $k_\tau^1(m) = 1$ for all $m \in \mathbb{N}$, we get

$$\tau A \sum_{j=1}^{n} T_\tau^j x = \sum_{j=0}^{\infty} k_\tau^{j+1}(n-1)A^j x - k_\tau^1(n-1)x = \sum_{j=0}^{\infty} k_\tau^{j+1}(n-1)A^j x - x = T_\tau^n x - x.$$

□

Next, we recall that for a given bounded operator $L \in \mathcal{B}(X)$, the sequence $\mathcal{L} : \mathbb{N}_0 \to \mathcal{B}(X)$ given by $\mathcal{L}(n) := L^n$ defines a time-discrete semigroup generated by $L - I$, which satisfies $\mathcal{L}(n+m) = \mathcal{L}(n)\mathcal{L}(m)$ for all $n, m \in \mathbb{N}_0$ and $\mathcal{L}(0) = I$, see [4] for more details. Sometimes the operator L is also called the generator of the discrete semigroup $\{\mathcal{L}(n)\}_{n\in\mathbb{N}_0}$, see for instance [14].

The next result relates τ-discrete semigroups and discrete semigroups in a Banach space.

Proposition 3.9. *Let $\tau > 0$. Let $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ be a τ-discrete semigroup generated by a closed linear operator A. Then, $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ is a discrete semigroup generated by $L := \tau^{-1}(\tau^{-1} - A)^{-1} = R_\tau$.*

Proof. We need to prove that $L^mL^n = L^{m+n}$ for all $n, m \in \mathbb{N}_0$. In fact, by Proposition 3.4, $L^m = T_\tau^m$ and $L^{m+n} = T_\tau^{m+n}$, and by Corollary 3.5 we get $L^mL^n = T_\tau^m T_\tau^n = T_\tau^{m+n} = L^{m+n}$. Moreover, clearly $L^0 = I$.

□

Remark 3.10. *We notice that if $L \in \mathcal{B}(X)$ is an invertible operator and $A := \frac{1}{\tau}(I - L^{-1})$, then the semigroup $T(t) := e^{tA}$ satisfies $L^n = \int_0^\infty q_n^\tau(t)T(t)dt$ for all $n \in \mathbb{N}$, where L^n is the n-power of L.*

4. C_0-semigroups and τ-discrete semigroups

In this section we study the relation of C_0-semigroups and τ-discrete semigroups generated by a closed linear operator A.

Proposition 4.11. *Let A be the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$. Then, A generates a τ-discrete semigroup $\{T_\tau^n\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$.*

Proof. Since A generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$, then (see [3, Proposition 3.1.9])

$$T(t)x = x + A\int_0^t T(s)xds = x + A(g_1 * T)(t)x, \quad x \in X, t \geq 0, \tag{4.15}$$

where $g_1(t) := 1$, for all $t \geq 0$. For each $x \in X$, we define $T_\tau^n x$, $n \in \mathbb{N}_0$, by $T_\tau^0 x := x$ and $T_\tau^n x := \int_0^\infty q_n^\tau(t)T(t)xdt$, for $n \in \mathbb{N}$. Multiplying (4.15) by $q_n^\tau(t)$ and integrating over $[0, \infty)$ we conclude by (2.9) that

$$T_\tau^n x = x + A\int_0^\infty q_n^\tau(t)(g_1 * T)(t)xdt = x + \tau A\sum_{j=1}^n T_\tau^j x.$$

Finally, as $T(t)Ax = AT(t)x$ for all $x \in D(A)$, (see [3, Proposition 3.1.9]) multiplying this equality by $q_n^\tau(t)$ and integrating over $[0, \infty)$, we get $T_\tau^n Ax = AT_\tau^n x$ for all $n \in \mathbb{N}_0$ and $x \in D(A)$. □

From the theory of C_0-semigroups, it is well known that, if A is the generator of C_0-semigroup in a Banach space X, $x_0 \in X$, and $f : [0, \infty) \to X$ belongs to $L^1([0, \infty), X)$, then the mild solution to the abstract Cauchy problem

$$\begin{cases} u'(t) &= Au(t) + f(t), \quad t > 0 \\ u(0) &= x_0, \end{cases} \tag{4.16}$$

is given by the variation of parameters formula

$$u(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds, \quad t \geq 0,$$

see for instance [3, Proposition 3.1.16] for further details.

In the next result we show the existence and uniqueness of the abstract discrete difference equation of first order, and we show that its solution can be written as a discrete variation of parameters formula, similarly to the continuous case.

Theorem 4.12. *Let $\tau > 0$. Let A be the generator of a τ-discrete semigroup $\{T_\tau^n\}_{n\in\mathbb{N}_0}$. If $x_0 \in X$ and $(f^n)_n$ is a bounded vector-valued sequence, then the discrete difference equation of first order*

$$\begin{cases} \nabla_\tau u^n &= Au^n + f^n, \quad n \geq 1 \\ u^0 &= x_0, \end{cases} \tag{4.17}$$

has a unique solution given by the sequence $(v^n)_{n\in\mathbb{N}_0}$, where

$$v^n := T_\tau^n x_0 + \tau\sum_{j=1}^n T_\tau^{n+1-j}f^j, \tag{4.18}$$

for all $n \geq 1$, and $v^0 := x_0$.

Proof. Since A generates the τ-discrete semigroup $\{T_\tau^n\}_{n\in\mathbb{N}_0}$, we have, by Proposition 3.4 that $T_\tau^n = R_\tau^n$ for all $n\in\mathbb{N}$. As $\tau^{-1}\in\rho(A)$ (see Proposition 3.4), the equation (4.17) is equivalent to

$$\tau(\tau^{-1}-A)u^n = u^{n-1}+\tau f^n, \quad n\in\mathbb{N},$$

that is, $u^n = R_\tau u^{n-1}+\tau R_\tau f^n$, for any $n\in\mathbb{N}$. The initial condition implies

$$u^n = R_\tau^n x_0 + \tau\sum_{j=1}^{n} R_\tau^{n+1-j} f^j, \quad n\in\mathbb{N},$$

which means that, $u^n = T_\tau^n x_0 + \tau\sum_{j=1}^n T_\tau^{n+1-j} f^j$, for all $n\in\mathbb{N}$. Therefore, the sequence $(v^n)_{n\in\mathbb{N}_0}$ defined by

$$v^n := \begin{cases} T_\tau^n x_0 + \tau\sum_{j=1}^n T_\tau^{n+1-j} f^j, & n\geq 1 \\ x_0, & n=0 \end{cases}$$

solves (4.17). The uniqueness follows from Proposition 3.6. □

The next theorem gives a subordination relation between the C_0-semigroup $\{T(t)\}_{t\geq 0}$ and a τ-discrete semigroup $\{T_\tau^n\}_{n\geq 0}$. For a related result, we refer to [13, Theorem 2] and [15].

Theorem 4.13. *Let A be the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$. For each $x\in X$, define $T_\tau^n x$ by*

$$T_\tau^0 x := x, \quad T_\tau^n x := \int_0^\infty q_n^\tau(t) T(t) x\, dt, \quad n\in\mathbb{N}.$$

Then,

$$T(t) = \lim_{\tau\to 0^+} \tau\sum_{n=1}^\infty q_n^\tau(t) T_\tau^n, \tag{4.19}$$

uniformly in t in compact subsets of $\mathbb{R}_+$.

Proof. We first notice that by Proposition 4.11 the sequence $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ is a τ-discrete semigroup generated by A. Let $x\in X$. By the inversion theorem for the Z-transform (see [4, Chapter 1]) and Proposition 2.2 we have

$$T_\tau^n x = \frac{1}{2\pi i}\int_\Gamma \tilde{T}(z) z^{n-1} x\, dz = \frac{1}{2\pi i}\int_\Gamma \left[\frac{1}{\tau}\hat{T}\left(\frac{1}{\tau}\left(1-\frac{1}{z}\right)\right)+I\right] z^{n-1} x\, dz,$$

where Γ is a simple closed contour containing the origin that encloses all the poles of $\tilde{T}(z)z^{n-1}$. We recall here that $\tilde{T}(z)$ denotes the Z-transform of the sequence $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ at z, and $\hat{T}$ denotes the Laplace transform of $t\mapsto T(t)$. Now, we introduce the change of variable $\lambda = \frac{1}{\tau}\left(1-\frac{1}{z}\right)$. Then,

$$T_\tau^n x = \frac{1}{2\pi i}\int_{\tilde{\Gamma}} \frac{1}{(1-\tau\lambda)^n}\hat{T}(\lambda) x\, d\lambda + \frac{1}{2\pi i}\int_{\tilde{\Gamma}} z^{n-1} x\, d\lambda,$$

where $\tilde{\Gamma}$ is the resulting path under this change of variable. Multiplying the last identity by $q_n^\tau(t)$ and summing up in $n\in\mathbb{N}$, we get

$$\tau\sum_{n=1}^\infty q_n^\tau(t) T_\tau^n x = \frac{1}{2\pi i}\int_{\tilde{\Gamma}} \tau\left(\sum_{n=1}^\infty \frac{1}{(1-\tau\lambda)^n} q_n^\tau(t)\right)\hat{T}(\lambda) x\, d\lambda + \frac{1}{2\pi i}\int_\Gamma \tau\sum_{n=1}^\infty z^{n-1} q_n^\tau(t) x\, d\lambda,$$

Now, we notice that $\tau\sum_{n=1}^\infty z^{n-1}q_n^\tau(t) = e^{-\frac{t}{\tau}}\sum_{n=1}^\infty \frac{\left(\frac{zt}{\tau}\right)^{n-1}}{(n-1)!} = e^{-\frac{t}{\tau}(1-z)}$, and $\lim_{\tau\to 0^+}\tau\sum_{n=1}^\infty z^{n-1}q_n^\tau(t) = 0$. For $E_\tau(t) := \tau\left(\sum_{n=1}^\infty \frac{1}{(1-\tau\lambda)^n} q_n^\tau(t)\right)$ where $t\geq 0$, it is easy to see that

$$\lim_{\tau\to 0^+} E_\tau(t) = e^{\lambda t}, \quad t\geq 0. \tag{4.20}$$

Therefore, by the inversion theorem for the Laplace transform, we obtain

$$\begin{aligned}\lim_{\tau\to 0^+}\left\|T(t)x-\tau\sum_{n=1}^{\infty}q_n^{\tau}(t)T_\tau^n x\right\| &= \lim_{\tau\to 0^+}\left\|\frac{1}{2\pi i}\left(\int_{\tilde\Gamma}e^{\lambda t}\hat T(\lambda)xd\lambda-\int_{\tilde\Gamma}E_\tau(t)\hat T(\lambda)xd\lambda-\int_{\Gamma}e^{-\frac{t}{\tau}(1-z)}xd\lambda\right)\right\| \\ &\leq \lim_{\tau\to 0^+}\frac{1}{2\pi}\int_{\tilde\Gamma}|e^{\lambda t}-E_\tau(t)|\|\hat T(\lambda)x\||d\lambda| + \lim_{\tau\to 0^+}\frac{1}{2\pi}\int_{\Gamma}|e^{-\frac{t}{\tau}(1-z)}|\|x\||d\lambda| \\ &= 0,\end{aligned}$$

for all $t\geq 0$ and $x\in X$. The proof is finished. □

The next Corollary gives a different proof to the Yosida's approximation theorem for C_0-semigroups.

Corollary 4.14 (Yosida's approximation)**.** *Let A be the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ with $D(A)$ dense in X. Let $T_\tau^n x$ defined as in Theorem 4.13. Then,*

$$T(t)x=\lim_{m\to\infty}e^{mtA(m-A)^{-1}}x.$$

for all $x\in X$, uniformly in t in compact subsets of $\mathbb{R}_+$.

Proof. By Proposition 3.4, $T_\tau^n=R_\tau^n$ for all $n\in\mathbb{N}_0$. Since $A(\tau^{-1}-A)^{-1}=R_\tau-I$ and R_τ is bounded, $A(\tau^{-1}-A)^{-1}$ is bounded, and therefore, if $\tau=1/m$ in Theorem 4.13, we obtain for $x\in D(A)$ that

$$\begin{aligned}T(t)x &= \lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{\infty}q_n^{\tau}(t)\left(m(m-A)^{-1}\right)^n x \\ &= \lim_{m\to\infty}e^{-tm}m(m-A)^{-1}e^{tmR_{\frac{1}{m}}}x \\ &= \lim_{m\to\infty}e^{tmA(m-A)^{-1}}A(m-A)^{-1}x+\lim_{m\to\infty}e^{tmA(m-A)^{-1}}x.\end{aligned}$$

Since $\lim_{m\to\infty}e^{tmA(m-A)^{-1}}A(m-A)^{-1}x=0$, for all $x\in D(A)$ (see for instance [12, Lemma 3.4]) and $\overline{D(A)}=X$, we conclude that $T(t)x=\lim_{m\to\infty}e^{tmA(m-A)^{-1}}x$, for all $x\in X$. □

We recall that a C_0-semigroup $\{T(t)\}_{t\geq 0}$ is called *uniformly exponentially stable* if there exist $M,\omega>0$ such that $\|T(t)\|\leq Me^{-\omega t}$ for all $t\geq 0$. The next result gives a characterization of uniform exponential stability of the C_0-semigroup $\{T(t)\}_{t\geq 0}$ in terms of the τ-discrete semigroup $\{T_\tau^n\}_{n\in\mathbb{N}_0}$.

Proposition 4.15. *Let A be the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$. Let $T_\tau^n x$ defined as in Theorem 4.13. Then, $\{T(t)\}_{t\geq 0}$ is uniformly exponentially stable with $\|T(t)\|\leq Me^{-\omega t}$, where $\omega,M>0$, if and only if*

$$\|T_\tau^n\|\leq\frac{M}{(1+\omega\tau)^n},\qquad \text{for all } n\in\mathbb{N}_0,\tau>0. \tag{4.21}$$

Proof. If $\|T(t)\|\leq Me^{-\omega t}$, then $\|T_\tau^n\|\leq\frac{M}{(1+\omega\tau)^n}$ for all $n\in\mathbb{N}_0$ as in the proof of [18, Proposition 3.7]. Conversely, assume that (4.21) holds. Let $x\in X$. By Theorem 4.13 we obtain for all $x\in X$ that

$$\|T(t)x\|\leq M\lim_{\tau\to 0^+}\tau\sum_{n=1}^{\infty}e^{-\frac{t}{\tau}}\left(\frac{t}{\tau}\right)^{n-1}\frac{1}{(n-1)!\tau}\frac{1}{(1+\omega\tau)^n}\|x\|=M\lim_{\tau\to 0^+}\frac{e^{-\frac{t\omega}{1+\omega\tau}}}{1+\omega\tau}\|x\|=Me^{-\omega t}\|x\|.$$

□

For a closed operator $A\in\operatorname{Sec}(\theta,M)$, we will consider the following path Γ_t : For $\frac{\pi}{2}<\theta<\pi$, we take ϕ such that $\frac{1}{2}\phi<\frac{\pi}{2}<\phi<\theta$. Next, we define $\Gamma\equiv\Gamma_t$ as the union $\Gamma_t^1\cup\Gamma_t^2$, where

$$\Gamma_t^1:=\left\{\frac{1}{t}e^{i\psi}:-\phi<\psi<\phi\right\}\quad\text{and}\quad\Gamma_t^2:=\left\{re^{\pm i\phi}:\frac{1}{t}\leq r\right\}.$$

The next result follow from [8, Lemma 2].

Lemma 4.16. *Let $A \in \mathrm{Sec}(\theta, M)$ and Γ be the complex path defined above. If $\mu \geq 0$, then there exists positive constant C, such that*

$$\int_\Gamma \left| \frac{e^{zt}}{z^\mu} \right| |dz| \leq C t^{\mu-1}$$

for all $t > 0$, where $C := \left(2\phi \int_{-\phi}^{\phi} e^{\cos(\psi)} d\psi + \frac{2}{-\cos(\phi)} \right)$.

The next theorem relates a τ-discrete semigroups and C_0-semigroup at $t_n = \tau n$ for all $n \in \mathbb{N}$.

Theorem 4.17. *Let $0 < \varepsilon < 1$. Suppose that $x \in D(A^\varepsilon)$. Let A be a sectorial operator which generates the C_0-semigroup $\{T(t)\}_{t \geq 0}$. Let $T_\tau^n x$ defined as in Theorem 4.13 and Γ be the complex path defined above. Then, for each $L > 0$ there exists a constant $D > 0$ such that, for $0 < t_n \leq L$,*

$$\|T(t_n)x - T_\tau^n x\| \leq D\tau t_n^{\varepsilon-1} \|x\|_\varepsilon. \tag{4.22}$$

Proof. Take a fixed $n \in \mathbb{N}$ such that $0 < t_n \leq L$, where $t_n := \tau n$. Then, we can write

$$T(t_n)x - T_\tau^n x = \int_0^\infty q_n^\tau(t)[T(t_n) - T(t)]x dt,$$

for all $x \in X$, and by the inversion of the Laplace transform, we get

$$T(t_n)x - T(t)x = \frac{1}{2\pi i} \int_\Gamma \frac{e^{zt_n} - e^{zt}}{z} z(z-A)^{-1} x dz.$$

As $z(z-A)^{-1} = A(z-A)^{-1} + I = A^{1-\varepsilon}(z-A)^{-1}A^\varepsilon + I$, we obtain

$$T(t_n)x - T(t)x = \frac{1}{2\pi i} \int_\Gamma \frac{e^{zt} - e^{zt_n}}{z} x dz + \frac{1}{2\pi i} \int_\Gamma \frac{e^{zt} - e^{zt_n}}{z} A^{1-\varepsilon}(z-A)^{-1} A^\varepsilon x dz.$$

It is easy to see that the first integral in the last identity is equal to zero. Since A is a sectorial operator, we have by (2.11) that $\|A^{1-\varepsilon}(z-A)^{-1}x\| \leq K(M+1)^{1-\varepsilon} \frac{\|x\|}{|z|^\varepsilon}$, for all $x \in X$ and therefore

$$\|T(t_n)x - T(t)x\| \leq \frac{K(M+1)^{1-\varepsilon}}{2\pi} \int_\Gamma \frac{|e^{zt} - e^{zt_n}|}{|z|} \frac{\|A^\varepsilon x\|}{|z|^\varepsilon} |dz|.$$

The mean value for complex-valued functions implies the existence of t_0, t_1 with $0 < t_n < t_0 < t_1 < t$ such that

$$\frac{|e^{zt} - e^{zt_n}|}{|z|} \leq (t - t_n) \left(|e^{t_0 z}| + |e^{t_1 z}| \right).$$

As $0 < \varepsilon < 1$ and $t_n < t_0 < t_1$ we have $t_1^{\varepsilon-1} < t_0^{\varepsilon-1} < t_n^{\varepsilon-1}$, and by Lemma 4.16 we get

$$\|T(t_n)x - T(t)x\| \leq \frac{K(M+1)^{1-\varepsilon}C}{\pi} (t - t_n) t_n^{\varepsilon-1} \|A^\varepsilon x\|.$$

It is easy to see that $\int_0^\infty q_n^\tau(t)(t - t_n)dt = \tau$, and therefore, we conclude that

$$\|T(t_n)x - T_\tau^n x\| \leq \int_0^\infty q_n^\tau(t) \|T(t_n)x - T(t)x\| dt \leq \frac{K(M+1)^{1-\varepsilon}C}{\pi} \tau t_n^{\varepsilon-1} \|A^\varepsilon x\| = D\tau t_n^{\varepsilon-1} \|A^\varepsilon x\|,$$

where $D = \frac{KC(M+1)^{1-\varepsilon}}{\pi}$.

□

Corollary 4.18. *Under the assumption of Theorem 4.17, we have*

$$\lim_{\tau \to 0^+} \|T(t_n)x - T_\tau^n x\| = 0.$$

5. Examples

In this section we give examples of τ-discrete semigroups generated by closed linear operators in some Banach spaces.

Example 5.19.

On $X = L^2(0,\pi)$ let $T(t)$ be defined by

$$T(t)(x(s)) := \sum_{k=0}^{\infty} a_k(x) e^{-k^2 t} \sin(ks), \quad x \in X,$$

where $a_k(x)$ are the Fourier coefficients of x, that is, $a_k(x) = \sqrt{\frac{2}{\pi}} \int_0^{\pi} x(r)\sin(kr)dr, x \in X$. Then, $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup of contractions on X whose generator is $Au = u''$ with domain $D(A) = \{x \in X : x \in H^2(0,\pi) \cap H_0^1(0,\pi)\}$. See for instance, [12, Section 2, Chapter II]. Then, for each $n \in \mathbb{N}$,

$$T_\tau^n(x(s)) = \int_0^{\infty} q_n^\tau(t) T(t)(x(s)) dt = \sum_{k=0}^{\infty} a_k(x)\sin(ks) \int_0^{\infty} q_n^\tau(t) e^{-k^2 t} dt = \sum_{k=0}^{\infty} \frac{1}{(1+k^2\tau)^n} a_k(x)\sin(ks).$$

Therefore,

$$T_\tau^n(x(s)) = \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \frac{1}{(1+k^2\tau)^n} \sin(ks) \int_0^{\pi} x(r)\sin(kr)dr, \quad x \in L^2(0,\pi), n \in \mathbb{N}.$$

By Proposition 4.11, $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ is a τ-discrete semigroup generated by A, and by Theorem 4.12, the unique solution to the semidiscrete problem

$$\nabla_\tau u^n(s) \quad = \quad Au^n(s) + f^n(s), \quad n \geq 1, s \in (0,\pi),$$

where $f^n \in L^2(0,\pi)$, under the initial condition $u^0 = x_0(s), x_0 \in L^2(0,\pi)$ is given by $v^n(s) = T_\tau^n(x_0(s)) + \tau \sum_{j=1}^{n} T_\tau^{n+1-j}(f^j(s))$, $n \geq 1$ and $v^0(s) = x_0(s)$, $s \in (0,\pi)$.

Example 5.20.

Let X be one of the spaces $L^p(\mathbb{R})$ (with $1 \leq p < \infty$), $C_0(\mathbb{R})$ or $BUC(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : f \text{ is uniformly continuous}\}$. For each $f \in X$, we define

$$(T(t)f)(s) := \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-r)^2}{4t}} f(r)dr, \quad s \in \mathbb{R}.$$

Then, $T(t)$ is a C_0-semigroup generated by $A = \frac{\partial^2}{\partial x^2}$, with domain $D(A) = \{f \in X : \frac{\partial^2 f}{\partial x^2} \in X\}$, see for instance [3, Example 3.7.6]. Now, we calculate T^n. By definition and Fubini's theorem we have

$$T_\tau^n(f(s)) = \int_0^{\infty} q_n^\tau(t)(T(t)f)(s)dt = \frac{1}{\sqrt{4\pi}} \frac{1}{\tau^n(n-1)!} \int_{-\infty}^{\infty}\int_0^{\infty} e^{-\frac{1}{\tau}t - \frac{(s-r)^2}{4}\frac{1}{t}} t^{(n-\frac{1}{2})-1} dt f(r)dr, \quad n \in \mathbb{N}.$$

By [16, Formula 9, p. 368] we have

$$\int_0^{\infty} e^{-\frac{1}{\tau}t - \frac{(s-r)^2}{4}\frac{1}{t}} t^{(n-\frac{1}{2})-1} dt = 2^{-n+\frac{1}{2}+1} \tau^{\frac{n}{2}-\frac{1}{4}} |s-r|^{n-\frac{1}{2}} K_{n-\frac{1}{2}}\left(\frac{|s-r|}{\tau}\right),$$

where $K_{n-\frac{1}{2}}$ is the modified Bessel function of second kind. By [16, Formula 8.468, p. 925] we can write

$$K_{n-\frac{1}{2}}\left(\frac{|s-r|}{\tau}\right) = \sqrt{\pi}\tau^{\frac{1}{2}} e^{-\frac{|s-r|}{\tau}} \sum_{k=0}^{n-1} \frac{(n-1+k)!}{k!(n-1-k)!} 2^{-k-\frac{1}{2}} |s-r|^{-k-\frac{1}{2}} \tau^k,$$

and, therefore

$$T_\tau^n(f(s)) = \frac{1}{2^{n-\frac{1}{2}}} \tau^{\frac{n}{2}-\frac{1}{2}} \sum_{k=0}^{n-1} \frac{1}{2^k} \binom{n-1+k}{k} \int_{-\infty}^{\infty} q_{n-k}^\tau(|s-r|) f(r)dr, \quad f \in X, s \in \mathbb{R}.$$

We conclude by Proposition 4.11 that $\{T_\tau^n\}_{n\in\mathbb{N}_0}$ is a τ-discrete semigroup generated by A, and thus, $T_\tau^{n+m} = T_\tau^n T_\tau^m$ for all $m, n \in \mathbb{N}_0$.

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