PERIODIC SOLUTIONS TO SECOND-ORDER DIFFERENTIAL EQUATIONS WITH FADING MEMORY

RODRIGO PONCE

Abstract. We characterize existence and uniqueness of periodic strong and mild solutions to an abstract second order differential equation with memory in Banach spaces. Using vector-valued Fourier multipliers we give necessary and sufficient conditions in order to ensure the well-posedness of this equation in Lebesgue, Hölder and Besov spaces.

1. Introduction

Let Ω be a bounded open set in $\mathbb{R}^n$ $(n = 1, 2, 3)$ with a smooth boundary $\partial\Omega$. Denote by $u(x, t)$ the temperature of the point $x \in \Omega$ at the time $t \in \mathbb{R}$. The heat conduction in materials with fading memory can be described by the integro-differential equation

$$(1.1)\quad cu_{tt}(x,t)+\alpha(0)u_t(x,t)+\int_{-\infty}^{t}\alpha'(t-s)u_t(x,s)ds = \beta(0)\Delta u(x,t)+\int_{-\infty}^{t}\beta'(t-s)\Delta u(x,s)ds+F(x,t),$$

where Δ is the Laplacian, $\alpha(t)$ and $\beta(t)$ are positive functions called respectively, the heat-flux relaxation and the energy relaxation functions, $c \neq 0$ is a constant (known as the heat capacity) and F is a suitable function, see for instance Gurtin and Pipkin [18]. Typically, the relaxation functions α and β are taken as

$$\alpha(t) = \sum_{j=1}^{m} \alpha_i e^{-p_i t}, \quad \beta(t) = \sum_{j=1}^{M} \beta_i e^{-q_i t},$$

where $\alpha_i, \beta_i, p_i, q_i > 0$. We observe that equation (1.1) can be written in the abstract form

$$(1.2)\qquad u''(t) + \lambda u'(t) + Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + \int_{-\infty}^{t} b(t-s)u(s)ds = f(t) \quad t \in \mathbb{R},$$

where $\lambda = \frac{\alpha(0)}{c}$, $A = \frac{1}{c}(\alpha'(0)I - \beta(0)\Delta)$, $a(t) = \frac{\beta(0)^{-1}}{c}\alpha'(t)$, $b(t) = \frac{1}{c}[\alpha''(t) - \beta^{-1}(0)\alpha'(0)\beta'(t)]$ and $f(t) = F(\cdot, t)$.

The existence of periodic solutions to integro-differential equations in the form of (1.2) has been studied by several authors. For instance, if $\Omega \subset H$, is a bounded set, where H is a Hilbert space, the existence and uniqueness of periodic solutions to equation (1.2) has been studied by Tiehu in [29] in terms of the resolvent operator

$$(-k^2 + i\lambda k + (1 + a_k)A + b_k I)^{-1} = \frac{-1}{1 + a_k}\left(\frac{k^2 - i\lambda k - b_k}{1 + a_k} - A\right)^{-1},$$

for all $k \in \mathbb{Z}$. In the context of general Banach spaces, the existence of periodic solutions to second order integro-differential equations has been studied recently by S. Bu and G. Cai in [7, 9, 10, 11, 12].

2020 *Mathematics Subject Classification.* Primary 45A05, Secondary 34G10, 34K30, 45M15.

Key words and phrases. Maximal regularity; Second order differential equations; Equations with fading memory; Operator-valued Fourier multipliers.

In this paper, we characterize the existence and uniqueness of strong and mild periodic solutions to

$$u''(t)+\lambda u'(t)+Au(t)+\int_{-\infty}^{t}a(t-s)Au(s)ds+\int_{-\infty}^{t}b(t-s)Bu(s)ds=f(t), \tag{1.3}$$

under the periodic initial conditions $u(0)=u(2\pi)$ and $u'(0)=u'(2\pi)$, where $\lambda\in\mathbb{R}$, $A:D(A)\subset X\to X$ and $B:D(B)\subset X\to X$ are closed linear operators defined in a Banach space $X\equiv(X,\|\cdot\|)$, the functions a,b are suitable kernels and the function f belongs to $L^p([0,2\pi],X)$. To achieve this, we use a method based in some results on vector-valued Fourier multipliers. We remark that this method has been considered by several authors to obtain necessary and sufficient conditions in order to ensure the existence and uniqueness of L^p-strong periodic solutions to a variety of abstract differential equations, see for instance [2, 8, 15, 19, 20, 21, 22, 23, 24, 25, 26, 27] and the references therein. On the other hand, the same method has been used by several authors to characterize the existence of mild periodic solutions to first, second and fractional order differential equations in Banach spaces, see for instance in [2, 5, 6, 20]. However, to the best of our knowledge, this problem has not been considered in the case of integro-differential equations in the form of (1.3).

The paper is organized as follow. In Section 2, we give the preliminaries and we recall some results on vector-valued Fourier multipliers and R-bounded sets. In Section 3, we study L^p-strong solutions to equation (1.3). In Section 4, we consider the existence and uniqueness of strong solutions in periodic Hölder and Besov spaces. Section 5 deals with periodic mild solutions to equation (1.3). Finally, in the last section we give some applications of the abstract results.

2. Preliminaries

For $1\le p<\infty$, $L^p([0,2\pi],X)$ denotes the space of all 2π-periodic Bochner measurable and p-integrable X-valued functions. For a function $f\in L^1([0,2\pi],X)$ we denote by $\hat f(k)$, the k-th Fourier coefficient of f, that is

$$\hat f(k)=\frac{1}{2\pi}\int_0^{2\pi}e^{-ikt}f(t)dt,$$

for all $k\in\mathbb{Z}$. Observe that the Fourier coefficients of f determine completely the function f, that is, $\hat f(k)=0$ for all $k\in\mathbb{Z}$ if and only if $f(t)=0$ a.e. Let X,Y be Banach spaces. We denote by $\mathcal{B}(X,Y)$ to the space of all bounded and linear operators from X into Y. If $X=Y$, then we write simply $\mathcal{B}(X)$. Finally, given a closed linear operator A defined on X, $D(A)$ and $\rho(A)$ denote respectively, its domain and its resolvent set. By $[D(A)]$ we denote the domain of A equipped with the graph norm.

Now, we recall some preliminaries about operator-valued Fourier multipliers.

Definition 2.1. [2] *For $1\le p<\infty$, we say that a sequence $\{M_k\}_{k\in\mathbb{Z}}\subset\mathcal{B}(X,Y)$ is an L^p-multiplier if, for each $f\in L^p([0,2\pi],X)$, there exists $u\in L^p([0,2\pi],Y)$ such that*

$$\hat u(k)=M_k\hat f(k) \text{ for all } k\in\mathbb{Z}.$$

Observe that from the uniqueness theorem of Fourier series it follows that u is uniquely determined by f. On the other hand, if $\{M_k\}_{k\in\mathbb{Z}}\subset\mathcal{B}(X,Y)$ is an L^p-multiplier, then there exists a unique bounded operator $\mathcal{M}:L^p([0,2\pi],X)\to L^p([0,2\pi],Y)$ such that $\widehat{\mathcal{M}f}(k)=M_k\hat f(k)$ for all $k\in\mathbb{Z}$ and $f\in L^p([0,2\pi],X)$. It is easy to see that the set of all Fourier multipliers is a vector space and if X,Y,Z are Banach spaces and, $\{M_k\}_{k\in\mathbb{Z}}\subset\mathcal{B}(X,Y)$ and $\{N_k\}_{k\in\mathbb{Z}}\subset\mathcal{B}(Y,Z)$ are L^p-multipliers, then $\{M_kN_k\}_{k\in\mathbb{Z}}\subset\mathcal{B}(X,Z)$ is an L^p-multiplier as well. Moreover, if $\{M_k\}_{k\in\mathbb{Z}},\{N_k\}_{k\in\mathbb{Z}}\subset\mathcal{B}(X,Y)$ are L^p-multipliers, then $\{M_k+N_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier as well.

For $j\in\mathbb{N}$, r_j denotes the j-th Rademacher function on $[0,1]$ i.e. $r_j(t)=\text{sgn}(\sin(2^j\pi t))$, where sgn is the sign function. For $x\in X$, $r_j\otimes x$, denotes the vector valued function $t\mapsto r_j(t)x$.

Definition 2.2. *A family of operators $\mathcal{T} \subset \mathcal{B}(X,Y)$ is called R-bounded, if there is a constant $C_p > 0$ and $p \in [1,\infty)$ such that for each $N \in \mathbb{N}, T_j \in \mathcal{T}, x_j \in X, j = 1,...,N$ the inequality*

$$\| \sum_{j=1}^{N} r_j \otimes T_j x_j \|_{L^p((0,1),Y)} \leq C_p \| \sum_{j=1}^{N} r_j \otimes x_j \|_{L^p((0,1),X)} \tag{2.4}$$

is valid.

From Kahane's inequality it follows that if (2.4) holds for some $p \in [1,\infty)$ then it holds for all $p \in [1,\infty)$, and therefore the definition of R-boundedness is independent of p. The smallest C_p in (2.4) is called R-bound of $\mathcal{T}$, and we denote it by $R_p(\mathcal{T})$.

We remark that the notion of R-boundedness is an important tool in the study of multipliers. Moreover, a large classes of classical operators are R-bounded, see for instance [17] and reference therein for more details. Hence, this assumption is not too restrictive for the applications that we consider in this article.

Remark 2.3.

Now, we recall some properties of R-bounded families of operators. We refer to the reader to [16, Section 3].

(*a*) If $\mathcal{T} \subset \mathcal{B}(X,Y)$ is R-bounded then it is uniformly bounded, with

$$\sup\{\|T\| : T \in \mathcal{T}\} \leq R_p(\mathcal{T}).$$

(*b*) When X and Y are Hilbert spaces, $\mathcal{T} \subset \mathcal{B}(X,Y)$ is R-bounded if and only if $\mathcal{T}$ is uniformly bounded.

(*c*) Let X, Y be Banach spaces and $\mathcal{T}, \mathcal{S} \subset \mathcal{B}(X,Y)$ be R-bounded. Then

$$\mathcal{T} + \mathcal{S} = \{T + S : T \in \mathcal{T}, S \in \mathcal{S}\}$$

is R-bounded as well, and $R_p(\mathcal{T} + \mathcal{S}) \leq R_p(\mathcal{T}) + R_p(\mathcal{S})$.

(*d*) Let X, Y, Z be Banach spaces, and $\mathcal{T} \subset \mathcal{B}(X,Y)$ and $\mathcal{S} \subset \mathcal{B}(Y,Z)$ be R-bounded. Then

$$\mathcal{S}\mathcal{T} = \{ST : T \in \mathcal{T}, S \in \mathcal{S}\}$$

is R-bounded, and $R_p(\mathcal{S}\mathcal{T}) \leq R_p(\mathcal{S}) R_p(\mathcal{T})$.

(*e*) Let X, Y be Banach spaces and $\mathcal{T} \subset \mathcal{B}(X,Y)$ be R-bounded. If $\{\alpha_k\}_{k\in\mathbb{Z}}$ is a bounded sequence, then $\{\alpha_k T : T \in \mathcal{T}\}$ is R-bounded.

The following result asserts that any L^p-multiplier is an R-bounded set.

Theorem 2.4. [2] *Let X be a Banach space and $\{M_k\}_{k\in\mathbb{Z}}$ be an L^p-multiplier, where $1 \leq p < \infty$. Then, the set $\{M_k : k \in \mathbb{Z}\}$ is R-bounded.*

Now, we recall a class of Banach spaces, the so-called UMD spaces, which share similar properties with Hilbert spaces and include also the L^p-spaces for $1 < p < \infty$. A Banach space X is said to be UMD, if the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for some (and then all) $p \in (1,\infty)$. Here the Hilbert transform $\mathcal{H}$ of a function $f \in \mathcal{S}(\mathbb{R}, X)$, the Schwartz space of rapidly decreasing X-valued functions, is defined by

$$(\mathcal{H}f)(t) := \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|y-t|>\varepsilon} \frac{f(y)}{t-y} dy.$$

These spaces are also called $\mathcal{HT}$ spaces. It is well known that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces. This has been shown by Bourgain [4] and Burkholder [13]. Some examples of UMD-spaces include the Hilbert spaces, Sobolev spaces $W_p^s(\Omega)$, $1 < p < \infty$, Lebesgue spaces $L^p(\Omega, \mu)$, $1 < p < \infty$, $L^p(\Omega, \mu; X)$, $1 < p < \infty$, when X is a UMD-space. Moreover, a UMD-space is reflexive and therefore, $L^1(\Omega, \mu), L^\infty(\Omega, \mu)$ and the Hölder space $C^s([0, 2\pi]; X)$ are not UMD. More information on UMD spaces can be found in [4, 13] and [14].

The next result, due to Arendt-Bu [2, Theorem 1.3], gives a converse of Theorem 2.4 and shows that under certain conditions, a set of operators is an L^p-multiplier in UMD spaces.

Theorem 2.5. [2] *Let X, Y be UMD spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$. If the sets $\{M_k\}_{k\in\mathbb{Z}}$ and $\{k(M_{k+1}-M_k)\}_{k\in\mathbb{Z}}$ are R-bounded, then $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1<p<\infty$.*

3. Periodic solutions on Lebesgue spaces

In this section we study the existence of L^p-strong solutions to equation (1.3). For a kernel a and a function g we introduce the following notation

$$(a\ast g)(t) := \int_{-\infty}^{t} a(t-s)g(s)ds.$$

With this notation, the equation (1.3) reads as

$$u''(t) + \lambda u'(t) + Au(t) + (a\ast Au)(t) + (b\ast Bu)(t) = f(t).$$

If $k \in \mathbb{Z}$, then it is easy to prove that $\widehat{(a\ast g)}(k) = \tilde{a}(ik)\hat{g}(k)$, where $\tilde{a}(ik)$ is the Laplace transform of a evaluated in ik. In what follows, we use the following notation:

$$a_k := \tilde{a}(ik) \quad \text{and} \quad b_k := \tilde{b}(ik), \quad k \in \mathbb{Z},$$

and we assume that $a_k \neq 1$ for all $k \in \mathbb{Z}$.

Remark 3.6. *Note that by the Riemann-Lebesgue lemma, we have that the sequences $\{a_k\}_{k\in\mathbb{Z}}$ and $\{\frac{1}{\alpha+a_k}\}_{k\in\mathbb{Z}}$ $(\alpha \neq 0)$ are bounded.*

Now, from [21] we recall the concept of 1 and 2-regular sequences. The general notion of n-regularity is the discrete analogue for the notion of n-regularity related to Volterra integral equations (see [28, Chapter I, Section 3.2]).

Definition 3.7. *A sequence $\{c_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}\setminus\{0\}$ is said to be*

(*a*) 1*-regular, if the sequence* $\left\{k\dfrac{(c_{k+1}-c_k)}{c_k}\right\}_{k\in\mathbb{Z}}$ *is bounded.*

(*b*) 2*-regular if it is 1-regular and the sequence* $\left\{k^2\dfrac{(c_{k+1}-2c_k+c_{k-1})}{c_k}\right\}_{k\in\mathbb{Z}}$ *is bounded.*

For example, if $a(t) = -te^{-\beta t}$ then the sequence defined by $a_k = \tilde{a}(ik) = -\frac{1}{(ik+\beta)^2}$ is 1-regular. On the other hand, if $a(t) := \frac{t^{m-1}}{\Gamma(m)}$ where m is an even integer, then $\{a_k\}_{k\in\mathbb{Z}}$ is a 2-regular sequence.

Remark 3.8. *Note that if $\{c_k\}_{k\in\mathbb{Z}}$ is* 1*-regular, then* $\lim_{|k|\to\infty} c_{k+1}/c_k = 1$. *On the other hand, $\{c_k\}_{k\in\mathbb{Z}}$ is* 1*-regular if and only if $\{1/c_k\}_{k\in\mathbb{Z}}$ is* 1*-regular* [21, Theorem 4.6], *which implies that* $\lim_{|k|\to\infty} c_k/c_{k+1} =$ 1 *and the sequence* $k\left\{\frac{(c_k-c_{k+1})}{c_{k+1}}\right\}_{k\in\mathbb{Z}}$ *is bounded. Finally, since $\{c_{k+1}/c_k\}_{k\in\mathbb{Z}}$ and $\{c_k/c_{k+1}\}_{k\in\mathbb{Z}}$ are bounded, we obtain that* $\lim_{|k|\to\infty}(c_{k+1}-c_k)/c_k = \lim_{|k|\to\infty}(c_{k+1}-c_k)/c_{k+1} = 0$, *and therefore, the sequences $\{(c_{k+1}-c_k)/c_k\}_{k\in\mathbb{Z}}$ and $\{(c_{k+1}-c_k)/c_{k+1}\}_{k\in\mathbb{Z}}$ are bounded as well.*

For $a, b \in L^1_{\text{loc}}(\mathbb{R}_+)$ we define the *resolvent set* $\rho_{a,b}(A,B)$ as

$$\begin{aligned}\rho_{a,b}(A,B) = \{\mu \in \mathbb{C} : (\mu^2 + \lambda\mu + (1+\tilde{a}(\mu))A + \tilde{b}(\mu)B) : D(A)\cap D(B) \to X \\ \text{is invertible and } (\mu^2+\lambda\mu+(1+\tilde{a}(\mu))A+\tilde{b}(\mu)B)^{-1} \in \mathcal{B}(X)\},\end{aligned}$$

where $\tilde{a}(\cdot)$ and $\tilde{b}(\cdot)$ denote the Laplace transform of a and b respectively.

Proposition 3.9. *Suppose that* $\{a_k\}_{k\in\mathbb{Z}}$ *and* $\{b_k\}_{k\in\mathbb{Z}}$ *are* 1*-regular sequences. Let* $A : D(A) \subset X \to X$ *and* $B : D(B) \subset X \to X$ *be closed linear operators defined on a UMD space* X *with* $D(A) \cap D(B) \neq \{0\}$*. Suppose that* $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{a,b}(A, B)$*. For each* $k \in \mathbb{Z}$*, define* $N_k := (-k^2 + i\lambda k + (1 + a_k)A + b_kB)^{-1}$*. Then, the following assertions are equivalent*

(*i*) *The families* $\{k^2N_k\}_{k\in\mathbb{Z}}$ *and* $\{b_kBN_k\}_{k\in\mathbb{Z}}$ *are* L^p*-multipliers for* $1 < p < \infty$*;*
(*ii*) *The families* $\{k^2N_k\}_{k\in\mathbb{Z}}$ *and* $\{b_kBN_k\}_{k\in\mathbb{Z}}$ *are* R*-bounded.*

Proof. By Theorem 2.4 it follows that (i) implies (ii). Conversely, let $M_k = -k^2N_k$, for $k \in \mathbb{Z}$. In order to prove that $\{k^2N_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier, we need to prove by Theorem 2.5 that $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ is R-bounded. In fact, the identity $(-k^2 + i\lambda k + (1 + a_k)A + b_kB)N_k = I$ implies that $(1 + a_k)AN_k = I + k^2N_k - i\lambda kN_k - b_kBN_k = I + k^2N_k + \frac{i\lambda}{k}M_k - b_kBN_k$ and by hypothesis $(1 + a_k)AN_k$ is R-bounded for all $k \in \mathbb{Z} \setminus \{0\}$. Now, if $k = 0$, then $(1 + a_0)AN_0 = I - b_0BN_0$ which is R-bounded by hypothesis. Therefore, the set $\{(1 + a_k)AN_k\}_{k\in\mathbb{Z}}$ is R-bounded.

Now, as $a_kAN_k = \frac{a_k}{1+a_k}(1 + a_k)AN_k$ and $AN_k = \frac{1}{1+a_k}(1 + a_k)AN_k$, we conclude that $\{a_kAN_k\}_{k\in\mathbb{Z}}$ are $\{AN_k\}_{k\in\mathbb{Z}}$ R-bounded sets by Remarks 2.3 and 3.6.

On the other hand, an easy computation shows that

$$\begin{aligned} k(M_{k+1} - M_k) &= -\frac{i\lambda}{k+1}M_kM_{k+1} + M_k\frac{k(a_{k+1} - a_k)}{a_k}a_{k+1}AN_{k+1}\frac{a_k}{a_{k+1}} - M_k(1 + a_{k+1})AN_{k+1} \\ &+ \frac{a_{k+1} - a_k}{a_{k+1}}a_{k+1}AN_{k+1}M_k - \frac{k}{k+1}M_{k+1}(1 + a_k)AN_k \\ &+ \frac{k(b_{k+1} - b_k)}{b_k}\frac{a_k}{a_{k+1}}M_kb_{k+1}N_{k+1}B - \frac{k}{k+1}M_{k+1}b_kBN_k, \end{aligned}$$

for all $k \neq -1$ and $-(M_0 - M_{-1}) = (-1)^2N_{-1}$. The hypothesis and Remark 3.8 show that $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ is R-bounded and therefore $\{k^2N_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. Similarly, to prove that $\{a_kAN_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier, we shall show that $\{k(R_{k+1} - R_k)\}_{k\in\mathbb{Z}}$ is R-bounded, where $R_k = a_kAN_k$. Indeed, an easy computation gives the identity

$$\begin{aligned} k(R_{k+1} - R_k) &= -\frac{k(a_{k+1} - a_k)}{a_{k+1}}a_{k+1}AN_{k+1}M_k + \frac{k(a_{k+1} - a_k)}{a_{k+1}}a_{k+1}AN_{k+1}i\lambda kN_k \\ &+ \frac{k(a_{k+1} - a_k)}{a_k}a_{k+1}AN_{k+1}AN_k + \frac{a_k}{a_{k+1}}a_{k+1}AN_{k+1}(1 - i\lambda)kN_k \\ &+ \frac{k(a_{k+1} - a_k)}{a_{k+1}}a_{k+1}AN_{k+1}b_kBN_k - \frac{a_k}{a_{k+1}}\frac{k(b_{k+1} - b_k)}{b_k}a_{k+1}AN_{k+1}b_kBN_k, \end{aligned}$$

for all $k \neq 0$. The 1-regularity of $\{a_k\}_{k\in\mathbb{Z}}, \{b_k\}_{k\in\mathbb{Z}}$, Remark 3.8 and hypothesis imply that $\{k(R_{k+1} - R_k)\}_{k\in\mathbb{Z}}$ is R-bounded and therefore $\{a_kAN_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier by Theorem 2.5. A similar proof shows that $\{b_kBN_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. □

For $n \geq 0$, the space $H^{n,p}([0, 2\pi], X)$ is defined by

$$H^{n,p}([0, 2\pi], X) := \{v \in L^p([0, 2\pi], X) : \exists w \in L^p([0, 2\pi], X) \text{ such that } \hat{w}(k) = (ik)^n\hat{v}(k) \text{ for all } k \in \mathbb{Z}\}.$$

Given a kernel a and a closed operator A we define the space

$$L^{a,p}_A([0, 2\pi], X) := \{v \in L^p([0, 2\pi], [D(A)]) : \exists w \in L^p([0, 2\pi], X) \text{ such that } \hat{w}(k) = a_kA\hat{v}(k) \text{ for all } k \in \mathbb{Z}\}.$$

Finally, we define the following solution space:

$$\mathcal{S} := H^{2,p}([0, 2\pi], X) \cap L^p([0, 2\pi], [D(A)]) \cap L^{a,p}_A([0, 2\pi], X) \cap L^{b,p}_B([0, 2\pi], X).$$

Definition 3.10. *We say that a function* $u \in \mathcal{S}$ *is a strong* L^p*-solution of* (1.3) *if* (1.3) *holds for almost every* $t \in [0, 2\pi]$*.*

Theorem 3.11. *Let $1 < p < \infty$. Suppose that $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ are 1-regular sequences. Let $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ be closed linear operators defined in a UMD space X with $D(A)\cap D(B) \neq \{0\}$. Then, the following assertions are equivalent*

(i) *For every $f \in L^p([0, 2\pi], X)$, there exists a unique strong L^p-solution of (1.3);*
(ii) *$\{ik\}_{k\in\mathbb{Z}} \subset \rho_{a,b}(A, B)$ and the families $\{k^2N_k\}_{k\in\mathbb{Z}}$ and $\{b_kBN_k\}_{k\in\mathbb{Z}}$ are R-bounded.*

Proof. (ii) $\Rightarrow$ (i). Let $f \in L^p([0, 2\pi], X)$. By Proposition 3.9 the set $\{k^2N_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier and therefore $\{N_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier as well. Therefore there exists $u \in L^p([0, 2\pi], X)$ such that

$$\hat{u}(k) = N_k\hat{f}(k), \text{ for all } k \in \mathbb{Z}. \tag{3.5}$$

In particular, we conclude that $\hat{u}(k) \in D(A) \cap D(B)$ for all $k \in \mathbb{Z}$. Moreover, $-k^2\hat{u}(k) = -k^2N_k\hat{f}(k)$ for all $k \in \mathbb{Z}$ and since $\{k^2N_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier, we have by [2, Lemma 2.1] that $u \in H^{2,p}([0, 2\pi], X)$. On the other, the identity $(1 + a_k)AN_k = I - b_kBN_k - \frac{i\lambda}{k}k^2N_k + k^2N_k$ and Proposition 3.9 imply that $\{(1 + a_k)AN_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier and therefore, there exists $v \in L^p([0, 2\pi], X)$ such that

$$\hat{v}(k) = (1 + a_k)AN_k\hat{f}(k) = (1 + a_k)A\hat{u}(k),$$

for all $k \in \mathbb{Z}$. In particular $u \in L^p([0, 2\pi], [D(A)]) \cap L^{a,p}_A([0, 2\pi], X)$. The uniqueness of the Fourier coefficients implies that $v(t) = Au(t) + (a\dot{*}Au)(t)$ a.e. $t \in [0, 2\pi]$. Similarly, since $\{b_kBN_k\}_{k\in Z}$ is an L^p-multiplier by Proposition 3.9, we have that there exists $w \in L^p([0, 2\pi], X)$ such that

$$\hat{w}(k) = b_kBN_k\hat{f}(k) = b_kB\hat{u}(k),$$

for all $k \in \mathbb{Z}$, which implies that $u \in L^{b,p}_B([0, 2\pi], X)$ and $w(t) = (b\dot{*}Bu)(t)$ a.e. $t \in [0, 2\pi]$. Finally, since $i\lambda kN_k = \frac{i\lambda}{k}k^2N_k$ is an L^p-multiplier, there exists $z \in L^p([0, 2\pi], X)$ such that

$$\hat{z}(k) = i\lambda kN_k\hat{f}(k) = i\lambda k\hat{u}(k),$$

for all $k \in \mathbb{Z}$. Moreover, by [2, Lemma 2.1] we have $z(t) = \lambda u'(t)$ a.e. $t \in [0, 2\pi]$. Now, the identity $(-k^2 + i\lambda k + (1 + a_k)A + b_kB)N_k\hat{f}(k) = \hat{f}(k)$ implies

$$-k^2N_k\hat{f}(k) + i\lambda kN_k\hat{f}(k) + (1 + a_k)AN_k\hat{f}(k) + b_kBN_k\hat{f}(k) = \hat{f}(k)$$

for all $k \in \mathbb{Z}$, that is

$$-k^2\hat{u}(k) + i\lambda k\hat{u}(k) + (1 + a_k)A\hat{u}(k) + b_kB\hat{u}(k) = \hat{f}(k),$$

which implies by the uniqueness theorem of Fourier coefficients that $u''(t) + \lambda u'(t) + Au(t) + (a\dot{*}Au)(t) + (b\dot{*}Bu)(t) = f(t)$ a.e. $t \in [0, 2\pi]$. Moreover, the above considerations show that $u \in \mathcal{S}$. In order to prove the uniqueness, let $u \in \mathcal{S}$ such that

$$u''(t) + \lambda u'(t) + Au(t) + (a\dot{*}Au)(t) + (b\dot{*}Bu)(t) = 0.$$

Thus, $(-k^2 + i\lambda k + (1 + a_k)A + b_kB)\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$. Since $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{a,b}(A, B)$ we conclude that $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$ and therefore $u(t) = 0$ a.e. $t \in [0, 2\pi]$, which proves uniqueness.

(i) $\Rightarrow$ (ii). Let $k \in \mathbb{Z}$ and $y \in X$. Define the function $f \in L^p([0, 2\pi], X)$ by $f(t) = e^{ikt}y$. By hypothesis, there exists $u \in \mathcal{S}$ such that

$$u''(t) + \lambda u'(t) + Au(t) + (a\dot{*}Au)(t) + (b\dot{*}Bu)(t) = f(t).$$

Hence

$$(-k^2 + i\lambda k + (1 + a_k)A + b_kB)\hat{u}(k) = \hat{f}(k) = y,$$

which means that $(-k^2 + i\lambda k + (1 + a_k)A + b_kB)$ is surjective. On the other hand, if $x \in D(A) \cap D(B)$ and $(-k^2 + i\lambda k + (1 + a_k)A + b_kB)x = 0$, then $u(t) = e^{ikt}x$ defines a strong L^p-solution of

$$u''(t) + \lambda u'(t) + Au(t) + (a\dot{*}Au)(t) + (b\dot{*}Bu)(t) = 0,$$

and by the uniqueness we have $u(t) = 0$, and therefore $x = 0$. We conclude that $(-k^2 + i\lambda k + (1 + a_k)A + b_kB)$ is injective. Now, we need to prove that $(-k^2 + i\lambda k + (1 + a_k)A + b_kB)^{-1}$ is a bounded operator for all $k \in \mathbb{Z}$. Take $y \in X$ and $k \in \mathbb{Z}$. Let $L : L^p([0, 2\pi], X) \to \mathcal{S}$ be the bounded linear operator which takes each $f \in L^p([0, 2\pi], X)$ to the unique solution $u \in \mathcal{S}$ of Equation (1.3). Given the function $f(t) = e^{ikt}y$, we claim that the function $u(t) := e^{ikt}x$, where $x = u(0) \in D(A) \cap D(B)$ defines such unique strong L^p-solution to (1.3). In fact, since $(a\dot{*}Au)(t) = \int_{-\infty}^{t} a(t-s)e^{iks}Axds = a_k Au(t)$, we have

$$\begin{aligned}(-k^2 + i\lambda k + (1 + a_k)A + b_k B)x &= -k^2u(0) + i\lambda ku(0) + Au(0) + a_k Au(0) + b_k Bu(0)\\ &= u''(0) + \lambda u'(0) + Au(0) + (a\dot{*}Au)(0) + (b\dot{*}Bu)(0)\\ &= f(0) = y.\end{aligned}$$

Since $(-k^2 + i\lambda k + (1 + a_k)A + b_k B)$ is surjective, we obtain $x = (-k^2 + i\lambda k + (1 + a_k)A + b_k B)^{-1}y$ and therefore

$$\|(-k^2 + i\lambda k + (1 + a_k)A + b_k B)^{-1}y\| = \|x\| = \|u(0)\| = \|Lf(0)\| \le \|L\| \, \|f(0)\| = \|L\| \, \|y\|,$$

which means that $(-k^2 + i\lambda k + (1 + a_k)A + b_k B)^{-1}$ is a bounded operator for all $k \in \mathbb{Z}$ and thus $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{a,b}(A, B)$. Next, we show that $\{k^2N_k\}_{k\in\mathbb{Z}}$ and $\{b_k BN_k\}_{k\in\mathbb{Z}}$ are R-bounded. By Proposition 3.9 we need to prove that $\{k^2N_k\}_{k\in\mathbb{Z}}$ and $\{b_k BN_k\}_{k\in\mathbb{Z}}$ are L^p-multipliers. In fact, let $f \in L^p([0, 2\pi], X)$. By hypothesis, there exists a unique $u \in \mathcal{S}$ such that

$$u''(t) + \lambda u'(t) + Au(t) + (a\dot{*}Au)(t) + (b\dot{*}Bu)(t) = f(t).$$

Thus, $\hat{u}(k) \in D(A) \cap D(B)$ and

$$(-k^2 + i\lambda k + (1 + a_k)A + b_k B)\hat{u}(k) = \hat{f}(k).$$

Since $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{a,b}(A, B)$ we obtain $\hat{u}(k) = (-k^2 + i\lambda k + (1 + a_k)A + b_k B)^{-1}\hat{f}(k) = N_k\hat{f}(k)$. Moreover, there exists $v \in L^p([0, 2\pi], X)$ such that $\hat{v}(k) = -k^2\hat{u}(k)$, because $u \in H^{2,p}([0, 2\pi], X)$ and thus $\hat{v}(k) = -k^2\hat{u}(k) = -k^2N_k\hat{f}(k)$ for all $k \in \mathbb{Z}$, which means that $\{k^2N_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. On the other hand, since $\hat{u}(k) \in D(A) \cap D(B)$ we have $b_k B\hat{u}(k) = b_k BN_k\hat{f}(k)$. Since $u \in L_B^{b,p}([0, 2\pi], X)$ we have that the function $w(t) := (b\dot{*}Bu)(t)$ belongs to $L^p([0, 2\pi], X)$ and $\hat{w}(k) = b_k BN_k\hat{f}(k)$, which implies that $\{b_k BN_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. The proof of the Theorem is complete.

□

Note that the solution u given in Theorem 3.11 satisfies the following maximal regularity property, which is consequence of the closed graph theorem.

Corollary 3.12. *In the context of Theorem 3.11, if condition (ii) is fulfilled, we have that u'', u', Au, $(a\dot{*}Au), (b\dot{*}Bu) \in L^p([0, 2\pi], X)$. Moreover, there exists a constant $C > 0$ independent of $f \in L^p([0, 2\pi], X)$ such that*

$$\|u''\|_{L^p} + |\lambda| \, \|u'\|_{L^p} + \|Au\|_{L^p} + \|a\dot{*}Au\|_{L^p} + \|b\dot{*}Bu\|_{L^p} \le C\|f\|_{L^p}.$$

Since in Hilbert spaces the concept of R-boundedness and boundedness are equivalent, we have the next Corollary.

Corollary 3.13. *Let $1 < p < \infty$. Suppose that $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ are 1-regular sequences. Let $A : D(A) \subset H \to H$ and $B : D(B) \subset H \to H$ be closed linear operators defined on a Hilbert space H with $D(A) \cap D(B) \neq \{0\}$. Then, the following assertions are equivalent*

- (i) *For every $f \in L^p([0, 2\pi], H)$, there exists a unique strong L^p-solution of (1.3);*
- (ii) *$\{ik\}_{k\in\mathbb{Z}} \subset \rho_{a,b}(A, B)$ and*

$$\sup_{k\in\mathbb{Z}} \|k^2N_k\| < \infty \quad \textit{and} \quad \sup_{k\in\mathbb{Z}} \|b_k BN_k\| < \infty.$$

On the other hand, the Fejer's Theorem (see [2, Proposition 1.1]) can be used to write the solution u given in Theorem 3.11. More precisely, if the condition (ii) holds the Theorem 3.11, then for $f \in L^p([0,2\pi],X)$, the solution $u \in \mathcal{S}$ of (1.3) is given by

$$u(t) = \lim_{n\to\infty} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikt} N_k \hat{f}(k),$$

where the convergence holds in $L^p([0,2\pi],X)$.

4. Periodic solutions on Hölder and Besov spaces

In this section, we present some analogous theorems to the above section, in the context of Hölder and Besov spaces $B^s_{p,q}([0,2\pi],X)$. Examples of Besov spaces include the Hölder-Zygmund spaces and the usual Hölder space $C^s([0,2\pi],X) = B^s_{\infty,\infty}([0,2\pi],X)$ for $0 < s < 1$. We first recall the notion of Besov spaces.

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space on $\mathbb{R}$. Let $\mathcal{S}'(\mathbb{R})$ be the space of all tempered distribution on $\mathbb{R}$ and $\mathcal{D}([0,2\pi])$ the space of all infinitely differentiable functions defined on $[0,2\pi]$ equipped with the locally convex topology given by the seminorms $\|f\|_n = \sup_{t\in[0,2\pi]} |f^{(n)}(t)|$ for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let X be a Banach space. Let $\mathcal{D}'([0,2\pi],X) := \mathcal{B}(\mathcal{D}([0,2\pi]),X)$ be the space of all X-valued 2π-periodic distributions. Now we consider the following dyadic-like subset of $\mathbb{R}$:

$$I_0 := \{x \in \mathbb{R} : |x| \le 2\} \quad \text{and} \quad I_n := \{x \in \mathbb{R} : 2^{n-1} < |x| \le 2^{n+1}\},$$

for $n \in \mathbb{N}$. By $\Phi(\mathbb{R})$ we denote the set of all systems $\phi = \{\phi_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R})$ such that $\text{supp}(\phi_j) \subset \overline{I_j}$ for each $j \in \mathbb{N}_0$, $\sum_{j\in\mathbb{N}_0} \phi_j(x) = 1$ for each $x \in \mathbb{R}$ and for $\alpha \in \mathbb{N}_0$, we have $\sup_{j\in\mathbb{N}_0, x\in\mathbb{R}} 2^{\alpha j} |\phi_j^{(\alpha)}(x)| < \infty$. Let $\phi = \{\phi_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R})$ be fixed. Denote by $(e_k \otimes \phi)$ the function defined by $(e_k \otimes \phi)(x) = e^{ikx}\phi(x)$. For $1 \le p, q \le \infty$, and $s > 0$, the X-valued periodic Besov space is defined by

$$B^s_{p,q}([0,2\pi],X) := \left\{ f \in \mathcal{D}'([0,2\pi],X) : \|f\|_{B^s_{p,q}} := \left(\sum_{j\in\mathbb{N}_0} 2^{sjq} \left\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k) \hat{f}(k) \right\|^q_{L^p} \right)^{1/q} < \infty \right\},$$

with the usual modifications in case $q = \infty$. The space $B^s_{p,q}([0,2\pi],X)$ is independent of the choice of the system ϕ and different choices of ϕ give equivalent norms to $\|\cdot\|_{B^s_{p,q}}$. We summarize here some useful properties of $B^s_{p,q}([0,2\pi],X)$. See [3, Section 2] for further details.

- (i) $(B^s_{p,q}([0,2\pi],X), \|\cdot\|_{B^s_{p,q}})$ is a Banach space;
- (ii) If $s > 0$, then $B^s_{p,q}([0,2\pi],X) \hookrightarrow L^p([0,2\pi],X)$, and the natural injection from $B^s_{p,q}([0,2\pi],X)$ into $L^p([0,2\pi],X)$ is a continuos linear operator;
- (iii) If $s_1 \le s_2$, then $B^{s_2}_{p,q}([0,2\pi],X) \subset B^{s_1}_{p,q}([0,2\pi],X)$;
- (iv) Let $s > 0$. Then $f \in B^{s+1}_{p,q}([0,2\pi],X)$ if and only if f is differentiable a.e. and $f' \in B^s_{p,q}([0,2\pi],X)$. This implies that if $u \in B^s_{p,q}([0,2\pi],X)$ is such that there exists $v \in B^s_{p,q}([0,2\pi],X)$ satisfying $\hat{v}(k) = ik\hat{u}(k)$ for all $k \in \mathbb{Z}$, then $u \in B^{s+1}_{p,q}([0,2\pi],X)$ and $u' = v$.

Now, we recall the definition of operator valued Fourier multipliers in the context of periodic Besov spaces.

Definition 4.14. *Let $1 \le p, q \le \infty, s > 0$. A sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ is a $B^s_{p,q}$-multiplier if for each $f \in B^s_{p,q}([0,2\pi],X)$ there exists a function $g \in B^s_{p,q}([0,2\pi],Y)$ such that*

$$\hat{g}(k) = M_k \hat{f}(k), \quad k \in \mathbb{Z}.$$

We recall the following operator-valued Fourier multiplier theorem in Besov spaces.

Theorem 4.15. [3] *Let X, Y be Banach spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$. Suppose that*

$$\sup_{k\in\mathbb{Z}} \|M_k\| < \infty,\ \sup_{k\in\mathbb{Z}} \|k(M_{k+1}-M_k)\| < \infty, \tag{4.6}$$

$$\sup_{k\in\mathbb{Z}} \|k^2(M_{k+1}-2M_k+M_{k-1})\| < \infty. \tag{4.7}$$

Then for $1 \le p,q \le \infty$, $s>0$, $\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier.

We remark that if X is a B-convex space (if X is for instance a UMD space), then the condition (4.6) is already sufficient for $\{M_k\}_{k\in\mathbb{Z}}$ to be a $B^s_{p,q}$-multiplier. As in the case of L^p-multipliers, we have the following properties

(*a*) If $\{M_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$ is a $B^s_{p,q}$-multiplier, then there exists a bounded operator $\mathcal{M}$, $\mathcal{M}: B^s_{p,q}([0,2\pi],X) \to B^s_{p,q}([0,2\pi],Y)$ such that $\widehat{\mathcal{M}f}(k) = M_k\hat{f}(k)$ for all $k\in\mathbb{Z}$. In particular, $\sup_{k\in\mathbb{Z}}\|M_k\| < \infty$.

(*b*) If $\{M_k\}_{k\in\mathbb{Z}}$ and $\{N_k\}_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-multipliers, then $\{M_k+N_k\}_{k\in\mathbb{Z}}$ and $\{M_kN_k\}_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-multipliers as well.

By using Theorem 4.15 we can prove the next result in the context of Besov spaces analogously to Proposition 3.9. We omit the details.

Proposition 4.16. *Let $1 \le p,q \le \infty$, and $s>0$. Suppose that $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ are 2-regular sequences. Let $A: D(A) \subset X \to X$ and $B: D(B) \subset X \to X$ be closed linear operators defined in a Banach space X with $D(A)\cap D(B) \neq \{0\}$. Suppose that $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{a,b}(A,B)$. For each $k\in\mathbb{Z}$, define $N_k := (-k^2+i\lambda k+(1+a_k)A+b_kB)^{-1}$. Then, the following assertions are equivalent*

(*i*) *The families $\{k^2N_k\}_{k\in\mathbb{Z}}$ and $\{b_kBN_k\}_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-multipliers;*

(*ii*) *$\{ik\}_{k\in\mathbb{Z}} \subset \rho_{a,b}(A,B)$ and*

$$\sup_{k\in\mathbb{Z}} \|k^2N_k\| < \infty \quad \textit{and} \quad \sup_{k\in\mathbb{Z}} \|b_kBN_k\| < \infty.$$

Now, we study the existence and uniqueness of solutions to equation (1.3) in $B^s_{p,q}([0,2\pi],X)$. Is remarkable that in this case, there are no geometrical conditions on the Banach space X. Contrary to the L^p case, the multiplier theorems established for vector-valued Besov spaces are valid for arbitrary Banach spaces X, see for instance [1] and [3].

Given a kernel a and a closed operator A we define the Besov-type space $B^s_{p,q,a,A}([0,2\pi],X)$ as

$$\begin{aligned} B^s_{p,q,a,A}([0,2\pi],X) \quad := \quad & \{v \in B^s_{p,q}([0,2\pi],[D(A)]) : \exists w \in B^s_{p,q}([0,2\pi],X) \text{ such that} \\ & \hat{w}(k) = a_kA\hat{v}(k) \text{ for all } k\in\mathbb{Z}\}. \end{aligned}$$

Note that if $v \in B^s_{p,q,a,A}([0,2\pi],X)$, then $(a \dot{*} Au) \in B^s_{p,q}([0,2\pi],X)$. Now, we define the following solution space:

$$\mathcal{S}_{p,q} := B^{s+2}_{p,q}([0,2\pi],X) \cap B^s_{p,q}([0,2\pi],[D(A)]) \cap B^s_{p,q,a,A}([0,2\pi],X) \cap B^s_{p,q,b,B}([0,2\pi],X).$$

Definition 4.17. *We say that a function $u \in \mathcal{S}_{p,q}$ is a strong $B^s_{p,q}$-solution of* (1.3) *if* (1.3) *holds for almost every $t\in[0,2\pi]$.*

The next result, compared with Theorem 3.11, does not require any restriction on the Banach space X. Its proof follows the same lines as in the proof of Theorem 3.11. We omit the details.

Theorem 4.18. *Let $1 \le p,q \le \infty$, and $s>0$. Suppose that $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ are 2-regular sequences. Let $A: D(A) \subset X \to X$ and $B: D(B) \subset X \to X$ be closed linear operators defined in a Banach space X with $D(A)\cap D(B) \neq \{0\}$. Then, the following assertions are equivalent*

(*i*) *For every $f \in B^s_{p,q}([0,2\pi],X)$, there exists a unique $B^s_{p,q}$-strong solution of (1.3);*

(*ii*) *$\{ik\}_{k\in\mathbb{Z}} \subset \rho_{a,b}(A,B)$ and*

$$\sup_{k\in\mathbb{Z}} \|k^2N_k\| < \infty \quad \textit{and} \quad \sup_{k\in\mathbb{Z}} \|b_kBN_k\| < \infty.$$

Moreover, we have also the next maximal regularity result.

Corollary 4.19. *In the context of Theorem 4.18, if condition* (ii) *is fulfilled, we have that the solution* u *to* (1.3) *satisfies* $u'', u', Au, (a\dot{*}Au), (b\dot{*}Bu) \in B^s_{p,q}([0,2\pi],X)$. *Moreover, there exists a constant* $C>0$ *such that*

$$\|u''\|_{B^s_{p,q}} + |\lambda|\,\|u'\|_{B^s_{p,q}} + \|Au\|_{B^s_{p,q}} + \|a\dot{*}Au\|_{B^s_{p,q}} + \|b\dot{*}Bu\|_{B^s_{p,q}} \le C\|f\|_{B^s_{p,q}}.$$

For $0<s<1$, we denote by $C^s(\mathbb{R},X)$ the Hölder space of all continuous functions $f:\mathbb{R}\to X$ such that

$$\|f(t)-f(s)\| \le c|t-s|^s$$

for all $t,s\in\mathbb{R}$ and some $c\ge 0$. By $C^s([0,2\pi],X)$ we denote the Hölder space of all 2π-periodic functions, that is $C^s([0,2\pi],X) = C^s(\mathbb{R},X)\cap C([0,2\pi],X)$, where $C([0,2\pi],X)$ is the space of all 2π-periodic continuous functions. Since $B^s_{\infty,\infty}([0,2\pi],X) = C^s([0,2\pi],X)$ we have the following result in Hölder spaces.

Corollary 4.20. *Let* $0<s<1$. *Suppose that* $\{a_k\}_{k\in\mathbb{Z}}$ *and* $\{b_k\}_{k\in\mathbb{Z}}$ *are 2-regular sequences. Let* $A: D(A)\subset X\to X$ *and* $B: D(B)\subset X\to X$ *be closed linear operators defined in a Banach space* X *with* $D(A)\cap D(B)\neq\{0\}$. *Then, the following assertions are equivalent*

(i) *For every* $f\in C^s([0,2\pi],X)$, *there exists a unique strong of (1.3) with* $u'', u', Au, (a\dot{*}Au), (b\dot{*}Bu) \in C^s([0,2\pi],X)$;

(ii) $\{ik\}_{k\in\mathbb{Z}}\subset\rho_{a,b}(A,B)$ *and*

$$\sup_{k\in\mathbb{Z}}\|k^2N_k\|<\infty \quad \textit{and} \quad \sup_{k\in\mathbb{Z}}\|b_kBN_k\|<\infty.$$

5. Mild periodic solutions

In this section, we study the existence of mild solutions to equation (1.3). The functions g_1 and g_2 are defined respectively by $g_1(t)=1$ and $g_2(t)=t$ for all $t\in[0,2\pi]$. The usual convolution between the functions f and g, denoted by $(f*g)(t)$, is defined by

$$(f*g)(t) = \int_0^t f(t-s)g(s)ds,$$

for all $t\in[0,2\pi]$. Observe that

$$(g_1*f)(t) = \int_0^t f(s)ds \quad \text{and} \quad (g_2*f)(t) = \int_0^t (t-s)f(s)ds,$$

and $(g_2*f)(t) = (g_1*g_1*f)(t)$ for all $t\in[0,2\pi]$.

Definition 5.21. *Let* $f\in L^1_{\text{loc}}(\mathbb{R},X)$. *A differentiable function* $u\in C([0,2\pi],X)$ *at* $t=0$ *is called a* mild solution *to (1.3) if* $(g_2*u)(t), (g_2*(a\dot{*}u))(t)\in D(A)$, $(g_2*(b\dot{*}u))(t)\in D(B)$, *for all* $t\in[0,2\pi]$ *and*

$$\begin{aligned} u(t) &= u(0)+tu'(0)+\lambda t u(0)-\lambda(g_1*u)(t)-A(g_2*u)(t)-A(g_2*(a\dot{*}u))(t) \\ &- B(g_2*(b\dot{*}u))(t)+(g_2*f)(t), \end{aligned} \tag{5.8}$$

for all $t\in[0,2\pi]$.

Observe that if $a(t)=b(t)=0$, for all t, $\lambda=0$, then this concept of mild solution is the same that in case of the second order problem $u''(t)+Au(t)=f(t)$.

It is clear that every L^p-strong solution to equation (1.3) is a mild solution, and conversely, if u is a mild solution to (1.3) and u belongs to the solution space $\mathcal{S}$, then u is an L^p-strong solution.

Lemma 5.22. *Let* $a\in L^1(\mathbb{R})$ *and a function* f. *Define the function* G^a_f *by* $G^a_f(t) := (g_2*(a\dot{*}f))(t)$, $t\in[0,2\pi]$. *Then, the Fourier coefficient of* G^a_f *are given by*

$$\widehat{G^a_f}(k) = -\frac{1}{2\pi ik}G^a_f(2\pi)+\frac{1}{k^2}a_0\hat{f}(0)-\frac{1}{k^2}a_k\hat{f}(k), \quad k\in\mathbb{Z}\setminus\{0\}.$$

Proof. It follows similarly to [20, Lemma 4.2]. □

Theorem 5.23. *Let $f \in L^1([0,2\pi],X)$ and $u \in C([0,2\pi],X)$ be a differentiable function at $t=0$. Assume that $\overline{D(A)} = X$. Then u is a mild solution to problem (1.3) satisfying $u'(0)=u'(2\pi)$ if and only if*

$$\hat{u}(k) \in D(A)\cap D(B) \quad \text{and} \quad (-k^2+ik\lambda+(1+a_k)A+b_kB)\hat{u}(k) = \hat{f}(k), \tag{5.9}$$

for all $k \in \mathbb{Z}$.

Proof. Assume that u is a mild solution to (1.3). From (5.8), we obtain (with $t=2\pi$) that

$$A(g_2 * u)(2\pi) + AG_u^a(2\pi) + BG_u^b(2\pi) = 2\pi\lambda u(0) + 2\pi u'(0) - \lambda(g_1 * u)(2\pi) + (g_2 * f)(2\pi). \tag{5.10}$$

Let $w(t) = u(t) - u(0) - tu'(0) - \lambda t u(0) + \lambda(g_1 * u)(t) - (g_2 * f)(t)$. From [2, Lemma 3.1] and hypothesis it follows that $\widehat{G_u^a}(k) \in D(A)$, $\widehat{G_u^b}(k) \in D(B)$, $\widehat{(g_2 * u)}(k) \in D(A)$ and

$$\hat{w}(k) = -A\widehat{G_u^a}(k) - B\widehat{G_u^b}(k) - A\widehat{(g_2 * u)}(k). \tag{5.11}$$

The Lemma 5.22 implies

$$\widehat{G_u^a}(k) = -\frac{1}{2\pi ik}G_u^a(2\pi) + \frac{a_0}{k^2}\hat{u}(0) - \frac{a_k}{k^2}\hat{u}(k) \quad \text{and} \quad \widehat{G_u^b}(k) = -\frac{1}{2\pi ik}G_u^b(2\pi) + \frac{b_0}{k^2}\hat{u}(0) - \frac{b_k}{k^2}\hat{u}(k).$$

Thus

$$A\widehat{G_u^a}(k) = -\frac{1}{2\pi ik}AG_u^a(2\pi) + \frac{a_0}{k^2}A\hat{u}(0) - \frac{a_k}{k^2}A\hat{u}(k), \tag{5.12}$$

and

$$B\widehat{G_u^b}(k) = -\frac{1}{2\pi ik}BG_u^b(2\pi) + \frac{b_0}{k^2}B\hat{u}(0) - \frac{b_k}{k^2}B\hat{u}(k). \tag{5.13}$$

Moreover, by [20, Lemma 4.2] we have

$$A\widehat{(g_2 * u)}(k) = -\frac{1}{2\pi ik}A(g_2 * u)(2\pi) + \frac{1}{k^2}A\hat{u}(0) - \frac{1}{k^2}A\hat{u}(k). \tag{5.14}$$

Since $\widehat{(g_1 * f)}(k) = -\frac{1}{ik}\hat{f}(0) + \frac{1}{ik}\hat{f}(k)$, we have by [20, Lemma 4.2]

$$\hat{w}(k) = \hat{u}(k) + \frac{1}{ik}u'(0) + \frac{\lambda}{ik}u(0) + \lambda\left[-\frac{1}{ik}\hat{f}(0) + \frac{1}{ik}\hat{f}(k)\right] - \left[-\frac{1}{2\pi ik}(g_2 * f)(2\pi) + \frac{1}{k^2}\hat{f}(0) - \frac{1}{k^2}\hat{f}(k)\right]. \tag{5.15}$$

On the other hand, the function u is differentiable and

$$u'(t) = u'(0) + \lambda u(0) - \lambda u(t) - A(g_1 * u)(t) - A(g_1 * (a\dot{*}u))(t) - B(g_1 * (b\dot{*}u))(t) + (g_1 * f)(t),$$

and if $t = 2\pi$, then

$$0 = -A(g_1 * u)(2\pi) - A(g_1 * (a\dot{*}u))(2\pi) - B(g_1 * (b\dot{*}u))(2\pi) + (g_1 * f)(2\pi),$$

that is,

$$(1+a_0)A\hat{u}(0) + b_0B\hat{u}(0) = \hat{f}(0). \tag{5.16}$$

Therefore, (5.12), (5.13) and (5.14) imply

$$\begin{aligned} -\frac{1}{k^2}(1+a_k)A\hat{u}(k) - \frac{1}{k^2}b_kB\hat{u}(k) &= A\widehat{G_u^a}(k) + B\widehat{G_u^b}(k) + A\widehat{(g_2 * u)}(k) \\ &+ \frac{1}{2\pi ik}\left[AG_u^a(2\pi) + BG_u^b(2\pi) + A(g_2 * u)(2\pi)\right] \\ &- \frac{1}{k^2}\left[a_0A\hat{u}(0) + b_0B\hat{u}(0) + A\hat{u}(0)\right]. \end{aligned}$$

By (5.10) and (5.15), we obtain

$$\begin{aligned} -\frac{1}{k^2}(1+a_k)A\hat{u}(k)-\frac{1}{k^2}b_kB\hat{u}(k) &= A\widehat{G^a_u}(k)+B\widehat{G^b_u}(k)+A\widehat{(g_2*u)}(k)\\ &+ \frac{1}{2\pi ik}\left[2\pi\lambda u(0)+2\pi u'(0)-\lambda(g_1*u)(2\pi)+(g_2*f)(2\pi)\right]\\ &- \frac{1}{k^2}\left[a_0A\hat{u}(0)+b_kB\hat{u}(0)+A\hat{u}(0)\right]\\ &= A\widehat{G^a_u}(k)+B\widehat{G^b_u}(k)+A\widehat{(g_2*u)}(k)\\ &+ \hat{w}(k)-\hat{u}(k)-\frac{\lambda}{ik}\hat{u}(k)+\frac{1}{k^2}\hat{f}(0)-\frac{1}{k^2}\hat{f}(k)\\ &- \frac{1}{k^2}\left[(1+a_0)A\hat{u}(0)+b_0B\hat{u}(0)\right]. \end{aligned}$$

By using the identities (5.11) and (5.16) we have

$$-\frac{1}{k^2}(1+a_k)A\hat{u}(k)-\frac{1}{k^2}b_kB\hat{u}(k) = -\hat{u}(k)-\frac{\lambda}{ik}\hat{u}(k)-\frac{1}{k^2}\hat{f}(k),$$

which implies $(-k^2+ik\lambda+(1+a_k)A+b_kB)\hat{u}(k)=\hat{f}(k)$, for all $k\in\mathbb{Z}$, $k\neq 0$ and this equality is also valid for $k=0$ by (5.16), and therefore the it holds for all $k\in\mathbb{Z}$.

Conversely, suppose that (5.9) holds for all $k\in\mathbb{Z}$. We shall prove that for all $x^*\in D(A^*)$, where A^* denotes the adjoint operator of A, we have

$$\begin{aligned} \langle (g_2*u)(t)+(g_2*(a\dot{*}u))(t),A^*x^*\rangle &= -\langle u(t),x^*\rangle+\langle u(0),x^*\rangle+\langle tu'(0),x^*\rangle+\langle \lambda tu(0),x^*\rangle\\ &\quad -\langle \lambda(g_1*u)(t),x^*\rangle-\langle B(g_2*(b\dot{*}u))(t),x^*\rangle+\langle (g_2*f)(t),x^*\rangle. \end{aligned}$$

If $w(t):=\langle u(t)+(a\dot{*}u)(t),A^*x^*\rangle+\langle B(b\dot{*}u)(t),x^*\rangle-\langle f(t),x^*\rangle$, then by (5.9) we have

$$\begin{aligned} \hat{w}(k) &= \langle (1+a_k)\hat{u}(k),A^*x^*\rangle+\langle b_kB\hat{u}(k),x^*\rangle-\langle \hat{f}(k),x^*\rangle\\ &= \langle k^2\hat{u}(k)-i\lambda k\hat{u}(k),x^*\rangle. \end{aligned}$$

Since $\hat{w}(0)=0$, the function $v(t):=(g_2*w)(t)+\langle u(t),x^*\rangle-t\langle u'(0),x^*\rangle-\lambda t\langle u(0),x^*\rangle+\lambda\langle (g_1*u)(t),x^*\rangle$ verifies by [20, Lemma 4.2]

$$\begin{aligned} \hat{v}(k) &= -\frac{1}{2\pi ik}(g_2*w)(2\pi)+\frac{1}{k^2}\hat{w}(0)-\frac{1}{k^2}\hat{w}(k)+\langle \hat{u}(k),x^*\rangle\\ &\quad +\frac{1}{ik}\langle u'(0),x^*\rangle+\frac{\lambda}{ik}\langle u(0),x^*\rangle-\frac{\lambda}{ik}\langle \hat{u}(0),x^*\rangle+\frac{\lambda}{ik}\langle \hat{u}(k),x^*\rangle\\ &= -\frac{1}{2\pi ik}(g_2*w)(2\pi)+\frac{1}{ik}\langle u'(0),x^*\rangle+\frac{\lambda}{ik}\langle u(0),x^*\rangle-\frac{\lambda}{ik}\langle \hat{u}(0),x^*\rangle, \end{aligned}$$

for all $k\neq 0$. Then, the function $z(t):=v(t)-\frac{t}{2\pi}(g_2*w)(2\pi)+t\langle u'(0),x^*\rangle+\lambda t\langle u(0),x^*\rangle-\lambda t\langle \hat{u}(0),x^*\rangle$ is constant. Since $v(0)=\langle u(0),x^*\rangle$, $z(0)=v(0)$ and $z(0)=z(t)$ for all t, we obtain

$$\langle u(0),x^*\rangle=v(0)=z(0)=z(t)=v(t)-\frac{t}{2\pi}(g_2*w)(2\pi)+t\langle u'(0),x^*\rangle+\lambda t\langle u(0),x^*\rangle-\lambda t\langle \hat{u}(0),x^*\rangle,$$

which implies

$$v(t)=\frac{t}{2\pi}(g_2*w)(2\pi)-t\langle u'(0),x^*\rangle+\langle u(0),x^*\rangle-\lambda t\langle u(0),x^*\rangle+\lambda t\langle \hat{u}(0),x^*\rangle.$$

From the definition of $v(t)$ we have

$$(g_2*w)(t)+\langle u(t),x^*\rangle+\lambda\langle (g_1*u)(t),x^*\rangle=\frac{t}{2\pi}(g_2*w)(2\pi)+\langle u(0),x^*\rangle+\lambda t\langle \hat{u}(0),x^*\rangle. \tag{5.17}$$

Since the function u is differentiable at $t=0$, we obtain

$$2\pi[\langle u'(0),x^*\rangle+\lambda\langle u(0),x^*\rangle-\lambda\langle \hat{u}(0),x^*\rangle]=(g_2*w)(2\pi), \tag{5.18}$$

and therefore, the equation (5.17) reads

$$(g_2 * w)(t) + \langle u(t), x^*\rangle + \lambda\langle (g_1 * u)(t), x^*\rangle = t\langle u'(0), x^*\rangle + \langle u(0), x^*\rangle + \lambda t\langle u(0), x^*\rangle,$$

which implies (by the definition of $w(t)$)

$$\begin{aligned}\langle u(t), x^*\rangle &= \langle u(0), x^*\rangle + t\langle u'(0), x^*\rangle + \lambda t\langle u(0), x^*\rangle - \lambda\langle (g_1 * u)(t), x^*\rangle - \langle A(g_2 * u)(t), x^*\rangle \\ &\quad -\langle A(g_2 * (a \tilde{*} u))(t), x^*\rangle - \langle B(g_2 * (b \tilde{*} u))(t), x^*\rangle + \langle (g_2 * f)(t), x^*\rangle,\end{aligned}$$

for all $x^* \in X^*$. Thus, u verifies (5.8). On the other hand, since u is differentiable at $t = 0$ (and therefore at $t = 2\pi$) we obtain from (5.17) and (5.18)

$$(g_1 * w)(2\pi) + \langle u'(2\pi), x^*\rangle = \langle u'(0), x^*\rangle$$

for all $x^* \in X^*$. But $(g_1 * w)(2\pi) = 2\pi\hat{w}(0) = 0$, which implies $\langle u'(2\pi), x^*\rangle = \langle u'(0), x^*\rangle$ for all $x^* \in X^*$. We conclude that $u'(0) = u'(2\pi)$. This finishes the proof of the Theorem. □

6. Some applications

In this section we discuss some applications to the abstract results presented in the previous sections. We consider the second order equation

$$u''(t) + \lambda u'(t) + Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + \int_{-\infty}^{t} b(t-s)u(s)ds = f(t), \tag{6.19}$$

where $\lambda \in \mathbb{R}$, A is self-adjoint dissipative operator defined in a Hilbert space H, the kernels $a, b \in L^1(\mathbb{R}_+)$ define the 1-regular sequences $\{a_k\}_{k\in\mathbb{Z}}$, $\{b_k\}_{k\in\mathbb{Z}}$ and $f \in L^p([0, 2\pi], H)$. We recall that $a_k = \tilde{a}(ik)$ and $b_k = \tilde{b}(ik)$ and we always assume that $a_k \neq 1$ for all $k \in \mathbb{Z}$. If $B = I$, that is, B is the identity operator in H, then

$$(-k^2 + i\lambda k + (1 + a_k)A + b_k I)^{-1} = \frac{-1}{1 + a_k}\left(\frac{k^2 - i\lambda k - b_k}{1 + a_k} - A\right)^{-1} \tag{6.20}$$

for all $k \in \mathbb{Z}$. Assume that $a(0) \neq \lambda$ and $a' \in L^1(\mathbb{R}_+)$. For each $k \in \mathbb{Z}$, define $\mu_k := \frac{k^2 - i\lambda k - b_k}{1 + a_k}$ and suppose that $\mu_k \notin \sigma(A)$ for all $k \in \mathbb{Z}$, where $\sigma(A)$ denotes the spectrum of A. Then, by [29, Lemma 2.2] there exists a constant $C > 0$ (independent of k) such that

$$\|(\mu_k - A)^{-1}\| \leq \frac{C}{1 + |k|}. \tag{6.21}$$

The next result gives a different approach to [29, Theorem 2.1] in the L^p-context.

Proposition 6.24. *Let $1 < p < \infty$ and assume the above conditions. Suppose that $\mathrm{Im}(\mu_k) \neq 0$ for all $k \in \mathbb{Z}$. If $f \in L^p([0, 2\pi], H)$, then there exists a unique L^p-strong solution to equation (6.19).*

Proof. According to Theorem 3.11 (or Corollary 3.13) we need to prove that $\sup_{k\in\mathbb{Z}} \|k^2 N_k\| < \infty$ and $\sup_{k\in\mathbb{Z}} \|b_k N_k\| < \infty$, where $N_k := (-k^2 + i\lambda k + (1 + a_k)A + b_k I)^{-1}$. In fact, we first notice that as $(1 + a_k)N_k = -(\mu_k - A)^{-1}$ and $A(\mu_k - A)^{-1} = \mu_k(\mu_k - A)^{-1} - I$ we obtain by (6.21) that

$$\|a_k A N_k\| = \frac{|a_k|}{|1 + a_k|}\|(1 + a_k)AN_k\| \leq \frac{|a_k|}{|1 + a_k|} + \frac{C}{|1 + a_k|}\frac{|a_k|\,|\mu_k|}{1 + |k|}.$$

By the Riemann-Lebesgue lemma we have

$$\|a_k A N_k\| \leq C_1 + C_2\frac{|a_k|\,|\mu_k|}{1 + |k|},$$

for all $k \in \mathbb{Z}$ and certain constants C_1, C_2. We write $a_k = \alpha_k + i\beta_k$ and $b_k = p_k + iq_k$ to estimate $(|a_k|\,|\mu_k|)/(1 + |k|)$. Observe that

$$\frac{|a_k|\,|\mu_k|}{1 + |k|} \leq \frac{|\alpha_k|\,|\mu_k|}{|k|} + \frac{|\beta_k|\,|\mu_k|}{|k|}.$$

To estimate $(|\alpha_k|\,|\mu_k|)/|k|$ we notice that

$$\frac{|\alpha_k|\,|\mu_k|}{|k|} \leq \frac{|\alpha_k|\,|\mathrm{Re}(\mu_k)|}{|k|} + \frac{|\alpha_k|\,|\mathrm{Im}(\mu_k)|}{|k|}$$

and by [29, Lemma 2.1]

$$\lim_{|k|\to\infty} \frac{|\mathrm{Im}(\mu_k)|}{|k|} = a(0) - \lambda$$

and therefore, by the Riemann-Lebesgue lemma the sequence $\{|\alpha_k|\,|\mathrm{Im}(\mu_k)|/|k|\}_{k\in\mathbb{Z}}$ is bounded. On the other hand,

$$\mathrm{Re}(\mu_k) = \frac{k^2(1+\alpha_k) - p_k(1+\alpha_k) - \lambda\beta_k k - \beta_k q_k}{(1+\alpha_k)^2 + \beta_k^2}$$

and therefore

$$|\alpha_k|\,\frac{|\mathrm{Re}(\mu_k)|}{|k|} \leq \frac{|k\alpha_k(1+\alpha_k)|}{(1+\alpha_k)^2+\beta_k^2} + |\alpha_k|\frac{|p_k(1+\alpha_k) - \lambda\beta_k k - \beta_k q_k|}{|k|((1+\alpha_k)^2+\beta_k^2)}.$$

Since $a' \in L^1(\mathbb{R}_+)$, the sequence $\{k\alpha_k\}_{k\in\mathbb{Z}}$ is bounded. Similarly, since $a, b \in L^1(\mathbb{R}_+)$, the sequences $\{\alpha_k\}_{k\in\mathbb{Z}}, \{\beta_k\}_{k\in\mathbb{Z}}, \{p_k\}_{k\in\mathbb{Z}}$ and $\{q_k\}_{k\in\mathbb{Z}}$ are bounded as well. We conclude that $\{|\alpha_k|\,|\mathrm{Re}(\mu_k)|/|k|\}_{k\in\mathbb{Z}}$ is bounded. Therefore

$$\sup_{k\in\mathbb{Z}} \|a_k A N_k\| < \infty.$$

Now, by the Riemann-Lebesgue lemma and (6.21) we have

$$\|b_k N_k\| \leq \frac{|b_k|}{|1+a_k|}\|(\mu_k - A)^{-1}\| \leq \frac{|b_k|C}{|1+a_k|(1+|k|)},$$

which implies

$$\sup_{k\in\mathbb{Z}} \|b_k N_k\| < \infty.$$

In order to prove that $k^2 N_k$ is uniformly bounded, we first notice the identity

$$-k^2 N_k = I - i\lambda k N_k - (1+a_k)AN_k - b_k N_k, \tag{6.22}$$

for all $k \in \mathbb{Z}$. Moreover, by (6.21) it follows that

$$\sup_{k\in\mathbb{Z}} \|\lambda k N_k\| < \infty.$$

On the other hand, since A is a self-adjoint dissipative operator, then A is sectorial operator with $\sigma(A) \subset (-\infty, 0]$. Since $\mathrm{Im}(\mu_k) \neq 0$ for all $k \in \mathbb{Z}$, there exists a constant M such that

$$\|\mu_k(\mu_k - A)^{-1}\| \leq M$$

for all $k \in \mathbb{Z}$. Moreover, the identity $(1+a_k)AN_k = -A(\mu_k - A)^{-1} = I - \mu_k(\mu_k - A)^{-1}$ implies

$$\|(1+a_k)AN_k\| \leq 1 + M.$$

Finally, by (6.22) we conclude that

$$\sup_{k\in\mathbb{Z}} \|k^2 N_k\| < \infty.$$

We conclude by Theorem 3.11 (or Corollary 3.13) that, if $f \in L^p([0,2\pi], H)$, (for $1 < p < \infty$) then there exists a unique L^p-strong solution u to (6.19). Moreover, by Corollary 3.12 the solution u verifies the regularity $u'', u', Au, (a{*}Au), (b{*}u) \in L^p([0,2\pi], H)$, and the maximal regularity property

$$\|u''\|_{L^p} + |\lambda|\,\|u'\|_{L^p} + \|Au\|_{L^p} + \|a{*}Au\|_{L^p} + \|b{*}u\|_{L^p} \leq C\|f\|_{L^p},$$

where $C > 0$ is a constant. $\square$

The proof of the next result follows similarly to Proposition 6.24, which is consequence of Corollary 4.20. We omit the details.

Proposition 6.25. *Let $1 < s < 1$ and assume the conditions in Proposition 6.24, but with $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ being 2-regular sequences. If $\mathrm{Im}(\mu_k) \neq 0$ for all $k \in \mathbb{Z}$ and $f \in C^s([0, 2\pi], H)$, then there exists a unique C^s-strong solution to equation (6.19).*

Example 6.26.

Let Ω be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. By [29], the equation of heat flow in materials with memory

$$cu_{tt}(x,t)+\alpha(0)u_t(x,t)+\int_{-\infty}^{t}\alpha'(t-s)u_t(x,s)ds = \beta(0)\Delta u(x,t)+\int_{-\infty}^{t}\beta'(t-s)\Delta u(x,s)ds+F(x,t), \tag{6.23}$$

with the boundary condition $u = 0$ in $\Omega \times \mathbb{R}$, can be written in the abstract form

$$u''(t) + \lambda u'(t) + Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + \int_{-\infty}^{t} b(t-s)u(s)ds = f(t) \quad t \in \mathbb{R}, \tag{6.24}$$

with $\lambda = \frac{\alpha(0)}{c}$, $A = \frac{1}{c}(\alpha'(0)I - \beta(0)\Delta)$, $a(t) = \frac{\beta(0)^{-1}}{c}\alpha'(t)$, $b(t) = \frac{1}{c}[\alpha''(t) - \beta^{-1}(0)\alpha'(0)\beta'(t)]$ and $f(t) = F(\cdot, t)$. Assume that $c > 0$ and $\alpha^{(j)}, \beta^{(j)} \in L^1(\mathbb{R}_+)$, $j = 0, 1, 2$, with

$$\begin{array}{ccc} \alpha(0) < 0, & \text{and} & \beta(0) > 0 \\ (-1)^j\alpha^{(j)}(t) \geq 0 & \text{and} & (-1)^j\beta^{(j)}(t) \geq 0, \quad t \in \mathbb{R}. \end{array}$$

Under these conditions, the operator A, with domain $D(A) = H^2(\Omega) \cap H^1(\Omega)$ is self-adjoint dissipative in $H = L^2(\Omega)$. Moreover, $\mathrm{Im}(\mu_k) \neq 0$ for all $k \in \mathbb{Z}$ (see [29, Theorem 2.1]). By Proposition 6.24, if $f \in L^p([0, 2\pi], L^2(\Omega))$, then the equation (6.23) has a unique L^p-strong solution u.

Example 6.27.

Now, we consider the following equation, which describes the one-dimensional longitudinal motions of a viscoelastic bar

$$u_{tt}(x,t) = \alpha(0)u_{xx}(x,t) + \int_{-\infty}^{t} \alpha'(t-s)u_{xx}(x,s)ds + f(x,t), \quad (x,t) \in (0,1) \times \mathbb{R}, \tag{6.25}$$

with boundary conditions $u(x, t) = 0$ for $x = 0, 1$ and $t \in \mathbb{R}$.

We assume that $\alpha(t) = \alpha_\infty + \alpha_1(t)$, where $\alpha_\infty > 0$ is a constant, $\alpha_1(0) > 0$, $\alpha^{(j)} \in L^1(\mathbb{R}_+)$, $j = 0, 1$, with $(-1)^j\alpha^{(j)}(t) \geq 0$. Under these assumptions, the equation (6.25) can be written as (see [29, Theorem 3.2])

$$u_{tt}(x,t) - c^2\Delta u(x,t) - \int_{-\infty}^{t} \alpha_1'(t-s)\Delta u(x,s)ds = f(x,t),$$

where $c^2 = \alpha_\infty + \alpha_1(0)$, and therefore $a(t) = c^{-2}\alpha_1'(t)$, $b(t) = \lambda = 0$, and $A = -c^2\Delta$ with A and $D(A)$ defined as in Example 6.26. Moreover $\mathrm{Im}(\mu_k) \neq 0$ for all k (see [29, Theorem 3.2]) and therefore by Proposition 6.24 the equation (6.25) has a unique L^p-strong solution u for each $f \in L^p([0, 2\pi], H)$.

Acknowledgements. The author thanks the reviewer for the detailed review and suggestions that have improved the previous version of the paper.

References

[1] H. Amann, *Operator-valued Fourier multipliers, vector-valued Besov spaces and applications,* Math. Nachr. **186** (1997), 5-56.

[2] W. Arendt, S. Bu, *The operator-valued Marcinkiewicz multiplier theorem and maximal regularity,* Math. Z. **240** (2002), 311-343.

[3] W. Arendt, S. Bu, *Operator-valued Fourier multiplier on periodic Besov spaces and applications,* Proc. Edin. Math. Soc. **47** (2) (2004), 15-33.

[4] J. Bourgain, *Some remarks on Banach spaces in which martingale differences sequences are unconditional,* Arkiv Math. **21** (1983), 163-168.

[5] S. Bu, *Mild well-posedness of vector-valued problems on the real line,* Arch. Math. **95** (2010), 63-73.

[6] S. Bu, *Mild well-posedness of equations with fractional derivative,* Math. Nachr., (2012) **285**, 202-209.

[7] S. Bu, G. Cai, *Well-posedness of second order degenerate integro-differential equations with infinite delay in vector-valued function spaces,* Math. Nachr., (2016) **289**, 436-451. doi:10.1002/mana.201400112.

[8] S. Bu, G. Cai, *Periodic solutions of third-order degenerate differential equations in vector-valued functional spaces,* Israel J. of Mathematics, (2016) **212**, 163-188.

[9] S. Bu, G. Cai, *Well-posedness of second-order degenerate differential equations with finite delay,* Proc. of the Edinburgh Math. Soc. (2017) 60(2), 349-360. doi:10.1017/S0013091516000262.

[10] S. Bu, G. Cai, *Well-posedness of second-order degenerate differential equations with finite delay in vector-valued function spaces,* Pacific J. of Math. **288** (2017), 27-46.

[11] S. Bu, G. Cai, *Periodic solutions of second order degenerate differential equations with finite delay in Banach spaces,* J. Fourier Anal. Appl. (2017), https://doi.org/10.1007/s00041-017-9560-8.

[12] S. Bu, G. Cai, *Periodic solutions of second order degenerate differential equations with delay in Banach spaces,* Canadian Math. Bull. (2017), http://dx.doi.org/10.4153/CMB-2017-057-6.

[13] D.L. Burkholder, *A geometrical condition that implies the existence of certain singular integrals on Banach-space-valued functions.* In: Proc. Conference on Harmonic Analysis in Honor of Antoni Zygmund, Chicago 1981, pp.270-286, Wadsworth, Belmont, CA, 1983.

[14] D. L. Burkholder, *Martingales and singular integrals in Banach spaces.* In: Handbook of the Geometry of Banach Spaces, Vol 1, pp.233-269, North-Holland, Amsterdam, (2001).

[15] A. Conejero, C. Lizama, M. Murillo, J. B. Seoane-Sepúlveda, *Well posedness for degenerate third order equations with delay and applications to inverse problems,* Israel J. Math. **229** (2019), no. 1, 219-254.

[16] R. Denk, M. Hieber, J. Prüss, *R-boundedness, Fourier multipliers and problems of elliptic and parabolic type,* Mem. Amer. Math. Soc. **166** (788), 2003.

[17] M. Girardi, L. Weis, *Criteria for R-boundedness of operator families,* Lecture Notes in Pure and Appl. Math., **234,** Dekker, New York, 2003, 203-221.

[18] M. Gurtin, A. Pipkin, *A general theory of heat conduction with finite wave speed,* Archs. ration. Mech. Analysis **31** (1968) 113-126.

[19] V. Keyantuo, C. Lizama, *Fourier multipliers and integro-differential equations in Banach spaces,* J. London Math. Soc. **69** (3) (2004), 737-750.

[20] V. Keyantuo, C. Lizama, *Periodic solutions of second order differential equations in Banach spaces,* Math. Z. **253** (2006), 489-514.

[21] V. Keyantuo, C. Lizama, V. Poblete, *Periodic solutions of integro-differential equations in vector-valued function spaces,* J. Differential Equations, **246** (3) (2009), 1007-1037.

[22] C. Lizama, V. Poblete, *Maximal regularity of delay equations in Banach spaces,* Studia Math. **175** (1) (2006), 91-102.

[23] C. Lizama, R. Ponce, *Periodic solutions of degenerate differential equations in vector-valued function spaces,* Studia Math. **202** (1) (2011), 49-63.

[24] C. Lizama, R. Ponce, *Maximal regularity for degenerate differential equations with infinite delay in periodic vector-valued function spaces,* Proc. Edin. Math. Soc. **56** (2013), **3**, 853-871.

[25] V. Poblete, *Solutions of second-order integro-differential equations on periodic Besov spaces,* Proc. Edin. Math. Soc. **56** (2007), **50**, 477-492.

[26] V. Poblete, R. Ponce, *Maximal L^p-regularity for fractional differential equations on the line,* Math. Narch. **290** (2017), No. 13, 2009-2023.

[27] R. Ponce, *On the well-posedness of degenerate fractional differential equations in vector valued function spaces,* Israel J. of Math. **219** (2017), 727-755.

[28] J. Prüss. *Evolutionary integral equations and applications*. Monographs Math., **87**, Birkhäuser Verlag, 1993.

[29] Q. Tiehu, *Periodic solutions of linear integro-differential equations in Hilbert spaces,* Funkcialaj Ekvacioj, **35** (1992), 31-40.

Universidad de Talca, Instituto de Matemáticas, Casilla 747, Talca-Chile.

Email address: rponce@utalca.cl, rponce@inst-mat.utalca.cl