APPROXIMATE CONTROLLABILITY OF ABSTRACT DISCRETE FRACTIONAL SYSTEMS OF ORDER $1 < \alpha < 2$ VIA RESOLVENT SEQUENCES.

RODRIGO PONCE

Abstract. We study the approximate controllability of the discrete fractional systems of order $1 < \alpha < 2$

$$(*) \quad {}_C\nabla^\alpha u^n = Au^n + Bv^n + f(n, u^n), \quad n \geq 2,$$

subject to the initial states $u^0 = x_0, u^1 = x_1$, where A is a closed linear operator defined on a Hilbert space X, B is a bounded linear operator from a Hilbert space U into X, $f : \mathbb{N}_0 \times X \to X$ is a given sequence and ${}_C\nabla^\alpha u^n$ is an approximation of the Caputo fractional derivative ∂_t^α of u at $t_n := \tau n$, where $\tau > 0$ is a given step size.

To do this, we first study resolvent sequences $\{S^n_{\alpha,\beta}\}_{n\in\mathbb{N}_0}$ generated by closed linear operators to obtain some subordination results. In addition, we discuss the existence of solutions to $(*)$ and next, we study the existence of optimal controls to obtain the approximate controllability of the discrete fractional system $(*)$ in terms of the resolvent sequence $\{S^n_{\alpha,\beta}\}_{n\in\mathbb{N}_0}$ for some $\alpha, \beta > 0$. Finally, we provide an example of a discrete fractional system to illustrate our results.

1. Introduction

Let $1 < \alpha < 2$. In recent years there has been increasing interest in the study of the controllability of discrete fractional systems in the form

$$\partial_t^\alpha u = Au + Bv + F, \quad t > 0, \tag{1.1}$$

subject to the initial conditions $u(0) = x_0, u'(0) = x_1$, where ∂_t^α denotes a fractional derivative (typically, in the sense of Caputo or Riemann-Liouville), $A : D(A) \subset X \to X$ is closed linear operator defined in a Hilbert or Banach space X, $B : U \subset X \to X$ is a bounded linear operator, v denotes the control of the system, F is a given linear or nonlinear function and $x_0, x_1 \in X$.

We observe that if $\alpha \to 1$, then $\partial_t^\alpha u$ corresponds to u' and the system (1.1) is transformed into the first-order problem

$$u'(t) = Au(t) + Bv(t) + F(t), \quad t > 0, u(0) = x_0. \tag{1.2}$$

In this case, the C_0-semigroup $\{T(t)\}_{t\geq 0}$ generated by A turns out to be a great tool for the study of the controllability of abstract systems in the form of (1.2) ([41, Chapter 11]), because the mild solution to (1.2) can be written in terms of $\{T(t)\}_{t\geq 0}$ as

$$u(t) = T(t)x_0 + \int_0^t T(t-s)[Bv(s) + F(s)]ds,$$

see for instance the monograph [41] for a detailed discussion on the controllability of this type of problems from different perspectives and tools.

Now, if $\alpha \to 2$, then $\partial_t^\alpha u$ is precisely the second derivative of u and the system (1.1) becomes

$$u''(t) = Au(t) + Bv(t) + F(t), \quad t > 0, \, u(0) = x_0, \, u'(0) = x_1. \tag{1.3}$$

2020 *Mathematics Subject Classification.* Primary 93C55, Secondary 39A12, 49M25, 26A33, 47D06.

Key words and phrases. fractional discrete systems, unbounded linear operators, cosine families, approximate controllability.

To study the approximate controllability of this second-order problem, a useful approach is the theory of cosine families generated by A ([40]). In this case, if A is the generator of the cosine familiy $\{C(t)\}_{t\in\mathbb{R}}$, then the mild solution to (1.3) is given by

$$u(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)[Bv(s) + F(s)]ds,$$

where $S(t) := \int_0^t C(s)ds$ is the corresponding sine family. The recent literature on the approximate controllability of second-order in infinite dimensional spaces is extensive. Just to mention a few: in the paper [21] the authors study approximate controllability of second-order implicit functional systems; the authors in [30] use the theory of cosine families to study the approximate controllability of stochastic systems of second order, and in [31] is studied the approximate controllability of differential inclusions via cosine families of operators.

More recently, the study of controllability of fractional system in the form of (1.1) has been widely studied. For $0 < \alpha < 1$, the authors in the paper [34] study the exact and approximate controllability of (1.1) in Banach spaces. In [38] is studied the approximate controllability of semilinear fractional differential systems. In these papers, the main hypothesis is that operator A generates a fractional resolvent of operators, that is, a family $\{S_{\alpha,1}(t)\}_{t\geq 0}$ of linear operators in X whose Laplace transform verifies $\widehat{S}_{\alpha,1}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}$ for all $\lambda^\alpha \in \rho(A)$. If A generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ and $0 < \alpha < 1$, the existence of $\{S_{\alpha,1}(t)\}_{t\geq 0}$ can be deduced thanks to a subordination result ([8]). In addition, the mild solution to the fractional system (1.1) can be written (for $0 < \alpha < 1$) in terms of $\{S_{\alpha,1}(t)\}_{t\geq 0}$ as

$$u(t) = S_{\alpha,1}(t)x_0 + \int_0^t S_{\alpha,\alpha}(t-s)[Bv(s) + F(s)]ds,$$

where, $\{S_{\alpha,1}(t)\}_{t\geq 0}$ and $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$ are given explicitly in terms of the semigroup $\{T(t)\}_{t\geq 0}$, as

$$S_{\alpha,1}(t)x = \int_0^\infty \Phi_\alpha(r)T(rt^\alpha)xdr, \quad t \geq 0, \quad S_{\alpha,\alpha}(t) = \alpha\int_0^\infty t^{\alpha-1}r\Phi_\alpha(r)T(rt^\alpha)dr, \quad t > 0,$$

where Φ_α is the Wright type function ([32, Appendix F], [27]).

In case $1 < \alpha < 2$, in the paper [9] is discussed the existence of optimal controls of fractional stochastic equations and in [10], the authors study the approximate controllability of fractional Sobolev-type differential equations. In this case, if A generates the fractional resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$, then the mild solution to (1.1) is given by

$$u(t) = S_{\alpha,1}(t)x_0 + S_{\alpha,2}(t)x_1 + \int_0^t S_{\alpha,\alpha}(t-s)[Bv(s) + F(s)]ds.$$

But, in this case it is necessary to assume the existence of such family $\{S_{\alpha,1}(t)\}_{t\geq 0}$ generated by A.

We notice that this family $\{S_{\alpha,1}(t)\}_{t\geq 0}$ corresponds, for $0 < \alpha < 2$, to a vector-valued version of the Mittag-Leffler function and it has been used in many recent works to study the controllability of fractional continuous systems, see for instance [7, 19, 23, 29, 38] and [39].

More recently, a different and interesting approach to study the approximate controllability of (1.1) for $1 < \alpha < 2$ has been introduced in the literature, whose main idea is very similar to the case $0 < \alpha < 1$: To assume that A is the generator of a cosine family $\{C(t)\}_{t\in\mathbb{R}}$ and, from here, to deduce (using some special functions) that A is also a generator of a fractional resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$ of linear operators in X whose Laplace transform verifies $\widehat{S}_{\alpha,1}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}$ for all $\lambda^\alpha \in \rho(A)$. In this case, the mild solution to (1.1) is given by

$$u(t) = S_{\alpha,1}(t)x_0 + S_{\alpha,2}(t)x_1 + \int_0^t S_{\alpha,\alpha}(t-s)[Bv(s) + F(s)]ds$$

where

$$S_{\alpha,1}(t)x = \int_0^\infty M_{\frac{\alpha}{2}}(r)C(rt^{\frac{\alpha}{2}})xdr,\ S_{\alpha,2}(t)x = \int_0^t S_{\alpha,1}(s)xds,\ S_{\alpha,\alpha}(t) = \frac{\alpha}{2}\int_0^\infty rM_{\frac{\alpha}{2}}(r)S(rt^{\frac{\alpha}{2}})dr, \tag{1.4}$$

where $S(t)$ is the corresponding sine family and $M_{\frac{\alpha}{2}}$ is the Mainardi type function, see [44] for a detailed discussion. These fractional resolvent families generated by A has been used very recently to study the approximate and finite-approximate controllability of impulsive system of order $1 < \alpha < 2$, see [4, 5, 6] for further details.

All the above shows that certain families of operators generated by A can be a great tool for the study of the controllability of the corresponding continuous systems.

On the other hand, there is a growing interest in discrete fractional systems, which can be understood as discretizations of the continuous case. Studies of the controllability of discrete systems of integer order have been widely discussed in recent decades. For example, in [24, 25] the authors study exact and approximate controllability of discrete semilinear systems, in [11, 12] is considered the controllability of delay discrete systems, the [43] focuses on relative controllability of delayed discrete system of order two, and in [20] the authors study the approximate controllability of abstract discrete systems.

Now, consider the discrete fractional system

$$\begin{cases} {}_C\nabla^\alpha u^n &= Au^n + Bv^n + f^n, \quad n \geq 1, \\ u^0 &= x_0, \\ u^1 &= x_1, \end{cases} \tag{1.5}$$

where $1 < \alpha < 2$, ${}_C\nabla^\alpha u^n$ is the discrete Caputo fractional derivative (see for instance [15, 26, 36]), $A : D(A) \subset X \to X$ is a closed linear operator in a Hilbert space X, $B : U \to X$ is a bounded linear operator, U is a Hilbert space, $(f^n)_{n\in\mathbb{N}_0}$ is a given sequence, the control v belongs to $\ell^2(\mathbb{N}_0, U)$ and the initial states x_0, x_1 belong to X.

We observe that the problem (1.5) can be seen as a time-discrete version of equation (1.1). As in the continuous case (1.1), the solution to (1.5) can be written in terms of certain family of sequences of operators. In fact, if A generates a resolvent sequence $\{S^n_{\alpha,1}\}_{n\in\mathbb{N}_0}$, (see Section 2) then the solution to (1.5) is given by

$$u^n = S^n_{\alpha,1}x_0 + S^n_{\alpha,2}x_1 + \tau\sum_{j=0}^n S^{n-j}_{\alpha,\alpha}[f^j + Bv^j], \quad n \geq 2, \tag{1.6}$$

and $u^0 = x_0$, $u^1 = x_1$, (see for instance [17, 28]), where

$$S^n_{\alpha,2}x = \tau\sum_{j=0}^n S^j_{\alpha,1}x, \quad \text{and} \quad S^n_{\alpha,\alpha} := \tau\sum_{j=0}^n k_\tau^{\alpha-1}(n-j)S^j_{\alpha,1}, \quad n \in \mathbb{N}_0,$$

and, for any $\beta > 0$ and a size step $\tau > 0$, k^β_τ is the sequence $k^\beta_\tau(n) := \frac{\tau^{\beta-1}\Gamma(\beta+n)}{\Gamma(\beta)\Gamma(n+1)}$, $n \in \mathbb{N}_0$, and $\Gamma(\cdot)$ stands for the Gamma function.

The controllability of fractional discrete systems in the form of (1.5) has been considered by many authors in the last years. For example, the paper [35] focuses on the controllability and observability of discrete-time fractional systems, the author in [22] studies the controllability of nonlinear discrete systems and the authors of [33] study the local controllability and observability of nonlinear discrete-time systems. However, all these papers take sequences $u, v : \mathbb{N}_0 \to \mathbb{R}^N$ and A, B as real-valued matrices.

Despite the growing interest in the study of the controllability of discrete fractional systems, the approximate controllability of systems with unbounded operators remains unstudied in the literature.

A natural question arise here: Since the families of linear operators generated by A (semigroups, cosine or fractional resolvent families) constitute a useful tool for the study of the controllability of systems (1.2), (1.3) and (1.1) in infinite dimensional spaces, and on the other hand, it is possible to write the solution to (1.5) as a discrete version of the formula for parameter variation in terms of the sequence $\{S^n_{\alpha,1}\}_{n\in\mathbb{N}_0}$, *Is* *it possible to use these sequences to study the controllability of discrete fractional systems in the form of* (1.5)*?*

This paper aims to give an answer to this question. Indeed, throughout this work we show that it is possible to obtain subordination theorems that guarantee that if A generates a discrete cosine, then A is also the generator of a discrete resolvent sequence $\{S^n_{\alpha,1}\}_{n\in\mathbb{N}_0}$. With this, we prove the approximate controllability of problem (1.5) from the approximate controllability of its homogeneous version. In addition, we show the existence of optimal controls and from this, we deduce its approximate controllability.

The rest of the paper is structured as follows. In Section 2 we give the preliminaries on discrete fractional calculus and discrete fractional resolvent sequences generated by a closed linear operator A. In addition, we give some subordination principles, which allows to find conditions on a closed operator A to be the generator of a resolvent sequence. To be more precise, we show that if A is a generator of a cosine family of linear operators, then A is the generator of a resolvent sequence $\{S^n_{\alpha,1}\}_{n\in\mathbb{N}_0}$. In this case we give an explicit representation of $\{S^n_{\alpha,1}\}_{n\in\mathbb{N}_0}$ in terms of some sequences derived from certain special functions.

Moreover, we show that $(u^n)_{n\in\mathbb{N}_0}$ given by (1.6) solves the discrete fractional system (1.5). In Section 3 we study the existence of solutions to the semilinear discrete fractional system

$$ {}_C\nabla^\alpha u^n = Au^n + Bv^n + f(n,u^n), \quad n \geq 2, \tag{1.7} $$

under the initial state $u^0 = x_0, u^1 = x_1$, where $f : \mathbb{N}_0 \times X \to X$ is a given term.

In Section 4 we focus on the approximate controllability of the discrete fractional system (1.7). Here, we study the existence of an optimal control v to (1.7). And, as a consequence of the previous results in this Section, we obtain the approximate controllability of (1.7).

Finally, in Section 5 we provide an example of a discrete fractional system to illustrate the results obtained in the previous sections.

2. Preliminaries

The set of non-negative integer numbers will be denoted by $\mathbb{N}_0$. For any $n \in \mathbb{N}_0$ and $\tau > 0$, we define the sequence of functions $\rho^\tau_n : [0,\infty) \to \mathbb{R}$, given by

$$ \rho^\tau_n(t) := e^{-\frac{t}{\tau}} \left(\frac{t}{\tau}\right)^n \frac{1}{\tau n!}, $$

We notice that ρ^τ_n are non-negative functions and $\int_0^\infty \rho^\tau_n(t)dt = 1$, for all $n \in \mathbb{N}_0$.

2.1. **Discrete fractional calculus and resolvent sequences.** Let X be a Banach space. The space of all vector-valued sequences $v : \mathbb{N}_0 \to X$ is denoted by $s(\mathbb{N}_0, X)$. For $n \in \mathbb{N}_0$, $\nabla_\tau : s(\mathbb{N}_0, X) \to s(\mathbb{N}_0, X)$, denotes *backward Euler operator of order one,* which is defined by

$$ \nabla_\tau v^n := \frac{v^n - v^{n-1}}{\tau}, \quad n \in \mathbb{N}. $$

For $m \geq 2$, we define the *backward difference operator of order* m, $\nabla^m_\tau : s(\mathbb{N}_0, X) \to s(\mathbb{N}_0, X)$, by

$$ (\nabla^m_\tau v)^n := \nabla^{m-1}_\tau (\nabla_\tau v)^n, \quad n \geq m. $$

Here ∇^1_τ is understood as $\nabla^1_\tau := \nabla_\tau$ and ∇^0_τ as the identity operator. We notice that if $v \in s(\mathbb{N}_0, X)$, then

$$ (\nabla^m_\tau v)^n = \frac{1}{\tau^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j v^{n-j}, \quad n \in \mathbb{N}. $$

Following [16, Chapter 1, Section 1.5] we adopt the convention

$$ \sum_{j=1}^{-k} v^j = 0, \quad k \in \mathbb{N}. \tag{2.8} $$

For any $\alpha > 0$, we define the sequence

$$k_\tau^\alpha(n) := \frac{\tau^{\alpha-1}\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)} = \int_0^\infty \rho_n^\tau(t) g_\alpha(t)dt, n \in \mathbb{N}_0, \tag{2.9}$$

where $g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $\Gamma(\cdot)$ is the Gamma function.

Definition 2.1. [36] *Let $\alpha > 0$. The $\alpha^{\text{th}}-$fractional sum of $v \in s(\mathbb{N}_0; X)$ is defined by*

$$(\nabla_\tau^{-\alpha} v)^n := \tau \sum_{j=0}^n k_\tau^\alpha(n-j)v^j, \quad n \in \mathbb{N}_0.$$

Definition 2.2. [36] *Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}_0$. The* Caputo fractional backward difference operator of order α, ${}_C\nabla^\alpha : s(\mathbb{R}_+; X) \to s(\mathbb{R}_+; X)$, *is defined by*

$$({}_C\nabla^\alpha v)^n := \nabla_\tau^{-(m-\alpha)}(\nabla_\tau^m v)^n, \quad n \in \mathbb{N},$$

where $m - 1 < \alpha < m$.

Let $\tau > 0$ be a given step-size. For $1 < \alpha < 2$, the sequence $({}_C\nabla^\alpha v)^n$ corresponds to an approximation of the Caputo fractional derivative $\partial_t^\alpha v$ of a function $v : \mathbb{R}_0 \to X$ at $t_n := \tau n$, that is, $({}_C\nabla^\alpha v)^n \approx \partial_t^\alpha v(t_n)$, where

$$\partial_t^\alpha v(t) := \int_0^t g_{2-\alpha}(t-s)v''(s)ds.$$

The Z-transform of a sequence $s \in s(\mathbb{N}_0, X)$, $\tilde{s}$, is defined by $\tilde{s}(z) := \sum_{j=0}^\infty z^{-j}s^j$, where $s^j := s(j)$ and $z \in \mathbb{C}$. For any $\alpha > 0$, the Z-transform of the sequence $\{k_\tau^\alpha(n)\}_{n\in\mathbb{N}_0}$ is given by

$$\widetilde{k_\tau^\alpha}(z) = \tau^{\alpha-1}\frac{z^\alpha}{(z-1)^\alpha}, \quad \text{for all } |z| > 1. \tag{2.10}$$

For a Hilbert space X, $\mathcal{B}(X)$ denotes the space of linear operators from X into X. Given a sequence of linear operators $\{S^n\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ and a scalar sequence $c = (c^n)_{n\in\mathbb{N}_0}$, we define its discrete convolution $c \star S$ as

$$(c \star S)^n := \sum_{k=0}^n c^{n-k}S^k, \quad n \in \mathbb{N}_0.$$

Similarly, for scalar valued sequences $b = (b^n)_{n\in\mathbb{N}_0}$ and $c = (c^n)_{n\in\mathbb{N}_0}$, we define $(b \star c \star S)^n := (b \star (c \star S))^n$ for all $n \in \mathbb{N}_0$.

From [36, Corollary 2.9], we have that if $\alpha, \beta > 0$, then we have the following semigroup property

$$k_\tau^{\alpha+\beta}(n) = \tau \sum_{j=0}^n k_\tau^\alpha(n-j)k_\tau^\beta(j) = \tau(k_\tau^\alpha \star k_\tau^\beta)^n, \quad \text{for any } n \in \mathbb{N}_0. \tag{2.11}$$

Definition 2.3. *Let $1 \le \alpha \le 2$, $0 < \beta \le 2$ and $\tau > 0$ be given. The closed linear operator A is called the generator of the (α, β)-resolvent sequence $\{S_{\alpha,\beta}^n\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ if it satisfies the following conditions*

(1) *$S_{\alpha,\beta}^n x \in D(A)$ for all $x \in X$ and $AS_{\alpha,\beta}^n x = S_{\alpha,\beta}^n Ax$ for all $x \in D(A)$, and $n \in \mathbb{N}_0$.*
(2) *For each $x \in X$ and $n \in \mathbb{N}_0$,*

$$S_{\alpha,\beta}^n x = k_\tau^\beta(n)x + \tau A(k_\tau^\alpha \star S_{\alpha,\beta})^n x = k_\tau^\beta(n)x + \tau A \sum_{j=0}^n k_\tau^\alpha(n-j)S_{\alpha,\beta}^j x. \tag{2.12}$$

From [17, Theorem 3.1] it follows that the discrete resolvent family $\{S_{\alpha,\beta}^n\}_{n\in\mathbb{N}_0}$ verifies the following functional equation

$$S_{\alpha,\beta}^m(k_\tau^\alpha \star S_{\alpha,\beta})^n - (k_\tau^\alpha \star S_{\alpha,\beta})^m S_{\alpha,\beta}^n = k_\tau^\beta(m)(k_\tau^\alpha \star S_{\alpha,\beta})^n - k_\tau^\beta(n)(k_\tau^\alpha \star S_{\alpha,\beta})^m,$$

for all $m,n\in\mathbb{N}_0$. On the other hand, by [17, Proposition 3.1], $\tau^{-\alpha}\in\rho(A)$ and

$$S^0_{\alpha,\beta}x=\tau^{\beta-1-\alpha}\left(\tau^{-\alpha}-A\right)^{-1}x,\quad x\in X.$$

In addition, from [17, Theorem 3.7] it follows that $\{S^n_{\alpha,\beta}\}_{n\in\mathbb{N}_0}$ can be written as

$$S^n_{\alpha,\beta}x=\sum_{j=1}^{n+1}a_{n,j}R^j_\tau x, \tag{2.13}$$

where, $(a_{n,l})$ is the sequence of real numbers defined by

$$a_{0,1}:=k^\beta_\tau(0),\quad a_{1,1}:=(k^\beta_\tau(1)k^\alpha_\tau(0)-k^\beta_\tau(0)k^\alpha_\tau(1))k^\alpha_\tau(0)^{-1},\quad a_{1,2}:=k^\beta_\tau(0)k^\alpha_\tau(1)k^\alpha_\tau(0)^{-1}$$

and for $n\geq 2$,

$$a_{n,n+1}:=k^\alpha_\tau(1)a_{n-1,n}k^\alpha_\tau(0)^{-1},\quad a_{n,1}:=\left(k^\beta_\tau(n)k^\alpha_\tau(0)-\sum_{j=0}^{n-1}k^\alpha_\tau(n-j)a_{j,1}\right)k^\alpha_\tau(0)^{-1},$$

and

$$a_{n,l}:=\left(\sum_{j=l-2}^{n-1}k^\alpha_\tau(n-j)a_{j,l-1}-\sum_{j=l-1}^{n-1}k^\alpha_\tau(n-j)a_{j,l}\right)k^\alpha_\tau(0)^{-1},\ 2\leq l\leq n,$$

where $R_\tau:X\to D(A)$ denotes the resolvent operator defined by

$$R_\tau:=\tau^{-\alpha}\left(\tau^{-\alpha}-A\right)^{-1}. \tag{2.14}$$

Finally, from [17] we have the following result for the Z-transform of $\{S^n_{\alpha,\beta}\}_{n\in\mathbb{N}_0}$.

Proposition 2.4. *Let $\{S^n_{\alpha,\beta}\}_{n\in\mathbb{N}_0}\subset B(X)$ be a discrete (α,β)-resolvent sequence generated by A. Then its Z-transform satisfies*

$$\widetilde{S_{\alpha,\beta}}(z)x=\frac{1}{\tau}\left(\frac{z-1}{\tau z}\right)^{\alpha-\beta}\left(\left(\frac{z-1}{\tau z}\right)^{\alpha}-A\right)^{-1}x,$$

for all $x\in X$.

2.2. **Subordination theorems.** In this subsection we give some subordination principles, which allow us to find conditions on a closed operator A to be the generator of a resolvent sequence.

Let $n\in\mathbb{N}_0$, $0<\alpha<1$ $\beta\geq 0$ and $\tau>0$. Following to [1, Definition 4.2] or [2, Definition 3.1], we define the *discrete scaled Wright function* $\varphi^\tau_{\alpha,\beta}$ as

$$\varphi^\tau_{\alpha,\beta}(n,j):=\frac{1}{2\pi i}\int_\Upsilon z^{n-1}\tau\widetilde{k}^\beta_\tau(z)\left(1-\frac{1}{\widetilde{k}^\alpha_\tau(z)}\right)^j dz,\quad j\in\mathbb{N}_0$$

where Υ is a path oriented counterclockwise that encloses all the singularities of the complex variable function $z\mapsto\left(1-\frac{1}{\widetilde{k}^\alpha_\tau(z)}\right)^j$. In addition, we define, for $0<\alpha<1$ and $\beta\geq 0$, the scaled Wright function in two variables $\varphi_{\alpha,\beta}$ by

$$\varphi_{\alpha,\beta}(t,s):=t^{\beta-1}W_{-\alpha,\beta}(-st^{-\alpha}),\quad t>0,s\in\mathbb{C},$$

where, for $\mu\geq 0$, $\lambda>-1$, and $z\in\mathbb{C}$, $W_{\lambda,\mu}$ is the Wright function defined by

$$W_{\lambda,\mu}(z):=\sum_{j=0}^\infty\frac{z^j}{j!\Gamma(\lambda j+\mu)}.$$

The next result gives some properties of $\varphi_{\alpha,\beta}$ and $\varphi^\tau_{\alpha,\beta}$. Its proof follows similarly to [1, Proposition 4.3] and therefore, we omit it.

Proposition 2.5. *Let $0<\alpha<1$, $\beta\geq 0$, $\tau>0$ and $n,j\in\mathbb{N}_0$. Then we have the following properties:*

(1) $\displaystyle\sum_{l=0}^{\infty}\varphi_{\alpha,\beta}^{\tau}(n,l)k_{\tau}^{\gamma}(l)=k_{\tau}^{\beta+\alpha\gamma}(n)$, *where* $\gamma>0$.

(2) $\displaystyle\varphi_{\alpha,\beta}^{\tau}(n,j)=\sum_{l=0}^{j}\binom{j}{l}(-1)^{l}\tau^{l+1}k_{\tau}^{\beta-\alpha l}(n)$.

(3) $\displaystyle\int_{0}^{\infty}\rho_{n}^{\tau}(t)\varphi_{\alpha,\beta}(t,s)dt=\sum_{l=0}^{\infty}\varphi_{\alpha,\beta}^{\tau}(n,l)\rho_{l}^{\tau}(s)$.

(4) $\displaystyle\widetilde{\varphi}_{\alpha,\beta}(z,j)=\sum_{l=0}^{j}\binom{j}{l}(-1)^{l}\tau^{l+1}\widetilde{k}_{\tau}^{\beta-\alpha l}(z)$.

(5) *For* $0<\tau\leq 1$, $\varphi_{\alpha,\beta}^{\tau}(n,j)\geq 0$.

Definition 2.6. *Let* $\tau>0$ *be given. The closed linear operator* A *is called the generator of a* cosine sequence *if it is the generator of a* $(2,1)$*-resolvent sequence* $\{S_{2,1}^{n}\}_{n\in\mathbb{N}_0}\subset\mathcal{B}(X)$*. We denote the cosine sequence* $\{S_{2,1}^{n}\}_{n\in\mathbb{N}_0}$ *simply as* $\{C^{n}\}_{n\in\mathbb{N}_0}$.

We notice that a cosine sequence $\{C^{n}\}_{n\in\mathbb{N}_0}$ verifies the following conditions

(1) $C^{n}x\in D(A)$ for all $x\in X$ and $AC^{n}x=C^{n}Ax$ for all $x\in D(A)$, and $n\in\mathbb{N}_0$.

(2) For each $x\in X$ and $n\in\mathbb{N}_0$,

$$C^{n}x=x+\tau A(k_{\tau}^{2}\star C)^{n}x=x+\tau A\sum_{j=0}^{n}k_{\tau}^{2}(n-j)C^{j}x. \tag{2.15}$$

The next results correspond to a discrete counterpart of the subordination theorems in [44, Theorem 3.1].

Theorem 2.7 (Subordination)**.** *Let* $1<\alpha<2$*. Assume that* A *is the generator of a cosine sequence* $\{C^{n}\}_{n\in\mathbb{N}_0}$*. Then* A *generates the* $(\alpha,1)$*-resolvent sequence* $\{S_{\alpha,1}^{n}\}_{n\in\mathbb{N}_0}$ *defined by*

$$S_{\alpha,1}^{n}x:=\sum_{j=0}^{\infty}\varphi_{\frac{\alpha}{2},1-\frac{\alpha}{2}}^{\tau}(n,j)C^{j}x,\quad x\in X. \tag{2.16}$$

Proof. Let $x\in X$. As A is the generator of a cosine sequence $\{C^{n}\}_{n\in\mathbb{N}_0}$, $AC^{n}x=C^{n}Ax$, for all $x\in D(A)$, and therefore $AS_{\alpha,1}^{n}x=S_{\alpha,1}^{n}Ax$ for all $x\in D(A)$ and $n\in\mathbb{N}_0$. By (4) in Proposition 2.5 and equation (2.10), the Z-transform of $\{S_{\alpha,1}^{n}\}_{n\in\mathbb{N}_0}$ satisfies

$$\begin{aligned}\widetilde{S_{\alpha,1}}(z) &= \sum_{j=0}^{\infty}\widetilde{\varphi}_{\frac{\alpha}{2},1-\frac{\alpha}{2}}(z,j)C^{j}x\\ &= \sum_{j=0}^{\infty}\sum_{l=0}^{j}\binom{j}{l}(-1)^{l}\tau^{l+1}\widetilde{k}_{\tau}^{1-\frac{\alpha}{2}-\frac{\alpha l}{2}}(z)C^{j}x\\ &= \left(\frac{\tau z}{z-1}\right)^{1-\frac{\alpha}{2}}\sum_{j=0}^{\infty}\sum_{l=0}^{j}\binom{j}{l}\left(-\tau\left(\frac{\tau z}{z-1}\right)^{-\frac{\alpha}{2}}\right)^{l}C^{j}x\\ &= \left(\frac{\tau z}{z-1}\right)^{1-\frac{\alpha}{2}}\sum_{j=0}^{\infty}\left(1-\tau\left(\frac{\tau z}{z-1}\right)^{-\frac{\alpha}{2}}\right)^{j}C^{j}x, \qquad (2.17)\end{aligned}$$

for all $|z|$ large enough. As

$$\left(1-\tau\left(\frac{\tau z}{z-1}\right)^{-\frac{\alpha}{2}}\right)^{j}=\left(\frac{(\tau z)^{\frac{\alpha}{2}}}{(\tau z)^{\frac{\alpha}{2}}-\tau(z-1)^{\frac{\alpha}{2}}}\right)^{-j},$$

and the Z-transform of the cosine sequence $\{C^n\}_{n\in\mathbb{N}_0}$, verifies (see Proposition 2.5)

$$\widetilde{C}(w)=\frac{1}{\tau}\left(\frac{w-1}{\tau w}\right)\left(\left(\frac{w-1}{\tau w}\right)^2-A\right)^{-1}x,$$

we have that the series in (2.17) corresponds precisely to the Z-transform of $\{C^n\}_{n\in\mathbb{N}_0}$ evaluated at

$$w:=\frac{1}{1-\tau\left(\frac{z-1}{\tau z}\right)^{\frac{\alpha}{2}}}.$$

Therefore,

$$\widetilde{S_{\alpha,1}}(z)=\left(\frac{z-1}{\tau z}\right)^{\frac{\alpha}{2}-1}\sum_{j=0}^{\infty}\left(\frac{(\tau z)^{\frac{\alpha}{2}}}{(\tau z)^{\frac{\alpha}{2}}-\tau(z-1)^{\frac{\alpha}{2}}}\right)^{-j}C^jx=\frac{1}{\tau}\left(\frac{z-1}{\tau z}\right)^{\alpha-1}\left(\left(\frac{z-1}{\tau z}\right)^{\alpha}-A\right)^{-1}x,$$

for all $x\in X$. By the uniqueness of the Z-transform, the last equality means that $\{S^n_{\alpha,1}\}_{n\in\mathbb{N}_0}$ defined in (2.16) is an $(\alpha,1)$-resolvent sequence generated by A.

□

Theorem 2.8 (Subordination)**.** *Let $1<\alpha<2$. Assume that A is the generator of a cosine sequence $\{C^n\}_{n\in\mathbb{N}_0}$. Then A generates the (α,α)-resolvent sequence $\{S^n_{\alpha,\alpha}\}_{n\in\mathbb{N}_0}$ given by*

$$S^n_{\alpha,\alpha}x=\sum_{j=0}^{\infty}\varphi^{\tau}_{\frac{\alpha}{2},\frac{\alpha}{2}}(n,j)C^jx,\quad x\in X. \tag{2.18}$$

Proof. The proof follows similary to the proof of Theorem 2.7 and therefore, we omit it. □

We notice that, from the uniqueness of the Z-transform, it follows that

$$S^n_{\alpha,2}x=\tau(k^1_\tau\star S_{\alpha,1})^n=\tau\sum_{j=0}^{n}S^j_{\alpha,1}x,\quad x\in X,n\in\mathbb{N}_0, \tag{2.19}$$

and

$$S^n_{\alpha,\alpha}x=\tau(k^{\alpha-1}_\tau\star S_{\alpha,1})^n=\tau\sum_{j=0}^{n}k^{\alpha-1}_\tau(n-j)S^j_{\alpha,1}x,\quad x\in X,n\in\mathbb{N}_0. \tag{2.20}$$

Lemma 2.9. [17, Theorem 3.5] *Assume that A generates a cosine family $\{C(t)\}_{t\geq 0}$ defined in X. If $1<\alpha<2$, then A is the generator of the $(\alpha,1)$-resolvent sequence $\{S^n_{\alpha,1}\}_{n\in\mathbb{N}_0}$ defined by*

$$S^n_{\alpha,1}x:=\int_0^\infty\int_0^\infty\rho^\tau_n(t)\varphi_{\frac{\alpha}{2},1-\frac{\alpha}{2}}(t,s)C(s)x\,dsdt,\quad t\geq 0,x\in X, \tag{2.21}$$

where $\varphi_{\frac{\alpha}{2},1-\frac{\alpha}{2}}$ is the Wright type function, which can be written as

$$\varphi_{\frac{\alpha}{2},1-\frac{\alpha}{2}}(t,s)=\frac{1}{\pi}\int_0^\infty\rho^{\frac{\alpha}{2}-1}e^{-s\rho^{\frac{\alpha}{2}}\cos\frac{\alpha}{2}(\pi-\theta)-t\rho\cos\theta}\sin\left(t\rho\sin\theta-s\rho^{\frac{\alpha}{2}}\sin\tfrac{\alpha}{2}(\pi-\theta)+\tfrac{\alpha}{2}(\pi-\theta)\right)d\rho,$$

for $\theta\in(\pi-\frac{2}{\alpha},\pi/2)$.

Lemma 2.10. [17, Theorem 3.6] *Let $1<\alpha<2$. Assume that A is the generator of a cosine family $\{C(t)\}_{t\in\mathbb{R}}$. Then A generates the (α,α)-resolvent sequence $\{S^n_{\alpha,\alpha}\}_{n\in\mathbb{N}_0}$ given by*

$$S^n_{\alpha,\alpha}x=\int_0^\infty\int_0^\infty\rho^\tau_n(t)\varphi_{\frac{\alpha}{2},\frac{\alpha}{2}}(t,s)C(s)x\,dsdt,\quad x\in X,$$

where $\varphi_{\frac{\alpha}{2},\frac{\alpha}{2}}$ is the Wright type function, which can be represented as

$$\varphi_{\frac{\alpha}{2},\frac{\alpha}{2}}(t,s)=(g_{\frac{\alpha}{2}}*\varphi_{\frac{\alpha}{2},0}(\cdot,s))(t), \tag{2.22}$$

where $\varphi_{\frac{\alpha}{2},0}(\cdot,s)$ *is given by*

$$\varphi_{\frac{\alpha}{2},0}(t,s)=\frac{1}{\pi}\int_0^\infty e^{t\rho\cos\theta-s\rho^{\frac{\alpha}{2}}\cos\alpha\theta}\cdot\sin\left(t\rho\sin\theta-s\rho\sin\tfrac{\alpha}{2}\theta+\theta\right)\,d\rho, \tag{2.23}$$

for $\pi/2<\theta<\pi$.

It is a well-known fact that if A is bounded operator, then

$$C(t):=\sum_{j=0}^{\infty}\frac{t^{2j}}{(2j!)}A^j \tag{2.24}$$

defines a cosine familiy of operators generated by A, (see for instance [3, Section 3.14]). By Lemma 2.9, A is the generator of the $(\alpha,1)$-resolvent sequence given by

$$S^n_{\alpha,1}x=\int_0^\infty\int_0^\infty \rho_n^\tau(t)\varphi_{\frac{\alpha}{2},1-\frac{\alpha}{2}}(t,s)C(s)x\,dsdt.$$

Since the series in (2.24) converges uniformly, we have by Fubini's theorem that

$$S^n_{\alpha,1}x=\sum_{j=0}^{\infty}\int_0^\infty\frac{s^{2j}}{(2j)!}\int_0^\infty \rho_n^\tau(t)\varphi_{\frac{\alpha}{2},1-\frac{\alpha}{2}}(t,s)\,dtdsA^jx.$$

By Proposition 2.5, we get

$$\begin{aligned}
S^n_{\alpha,1}x &= \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\varphi^\tau_{\frac{\alpha}{2},1-\frac{\alpha}{2}}(n,l)\int_0^\infty\rho_l^\tau(s)\frac{s^{2j}}{(2j)!}\,dsA^jx\\
&= \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\varphi^\tau_{\frac{\alpha}{2},1-\frac{\alpha}{2}}(n,l)k_\tau^{2j+1}(l)\int_0^\infty\rho^\tau_{l+2j}(s)\,dsA^jx\\
&= \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\varphi^\tau_{\frac{\alpha}{2},1-\frac{\alpha}{2}}(n,l)k_\tau^{2j+1}(l)A^jx\\
&= \sum_{j=0}^{\infty}k_\tau^{\alpha j+1}(n)A^jx.
\end{aligned}$$

Similarly, if A is a bounded operator, then A generates the (α,α)-resolvent sequence given by

$$S^n_{\alpha,\alpha}x=\sum_{j=0}^{\infty}k_\tau^{\alpha j+\alpha}(n)A^jx.$$

Example 2.11. *If* $X=\mathbb{R}$, *and* $A:\mathbb{R}\to\mathbb{R}$ *is defined by* $Ax=-ax$, *then* $C(t)=\cos(t\sqrt{a})$ *is a strongly continuous cosine function generated by* A. *In this case, the* $(\alpha,1)$*-resolvent sequence generated by* A *is given by*

$$S^n_{\alpha,1}x=\sum_{j=0}^{\infty}k_\tau^{\alpha j+1}(n)a^jx=\frac{1}{\tau}\mathcal{E}^\tau_{\alpha,1}(a,n+1),n\in\mathbb{N}_0,$$

where, for $\alpha,\beta>0$, $\mathcal{E}^\tau_{\alpha,\beta}$ *is the Mittag-Leffler sequence defined in* [1, Section 4].

Example 2.12. *Now, if* A *is a selfadjoint operator on a Hilbert space* X, *and* A *is bounded above; that is,* $\langle Ax,x\rangle_X\le\omega\|x\|^2$ *for all* $x\in D(A)$ *and some* $\omega\in\mathbb{R}$. *Then* A *generates a cosine function. See for instance* [3, Example 3.14.16]. *Then it also generates an* $(\alpha,1)$*-resolvent sequence.*

Proposition 2.13. *Let A be a closed operator that generates a bounded cosine sequence $\{C^n\}_{n\in\mathbb{N}_0}$ with $\|C^n\| \leq M$ for all $n \in \mathbb{N}_0$. Let $1 < \alpha < 2$ and $\{S^n_{\alpha,1}\}_{n\in\mathbb{N}_0}$, $\{S^n_{\alpha,2}\}_{n\in\mathbb{N}_0}$ and $\{S^n_{\alpha,\alpha}\}_{n\in\mathbb{N}_0}$ be the resolvent sequences given, respectively, in* (2.16), (2.19) *and* (2.18). *Then*

$$\|S^n_{\alpha,1}x\| \leq M\|x\|, \quad \|S^n_{\alpha,\alpha}x\| \leq Mk^\alpha_\tau(n)\|x\| \quad \text{and} \quad \|S^n_{\alpha,2}x\| \leq M(n+1)\tau\|x\|.$$

for any $n \in \mathbb{N}_0$ and $x \in X$.

Proof. Let $x \in X$. Since $\|C^n\| \leq M$, we have by Proposition 2.5 that

$$\|S^n_{\alpha,1}x\| \leq M\sum_{l=0}^{\infty}\varphi^\tau_{\frac{\alpha}{2},1-\frac{\alpha}{2}}(n,l)\|x\| = Mk^1_\tau(n)\|x\| = M\|x\|,$$

for any $x \in X$, $n \in \mathbb{N}_0$, and, similarly,

$$\|S^n_{\alpha,\alpha}x\| \leq M\sum_{l=0}^{\infty}\varphi^\tau_{\frac{\alpha}{2},\frac{\alpha}{2}}(n,l)\|x\| = Mk^\alpha_\tau(n)\|x\|.$$

Finally, by (2.19)

$$\|S^n_{\alpha,2}x\| \leq \tau\sum_{j=0}^{n}\|S^j_{\alpha,1}x\| \leq M(n+1)\tau\|x\|.$$

□

Remark 2.14. *Suppose that A is the generator of an $(\alpha,1)$-resolvent sequence $\{S^n_{\alpha,1}\}_{n\in\mathbb{N}_0}$ such that $\|S^n_{\alpha,1}\| \leq M$ for all $n \in \mathbb{N}_0$. Since $0 < \alpha - 1 < 1$ and $n \geq 0$, we have (by* [14, Inequality (1.1)]*) that*

$$k^\alpha_\tau(n) = \frac{\tau^{\alpha-1}\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+2)}(n+1) < \frac{(\tau(n+1))^{\alpha-1}}{\Gamma(\alpha)},$$

and, thus the semigroup property (2.11) *implies that*

$$\|S^n_{\alpha,\alpha}\| \leq \tau\sum_{j=0}^{n}k^{\alpha-1}_\tau(n-j)\|S^j_{\alpha,1}\| \leq Mk^{\alpha-1}_\tau(n) \leq M\frac{(\tau(n+1))^{\alpha-1}}{\Gamma(\alpha)}. \tag{2.25}$$

Proposition 2.15. *If the resolvent operator R_τ defined in* (2.14) *is compact for all $\tau^{-\alpha} \in \rho(A)$, then $S^n_{\alpha,\beta}$ is a compact operator for all $n \in \mathbb{N}_0$.*

Proof. The proof follows from the representation as a finite sum of $S^n_{\alpha,\beta}$ given in (2.13). □

3. Existence of solutions.

In this section we study the existence of solutions to discrete fractional system of order $1 < \alpha < 2$. Consider the system

$$\begin{cases} {}_C\nabla^\alpha u^n &= Au^n + Bv^n, \quad n \geq 2, \\ u^0 &= x_0 \\ u^1 &= x_1, \end{cases} \tag{3.26}$$

where $A : D(A) \subset X \to X$ is a closed linear operator in X, $B : U \to X$ is a bounded linear operator, the control $v \in \ell^2(\mathbb{N}_0, U)$ and $x_0, x_1 \in X$.

Proposition 3.16. *Let $1 < \alpha \leq 2$. Assume that A is the generator of an $(\alpha,1)$-resolvent sequence $\{S^n_{\alpha,1}\}_{n\in\mathbb{N}_0}$. If $x_0, x_1 \in X$, then the discrete fractional system* (3.26) *has a unique solution for each control $v \in \ell^2(\mathbb{N}_0, U)$ given by the sequence*

$$u^n := S^n_{\alpha,1}x_0 + \tau(k^1_\tau \star S_{\alpha,1})^n x_1 + \tau^2(k^{\alpha-1}_\tau \star S_{\alpha,1} \star Bv)^n, \tag{3.27}$$

for all $n \geq 2$ and $u^0 := x_0$, $u^1 := x_1$.

Proof. Since $\{S^n_{\alpha,1}\}_{t\geq 0}$ is an $(\alpha,1)$-resolvent sequence, u^n defined in (4) belongs to $D(A)$ for all $n\in\mathbb{N}_0$, and

$$S^n_{\alpha,1}x = x+\tau A\sum_{j=0}^{n} k^{\alpha}_{\tau}(n-j)S^j_{\alpha,1}x, \quad n\geq 0, x\in X. \tag{3.28}$$

Now, for $n\geq 2$, we have

$$_C\nabla^{\alpha}(S_{\alpha,1}x)^n = \nabla_{\tau}^{-(2-\alpha)}\nabla^2_{\tau}(S_{\alpha,1}x)^n = \tau\sum_{j=0}^{n} k^{2-\alpha}_{\tau}(n-j)(\nabla^2_{\tau}S_{\alpha,1}x)^j. \tag{3.29}$$

By (3.28), we obtain

$$\begin{aligned}(\nabla^2_{\tau}S_{\alpha,1}x)^j &= \frac{1}{\tau^2}(S^j_{\alpha,1}x-2S^{j-1}_{\alpha,1}x+S^{j-2}_{\alpha,1}x)\\ &= \frac{A}{\tau}\Big[\sum_{l=0}^{j}k^{\alpha}_{\tau}(j-l)S^l_{\alpha,1}x-2\sum_{l=0}^{j-1}k^{\alpha}_{\tau}(j-1-l)S^l_{\alpha,1}x+\sum_{l=0}^{j-2}k^{\alpha}_{\tau}(j-2-l)S^l_{\alpha,1}x\Big],\end{aligned}$$

for all $j\geq 2$. On the other hand, by (2.11) we obtain that

$$\tau\sum_{j=0}^{n}k^{2-\alpha}_{\tau}(n-j)\sum_{l=0}^{j}k^{\alpha}_{\tau}(j-l)S^l_{\alpha,1}x = \tau(k^{2-\alpha}_{\tau}\star k^{\alpha}_{\tau}\star S_{\alpha,1})^n = \tau(k^1_{\tau}\star k^1_{\tau}\star S_{\alpha,1})^n x = \tau\sum_{j=0}^{n}\sum_{l=0}^{j}S^l_{\alpha,1}x, \tag{3.30}$$

for all $n\geq 2$. Similarly, and by convention (2.8), we get that

$$\sum_{j=0}^{n}k^{2-\alpha}_{\tau}(n-j)\sum_{l=0}^{j-1}k^{\alpha}_{\tau}(j-1-l)S^l_{\alpha,1}x = \sum_{j=0}^{n-1}\sum_{l=0}^{j}S^l_{\alpha,1}x \tag{3.31}$$

and

$$\sum_{j=0}^{n}k^{2-\alpha}_{\tau}(n-j)\sum_{l=0}^{j-2}k^{\alpha}_{\tau}(j-2-l)S^l_{\alpha,1}x = \sum_{j=0}^{n-2}\sum_{l=0}^{j}S^l_{\alpha,1}x. \tag{3.32}$$

Hence, the equations (3.29)–(3.32) imply that

$$\begin{aligned}_C\nabla^{\alpha}(S_{\alpha,1}x)^n &= A\sum_{j=0}^{n}k^{2-\alpha}_{\tau}(n-j)\left[\sum_{l=0}^{j}k^{\alpha}_{\tau}(j-l)S^l_{\alpha,1}x-2\sum_{l=0}^{j-1}k^{\alpha}_{\tau}(j-1-l)S^l_{\alpha,1}x+\sum_{l=0}^{j-2}k^{\alpha}_{\tau}(j-2-l)S^l_{\alpha,1}x\right]\\ &= A\left[\sum_{j=0}^{n}\sum_{l=0}^{j}S^l_{\alpha}x-2\sum_{j=0}^{n-1}\sum_{l=0}^{j}S^l_{\alpha,1}x+\sum_{j=0}^{n-2}\sum_{l=0}^{j}S^l_{\alpha,1}x\right]\\ &= AS^n_{\alpha}x,\end{aligned}$$

for all $n\geq 2$ and $x\in X$. Therefore,

$$_C\nabla^{\alpha}S^n_{\alpha,1}x_0 = AS^n_{\alpha,1}x_0. \tag{3.33}$$

On the other hand, by definition we can write

$$_C\nabla^{\alpha}(\tau(k^1_{\tau}\star S_{\alpha,1})^n)x = \tau^2\sum_{j=0}^{n}k^{2-\alpha}_{\tau}(n-j)\nabla^2_{\tau}(k^1_{\tau}\star S_{\alpha,1})^j x. \tag{3.34}$$

As $k_\tau^1(n)=1$ for all $n\in\mathbb{N}_0$, we have

$$\begin{aligned}\nabla_\tau^2(k_\tau^1\star S_{\alpha,1})^j &= \frac{1}{\tau^2}\left[\sum_{l=0}^{j} S_{\alpha,1}^l x - 2\sum_{l=0}^{j-1} S_{\alpha,1}^l x + \sum_{l=0}^{j-2} S_{\alpha,1}^l x\right]\\ &= \frac{1}{\tau^2}[S_{\alpha,1}^j x - S_{\alpha,1}^{j-1}x]\\ &= \frac{A}{\tau}\left[\sum_{l=0}^{j} k_\tau^\alpha(j-l)S_{\alpha,1}^l x - \sum_{l=0}^{j-1} k_\tau^\alpha(j-1-l)S_{\alpha,1}x\right],\end{aligned}$$

and, by (3.34), (3.30),(3.31) and (3.32) we have

$$\begin{aligned}{}_C\nabla^\alpha(\tau(k_\tau^1\star S_{\alpha,1})^n)x &= \tau A\left[\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\sum_{l=0}^{j} k_\tau^\alpha(j-l)S_{\alpha,1}^l x - \sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\sum_{l=0}^{j-1} k_\tau^\alpha(j-l)S_{\alpha,1}^l x\right]\\ &= \tau A\left[\sum_{j=0}^{n}\sum_{l=0}^{j} S_{\alpha,1}^l x - \sum_{j=0}^{n-1}\sum_{l=0}^{j} S_{\alpha,1}^l x\right]\\ &= A(\tau(k_\tau^1\star S_{\alpha,1})^n x),\quad x\in X.\end{aligned}$$

This implies that

$$ {}_C\nabla^\alpha(\tau(k_\tau^1\star S_{\alpha,1})^n)x_1 = A(\tau(k_\tau^1\star S_{\alpha,1})^n)x_1,\quad n\geq 2. \tag{3.35}$$

Finally, by definition we have

$$\begin{aligned}{}_C\nabla^\alpha(\tau^2(k_\tau^{\alpha-1}\star S_{\alpha,1}\star Bv)^n) &= \nabla_\tau^{-(2-\alpha)}\nabla_\tau^2(\tau^2(k_\tau^{\alpha-1}\star S_{\alpha,1}\star Bv))^n\\ &= \tau\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\nabla_\tau^2(\tau^2(k_\tau^{\alpha-1}\star S_{\alpha,1}\star Bv)^j), \qquad (3.36)\end{aligned}$$

for all $n\geq 2$. As $\{S_{\alpha,1}^n\}_{n\in\mathbb{N}_0}$ is an $(\alpha,1)$-resolvent sequence, we get by (3.28)

$$(k_\tau^{\alpha-1}\star S_{\alpha,1}\star Bv)^n = \frac{1}{\tau}\sum_{j=0}^{n} k_\tau^\alpha(n-j)Bv^j + \tau A\sum_{j=0}^{n} k_\tau^\alpha(n-j)(k_\tau^{\alpha-1}\star S_{\alpha,1}\star Bv)^j,\quad n\geq 2. \tag{3.37}$$

By (3.36),(3.37), and (3.30)–(3.32) we have

$$\begin{aligned}{}_C\nabla^\alpha(\tau^2(k_\tau^{\alpha-1}\star S_{\alpha,1}\star Bv)^n) &= \nabla_\tau^{-(2-\alpha)}\nabla_\tau^2(\tau^2(k_\tau^{\alpha-1}\star S_{\alpha,1}\star Bv))^n\\ &= \sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\left[\sum_{l=0}^{j} k_\tau^\alpha(j-l)Bv^l - 2\sum_{l=0}^{j-1} k_\tau^\alpha(j-1-l)Bv^l\right.\\ &\quad\left.+\sum_{l=0}^{j-2} k_\tau^\alpha(j-2-l)Bv^l\right]\\ &\quad+\tau^2 A\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\left[\sum_{l=0}^{j} k_\tau^\alpha(j-l)(k_\tau^{\alpha-1}\star S_{\alpha,1}\star Bv)^l\right.\\ &\quad\left.-2\sum_{l=0}^{j-1} k_\tau^\alpha(j-1-l)(k_\tau^{\alpha-1}\star S_{\alpha,1}\star Bv)^l + \sum_{l=0}^{j-2} k_\tau^\alpha(j-2-l)(k_\tau^{\alpha-1}\star S_{\alpha,1}\star Bv)^l\right]\\ &= \tau^2(k_\tau^{\alpha-1}\star S_{\alpha,1}\star Bv)^n + Bv^n. \qquad (3.38)\end{aligned}$$

Now, we define the sequence $(u^n)_{n\in\mathbb{N}_0}$ by $u^n := S^n_{\alpha,1}x_0 + \tau(k^1_\tau \star S_{\alpha,1})^n x_1 + \tau^2(k^{\alpha-1}_\tau \star S_{\alpha,1} \star Bv)^n)$, for $n \geq 2$ and $u^0 := x_0$, $u^1 := x_1$, then by (3.33), (3.35) and (3.38) we have that

$$\begin{aligned} {}_C\nabla^\alpha(u^n) &= {}_C\nabla^\alpha \left(S^n_{\alpha,1}x_0 + \tau(k^1_\tau \star S_{\alpha,1})^n x_1 + \tau^2(k^{\alpha-1}_\tau \star S_{\alpha,1} \star Bv)^n)\right) \\ &= AS^n_{\alpha,1}x_0 + A(\tau(k^1_\tau \star S_{\alpha,1})^n x_1) + f^n + A(\tau^2(k^{\alpha-1}_\tau \star S_{\alpha,1} \star Bv)^n) \\ &= Au^n + Bv^n, \end{aligned}$$

for all $n \geq 2$, that is, $(u^n)_{n\in\mathbb{N}_0}$ solves the equation

$${}_C\nabla^\alpha u^n = Au^n + Bv^n, \quad n \geq 2,$$

under the initial conditions $u^0 = x_0$, and $u^1 = x_1$. The uniqueness, follows from [17, Proposition 3.3]. □

Remark 3.17. *From* (2.19) *and* (2.20) *it follows that that solution to Equation* (3.26) *can be written as*

$$u^n := S^n_{\alpha,1}x_0 + S^n_{\alpha,2}x_1 + \tau(S_{\alpha,\alpha} \star Bv)^n, n \geq 2.$$

Similarly to Proposition 3.16, we can prove the following result.

Proposition 3.18. *Assume that A is the generator of an $(\alpha,1)$-resolvent sequence $\{S^n_{\alpha,1}\}_{n\in\mathbb{N}_0}$. If $(f^j)_{j\in\mathbb{N}_0}$ is a given sequence and $x_0, x_1 \in X$, then the discrete fractional system* (1.5) *has a unique solution for each control $v \in \ell^2(\mathbb{N}_0, U)$ given by the sequence*

$$u^n := S^n_{\alpha,1}x_0 + \tau(k^1_\tau \star S_{\alpha,1})^n x_1 + \tau^2(k^{\alpha-1}_\tau \star S_{\alpha,1} \star Bv)^n + \tau^2(k^{\alpha-1}_\tau \star S_{\alpha,1} \star f)^n,$$

for all $n \geq 2$ and $u^0 := x_0, u^1 := x_1$.

Finally, let us consider the following discrete fractional system

$$\left\{ \begin{array}{rcl} {}_C\nabla^\alpha u^n &=& Au^n + f(n,u^n) + Bv^n, \quad n \geq 2, \\ u^0 &=& x_0 \\ u^1 &=& x_1, \end{array} \right. \tag{3.39}$$

where $A : D(A) \subset X \to X$ is a closed linear operator in X, $B : U \to X$ is a bounded linear operator, the control $v \in \ell^2(\mathbb{N}_0, U)$, $x_0, x_1 \in X$, and $f : \mathbb{N}_0 \times X \to X$.

Inspired in Propositions 3.16 and 3.18 we introduce the following definition of solution to Problem (3.39).

Definition 3.19. *A sequence $(u^n)_{n\in\mathbb{N}_0}$ is called a* solution *of the discrete fractional system* (3.39) *if it satisfies*

$$u^n = S^n_{\alpha,1}x_0 + S^n_{\alpha,2}x_1 + \tau\sum_{j=0}^{n} S^{n-j}_{\alpha,\alpha}[f(j,u^j) + Bv^j],$$

for all $n \geq 2$ and $u^0 := x_0, u^1 := x_1$, for $x_0, x_1 \in X$ and $v \in \ell^2(\mathbb{N}_0, U)$.

4. Approximate controllability

In this section we study the approximate controllability of discrete fractional systems (3.26) and (3.39) of order $1 < \alpha < 2$. By Proposition 3.16 and Remark 3.17, the Problem (3.26) has a unique solution $(u^n)_{n\in\mathbb{N}_0}$ given by

$$u^n := S^n_{\alpha,1}x_0 + S^n_{\alpha,2}x_1 + \tau(S_{\alpha,\alpha} \star Bv)^n,$$

for all $n \geq 2$ and $u^0 := x_0$, $u^1 := x_1$.

Let $(u^n)_{n\in\mathbb{N}_0}$ be the solution to the discrete fractional system (3.26) corresponding to the control v.

Definition 4.20. *The system* (3.26) *(respectively,* (3.39)*) is said to be* approximately controllable *on $[0,N_0]_{\mathbb{N}_0} := [0,N_0] \cap \mathbb{N}_0$ if for any $x_0, x_1 \in X$, $\varepsilon > 0$ and every desired final state $x_{N_0} \in X$, there exists a control $v \in \ell^2(\mathbb{N}_0, U)$ such that, the corresponding solution $(u^n)_{n\in\mathbb{N}_0}$ of* (3.26) *(respectively,* (3.39)*) satisfies $\|u^{N_0} - x_{N_0}\| < \varepsilon$.*

To study the approximate controllability of the system (3.26) we need to introduce the following operators:

For any fixed $n \in \mathbb{N}_0$, we define $\Gamma_{n,\tau} : \ell^2(\mathbb{N}_0, U) \to X$ by

$$\Gamma_{n,\tau} v := \tau \sum_{j=0}^{n} S_{\alpha,\alpha}^{n-j} B v^j, \quad v = (v^j)_{j\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0, U),$$

and the *grammian map* $L_{n,\tau} : X \to X$ by $L_{n,\tau} := \Gamma_{n,\tau}\Gamma_{n,\tau}^*$, where $*$ is used to denote the adjoint. In addition, we define the resolvent operator $R(\lambda, L_{n,\tau}) := (\lambda I + L_{n,\tau})^{-1}$, for $\lambda > 0$.

Proposition 4.21. *The adjoint $\Gamma_{n,\tau}^* : X \to \ell^2(\mathbb{N}_0, U)$ of the operator $\Gamma_{n,\tau}$ is given by*

$$(\Gamma_{n,\tau}^* x)^j = \begin{cases} \tau B^* S_{\alpha,\alpha}^{n-j^*} x, & 0 \le j \le n, \\ 0, & j > n. \end{cases}$$

for each $n \in \mathbb{N}_0$ and $x \in X$. Moreover, $L_{n,\tau} x = \tau^2 \sum_{j=0}^{n} S_{\alpha,\alpha}^{n-j} B B^* S_{\alpha,\alpha}^{n-j^*} x$, for $n \in \mathbb{N}_0, x \in X$.

Proof. Let $x \in X$ and $v \in \ell^2(\mathbb{N}_0, U)$. If $n \in \mathbb{N}_0$, then

$$\langle \Gamma_{n,\tau} v, x\rangle_X = \tau \sum_{j=0}^{n} \langle S_{\alpha,\alpha}^{n-j} B v^j, x\rangle_U = \sum_{j=0}^{n} \langle v^j, \tau B^* S_{\alpha,\alpha}^{n-j^*} x\rangle_U = \langle v, \Gamma_{n,\tau}^* x\rangle_{\ell^2(\mathbb{N}_0,U)}.$$

□

By the representation of $S_{\alpha,\alpha}^n$ as a finite combination of the resolvent operator R_τ given in (2.13), we observe that if A is a closed and densely defined operator on X, then the adjoint $S_{\alpha,\alpha}^{n^*} : X \to X$ of $S_{\alpha,\alpha}^n$ is given by

$$S_{\alpha,\alpha}^{n^*} x = \sum_{j=1}^{n+1} a_{n,j} (R_\tau^j)^* x = \sum_{j=1}^{n+1} a_{n,j} \tau^{-\alpha j} \left(\tau^{-\alpha} - A^*\right)^{-j} x, \tag{4.40}$$

for each $n \in \mathbb{N}_0$ and $x \in X$.

The next result follows similarly to [25, Theorem 2.1]. We omit its proof.

Proposition 4.22. *The system* (3.26) *is approximately controllably on* $[0, N_0]_{\mathbb{N}_0}$ *if and only if, one of the following statements holds*

(1) $\overline{\mathrm{Rang}(\Gamma_{N_0,\tau})} = X$.
(2) $\ker(\Gamma_{N_0,\tau}^*) = \{0\}$.
(3) $\langle L_{N_0,\tau} x, x\rangle > 0$ *for all* $0 \neq x \in X$.
(4) *If* $B^* S_{\alpha,\alpha}^{N_0-j^*} x = 0$ *for all* $0 \le j \le N_0$*, then* $x = 0$.
(5) $\lim_{\lambda\to 0^+} \lambda(\lambda I + L_{N_0,\tau})^{-1} = 0$ *in the strong operator topology.*
(6) *For all* $x \in X$ *we have that* $\Gamma_{N_0,\tau} v_\lambda = x - \lambda(\lambda I + L_{N_0,\tau})^{-1} x$*, where*

$$v_\lambda = v_{N_0,\lambda} := \Gamma_{N_0,\tau}^* (\lambda I + L_{N_0,\tau})^{-1} x \in \ell^2(\mathbb{N}_0, U), \quad \lambda \in (0, 1].$$

By (5) and (6) in Proposition 4.22 we conclude that

$$\lim_{\lambda\to 0^+} \Gamma_{N_0,\tau} v_\lambda = x, \quad \text{for all } x \in X. \tag{4.41}$$

This means that the family of operators $T_{N_0,\lambda} : X \to \ell^2(\mathbb{N}_0, U)$ defined by

$$T_{N_0,\lambda} x := \Gamma_{N_0,\tau}^* (\lambda I + L_{N_0,\tau})^{-1} x, \quad \lambda \in (0, 1],$$

is an approximate inverse of $\Gamma_{N_0,\tau}$, that is,

$$\lim_{\lambda\to 0^+} \Gamma_{N_0,\tau} T_{N_0,\lambda} = I. \tag{4.42}$$

Now, we study the problem of finding a control v to Problem (3.26). Define the cost functional $J : \ell^\infty(\mathbb{N}_0, X) \times \ell^2(\mathbb{N}_0, U) \to \mathbb{R}$ by

$$J(u,v) := \|u^{N_0} - x_{N_0}\|^2 + \lambda \sum_{j=0}^{N_0} \|v^j\|^2, \tag{4.43}$$

where $\lambda > 0$, $u = (u^n)_{n\in\mathbb{N}_0}$ is the solution to (3.26) with the control $v \in \ell^2(\mathbb{N}_0, U)$ and desired state $x_{N_0} \in X$.

We define the admissible class $\mathcal{A}_{\rm ad}$ by

$$\mathcal{A}_{\rm ad} := \{(u,v) : u \text{ is the unique solution to (3.26) with control } v \in \mathcal{U}_{\rm ad}\}$$

where $\mathcal{U}_{\rm ad}$ is the *admissible control class*, that is $\mathcal{U}_{\rm ad} = \ell^2(\mathbb{N}_0; U)$. By Proposition 3.16 we notice that if A generates an $(\alpha, 1)$-resolvent sequence then $\mathcal{A}_{\rm ad} \neq \emptyset$.

Let us to consider the optimal control problem

$$\min_{(u,v)\in\mathcal{A}_{\rm ad}} J(u,v). \tag{4.44}$$

An *optimal solution* to Problem (4.44) is a solution (u_0, v_0) (known as optimal pair) of (4.44). The control v_0 is called an *optimal control*.

The next result is a discrete version of [37, Theorem 3.1] and gives a solution to the Problem (4.44).

Theorem 4.23. *Let $\tau > 0$. Assume that A is the generator of an $(\alpha, 1)$-resolvent sequence $\{S^n_{\alpha,1}\}_{n\in\mathbb{N}_0}$ such that $\|S^n_{\alpha,1}\| \le M$ for all $n \in \mathbb{N}_0$. Suppose that $(\tau^{-\alpha} - A)^{-1}$ is a compact operator for $\tau^{-\alpha} \in \rho(A)$. Let $x^0, x^1 \in X$ be given. Then there exists at least one pair $(\overline{u}_0, \overline{v}_0) \in \mathcal{A}_{\rm ad}$ such that $J(\cdot,\cdot)$ attains its minimum at $(\overline{u}_0, \overline{v}_0)$.*

Proof. Let $J := \inf_{v\in\mathcal{U}_{\rm ad}} J(u,v)$. As $0 \le J < \infty$, by the definition of infimum, there exists a sequence $v_n \in \mathcal{U}_{\rm ad}$ such that

$$\lim_{n\to\infty} J(u_n, v_n) = J,$$

where u_n is the unique solution to (3.26) with control v_n and $u^0_n = x_0$, $u^1_n = x_1$ and the desired state $x_{N_0} \in X$. By Proposition 3.16 and Remark 3.17, u_n is given by

$$u^m_n = S^m_{\alpha,1} x_0 + S^m_{\alpha,2} x_1 + \tau \sum_{j=0}^{m} S^{m-j}_{\alpha,\alpha} B v^j_n, \quad m \ge 2. \tag{4.45}$$

Since $0 \in \mathcal{U}_{\rm ad}$ we may assume that $J(u_n, v_n) \le J(u, 0)$. By the definition of $J(\cdot,\cdot)$ we have

$$J(u_n, v_n) = \|u^{N_0}_n - x_{N_0}\|^2 + \lambda \sum_{j=0}^{N_0} \|v^j_n\|^2 \le \|u^{N_0} - x_{N_0}\|^2 \le 2(\|u^{N_0}\|^2 + \|x_{N_0}\|^2).$$

Therefore, there exists $R > 0$ (large enough) such that $0 \le J(u_n, v_n) \le R$ for all $n \in \mathbb{N}_0$. This also implies that

$$\sum_{j=0}^{N_0} \|v^j_n\|^2 \le C, \tag{4.46}$$

for all $n \in \mathbb{N}_0$, where C is a positive constant. By (4.45) and (2.19) we may proceed as in the proof of (2.25) to obtain

$$\begin{aligned}\|u_n^m\| &\leq \|S_{\alpha,1}^m x_0\| + \|S_{\alpha,2}^m x_1\| + \tau \sum_{j=0}^{m} \|S_{\alpha,\alpha}^{m-j} B v_n^j\| \\ &\leq M\|x_0\| + M(m+1)\tau\|x_1\| + \frac{\tau^\alpha M(m+1)^\alpha m^{1/2}\|B\|}{\Gamma(\alpha)} \left(\sum_{j=0}^{m} \|v_n^j\|^2\right)^{1/2},\end{aligned}$$

for all $0 \leq m \leq N_0$. This means that $(u_n)_{n\in\mathbb{N}_0}$ is a bounded sequence. Hence, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}_0}$ of $(u_n)_{n\in\mathbb{N}_0}$ which converges weakly to $\overline{u}_0$ in $\ell^2(\mathbb{N}_0, X)$. Similarly, from (4.46) we can find a subsequence $(v_{n_k})_{k\in\mathbb{N}_0}$ of $(v_n)_{n\in\mathbb{N}_0}$ converging weakly to $\overline{v}_0$ in $\ell^2(\mathbb{N}_0, U)$, and as B is a bounded operator, Bv_{n_k} converges weakly to $B\overline{v}_0$ as $k \to \infty$.

Since $(\tau^{-\alpha} - A)^{-1}$ is a compact operator, the representation (2.13) of $S_{\alpha,\alpha}^m$ implies that $S_{\alpha,\alpha}^m$ is a compact operator for all $m \in \mathbb{N}_0$. By [42, Corollary 2.3], the operator $T : \ell^2(\mathbb{N}_0, X) \to \ell^\infty(\mathbb{N}_0, X)$ given by $T(g)(m) := \sum_{j=0}^m S_{\alpha,\alpha}^{m-j} g^j$, for $g = (g^m)_{m\in\mathbb{N}_0}$ is also compact. Hence,

$$\left\| \sum_{j=0}^{m} S_{\alpha,\alpha}^{m-j} B v_{n_k}^j - \sum_{j=0}^{m} S_{\alpha,\alpha}^{m-j} B \overline{v}_0^j \right\| \to 0, \quad \text{as} \quad k \to \infty, \tag{4.47}$$

for all $0 \leq m \leq N_0$. Let $\overline{w}_0 = (\overline{w}_0^j)_{j\in\mathbb{N}_0}$ be the solution to (3.26) with control $\overline{v}_0 = (\overline{v}_0^j)_{j\in\mathbb{N}_0}$ (which exists and it is unique by Proposition 3.16). Since u_{n_k} is the unique solution to (3.26) with control v_{n_k}, by (4.45) and (4.47) we obtain

$$\|u_{n_k}^m - \overline{w}_0^m\| \to 0 \quad \text{as} \quad k \to \infty,$$

for all $0 \leq m \leq N_0$. As $(u_{n_k})_{k\in\mathbb{N}_0}$ converges weakly to $\overline{u}_0$, we have $\overline{w}_0 = \overline{u}_0$, and

$$\overline{u}_0^m = S_{\alpha,1}^m x_0 + S_{\alpha,2}^m x_1 + \tau \sum_{j=0}^{m} S_{\alpha,\alpha}^{m-j} B \overline{v}_0^j, \quad m \geq 2, \tag{4.48}$$

where $\overline{u}_0^0 = x_0$ and $\overline{u}_0^1 = x_1$. This means that the sequence $\overline{u}_0 = (\overline{u}_0^m)_{m\in\mathbb{N}_0}$ is a solution to (3.26) with control $\overline{v}_0 = (\overline{v}_0^m)_{m\in\mathbb{N}_0}$ and by Proposition 3.16 it is the unique solution to (3.26). This means that $(\overline{u}_0, \overline{v}_0) \in \mathcal{A}_{\text{ad}}$.

Finally, we claim that $(\overline{u}_0, \overline{v}_0)$ is a minimizer, that is $J = J(\overline{u}_0, \overline{v}_0)$. In fact, by [13, Propositions III.1.6, III.1.10 and II.4.5] $J(\cdot,\cdot)$ is continuous, convex on $\ell^2(\mathbb{N}_0, X) \times \ell^2(\mathbb{N}_0, U)$ and weakly lower semi-continuous. Since (u_{n_k}, v_{n_k}) converges weakly to $(\overline{u}_0, \overline{v}_0)$ in $\ell^2(\mathbb{N}_0, X) \times \ell^2(\mathbb{N}_0, U)$ we have

$$J \leq J(\overline{u}_0, \overline{v}_0) \leq \liminf_{k\to\infty} J(u_{n_k}, v_{n_k}) = \lim_{k\to\infty} J(u_{n_k}, v_{n_k}) = J,$$

and therefore $J = J(\overline{u}_0, \overline{v}_0)$.

□

Remark 4.24. As J defined in (4.43) *is convex, the system* (3.26) *is linear and the admissible control class $\mathcal{U}_{\text{ad}} = \ell^2(\mathbb{N}_0; U)$ is convex, then the optimal control $\overline{v}_0$ Theorem 4.23 is unique.*

Proposition 4.25. *Suppose that $v = (v^j)_{j\in\mathbb{N}_0}$ is the optimal control of* (4.43)*, then*

$$v^j = \tau B^* S_{\alpha,\alpha}^{N_0-j^*} R(\lambda, L_{N_0,\tau}) p(u), \quad 0 \leq j \leq N_0,$$

where

$$p(u) := x_{N_0} - S_{\alpha,1}^{N_0} x_0 - S_{\alpha,2}^{N_0} x_1. \tag{4.49}$$

Proof. Let v the optimal control of (4.43). Define $I(\varepsilon) := J(u_{v+\varepsilon w}, v+\varepsilon w)$ where $w \in \ell^2(\mathbb{N}_0, U)$ and $u_{v+\varepsilon w}$ is the unique solution of (3.26) with respect to the control $v + \varepsilon w$. Then $u_{v+\varepsilon w}$ verifies

$$u^n_{v+\varepsilon w} = S^n_{\alpha,1}x_0 + S^n_{\alpha,2}x_1 + \tau\sum_{j=0}^{n} S^{n-j}_{\alpha,\alpha}B(v^j + \varepsilon w^j), \quad n \geq 2.$$

Computing the variation of J, we get

$$\frac{d}{d\varepsilon}I(\varepsilon)\big|_{\varepsilon=0} = 2\tau\sum_{j=0}^{N_0}\left\langle u^{N_0} - x_{N_0}, S^{N_0-j}_{\alpha,\alpha}Bw^j\right\rangle + 2\sum_{j=0}^{N_0}\lambda\left\langle v^j, w^j\right\rangle = 2\sum_{j=0}^{N_0}\left\langle \tau B^* S^{N_0-j^*}_{\alpha,\alpha}(u^{N_0} - x_{N_0}) + \lambda v^j, w^j\right\rangle.$$

As $\varepsilon = 0$ is a critical point of I and $w \in \ell^2(\mathbb{N}_0, U)$ is an arbitrary element, we get

$$v^j = -\lambda^{-1}\tau B^* S^{N_0-j^*}_{\alpha,\alpha}(u^{N_0} - x_{N_0}), \quad 0 \leq j \leq N_0. \tag{4.50}$$

Then

$$u^{N_0} = S^{N_0}_{\alpha,1}x_0 + S^{N_0}_{\alpha,2}x_1 - \tau^2\lambda^{-1}\sum_{j=0}^{N_0} S^{N_0-j}_{\alpha,\alpha}BB^*S^{N_0-j^*}_{\alpha,\alpha}(u^{N_0} - x_{N_0}) = S^{N_0}_{\alpha,1}x_0 + S^{N_0}_{\alpha,2}x_1 - \lambda^{-1}L_{N_0,\tau}(u^{N_0} - x_{N_0}).$$

For p defined in (4.49) we have

$$u^{N_0} - x_{N_0} = -p(u) - \lambda^{-1}L_{N_0,\tau}(u^{N_0} - x_{N_0}),$$

which implies that

$$u^{N_0} - x_{N_0} = -\lambda(\lambda I + L_{N_0,\tau})^{-1}p(u) = -\lambda R(\lambda, L_{N_0,\tau})p(u).$$

From (4.50) we conclude that

$$v^j = \tau B^* S^{N_0-j^*}_{\alpha,\alpha}R(\lambda, L_{N_0,\tau})p(u), \quad 0 \leq j \leq N_0.$$

□

Motivated by the linear case, for any $\lambda > 0$ and $x_{N_0} \in X$, consider the system

$$\begin{cases} u^n &= S^n_{\alpha,1}x_0 + S^n_{\alpha,2}x_1 + \tau\sum_{j=0}^{n} S^{n-j}_{\alpha,\alpha}[f(j,u^j) + Bv^j], \quad 2 \leq n \leq N_0, \text{ and } u^0 = x_0, u^1 = x_1, \\ v^j &= \tau B^* S^{N_0-j^*}_{\alpha,\alpha}R(\lambda, L_{N_0,\tau})\overline{p}(u), \quad 0 \leq j \leq N_0, \\ \overline{p}(u) &= x_{N_0} - S^{N_0}_{\alpha,1}x_0 - S^{N_0}_{\alpha,2}x_1 - \tau\sum_{j=0}^{N_0} S^{N_0-j}_{\alpha,\alpha}f(j,u^j). \end{cases} \tag{4.51}$$

where $x_0, x_1 \in X$ and $v = (v^j)_{j\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0, U)$. To show the approximate controllability of the discrete fractional system (3.39) we will use (4.51). More precisely, under suitable conditions, we will first show that for any $\lambda > 0$ and $x_{N_0} \in X$, the system (3.39) has at least one solution $u_\lambda = (u^j_\lambda)_{j\in\mathbb{N}_0}$ for any control $v = (v^j)_{j\in\mathbb{N}_0}$ satisfying (4.51). And, then for any $x_{N_0} \in X$, we will use u_λ to approximate x_{N_0}.

For $r > 0$ we define $W_r := \{u \in \ell^\infty(\mathbb{N}_0, X) : \|u\| \leq r\}$. Clearly, W_r is a closed, bounded and convex set. Consider the following assumptions

- **H1:** For $f : \mathbb{N}_0 \times X \to X$ there exists a constant K such that $\|f(j,x)\| \leq K$ for all $(j,x) \in \mathbb{N}_0 \times X$.
- **H2:** $(\tau^{-\alpha} - A)^{-1}$ is a compact operator for all $\tau^{-\alpha} \in \rho(A)$ and A generates an $(\alpha,1)$-resolvent sequence such that $\|S^n_{\alpha,1}\| \leq M$ for all $n \in \mathbb{N}_0$.
- **H3:** $\|R(\lambda, L_{N_0,\tau})\| \leq \frac{1}{\lambda}$ for all $\lambda > 0$.

Theorem 4.26. *Assume that conditions* $(H1)-(H3)$ *are satisfied. Then the system* (3.39) *with the control* $v = (v^j)_{j\in\mathbb{N}_0}$ *given in* (4.51) *has at least one solution.*

Proof. For a fixed $x_{N_0} \in X$ and $\lambda > 0$, consider the solution operator $Q : \ell^\infty(\mathbb{N}_0, X) \to \ell^\infty(\mathbb{N}_0, X)$ defined by

$$(Qu)^n = \begin{cases} S^n_{\alpha,1}x_0 + S^n_{\alpha,2}x_1 + \tau \sum_{j=0}^{n} S^{n-j}_{\alpha,\alpha}[f(j,u^j) + Bv^j], & 2 \le n \le N_0, \\ 0, & n > N_0, \end{cases}$$

and $(Qu)^0 = x_0, (Qu)^1 = x_1$, where $u = (u^j)_{j\in\mathbb{N}_0} \in \ell^\infty(\mathbb{N}_0, X)$ and the control v is given in (4.51) by

$$v^j = \tau B^* S^{N_0-j^*}_{\alpha,\alpha} R(\lambda, L_{N_0,\tau})\overline{p}(u), \quad 0 \le j \le N_0,$$

with

$$\overline{p}(u) := x_{N_0} - S^{N_0}_{\alpha,1}x_0 - S^{N_0}_{\alpha,2}x_1 - \tau \sum_{j=0}^{N_0} S^{N_0-j}_{\alpha,\alpha} f(j,u^j).$$

As $(\tau^{-\alpha} - A)^{-1}$ is a compact operator, $S^n_{\alpha,1}, S^n_{\alpha,2}$ and $S^n_{\alpha,\alpha}$ are compact operators (by (2.13)). Since Q has finite rank, Q is a compact operator.

We will show that there exists a positive number $r_0 > 0$ such that $QW_{r_0} \subset W_{r_0}$. In fact, $(H2)$, (2.19), (2.20) and (2.25) imply that

$$\|S^n_{\alpha,1}\| \le M, \quad \|S^n_{\alpha,1}\| \le M(n+1)\tau, \quad \|S^n_{\alpha,\alpha}\| \le M\frac{(\tau(n+1))^{\alpha-1}}{\Gamma(\alpha)}, \tag{4.52}$$

for all $n \in \mathbb{N}_0$. Thus, for any $0 \le n \le N_0$ we have

$$\|(Qu)^n\| \le M\|x_0\| + M(n+1)\tau\|x_1\| + K\overline{C}_\alpha + \tau \sum_{j=0}^{n} \|S^{n-j}_{\alpha,\alpha}Bv^j\|,$$

where $\overline{C}_\alpha := \frac{\tau^\alpha M(N_0+1)^\alpha}{\Gamma(\alpha)}$. In addition, as

$$\|\overline{p}(u)\| \le \|x_{N_0}\| + M\|x_0\| + M(N_0+1)\tau\|x_1\| + K\overline{C}_\alpha,$$

we have

$$\begin{aligned}\tau \sum_{j=0}^{n} \|S^{n-j}_{\alpha,\alpha}Bv^j\| &\le \tau^2\|B\|^2\|S^{n-j}_{\alpha,\alpha}\|\|S^{N_0-j}_{\alpha,\alpha}\|\|R(\lambda, L_{N_0,\tau})\|\|\overline{p}(u)\| \\ &\le \frac{\tau^2\overline{C}^2_\alpha\|B\|^2}{\lambda}\left[\|x_{N_0}\| + M\|x_0\| + M(N_0+1)\tau\|x_1\| + K\overline{C}_\alpha\right].\end{aligned}$$

Therefore, for $r_0 > 0$ large enough we have $QW_{r_0} \subset W_{r_0}$. By the Schauder's fixed point theorem, the operator Q has a fixed point in W_{r_0}, which is a solution to the system (3.39). □

Finally, we consider the following hypothesis:

H4: $\lambda R(\lambda, L_{N_0,\tau}) \to 0$ as $\lambda \to 0^+$ in the strong operator topology.

Theorem 4.27. *Assume that conditions* $(H1)-(H4)$ *are satisfied. Then the system* (3.39) *is approximately controllable on* $[0, N_0]_{\mathbb{N}_0}$.

Proof. By Theorem 4.26, we have that for every $\lambda > 0$ and $x_{N_0} \in X$, there exists a solution $u_\lambda = (u^j_\lambda)_{j\in\mathbb{N}_0}$ of the system (3.39) with the control

$$v^j_\lambda = \tau B^* S^{N_0-j^*}_{\alpha,\alpha} R(\lambda, L_{N_0,\tau})\overline{p}(u_\lambda), \quad 0 \le j \le N_0,$$

where $\overline{p}$ is defined in (4.51). Then

$$\begin{aligned}
u_\lambda^{N_0} &= S_{\alpha,1}^{N_0}x_0 + S_{\alpha,2}^{N_0}x_1 + \tau\sum_{j=0}^{N_0} S_{\alpha,\alpha}^{N_0-j} f(j,u_\lambda^j) + \tau\sum_{j=0}^{N_0} S_{\alpha,\alpha}^{N_0-j} Bv_\lambda^j \\
&= S_{\alpha,1}^{N_0}x_0 + S_{\alpha,2}^{N_0}x_1 + \tau\sum_{j=0}^{N_0} S_{\alpha,\alpha}^{N_0-j} f(j,u_\lambda^j) + L_{N_0,\tau}R(\lambda, L_{N_0,\tau})\overline{p}(u_\lambda) \\
&= x_{N_0} - \overline{p}(u_\lambda) + L_{N_0,\tau}R(\lambda, L_{N_0,\tau})\overline{p}(u_\lambda) \\
&= x_{N_0} - \lambda R(\lambda, L_{N_0,\tau})\overline{p}(u_\lambda). \qquad (4.53)
\end{aligned}$$

By $(H1)$, we have

$$\tau\sum_{j=0}^{N_0} \|f(j,u_\lambda^j)\| \le KN_0,$$

and therefore, there exists a subsequence of $f(\cdot, u_\lambda^{\cdot})$, denoted again by $f(\cdot, u_\lambda^{\cdot})$, which converges weakly to an element $w = (w^j)_{j\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0, X)$. Now, we define

$$z := x_{N_0} - S_{\alpha,1}^{N_0}x_0 - S_{\alpha,2}^{N_0}x_1 + \tau\sum_{j=0}^{N_0} S_{\alpha,\alpha}^{N_0-j} w^j.$$

Hence,

$$\|\overline{p}(u_\lambda) - z\| \le \tau\left\|\sum_{j=0}^{N_0} S_{\alpha,\alpha}^{N_0-j}[f(j,u_\lambda^j) - w^j]\right\|. \qquad (4.54)$$

Since $(\tau^{-\alpha} - A)^{-1}$ is a compact operator, $S_{\alpha,\alpha}^n$ is a compact operator for all $n \in \mathbb{N}_0$, and therefore the operator $T : \ell^2(\mathbb{N}_0, X) \to \ell^\infty(\mathbb{N}_0, X)$ defined by $T(g)(m) := \sum_{j=0}^m S_{\alpha,\alpha}^{m-j} g^j$, for $g = (g^m)_{m\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0, X)$ is compact. This implies that

$$\left\|\sum_{j=0}^{N_0} S_{\alpha,\alpha}^{N_0-j}[f(j,u_\lambda^j) - w^j]\right\| \to 0, \quad \text{as} \quad \lambda \to 0^+.$$

From (4.53), (4.54) and $(H4)$ it follows that

$$\|u_\lambda^{N_0} - x_{N_0}\| \le \|\lambda R(\lambda, L_{N_0,\tau})\overline{p}(u_\lambda)\| \le \|\lambda R(\lambda, L_{N_0,\tau})\|\|\overline{p}(u_\lambda) - z\| + \|\lambda R(\lambda, L_{N_0,\tau})z\| \to 0, \quad \text{as} \quad \lambda \to 0^+.$$

We conclude that the system (3.39) is approximately controllable on $[0, N_0]_{\mathbb{N}_0}$. □

5. Application

In this section, we discuss the approximate controllability of a fractional discrete system of order $1 < \alpha < 2$.

On the space $X = L^2([0,\pi])$ we define the operator $A : D(A) \subset X \to X$ by $Aw(x) = w''(x)$ with domain $D(A) := H^2([0,\pi]) \cap H_0^1([0,\pi])$. Then A is a self-adjoint operator, the spectrum of A is given by $\sigma(A) = \{-m^2 : m \in \mathbb{N}\}$, and the corresponding eigenfunctions are given by $\phi_m(x) = \sqrt{\frac{2}{\pi}}\sin(mx)$, for $x \in [0,\pi]$. Moreover, A can be written as

$$Aw = \sum_{m=1}^{\infty} -m^2\langle w, \phi_m\rangle\phi_m, \quad w \in D(A). \qquad (5.55)$$

Let us consider the problem

$$(5.56)\qquad \begin{cases} {}_C\nabla^\alpha u^n(s) &=& Au^n(s)+f(n,u^n(s))+v^n(s), \quad n\geq 2,\\ u^0(s) &=& x_0(s)\\ u^1(s) &=& x_1(s), \end{cases}$$

where $s\in[0,\pi]$, A is defined above, $B:U\to X$ is the bounded linear operator defined by $(Bv)^j=v^j$, for $v=(v^j)_{j\mathbb{N}_0}\in\ell^2(\mathbb{N}_0,U)$, f verifies the condition $(H1)$ and $x_0,x_1\in X$.

Writing $u^n:=u^n(\cdot), v^n:=v^n(\cdot), x_0=x_0(\cdot), x_1=x_1(\cdot)$, the system (5.56) can be expressed in the abstract form (3.39).

We notice that this problem can be seen as a time-discretization of the problem

$$(5.57)\qquad \begin{cases} \partial_t^\alpha u(t,s) &=& \dfrac{\partial^2 u(t,s)}{\partial s^2}+v(t,s)+f(t,u(t,s)), \quad t\in[0,T],\\ u(t,0) &=& u(t,\pi)=0, \quad t\in[0,T]\\ u(0,s) &=& \varphi_0(s), \quad s\in[0,\pi]\\ u_t(0,s) &=& \varphi_1(s), \quad s\in[0,\pi], \end{cases}$$

where $T>0$ and $\varphi_0,\varphi_1\in X$ are given functions. In fact, writing $u(t):=u(t,\cdot)$, $v(t):=v(t,\cdot)$ and $f(t)=f(t,\cdot)$, the problem (5.57) can be written in the abstract form

$$(5.58)\qquad \begin{cases} \partial_t^\alpha u(t) &=& Au(t)+v(t)+f(t,u(t)), \quad t\in[0,T],\\ u(0) &=& \varphi_0,\\ u_t(0) &=& \varphi_1 \end{cases}$$

Multiplying (5.58) by $\rho_n^\tau(t)$ (for a fixed $\tau>0$) and integrating over $\mathbb{R}$ we obtain (similarly to [36, Section 2]) the problem (5.56), where ${}_C\nabla^\alpha u^n(s)=\int_0^\infty \rho_n^\tau(t)\partial_t^\alpha u(t,s)dt$, $u^n(s)=\int_0^\infty\rho_n^\tau(t)u(t,s)dt$, (analogously for v^n,f^n) and $x_0=\varphi_0$, $x_1=\varphi_1$. We notice that ${}_C\nabla^\alpha u^n$ and u^n correspond to an approximation of the Caputo fractional derivative $\partial_t^\alpha u(t)$ and $u(t)$, respectively, evaluated at $t_n:=n\tau$.

On the other hand, it is a well-known fact that A generates the cosine family $\{C(t)\}_{t\in\mathbb{R}}$ given by

$$(5.59)\qquad C(t)x=\sum_{m=1}^\infty \cos(mt)\langle x,\phi_m\rangle\phi_m, \quad x\in X.$$

Then A generates the cosine sequence C^n defined by

$$C^n x=\int_0^\infty \rho_n^\tau(t)C(t)x\,dt, \quad n\in\mathbb{N}_0.$$

In fact, as A generates the cosine family $\{C(t)\}_{t\in\mathbb{R}}$, we have

$$(5.60)\qquad C(t)x=x+A\int_0^t (t-s)C(s)x\,ds, \quad t>0, x\in X.$$

Multiplying (5.60) by $\rho_n^\tau(t)$ and integrating over $\mathbb{R}$, we get (2.15).

We notice that by [18, Formula 3.944-6, p.498], C^n can be computed explicitly:

$$\begin{aligned} C^n x &= \sum_{m=1}^\infty \frac{1}{\tau^{n+1}n!}\int_0^\infty e^{-\frac{t}{\tau}}t^n\cos(mt)\,dt\langle x,\phi_m\rangle\phi_m\\ &= \sum_{m=1}^\infty \frac{1}{\tau^{n+1}}\frac{1}{(m^2+\tau^{-2})^{(n+1)/2}}\cos\left((n+1)\arctan(\tau m)\right)\langle x,\phi_m\rangle\phi_m. \end{aligned}$$

This implies that (for $n\geq 1$)

$$\|C^n x\|\leq \frac{1}{\tau^{n+1}}\sum_{m=1}^\infty\frac{1}{m^2}\|x\|\leq\frac{\pi^2}{6\tau^2}\|x\|.$$

As $\cos(\arctan(ar)) = 1/\sqrt{a^2r^2+1}$ we obtain for $n = 0$, that

$$C^0 x = \sum_{m=1}^{\infty} \frac{1}{(m^2+\tau^{-2})^{1/2}} \cos(\arctan(\tau m)) \langle x, \phi_m\rangle \phi_m \leq \frac{1}{\tau}\sum_{m=1}^{\infty} \frac{1}{m^2}\langle x, \phi_m\rangle \phi_m \leq \frac{\pi^2}{6\tau}\|x\|.$$

By Theorem 2.7, A generates the $(\alpha, 1)$-resolvent sequence $\{S^n_{\alpha,1}\}_{n\in\mathbb{N}_0}$ given by

$$S^n_{\alpha,1} x := \sum_{j=0}^{\infty} \varphi^{\tau}_{\frac{\alpha}{2},1-\frac{\alpha}{2}}(n,j) C^j x, \quad x \in X,$$

and by Propostion 2.5 we obtain

$$\|S^n_{\alpha,1}x\| \leq \sum_{j=0}^{\infty} \varphi^{\tau}_{\frac{\alpha}{2},1-\frac{\alpha}{2}}(n,j)\|C^j x\| \leq \frac{\pi^2}{6}\max\{1/\tau, 1/\tau^2\}\sum_{j=0}^{\infty} \varphi^{\tau}_{\frac{\alpha}{2},1-\frac{\alpha}{2}}(n,j) = \frac{\pi^2}{6}\max\{1/\tau, 1/\tau^2\} =: M,$$

for all $n \in \mathbb{N}_0$. This means that $\{S^n_{\alpha,1}\}_{n\in\mathbb{N}_0}$ verifies the hypothesis $(H2)$. We notice that by (2.11) and (5.55), the resolvent sequence $\{S^n_{\alpha,1}\}_{n\in\mathbb{N}_0}$ can be written as

$$S^n_{\alpha,1} x = \sum_{m=1}^{\infty}\sum_{j=0}^{\infty} k^{\alpha j+1}_{\tau}(n) m^j \langle x, \phi_m\rangle \phi_m, \quad x \in X.$$

On other hand, as A is a self-adjoint operator, by (4.40) we obtain that if $x \in X$ then

$$\langle L_{N_0,\tau} x, x\rangle = \tau^2 \sum_{j=0}^{N_0} \langle S^{N_0-j}_{\alpha,\alpha} S^{N_0-j^*}_{\alpha,\alpha} x, x\rangle = \tau^2 \sum_{j=0}^{N_0} \|S^{N_0-j}_{\alpha,\alpha} x\|^2.$$

If $\langle L_{N_0,\tau} x, x\rangle = 0$, then $S^j_{\alpha,\alpha} x = 0$ for all $0 \leq j \leq N_0$. By (2.12) we have

$$0 = S^j_{\alpha,\alpha} x = k^{\alpha}_{\tau}(j)x + \tau A(k^{\alpha}_{\tau} \star S_{\alpha,\beta})^j x = k^{\alpha}_{\tau}(j)x, \quad 0 \leq j \leq N_0,$$

which implies that $x = 0$. We conclude that $\langle L_{N_0,\tau} x, x\rangle > 0$, for all $x \neq 0$. By Proposition 4.22, the system (3.26) is approximately controllable, and therefore $\lambda R(\lambda, L_{N_0,\tau}) \to 0$ as $\lambda \to 0^+$ in the strong operator topology. This means that the assumption $(H4)$ holds true.

By Theorem 4.27, the system (5.56) is approximately controllable on $[0, N_0]_{\mathbb{N}_0}$.

5.1. **Conclusions.** In this paper we introduced a method based on resolvent sequences generated by a closed linear operator A to study the approximate controllability of an abstract fractional discrete system of order $1 < \alpha < 2$. The main results extend some previously obtained in the finite dimensional case, and the method used here shows that it is possible to use these sequences of linear operators similarly to the continuous case, which provides an interesting tool that could be used for the study of the controllability of other discrete infinite dimensional systems.

In future, one can study the connections between the approximate controllability of a discrete system and its continuous counterpart.

Conflict of interest. This work does not have any conflicts of interest.

Acknowledgements. The author thanks to the referees for the carefully reading of the manuscript and for making suggestions which have improved the previous version of this paper.

References

[1] L. Abadias, E. Álvarez, S. Díaz, *Subordination principle, Wright functions and large-time behaviour for the discrete in time fractional diffusion equation,* J. Math. Anal. Appl. **507** (2022) 125741.

[2] E. Álvarez, S. Díaz, C. Lizama, *C-Semigroups, subordination principle and the Lévy α-stable distribution on discrete time,* Comm. in Contemporary Mathematics, (2020) 2050063 (32 pages).

[3] W. Arendt, C. Batty, M. Hieber, F. Neubrander, *Vector-Valued Laplace transforms and Cauchy problems.* Monogr. Math., vol. **96**, Birkhäuser, Basel, 2011.

[4] S. Arora, M. Mohan, J. Dabas, *Approximate controllability of fractional order non-instantaneous impulsive functional evolution equations with state-dependent delay in Banach spaces,* IMA J. Math. Control Inform. 39 (2022), no. 4, 1103-1142.

[5] S. Arora, M. Mohan, J. Dabas, *Finite-approximate controllability of impulsive fractional functional evolution equations of order* $1 < \alpha < 2$, J. Optim. Theory Appl. (2023) **197**, 855-890.

[6] S. Arora, M. Mohan, J. Dabas, *Approximate controllability of impulsive fractional evolution equations of order* $1 < \alpha < 2$ *with state-dependent delay in Banach spaces,* Math. Methods in Appl. Sciences, **46** (1), (2023), 531-559.

[7] K. Balachandran, J. Park, *Controllability of fractional integrodifferential systems in Banach spaces,* Nonlinear Analysis: Hybrid Systems **3** (2009) 363-367.

[8] E. Bazhlekova, *Fractional evolution equations in Banach spaces,* Ph.D. thesis, Eindhoven University of Technology, 2001.

[9] Y. K. Chang, A. Pereira, R. Ponce, *Approximate controllability for fractional differential equations of Sobolev type via properties on resolvent operators,* Fract. Calc. Appl. Anal. **20**, (2017) 963–987.

[10] Y. K. Chang, Y. Pei, R. Ponce, *Existence and optimal controls for fractional stochastic evolution equations of Sobolev type via fractional resolvent operators,* J. Optim. Theory Appl. **182** (2019), no. 2, 558-572.

[11] J. Diblík, M. Fečkan, M. Pospísil, *On the new control functions for linear discrete delay systems,* SIAM J. Control Optim, Vol 52, No. 3, (2014), 1745-1760.

[12] J. Diblík, D. Khusainov, M. Ruzicková, *Controllability of Linear Discrete Systems with Constant Coefficients and Pure Delay,* SIAM J. Control Optim. Vol. 47, No. 3, (2008), 1140-1149.

[13] I. Ekeland, T. Turnbull, *Infinite-dimensional Optimization and Convexity,* The University of Chicago press, Chicago and London, 1983.

[14] C. Giordano, A. Laforgia, *Inequalities and monotonicity properties for the gamma function,* Proceedings of the Fifth International Symposium on Orthogonal Polynomials, Special Functions and their Applications (Patras, 1999). J. Comput. Appl. Math. 133 (2001), no. 1-2, 387-396.

[15] C. Goodrich, C. Lizama, *A transference principle for nonlocal operators using a convolutional approach: Fractional monotonicity and convexity,* Israel J of Mathematics, **236** (2) (2020), 533-589.

[16] C. Goodrich, A. Peterson, *Discrete fractional calculus,* Springer, Cham, 2015.

[17] J. González-Camus, R. Ponce, *Explicit representation of discrete fractional resolvent families in Banach spaces,* Fract. Calc. Appl. Anal., Vol. 24, No 6 (2021), pp. 1853–1878.

[18] I. Gradshteyn, I. Ryzhik, *Table of integrals, series and products,* Academic Press, New York, 2000.

[19] T. Guo, *Controllability and observability of impulsive fractional linear time-invariant system,* Comp. Math. Appl. **64** (2012) 3171-3182.

[20] H. Henríquez, C. Cuevas, *Approximate controllability of abstract discrete-time systems,* Advances in Difference Equations, Volume 2010, Article ID 695290.

[21] H. Henríquez, E. Hernández, *Approximate controllability of second-order distributed implicit functional systems,* Nonlinear Analysis **70** (2009) 1023-1039.

[22] J. Klamka, *Local controllability of fractional discrete-time semilinear systems,* Acta Mechanica et Automatica, **5**(2), 55–58, 2011.

[23] S. Kumar, N. Sukavanam, *Approximate controllability of fractional order semilinear systems with bounded delay.* J. Differential Equations 252 (2012), no. 11, 6163-6174.

[24] H. Leiva, J. Uzcategui, *Exact controllability for semilinear difference equation and application,* J. Difference Equ. Appl. **14** (2008), no. 7, 671–679.

[25] H. Leiva, J. Uzcategui, *Approximate controllability of discrete semilinear systems and applications,* Disc. Cont. Dyn. Systems, Series B, **21** (2016), no. 6, 1803-1812.

[26] C. Lizama, *The Poisson distribution, abstract fractional difference equations, and stability,* Proc. Amer. Math. Soc. **145** (2017), no. 9, 3809–3827.

[27] C. Lizama, *Abstract Linear Fractional Evolution Equations, in: Handbook of Fractional Calculus with Applications. Volume 2: Fractional Differential Equations.* Ed. by A. Kochubei and Y. Luchko, De Gruyter, Berlin, 2019, pp. 465-498.

[28] C. Lizama, R. Ponce, *Time discretization and convergence to superdiffusion equations via Poisson distribution,* Comm. in Pure and Applied Analysis, 22 (2) (2023), 572–596.

[29] N. Mahmudov, *Approximate controllability of fractional Sobolev-type evolution equations in Banach Spaces,* Abstract and Appl. Analysis, Volume 2013, Article ID 502839.

[30] N. Mahmudov, M. McKibben, *Approximate controllability of second-order neutral stochastic evolution equations,* Dyn. Contin. Discrete Impuls. Syst. **13**(5), 619–634 (2006).
[31] N. Mahmudov, V. Vijayakumar, R. Murugesu, *Approximate controllability of second-order evolution differential inclusions in Hilbert spaces,* Mediterr. J. Math. **13** (2016), 3433-3454.
[32] F. Mainardi, *Fractional calculus and waves in linear viscoelasticity: An introduction to mathematical models,* Imperial College Press, 2010.
[33] D. Mozyrska, E. Pawłuszewicz, M Wyrwas, *Local observability and controllability of nonlinear discrete-time fractional order systems based on their linearisation.* Int. Journal of Systems Science, **48** (4), 788-794, (2017).
[34] T. Mur, H. Henríquez, *Controllability of abstract systems of fractional order,* Frac. Calculus and Appl. Analysis, Vol. **18**, No 6 (2015), pp. 1379-1398.
[35] M. Ortigueira, *Fractional discrete-time linear systems,* In Proceedings of the 1997 IEEE International Conference on Acoustics, speech, and Signal Processing (ICASSP 97) (Vol. 3, pp. 2241–2244). Munich: IEEE Computer Society, (1997).
[36] R. Ponce, *Time discretization of fractional subdiffusion equations via fractional resolvent operators.* Comput. Math. Appl. **80** (2020), no. 4, 69-92.
[37] K. Ravikumar, M. Mohan, A. Anguraj, *Approximate controllability of a non-autonomous evolution equation in Banach spaces,* Numer. Algebra Control Optim. **11** (2021), no. 3, 461-485.
[38] R. Sakthivel, Y. Ren, N. Mahmudov. *On the approximate controllability of semilinear fractional differential systems,* Comput. Math. Appl. **62** (2011), 1451-1459.
[39] N. Sukavanam, S. Kumar, *Approximate controllability of fractional order semilinear delay systems,* J. Optim. Theory Appl. (2011) **151**, 373-384.
[40] C. Travis, G. Webb, *Cosine families and abstract nonlinear second order differential equations,* Acta Math. Acad. Sci. Hungaricae 32 (1978) 76-96.
[41] M. Tucsnak, G. Weiss, *Observation and control for operator semigroups,* Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Advanced Texts: Basel Textbooks. Birkhäuser Verlag, Basel, 2009. xii+483 pp.
[42] L. Weis, *A generalization of the Vidav-Jorgens perturbation theorem for semigroups and its application to transport theory,* J. of Mathematical Analysis and Applications, vol. 129, no. 1, pp. 6–23, 1988.
[43] M. Yang, M. Fečkan, J. Wang, *Relative controllability for delayed linear discrete system with second-order differences,* Qual. Theory of Dyn. Systems (2022) **21**, 113.
[44] Y. Zhou, J. He, *New results on controllability of fractional evolution systems with order $\alpha \in (1,2)$.* Evol. Equ. Control Theory 10(3), 491–509 (2021).

Universidad de Talca, Instituto de Matemáticas, Casilla 747, Talca-Chile.
Email address: `rponce@utalca.cl, rponce@inst-mat.utalca.cl`